

ON BANACH-MAZUR COMPACTA

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Abstract

We study Banach-Mazur compacta $Q(n)$, that is, the sets of all isometry classes of n -dimensional Banach spaces topologized by the Banach-Mazur metric. Our main result is that $Q(2)$ is homeomorphic to the compactification of a Hilbert cube manifold by a point, for we prove that $Q_{\epsilon}(2) = Q(2) \setminus \{\text{Eucl.}\}$ is a Hilbert cube manifold. As a corollary it follows that $Q(2)$ is not homogeneous.

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1. Introduction

This paper studies topological properties of Banach-Mazur compacta $Q(n)$, that is, the sets of all isometry classes of n -dimensional Banach spaces topologized by the Banach-Mazur metric. Recently, substantial progress was made concerning these spaces. It was proved in [14] that $Q(2)$ is an absolute extensor (defined below). Later this result was generalized to all $n \geq 2$ (see [5]). The long-standing problem about topological equivalence of $Q(n)$ and the Hilbert cube I^{∞} was finally solved negatively for $n = 2$ in [4].

THEOREM 1.1. *$Q(2)$ and I^{∞} are not homeomorphic.*

For any space X to be homeomorphic to the Hilbert cube I^{∞} , the following necessary conditions must be satisfied for every point $x \in X$:

- (a) $X \setminus \{x\}$ must be homotopically trivial; and
- (b) $X \setminus \{x\}$ must be a Hilbert cube manifold.

The key idea of the proof of Theorem 1.1 was to show that $Q(2)$ fails to possess the property (a) at the Euclidean point $\{\text{Eucl.}\}$, which corresponds to the isometry class of the Euclidean space. On the other hand, the main result of this paper, Theorem 1.2 stated below, implies that the complement $Q(2) \setminus \{x\}$ of every other point $x \in Q(2)$ turns out to be homotopically trivial. Furthermore, Theorem 1.2 demonstrates that as far as the property (b) is concerned, everything turns out to be exactly the opposite: $Q(2) \setminus \{\text{Eucl.}\}$ is a Hilbert cube manifold, while the complement $Q(2) \setminus \{x\}$ of every other point $x \in Q(2)$ is not.

THEOREM 1.2. $Q_{\mathcal{E}}(2) = Q(2) \setminus \{\text{Eucl.}\}$ is a Hilbert cube manifold.

As a corollary we prove that $Q(2)$ is not homogeneous (recall that a space X is said to be *homogeneous* if for every pair of points $x_1, x_2 \in X$ there exists a homeomorphism $h : X \rightarrow X$ such that $h(x_1) = x_2$).

COROLLARY 1.3. $Q(2)$ is not a homogeneous space.

PROOF OF COROLLARY 1.3. By [4], $\{\text{Eucl.}\}$ is not a Z -set in $Q(2)$. On the other hand, it follows by our Theorem 1.2 above that for every point $x \in Q(2) \setminus \{\text{Eucl.}\}$, $\{x\}$ is a Z -set in $Q(2) \setminus \{\text{Eucl.}\}$, hence also a Z -set in $Q(2)$. Therefore $(Q(2), \{\text{Eucl.}\}) \not\cong (Q(2), \{x\})$. \square

2. Preliminaries

We identify the set $\text{BAN}(n)$ of all n -dimensional Banach spaces with the set of all norms in \mathbb{R}^n . The *Banach-Mazur* distance $\rho(X, Y)$ between spaces $X = \{\mathbb{R}^n, \|\cdot\|_X\}$ and $Y = \{\mathbb{R}^n, \|\cdot\|_Y\} \in \text{BAN}(n)$ is defined as follows:

$$\rho(X, Y) = \inf \{ \|T\| \cdot \|T^{-1}\| \mid T : X \rightarrow Y \text{ is an isomorphism} \},$$

where $\|T\|, \|T^{-1}\|$ are norms of the operators T and T^{-1} , respectively. It is well-known that for every $X, Y, Z \in \text{BAN}(n)$, the following properties hold:

- (1) $\rho(X, Z) \leq \rho(X, Y)\rho(Y, Z)$;
- (2) $\rho(X, Y) = \rho(Y, X) \geq 1$; and
- (3) $\rho(X, Y) = 1$ if and only if X and Y are *isometric*, $X \cong Y$, that is, there exists an isomorphism $T : X \rightarrow Y$ which preserves the norm $\|x\|_X = \|T(x)\|_Y$ for every $x \in X$.

It follows that the function $\ln \rho(X, Y)$ is a pseudometric on the space $\text{BAN}(n)$, which in the decomposition space $Q(n) = \text{BAN}(n) / \cong$ becomes the metric $d([X], [Y]) = \ln \rho(X, Y)$, where

$$X \cong Y \iff \rho(X, Y) = 1 \iff \ln \rho(X, Y) = 0.$$

The resulting metric space $(Q(n), d)$ of all isometry classes of n -dimensional Banach spaces is called the *Banach-Mazur compactum*.

This compactum allows for a different, more suitable presentation as a decomposition of the space $C(n)$ of all compact convex symmetric (rel 0) bodies in \mathbb{R}^n . If one measures the distance between subsets of \mathbb{R}^n by the Hausdorff metric $\rho_H(A, B)$ and defines the linear combination $\sum_{i=0}^n \lambda_i A_i$ by means of the Minkowski operation, then $(C(n), \rho_H)$ becomes a locally compact convex space.

Moreover, $C(n)$ can be equipped with an action of the general linear group $GL(n) \times C(n) \rightarrow C(n)$, given by $T \cdot V = T(V)$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}^n \in GL(n)$ and $V \in C(n)$, which agrees with the convex structure on $C(n)$. We show that the orbit space $C(n)/GL(n)$ is naturally homeomorphic to the Banach-Mazur compactum.

Indeed, for an arbitrary body $V \in C(n)$, the Minkowski functional $p_V(x) = \inf\{t^{-1} \mid tx \in V\}$ defines a norm on \mathbb{R}^n and consequently, induces a continuous bijection $M : C(n) \rightarrow \text{BAN}(n)$ defined by $M(V) = (\mathbb{R}^n, p_V)$. Since it is well-known that Banach spaces $M(V)$ and $M(W)$ are isomorphic if and only if $V = T \cdot W$ for some $T \in GL(n)$, it follows that M induces a continuous bijection of the quotient spaces

$$\tilde{M} : C(n)/GL(n) \rightarrow Q(n) = \text{BAN}(n)/\cong,$$

which is a homeomorphism.

Hereafter, we shall consider only locally compact Lie groups (for example $GL(n)$), metric spaces and continuous maps, unless otherwise specified. An *action* of G on a space X is a homeomorphism $T : G \rightarrow \text{Aut } X$ of the group G into the group $\text{Aut } X$ of all autohomeomorphisms of X such that the map $G \times X \rightarrow X$, given by $(g, x) \mapsto T(g)(x) = gx$, is continuous. A space X with a fixed action of G is called a *G-space*.

For any point $x \in X$, the *isotropy subgroup* of x , or the *stabilizer* of x , is defined as $G_x = \{g \in G \mid gx = x\}$, and the *orbit* of x as $G(x) = \{gx \mid g \in G\}$. The space of all orbits is denoted by X/G and the natural map $\pi : X \rightarrow X/G$, given by $\pi(x) = G(x)$, is called the *orbit projection*. The orbit space X/G is equipped with the quotient topology, induced by π .

Actions of noncompact groups G do not agree very well with the orbit structure of X : the orbit of a point x can be dense in X , the orbit space X/G can be non-Hausdorff, two orbits with the same stabilizer can be nonhomeomorphic, etc. Palais [22] singled out a class of G -spaces with the action of a locally compact group which do not have such deficiencies—he called such spaces *proper*.

DEFINITION 2.1. (a) Given subsets $A, B \subset X$ consider the following subset of the group G :

$$((A, B)) = \{g \in G \mid gA \cap B \neq \emptyset\}.$$

Then A is said to be *thin* with respect to B , if $((A, B))$ is precompact, that is, it lies in a compact subset of G . Since $((A, B)) = ((B, A))^{-1}$, it follows that B is also thin with respect to A .

(b) $A \subset X$ is said to be *small* if for every point $x \in X$, there exists a neighbourhood $O(x) \subset X$ of x , which is thin with respect to A .

(c) A G -space X is said to be *proper* if it possesses a basis, consisting of small neighbourhoods.

In general, the orbit projection $\pi : X \rightarrow X/G$ of a proper G -space X fails to be a closed map. This forces us to seek those closed subsets $F \subset X$ of X for which the restriction $\pi|_F : F \rightarrow X/G$ is closed.

DEFINITION 2.2. A closed subset $F \subset Z$ of a G -space Z is said to be *fundamental* if F is small in Z and intersects every orbit, that is, $F \cap G(z) \neq \emptyset$ for every $z \in Z$.

PROPOSITION 2.3. *Suppose that a G -space Z is proper and that the orbit space Z/G is metrizable. Then*

(d) *there exists a fundamental subset $F \subset Z$; and*

(e) *for every fundamental subset $F \subset Z$, the restriction $\pi|_F : F \rightarrow Z/G$ is a proper map.*

DEFINITION 2.4. An *exact slice* at the point $x \in X$ is a G -map $\varphi : U \rightarrow G(x)$ of some G -neighbourhood $U \subset X$ (that is, $G \cdot U = U$) of the orbit $G(x)$, such that $\varphi(x) = x$. The preimage $\varphi^{-1}(x)$ of the point x is also called a *slice* or a G_x -*kernel*.

The principal results concerning slices belong to Abels [1] and Palais [22].

THEOREM 2.5 (Palais). *A proper completely regular G -space X has a slice at every point x .*

THEOREM 2.6 (Abels). *Let X be a proper G -space with a paracompact orbit space and K a maximal compact subgroup of G . Then there exists a G -map $f : X \rightarrow G/K$ (a so-called global K -slice). Conversely, if there exists a global K -slice, then X is a proper G -space.*

In the sequel, we shall work in the class \mathcal{G} of all metric proper G -spaces, whose orbit space is also metric. The following properties of the class \mathcal{G} are well known (see [22]).

PROPOSITION 2.7. *Let $X \in \mathcal{G}$ and let Y be a separable metric G -space. Then the following properties hold:*

- (f) *The orbit $G(x)$ is closed in X , the stabilizer G_x is compact and the natural map $G/G_x \rightarrow G(x)$, given by $g \cdot G_x \mapsto gx$, is a homeomorphism.*
- (g) *X can be equipped with an invariant metric, that is, $d(gx, gx') = d(x, x')$, for every $g \in G$ and $X \times Y \in \mathcal{G}$.*
- (h) *$G/L \in \mathcal{G}$, for every compact subgroup $L < G$.*

Next, we introduce several notions connected with the property of absolute extendability of maps. A space X is called an *absolute neighbourhood extensor*, $X \in ANE$, if every map $\varphi : A \rightarrow X$, defined on a closed subset $A \subset Z$ of a metric space Z , and called a *partial map*, can be extended over some neighbourhood $U \subset Z$ of A , $\hat{\varphi} : U \rightarrow X$, $\hat{\varphi}|_A = \varphi$. If we can always take $U = Z$, then X is called an *absolute extensor*, $X \in AE$. We note that in the case when X is a metric space, the concepts of the absolute (neighbourhood) retract and the absolute (neighbourhood) extensor coincide.

If $X \in A[N]E$, Z is a G -space from the class \mathcal{G} and φ is a G -map (which in this case means that φ is constant along every orbit), then the extension $\hat{\varphi}$ can also be chosen to be a G -map. This follows from the closedness of A/G in Z/G (which, in turn follows by the closedness of A in Z). In connection with this example we introduce some more general concepts.

DEFINITION 2.8. A G -space X is called an *equivariant absolute neighbourhood extensor*, $X \in G-ANE$, if every partial G -map $Z \leftarrow A \xrightarrow{\varphi} X$, where Z is a G -space from the class \mathcal{G} , can be extended to a G -map $\hat{\varphi} : U \rightarrow X$, defined on some G -neighbourhood $U \subset Z$ of A . If we can always take $U = Z$ then X is called an *equivariant absolute extensor*, $X \in G-AE$.

DEFINITION 2.9. A G -space X is called an *approximate $G-A[N]E$ -space*, $X \in G-AA[N]E$, if for every G -space Z from the class \mathcal{G} , every fundamental subset F of Z , and every covering $\omega \in \text{cov}(X)$, the following holds: For every partial G -map $Z \leftarrow A \xrightarrow{\varphi} X$ there is an ‘approximate’ G -extension $\tilde{\varphi} : Z \rightarrow X$ [respectively $\tilde{\varphi} : U \rightarrow X$, where $U \subset Z$ is a G -neighbourhood of A] such that the restrictions $\varphi|_{A \cap F}$ and $\tilde{\varphi}|_{A \cap F}$ are ω -close, that is, $(\varphi|_{A \cap F}, \tilde{\varphi}|_{A \cap F}) < \omega$ (see [13]).

3. Equivariant extensors for locally compact Lie groups

For our purposes, the most important example of a proper $GL(n)$ -space is the space $C(n)$ of all convex bodies.

PROPOSITION 3.1. For every n , $C(n)$ is a proper $GL(n)$ -space.

PROOF. It suffices to prove that the following closed set

$$C(r, R) = \{V \in C(n) \mid B^n(r) \subset V \subset B^n(R)\},$$

where $B^n(a)$ denotes the closed ball with center at 0 and of radius a , is thin for every $0 < r < R < \infty$, that is, that the set

$$\mathcal{A} = ((C(r, R), C(r, R))) = \{g \in GL(n) \mid gC(r, R) \cap C(r, R) \neq \emptyset\}$$

is precompact.

Suppose not. Then for some sequence $g_n = \|g_{ij}^n\| \in \mathcal{A}$ and some indices (i_0, j_0) , one of following cases occur

(α) $g_{i_0 j_0}^n \rightarrow \infty$; or

(β) $\det \|g_{ij}^n\| \rightarrow 0$.

Suppose that $g_n V_n \in C(r, R)$ for some $V_n \in C(r, R)$. Since the point A , for which only the j_0 -th coordinate is equal to r , while all others are 0, lies in V_n , it follows that $g_n A \in B^n(R)$. But the i_0 -th coordinate of $g_n A$ is equal to $g_{i_0 j_0}^n r$ and $g_{i_0 j_0}^n$ does not converge to ∞ . On the other hand,

$$0 < \text{vol } B^n(r) \leq \text{vol}(g_n V_n)$$

and

$$\text{vol}(g_n V_n) = \det \|g_{ij}^n\| \text{vol } V_n \leq \det \|g_{ij}^n\| \text{vol } B^n(R).$$

Therefore, $\det \|g_{ij}^n\|$ does not converge to 0. □

The orthogonal group $O(n)$ is a maximal compact subgroup of $GL(n)$. By Theorem 2.6 there exists a global $O(n)$ -slice $f : C(n) \rightarrow GL(n)/O(n)$.

PROPOSITION 3.2. *Let X be a proper G -ANE-space. Then*

(γ) *For every G -neighbourhood U of the orbit $G(x)$, there exist a G -neighbourhood V and a G -map $H : V \times [0, 1] \rightarrow U$ such that $H_0 = \text{Id}$, $\text{Im}(H_1) \subset G(x)$, and $H_t|_{G(x)} = \text{Id}$ for all $t \in I$.*

PROOF. Consider in the proper G -space $X \times [0, 1]$ the partial G -map

$$X \times [0, 1] \leftarrow X \times \{0\} \cup G(x) \times [0, 1] \cup U_1 \times \{1\} \xrightarrow{\varphi} X$$

such that $\varphi|_{X \times \{0\}} = \text{Id}$, $\varphi|_{G(x) \times [0, 1]} = \text{Id}$, and $\varphi|_{U_1 \times \{1\}}$ is the existing retraction (provided by Theorem 2.6) $r : U_1 \rightarrow G(x)$ of some G -neighbourhood $U_1 \subset U$.

Let $\hat{\varphi} : W \rightarrow X$ be any extension of φ onto the G -neighbourhood W , which contains a G -neighbourhood of the type $V \times I \supset G(x)$. We get the desired map H by restricting $\hat{\varphi}$ onto $V \times I$. □

The following theorem of Abels [2, 4.4] allows us to reduce the studying of non-compact group actions to compact ones.

THEOREM 3.3. *For every $X \in \mathcal{G}$, $X \in G\text{-}A[N]E$ if and only if $X \in L\text{-}A[N]E$ for every compact subgroup $L < G$.*

THEOREM 3.4. *For every n , $C(n)$ is a $GL(n)\text{-}AE$ space.*

By Theorem 3.3, $C(n) \in GL(n)\text{-}AE$ if and only if $C(n) \in L\text{-}AE$, for every compact subgroup $L < GL(n)$. Another theorem of Abels [2, 4.2] asserts that every locally convex complete topological vector G -space is $G\text{-}AE$, for every compact group G . Let us apply the argument from this paper to prove that $C(n) \in L\text{-}AE$.

Since $C(n)$ is convex (with respect to the Minkowski linear combination of convex bodies), Dugunji’s theorem implies that $C(n) \in AE$. Therefore every partial L -map $Z \leftarrow A \xrightarrow{f} C(n)$ can be continuously extended over Z , $F : Z \rightarrow C(n)$. Now define

$$\hat{F}(z) = \int_L g^{-1} \cdot F(gz) \partial\mu,$$

where $\partial\mu$ is the normalized Haar measure and \int_L means the integral of the set-valued mapping [9]:

$$\Phi_z : L \rightsquigarrow \mathbb{R}^n, \Phi_z(g) = g^{-1} \cdot F(gz) \subset \mathbb{R}^n.$$

On account of the continuous dependence $\Phi_z(g)$ on z and g , the convexity and the closeness of its images, \hat{F} is a continuous map with closed convex values [9]. It is easy to see that \hat{F} is an L -map from Z into $C(n)$ and that $\hat{F} \upharpoonright A = f$. □

Let (X, d) be a metric G -space of diameter 1 from \mathcal{G} . Then we can introduce a metric on the cone $\text{Con } X = X \times [0, 1]/X \times \{0\}$ as follows:

$$\rho((x, t), (x', t')) = \sqrt{t^2 + (t')^2 - 2tt' \cos \gamma}, \quad \text{where } \cos \gamma = \frac{2 - d^2(x, x')}{2}.$$

It is easy to see that $(\text{Con } X, \rho)$ is a metric G -space (the group G acts along X) and the natural embedding $X \mapsto X \times \{1\} \hookrightarrow \text{Con } X$ is an isometry, while $\text{Con } X$ is not a proper space.

PROPOSITION 3.5. *If a metric G -space X is a $G\text{-}ANE$ space, then $\text{Con } X$ is a $G\text{-}AE$ space.*

PROOF. Suppose that a proper G -space $Z \in \mathcal{G}$ and a partial G -map $Z \leftarrow A \xrightarrow{f} \text{Con } X$ are given. Let $A_0 = \varphi^{-1}(*) \subset A$, where $(*)$ is the vertex of $\text{Con } X$. Then

for every $a \in A \setminus A_0$, $\varphi(a)$ can be represented in the form $(\varphi_1(a), \varphi_2(a))$, where $\varphi_1 : A \setminus A_0 \rightarrow X$ is a continuous G -map and $\varphi_2 : A \rightarrow [0, 1]$ is a continuous function, constant on the orbits and such that $\varphi_2(A \setminus A_0) \subset (0, 1]$ and $\varphi_2(A_0) = 0$.

Since $X \in G\text{-ANE}$, the map $\varphi_1(a)$ can be extended to a G -map $\psi : U \rightarrow X$, defined on an open subset U of Z/G , $Z \setminus A_0 \supset U \supset A \setminus A_0$. Since the orbit space Z/G is metrizable, there exists a continuous function $\xi : Z \rightarrow [0, 1]$, constant on orbits, such that $\xi|_A = \varphi_2$ and $\xi|_{Z \setminus U} = 0$ by the Urysohn theorem. The desired extension $\hat{\varphi} : Z \rightarrow \text{Con } X$ of the G -map φ is then defined by the formula:

$$\hat{\varphi} = \begin{cases} (\psi(z), \xi(z)) & z \in U; \\ (*) & z \notin U. \end{cases} \quad \square$$

PROPOSITION 3.6. *Let H be a compact subgroup of the locally compact Lie group G . Then G/H is a $G\text{-ANE-space}$.*

PROOF. Every compact subgroup $H < G$ smoothly acts on the differentiable manifold G/H . By [21, 1.6.6], $G \in H\text{-ANE}$. By Theorem 3.3, $G \in G\text{-ANE}$. □

It is convenient to reduce the studying of the equivariant extensors to the corresponding easier problem for approximate equivariant extensors. For example, if some class \mathcal{B} of G -spaces is invariant under the product on the semiopen segment $J = [0, 1)$, then \mathcal{B} is contained in the class $G\text{-A}[N]E$ if and only if \mathcal{B} is contained in the class of the approximate $G\text{-A}[N]E$.

THEOREM 3.7. *Suppose that the product $X \times J$ of a metric G -space X and $J = [0, 1)$ is a $G\text{-AANE-space}$. Then X is a $G\text{-ANE-space}$.*

For the trivial group G this is a well-known fact, which follows from [12] and [18].

PROOF OF THEOREM 3.7. First, we consider any (not necessarily locally finite) covering $\omega \in \text{cov}(X \times J)$ adjoining to the subset $X \times \{1\}$ of $X \times [0, 1]$. The latter means by definition that:

(δ) For every neighbourhood $U(x, 1)$ of the point $(x, 1) \in X \times \{1\}$ in $X \times [0, 1]$, there exists a smaller neighbourhood $V(x, 1)$ such that $W \subset U(x, 1)$, for every $W \in \omega$ such that $W \cap V(x, 1) \neq \emptyset$.

Let F be a fundamental set of Z (see Proposition 2.3). Then $F \times J$ is a fundamental set of $Z \times J$. After these preliminaries, we begin the extending of the partial G -map $Z \leftarrow A \xrightarrow{\varphi} X$. Recall that $X \times J \in G\text{-AA}[N]E$ and construct for the other partial G -map

$$Z \times J \leftarrow A \times J \xrightarrow{\psi = \varphi \times \text{Id}_J} X \times J$$

a G -map $\tilde{\psi} : Z \times J \rightarrow X \times J$ [respectively $\tilde{\psi} : U \rightarrow X \times J$] such that

$$\left(\varphi|_{(A \cap F) \times J}, \tilde{\psi}|_{(A \cap F \times J)}\right) \prec \omega.$$

We give all details of the proof only for the case when $X \times J \in G\text{-AAE}$. The case when $X \times J \in G\text{-AANE}$ is dealt with similarly. Extending $\tilde{\psi}$ over $A \times \{1\}$ by the formula $\tilde{\psi}(a, 1) = (\varphi(a), 1)$, we obtain a G -map (which we denote by the same letter) $\tilde{\psi} : Z \times J \cup A \times [0, 1] \rightarrow X \times J$, the restrictions of which onto the closed G -set $A \times [0, 1]$ and the open G -set $Z \times J$ are continuous. Now we apply the following lemma.

LEMMA 3.8. *Suppose that a G -map $f : H \cup E \rightarrow Y$ is defined on the union $H \cup E$ of a closed G -space $H \in \mathcal{G}$ and open G -subset E of a proper G -space $T \in \mathcal{G}$, such that $f|_H$ and $f|_E$ are continuous. Then there exists a closed G -subspace $K \subset T$ such that $H \subset K \subset H \cup E$, $H \cap E \subset \text{Int}(K)$ and $f|_K$ is a continuous G -map.*

Apply Lemma 3.8 for $T = Z \times [0, 1]$, $H = A \times [0, 1]$, $E = Z \times J$ and $f = \tilde{\psi}$. We get a closed G -subset L of $Z \times [0, 1]$ such that $A \times [0, 1] \subset L$, $A \times [0, 1) \subset \text{Int } L$ and $\tilde{\psi}|_L$ is a continuous G -map.

Next, we construct a decreasing sequence $L = U_1 \supset \text{Cl } U_2 \supset \dots$ of open G -neighbourhoods of the set A and a monotone sequence of numbers $0 = t_1 < t_2 < \dots$, such that $\lim_{i \rightarrow \infty} t_i = 1$ and $U_k \times [0, t_k] \subset L$.

Let $\xi : Z \rightarrow [0, 1]$ be a continuous real-valued function, constant on the orbits and such that $\xi(U_1 \setminus U_2) = 0$, $\xi(U_i \setminus U_{i+1}) \subset [t_{i-1}, t_i]$ for every $i \geq 2$, and $\xi(A) = 1$. Clearly, the graph $\text{GR} = \{(z, \xi(z)) \mid z \in Z\}$ of f lies in L and the restriction of $\tilde{\psi}$ onto GR is a continuous G -map. The desired extension is now given by the formula:

$$\hat{\varphi}(z) \text{ is the projection of } \tilde{\psi}(z, \xi(z)) \in X \times [0, 1] \text{ onto } X.$$

Using $\text{diam } W_n \rightarrow 0$, $W_n \in \omega$, whenever $\text{dist}((x, 1), W_n) \rightarrow 0$ for some point $(x, 1) \in X \times \{1\}$, it is easy to check the continuity of $\hat{\varphi}$. □

4. Orbit spaces of equivariant absolute extensors

This section is dedicated to a proof of the following result.

THEOREM 4.1. *Let G be a locally compact Lie group and X a proper $G\text{-A}[N]E$ from \mathcal{G} . Then the orbit space X/G is an absolute [neighbourhood] extensor.*

Since $C(n)$ is a proper $\text{GL}(n)$ -space from \mathcal{G} which is an equivariant absolute extensor, we obtain as an immediate corollary of Theorem 4.1 that for every closed

subgroup $H < GL(n)$, the orbit space $C(n)/H$ belongs to the class of absolute extensors.

We begin with the following embedding theorem.

PROPOSITION 4.2. *Let $X \in \mathcal{G}$. Then there exist a countable number of finite-dimensional G -ANE-spaces R_{nm} ($n, m \in \mathbb{Z}^+$), from the class \mathcal{G} , and a topological G -embedding $i : X \hookrightarrow \prod_{n,m}^{\infty} \text{Con } R_{n,m}$.*

Let X be equipped by the invariant metric (see Proposition 2.7 (g)). For every point x and every $\varepsilon > 0$, we fix a G -map $\varphi_{x\varepsilon} : X \rightarrow \text{Con}(G(x))$ satisfying the properties of the following proposition.

PROPOSITION 4.3. *Let $X \in \mathcal{G}$. Then for every point $x \in X$ and every $\varepsilon > 0$, there exists a G -map $\varphi : X \rightarrow \text{Con}(G(x))$ with $\varphi(x) = x$, such that*

(5) $\text{diam } \varphi^{-1}((V \cdot x) \times (0, 1]) < \varepsilon$, for some neighbourhood V of the stabilizer G_x in G .

PROOF. Let $r : U(x) \rightarrow G(x)$ be a G -retraction. We may assume that not only does the G_x -kernel $r^{-1}(x)$ have diameter less than ε , but also $\text{diam}(V \cdot r^{-1}(x)) < \varepsilon$, for some neighbourhood V of the compact stabilizer G_x . This is possible by Theorem 2.5 and the following lemma.

LEMMA 4.4. *For every neighbourhood $O(x) \subset X$, there exists a smaller neighbourhood $O_1(x)$ such that*

- (6) $G_x \cap \text{cl}\{g \mid g O_1(x) \setminus O(x) \neq \emptyset\} = \emptyset$; and
- (7) $G \cdot O_1(x) \cap r^{-1}(x) \subset O(x)$.

The desired G -map of X is then given by the formula:

$$\varphi(x') = \begin{cases} (r(x'), \xi(x')) & x' \in U(x); \\ (*) & x' \notin U(x). \end{cases}$$

Here, the function $\xi : X \rightarrow [0, 1]$ is constant on orbits, $\xi(x)=1$ and $\xi(X \setminus U(x))=0$. \square

Since by hypothesis X/G is metrizable, there exists a σ -disjoint basis $\mathcal{B} = \{W_\mu\}_{\mu \in M}$ of open subsets, such that $\mathcal{B} = \bigsqcup \mathcal{B}_n$, where $\mathcal{B}_n = \{W_\mu\}_{\mu \in M_n \subset M}$ is a disjoint family and $\bigsqcup_{n=1}^{\infty} M_n = M$.

DEFINITION 4.5. A pair $\nu = (\mu_1, \mu_2) \in M \times M$ of indices is said to be *canonical*, if

- (8) $W_{\mu_1} \subseteq W_{\mu_2}$ (that is, $\overline{W_{\mu_1}} \subset W_{\mu_2}$); and
- there exist $x \in X$ and $\varepsilon > 0$ such that:

(9) $x \in \pi^{-1} W_{\mu_1} \subset V_{x\varepsilon}$ and $U_{x\varepsilon} \subset \pi^{-1} W_{\mu_1}$, where

$$V_{x\varepsilon} = \varphi_{x\varepsilon}^{-1}(G_x \times (1/2, 1]) \quad \text{and} \quad U_{x\varepsilon} = \varphi_{x\varepsilon}^{-1}(G_x \times (0, 1]),$$

and $\pi : X \rightarrow X/G$ is the orbit projection.

We denote the set of all canonical pairs by $K \subset M \times M$.

PROPOSITION 4.6. *There exists a correspondence $\nu \in K \mapsto (x_\nu, \varepsilon_\nu) \in X \times \mathbb{R}^+$ such that (x_ν, ε_ν) satisfies (9) and*

(10) *For every closed subset $F \subset X$ and $x \notin F$ there exists a canonical pair $\nu \in K$ with $\varphi_{x_\nu \varepsilon_\nu}(x) \notin \varphi_{x_\nu \varepsilon_\nu}(F)$ (that is, $\varphi_{x_\nu \varepsilon_\nu}$ separates the point x from the closed subset F).*

PROOF. Let

$$i(\nu) = \inf\{\varepsilon > 0 \mid (x, \varepsilon) \text{ satisfies (9) for some point } x \in X\}.$$

It is evident that $i(\nu) > 0$. Therefore, every $\nu \in K$ yields a pair (x_ν, ε_ν) possessing (9) and such that

(11) $\varepsilon_\nu < 2i(\nu)$.

Let $4a = \rho(x, F)$. Since \mathcal{B} is a basis, there exist $\nu = (\mu_1, \mu_2) \in K$ and $\varepsilon < a$ such that

$$x \in \pi^{-1} W_{\mu_1} \subset V_{x\varepsilon} \subset U_{x\varepsilon} \subset \pi^{-1} W_{\mu_1}.$$

It follows from (11) that $\varepsilon_\nu < 2a$.

Let us prove that ν is a desired pair. Suppose that a neighbourhood V of G_{x_ν} satisfies the hypotheses of Proposition 4.3:

$$\text{diam } \varphi_{x_\nu \varepsilon_\nu}^{-1}((Vx_\nu) \times (0, 1]) < \varepsilon_\nu < 2a.$$

Since $x \in V_{x_\nu \varepsilon_\nu}$, it follows that $\varphi_{x_\nu \varepsilon_\nu}(x) = (gx_\nu, t)$, $t > 1/2$.

Pick a neighbourhood $W = gVg^{-1}$ of $e \in G$. Then

$$\varphi_{x_\nu \varepsilon_\nu}(x) \in (W \cdot gx_\mu) \times (1/2, 1]$$

and

$$\begin{aligned} A &= \varphi_{x_\nu \varepsilon_\nu}^{-1}(W \cdot gx_\mu \times (1/2, 1]) \\ &\subset \varphi_{x_\nu \varepsilon_\nu}^{-1}(g \cdot V \cdot x_\mu \times (1/2, 1]) = g \cdot \varphi_{x_\nu \varepsilon_\nu}^{-1}(V \cdot x_\mu \times (1/2, 1]). \end{aligned}$$

By the invariance of the metric, the latter set has diameter smaller than $2a$, hence the diameter of the open neighbourhood A of x is also less than $2a$. As a consequence, it follows that $A \cap F = \emptyset$ and $\varphi_{x_\nu \varepsilon_\nu}(x) \notin \varphi_{x_\nu \varepsilon_\nu}(F)$. □

PROOF OF PROPOSITION 4.2. Let us introduce a countable family of spaces:

$$R_{nm} = \coprod \{ G(x_\nu) \mid \nu = (\mu_1, \mu_2) \in K, \mu_1 \in \mathcal{B}_n, \mu_2 \in \mathcal{B}_m \}.$$

Since $G(x_\nu) \in G\text{-ANE}$, R_{nm} is also a $G\text{-ANE}$. Since \mathcal{B}_m is a disjoint family and

$$\varphi_{x_\nu, \varepsilon_\nu}|_{(X \setminus \pi^{-1}W_{\mu_2})} = (*) \in \text{Con}(G(x_\nu)),$$

we obtain that

$$\psi_{nm} : X \rightarrow \text{Con } R_{nm}, \quad \psi_{nm}|_{\pi^{-1}W_{\mu_2}} = \varphi_{x_\nu, \varepsilon_\nu}, \quad \psi_{nm}|_{X \setminus \cup \pi^{-1}W_{\mu_2}} = (*)$$

is a well-defined G -map. Since $\{\psi_{nm}\}$ separates points from closed subsets, the diagonal product

$$\Delta\psi_{nm} : X \rightarrow \prod_{n,m} \text{Con } R_{nm}$$

is a topological G -embedding. □

PROPOSITION 4.7. Suppose that a G -space H is the limit of the inverse spectrum $\{H_1 \xleftarrow{q_1} H_2 \xleftarrow{q_2} H_3 \leftarrow \dots\}$ of G -spaces H_i and G -maps q_i , and that

(12) The stabilizer G_h of any point $h \in H_i \setminus H_i^G$ is compact.

Then the orbit spaces H/G and $\varprojlim \{H_1/G \xleftarrow{\tilde{q}_1} H_2/G \xleftarrow{\tilde{q}_2} H_3/G \leftarrow \dots\}$ are homeomorphic.

PROOF. The homeomorphism $\varphi : H/G \rightarrow \varprojlim \{H_i/G, \tilde{q}_i\}$ is given by the formula:

$$\varphi([h]) = ([h_1], [h_2], \dots), \text{ where } h = (h_i) \in H.$$

It is easy to verify that φ is continuous and surjective. We shall thus only verify that φ is injective. Assume that $[h] \neq [e]$, where $h = (h_i), e = (e_i) \in H$ and let us show that then $\varphi([h]) \neq \varphi([e])$. It suffices to prove the following lemma. □

LEMMA 4.8. There exists an integer i such that $e_i \notin G(h_i)$.

PROOF. If $e, h \in H^G$, then $e_i \neq h_i = G \cdot h_i$, for some i . So we may assume that $h \notin H^G$, that is, $G_h = \bigcap G_{h_i} \neq G$. By (12) and inclusion $G_{h_{i+1}} \subset G_{h_i}$, almost all G_{h_i} 's differ from G and almost all G_{h_i} are compact.

Suppose to the contrary, that $e_i = g_i h_i, g_i \in G$ for every i . It is easy to show that then

$$e_k = g_k h_k = g_{k+1} h_k = \dots = g_i h_k$$

for every $k \leq l$. Therefore, $g_l \in g_k \cdot G_{h_k}$, for every $k \leq l$.

Since the stabilizer G_{h_m} is compact for some m , it follows that the sequence $\{g_l\}_{l \geq m} \subset g_m \cdot G_{h_m}$ converges to $g_0 \in g_m \cdot G_{h_m}$. Analogously, one can show that $g_0 \in g_p \cdot G_{h_p}$, for all $p \geq m$. Consequently, $g_0 h_p = g_p h_p = e_p$, for all $p \geq m$, that is, $e = g_0 h$. Contradiction. □

PROOF OF THEOREM 4.1. Using the hypotheses, let us fix a topological G -embedding (Proposition 4.2):

$$i : X \hookrightarrow \prod_{n,m} \text{Con } R_{nm} = D$$

and a closed topological embedding $j : X/G \hookrightarrow L$ of the orbit space X/G into a linear normed space L . It is obvious that

$$i \times (j \circ \pi) = e : X \hookrightarrow L \times D$$

is a closed topological G -embedding. Since the image $e(X)$ does not contain points with a noncompact stabilizer, $e(X)$ does not intersect the closed set $L \times \{*\}$, where $\{*\}$ is the product of the vertices of the cone-factors of D . Therefore, $e(X)$ lies in the proper open G -space $U' = L \times (D \setminus \{*\})$.

Since $L \times D \in G\text{-AE}$, it follows that $U' \in G\text{-ANE}$. Since $X \in G\text{-ANE}$, there exists a G -retraction $r : U \rightarrow X$ of some G -neighbourhood U , $e(X) \subset U \subset U'$. Hence, $\tilde{r} : U/G \rightarrow X/G$ is a retraction and the inclusion $X/G \in \text{ANE}$ is reduced to another inclusion $U/G \in \text{ANE}$.

If we now prove that $D/G \in \text{AE}$, then $(L \times D)/G = L \times (D/G) \in \text{AE}$, and therefore, $U/G \in \text{ANE}$ as an open subset of the orbit space. To complete the proof of the theorem, it thus remains to verify that $D/G \in \text{AE}$.

Let us introduce the following notations: $D_p = \prod_{n+m \leq p} \text{Con } R_{nm}$ and $q_r : D_{r+1} \rightarrow D_r$ is a projection. Since R_{nm} is metrized by a complete invariant metric, it follows that $\text{Con } R_{nm}$ and D_r are also metrized by a complete invariant metric. Thus, the orbit space D_r/G is also metrized by a complete metric. It follows from $D_r \in G\text{-AE}$ and Proposition 3.2 that $D_r/G \in \text{LC} \cap \text{C}$. Due to its countable-dimensionality and the Haver theorem [15] we obtain that $D_r/G \in \text{AE}$.

Since $\text{Con } R_{r+1} \in \text{AE}$, the projection q_m is a fiberwise G -contractible map, that is, there exist fiberwise G -maps $s : D_r \rightarrow D_{r+1}$, $q_r \circ s = \text{Id}$ and $H : D_{r+1} \times [0, 1] \rightarrow D_{r+1}$, $q_r \circ H = q_r$, such that $H_0 = \text{Id}$ and $\text{Im}(H_1) = \text{Im}(s)$. Passing to the orbit spaces we obtain fiberwise contractible maps $\tilde{q}_r : D_{r+1}/G \rightarrow D_r/G$, that is, \tilde{q}_r is a fine homotopy equivalence. Since all the conditions of Curtis's theorem [11] are satisfied, we conclude that $\varprojlim \{D_i/G, q_i\}$ is an AE . But by Proposition 4.7 this inverse limit coincides with the orbit space D/G . □

5. Proof of Theorem 1.2

By Theorem 2.6 and Proposition 3.1, there exists a $GL(n)$ -retraction $r : C(n) \rightarrow GL(n)/O(n) = \mathfrak{E}$, which is nevertheless unacceptable for us because of its nonconstructibility. Another geometric $GL(n)$ -retraction, generated by the Löwner ellipsoid, will be more convenient.

THEOREM 5.1 (see [17]). *For every convex body $V \in C(n)$, there exists a unique ellipsoid $E_V \in C(n)$, which contains V and has the minimal Euclidean volume.*

The $GL(n)$ -invariance of E_V (that is, $E_{AV} = AE_V$ for all $A \in GL(N)$) then follows by minimality of the volume. A continuous dependence E_V on V with respect to the Hausdorff metric was proved in [5]. Therefore, $\mathcal{L} : C(n) \rightarrow \mathfrak{E}$, $\mathcal{L}(V) = E_V$, is a $GL(n)$ -retraction of $C(n)$ onto the *ellipsoid orbit* \mathfrak{E} (\mathcal{L} is called the *Löwner retraction*).

Since the symmetry group Sym_{B^n} of B^n is $O(n)$, the $O(n)$ -slice $L(n) = \mathcal{L}^{-1}(B^n)$ is an $O(n)$ -space. In other words, $L(n)$ consists of all bodies $V \in C(n)$ whose minimal Löwner ellipsoid coincides with B^n . The orbit space $Q(n) = C(n)/GL(n)$ is homeomorphic to $L(n)/O(n)$. Therefore, by Theorem 4.1,

$$L(n)/O(n) = Q(n) \in AE \quad \text{and} \quad Q_\varepsilon = Q(n) \setminus \{\text{Eucl.}\} = L_\varepsilon(n)/O(n) \in ANE,$$

where $L_\varepsilon = L(n) \setminus \{B^n\}$, and so Theorem 1.2 is reduced to the following:

THEOREM 5.2. *$Q_\varepsilon(2) = L_\varepsilon(2)/O(2)$ is a Hilbert cube manifold.*

We prove Theorem 5.2 in three main steps which are carefully outlined below.

Step 1. Reduction of Theorem 5.2 to Proposition 5.3 and the Toruńczyk characterization for Q -manifolds

PROPOSITION 5.3. *For every integer $n \geq 2$ and every $\delta > 0$, there exist $O(n)$ -maps $f_i : L_\varepsilon(n) \rightarrow L_\varepsilon(n)$, $i \in \{1, 2\}$, such that*

- (1) *f_i and $\text{Id}_{L_\varepsilon(n)}$ are δ -close; and*
- (2) *if $n = 2$ then $\text{Im } f_1 \cap \text{Im } f_2 = \emptyset$.*

PROOF OF THEOREM 5.2. According to the Toruńczyk characterization criterion [19], in order to prove Theorem 5.2, it suffices to check that for every $\varepsilon > 0$ and for all pairs of maps $\varphi_i : I^\infty \rightarrow Q_\varepsilon(n)$, $i \in \{1, 2\}$, there are continuous maps $g_i : I^\infty \rightarrow Q_\varepsilon(n)$, ε -close to φ_i , $i \in \{1, 2\}$, such that if $n = 2$ then $\text{Im } g_1 \cap \text{Im } g_2 = \emptyset$.

Since $F = \cup \text{Im } \varphi_i$ and $F_1 = \pi^{-1}(F)$ are compact (here $\pi : L_{\mathcal{E}}(n) \rightarrow L_{\mathcal{E}}(n)/O(n)$ is the orbit projection), there exists $\delta > 0$ such that $\text{dist}(\pi(a), \pi(b)) < \varepsilon$, for every $a, b \in F_1$, with $\text{dist}(a, b) < \delta$.

By Proposition 5.3 for every $n \geq 2$, there are $O(n)$ -maps $f_i : L_{\mathcal{E}}(n) \rightarrow L_{\mathcal{E}}(n)$, $i \in \{1, 2\}$, satisfying (1) for $\delta > 0$ and (2) for $n = 2$. The induced maps \tilde{f}_i of the orbit spaces, $i \in \{1, 2\}$, have the following properties for $n = 2$:

$$\rho(\tilde{f}_i|_F, \text{Id}_F) < \varepsilon \quad \text{and} \quad \cap \text{Im } \tilde{f}_i = \emptyset.$$

Finally, the desired maps $g_i : I^\infty \rightarrow Q_{\mathcal{E}}(2)$, $i \in \{1, 2\}$, are defined by the formula $g_i = \tilde{f}_i \circ \varphi_i$. □

Step 2. Construction of f_1

Let us consider so-called *contact map* $\alpha : L(n) \rightarrow \exp(S^{n-1})$, defined by $\alpha(V) = V \cap S^{n-1}$. The following lemma, whose routine verification is omitted, records several basic properties of α .

- LEMMA 5.4. (3) α preserves the action of $O(n)$, $\alpha(A \cdot V) = A \cdot \alpha(V)$, for every $A \in O(n)$;
 (4) $\alpha(V) \neq \emptyset$, for every $V \in L(n)$;
 (5) $\alpha(V)$ is a central symmetric subset of S^{n-1} ; and
 (6) $\alpha(V) = S^{n-1}$ if and only if $V = B^n$.

LEMMA 5.5. (7) Let $V \subseteq W \subseteq B^n$, where $V \in L(n)$ and $W \in C(n)$. Then $W \in L(n)$.

- (8) For every subset $A \subseteq B^n$, $\alpha(\text{Conv}(A)) = \text{Conv}(A) \cap S^{n-1} = A \cap S^{n-1}$.

PROOF. (7) The minimal Löwner ellipsoid for W and V coincides with B^n . Hence $W \in L(n)$.

In order to prove (8), it suffices to observe that every point $s \in \text{Conv}(A) \cap S^{n-1}$ is an extreme point of B^n and therefore is also an extreme point of $\text{Conv}(A) \subseteq B^n$. But all extreme points of $\text{Conv}(A)$ are contained in A . Therefore $s \in A$. □

Unfortunately, the contact map α is discontinuous. The following reasoning compensates for this unpleasant moment. Let us denote by $\widehat{x0y}$ the nonoriented angle between the rays $[0x)$ and $[0y)$, where $x, y \in B^n$ and $x, y \neq 0$. Next, we introduce a version of the closed ε -neighbourhood of a set, which will be convenient for us. Let $\varepsilon > 0$ and $V \in L(n)$. By V_ε we denote

$$V \cup \{x \in B^n \setminus \{0\} \mid \text{there exists } y \in V \text{ with } \|x\| = \|y\| \text{ and } \widehat{x0y} \leq \varepsilon\}.$$

It is clear that V_ε preserves the action of $O(n) : (g \cdot V)_\varepsilon = g \cdot V_\varepsilon$, for every $g \in O(n)$, $V \in L_\varepsilon(n)$. The compactness of V implies that V_ε is compact; the inequality $\|x - y\| < \widehat{x0y}$, for every $\|x\| = \|y\|$, implies that

$$(9) \quad V \subseteq V_\varepsilon \subseteq \overline{N}(V; \varepsilon), \text{ where } \overline{N}(V; \varepsilon) \text{ is a closed } \varepsilon\text{-neighbourhood of } V \text{ in } B^n.$$

We need V_ε to be continuously dependent on V and ε .

PROPOSITION 5.6. *Let $\varepsilon_k \rightarrow \varepsilon > 0$ and $V_k \in L(n) \rightarrow V$. Then $(V_k)_{\varepsilon_k} \rightarrow V_\varepsilon$.*

PROOF. Let $R_k = (V_k)_{\varepsilon_k}$ and $R = V_\varepsilon$. Suppose that the assertion of the proposition is false, that is, that $R_k \not\rightarrow R$. Then there exist $\alpha > 0$ and a sequence $k_i \rightarrow \infty$ such that

- (10) $x_0 \notin N(R_{k_i}; \alpha)$, for some $x_0 \in R$; or
- (11) there exists $x_i \in R_{k_i}$, $i \geq 1$, with $x_i \notin N(R; \alpha)$.

In the first case, $\widehat{x_0 0 y_0} \leq \varepsilon$, for some $y_0 \in V$, with $\|y_0\| = \|x_0\|$. Since $V_k \rightarrow V$, there exists a sequence $y_k \in V_k \rightarrow y_0$. It is easy to see that there exists a sequence $x_k \in B^n \rightarrow x_0$, $\widehat{x_k 0 y_k} \leq \varepsilon_k$, $\|x_k\| = \|y_k\|$. It means that $x_k \in (V_k)_{\varepsilon_k} = R_k$, for every k and the limit point x_0 of $\{x_k\}$ belongs to $N(R_{k_i}; \alpha)$, for some k_i . This contradicts (10).

In the second case, there exists a sequence $\{y_i \in V_{k_i}\}$ such that $\|y_i\| = \|x_i\|$ and $\widehat{y_i 0 x_i} \leq \varepsilon_{k_i}$. By compactness of B^n , we can suppose that there exist the limits $y_i \rightarrow y \in V$ and $x_i \rightarrow x \in B^n$. Then $\|y\| = \|x\|$ and $\widehat{x 0 y} \leq \varepsilon$. Therefore, $x \in V_\varepsilon = R$. This contradicts the fact that $x_i \notin N(R; \alpha)$. □

Consider the following set-valued map:

$$F : L_\varepsilon(n) \rightsquigarrow \mathbb{R}^+, \quad F(V) \stackrel{\text{def}}{=} \{t > 0 \mid B^n \setminus N(V; t) \neq \emptyset\},$$

where $N(V; t)$ is the open t -neighbourhood of V in B^n .

Since $N(V; t)$ is a continuous set-valued map from $L_\varepsilon(n) \times \mathbb{R}^+$ into B^n (in the Hausdorff metric) and $B^n \setminus V \neq \emptyset$, the map F is lower semicontinuous and has domain $L_\varepsilon(n)$. Let us consider the function $f : \text{Graph}(F) \rightarrow \mathbb{R}^+$ given by $f(V, t) = t$ and defined on the graph F . Then the function $g : L_\varepsilon(n) \rightarrow \mathbb{R}^+$, defined by

$$g(V) = \sup\{t > 0 \mid B^n \setminus N(V; t) \neq \emptyset\} = \sup\{f(V, t) \mid (V, t) \in \text{Graph}(F)\}$$

is well defined and lower semi-continuous [9, page 48] (in set-valued analysis g is called a *marginal function* [24]).

By the Dowker theorem [13], there exists a continuous function $\gamma : L_\varepsilon(n) \rightarrow \mathbb{R}^+$ with $\gamma(V) < \delta \cdot g(V)$, $V \in L_\varepsilon(n)$. By Proposition 5.6, it is clear that $V_{\gamma(V)}$ continuously depends on $V \in L_\varepsilon(n)$. The desired continuous $O(n)$ -map $f_1 : L_\varepsilon(n) \rightarrow L_\varepsilon(n)$ is defined by setting $f_1(V) = \text{Conv}(V_{\gamma(V)})$. By (9), f_1 and $\text{Id}_{L_\varepsilon(n)}$ are δ -close.

Let $\text{dist}(v, w)$ be the spherical distance between $v, w \in S^{n-1}$ and $\overline{N}_{\text{sph}}(A; R)$ be the closed R -neighbourhood of the subset $A \subset S^{n-1}$ with respect to the spherical distance. By Lemma 5.5 (8),

$$\alpha \circ f_1(V) = \text{Conv}(V_{\gamma(V)}) \cap S^{n-1} = V_{\gamma(V)} \cap S^{n-1} = \overline{N}_{\text{sph}}(V; \gamma(V)).$$

The last equality means the boundary of $f_1(V)$ to contain an open (nonempty) subset S^{n-1} , for every $V \in L_{\mathcal{G}}(n)$. The mapping f_2 will be constructed without such property and therefore $\text{Im } f_1 \cap \text{Im } f_2 = \emptyset$.

Step 3. Construction of f_2

THEOREM 5.7. *For every $\sigma > 0$, there exists an $O(n)$ -mapping $F : L_{\mathcal{G}}(n) \rightarrow C(n)$ such that*

(12) $\rho(F, \text{Id}_{L_{\mathcal{G}}(n)}) < \sigma$; and

(13) *for every $V \in L_{\mathcal{G}}(n)$, $F(V) = \text{Conv}(\sum_{i=1}^m \lambda_i D_i)$, where D_i is an H_i -orbit, H_i is a proper subgroup of $O(n)$ and $\sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0$.*

In connection with this theorem we formulate a geometric conjecture, which is trivially true in dimension 2. If Conjecture 5.8 is valid then our proof of Theorem 1.2 immediately generalizes to arbitrary $n \geq 2$.

CONJECTURE 5.8. *The body $\sum_{i=1}^m \lambda_i D_i$ (hence also $\text{Conv}(\sum_{i=1}^m \lambda_i D_i)$) in the formula (13) ‘essentially differs’ from the ball, that is, its boundary does not contain open subsets of the sphere.*

PROOF. By the Palais theorem (Theorem 2.5) any orbit $O(n)V, V \in L_{\mathcal{G}}(n)$, allows an $O(n)$ -retraction $r'_V : \mathcal{U}_V \rightarrow O(n)V, r'_V(V) = V$. Here we can assume that:

(14) $\rho_H(W, r'_V(W)) < \sigma/2$, for all $W \in \mathcal{U}_V$.

LEMMA 5.9. *For every $\theta > 0$ there exists a finite set $K \subset \text{Bd } V$ such that:*

- (i) $W = \text{Conv}(\text{St}_V K)$ and V have equal stabilizers; and
- (ii) $\rho_H(V, W) < \theta$.

PROOF. It follows from the existence of slices that for some numbers $\theta > \theta_1 > 0$ from $\rho_H(V, V') < \theta$ and $\text{St}_{V'} \supseteq \text{St}_V$, it always follows that $\text{St}_{V'} = \text{St}_V$. Consider a discrete subset $K \subset \text{Bd } V$ such that $\rho_H(V, \text{Conv } K) < \theta_1$. Then

$$V \supseteq \text{Conv}(\text{St}_V K) = W \supseteq \text{Conv } K$$

and therefore $\rho_H(V, W) < \theta_1$. Next, it follows from $\text{St}_W = \text{St}_{\text{Conv}(\text{St}_V K)} = \text{St}_{\text{St}_V K} \supseteq \text{St}_V$ and $\rho(V, W) < \theta_1$ that $\text{St}_W = \text{St}_V$. □

For every $V \in L_{\mathcal{L}}(n)$, fix $V' = \text{Conv}(HK_V) \in C(n)$ such that $H = \text{St}_V, K_V \subset \text{Bd } V, |K_V| < \infty$ and $\rho_H(V, V') < \sigma/2$. Let us introduce the composition

$$r = h_V \circ r' : \mathcal{U}_V \rightarrow O(n)V \rightarrow O(n)V',$$

where $h_V(gV) = gV'$ is an $O(n)$ -homeomorphism.

If we get V' sufficiently close to V then we obtain the following:

(15) $\text{dist}(W, r_V W) < \sigma$, for every $W \in \mathcal{U}_V$.

We inscribe a locally finite cover $\{T_\mu\}$ into the open cover $\{\mathcal{U}_V/O(n)\}$ of the orbit space $L_{\mathcal{L}}(n)/O(n) = Q_{\mathcal{L}}(n)$. Let $T_\mu \subset \pi(U_{V_\mu})$.

We now define the desired $O(n)$ -map $F : L_{\mathcal{L}}(n) \rightarrow C(n)$ as follows:

$$F(W) = \sum_{\mu} \gamma_{\mu}(\pi W) \cdot r_{V_{\mu}}(W), \quad W \in L_{\mathcal{L}}(n),$$

where $\{\gamma_{\mu}(\cdot)\}$ is a continuous partition of unity, subordinate to the cover $\{T_{\mu}\}$.

We verify the conditions (12) and (13) of Theorem 5.7. Let $T_1, \dots, T_m \in \{T_{\mu}\}$ be all the elements which contain πW and let $T_i \subset \pi(U_{V_i})$. It follows by (4) that $\rho_H(W, r_{V_i}(W)) < \sigma$, for all i . Then by convexity of the ball of radius σ at $C(n)$ we have that $\text{dist}(W, FW) < \sigma$. Thus (12) has been verified.

Condition (13) follows, since $H \text{ Conv } K$ is a union of a finite number of H -orbits for every proper subgroup $H < O(n)$ and finite K . □

It is well known [2] that there exists a $O(n)$ -retraction $R : C(n) \rightarrow L(n)$ which takes $C_{\mathcal{L}}(n)$ into $L_{\mathcal{L}}(n)$. But we need the following precise result which follows from geometric considerations:

THEOREM 5.10. *There exists a continuous $O(n)$ -retraction $R : C(n) \rightarrow L(n)$, such that V and $R(V)$ are affinely equivalent, for every $V \in C(n)$.*

PROOF. Let $L(V)$ be the Löwner ellipsoid, circumscribed around $V, g \in \text{GL}(n), g(L(V)) = B^n$. As is well known, g can be represented as $g = g_2 \circ g_1$, where $g_2 \in O(n)$ and g_1 is self-adjoint. Here $R(V) = g_1(V)$. □

Since $L(n)$ is compact, for every $\delta > 0$ there exists $\sigma > 0, \sigma < \delta/2$, such that for every $V \in L(n)$ and every $W \in C(n)$,

$$\rho_H(V, W) < \sigma \Rightarrow \rho_H(W, R(W)) < \delta/2.$$

By Theorem 5.7 there is a mapping $F : L_{\mathcal{L}}(n) \rightarrow C(n)$ such that $\rho(F, \text{Id}_{L_{\mathcal{L}}(n)}) < \sigma$. The desired map f_2 is $R \circ F$.

Indeed,

$$\begin{aligned} \rho_H(V, f_2 V) &= \rho_H(V, R \circ F(V)) \\ &< \rho_H(V, F(V)) + \rho_H(F(V), R(FV)) < \sigma + \delta/2 < \delta. \end{aligned}$$

Since for $n = 2$, the boundary $F(V)$, $V \in L_{\mathcal{E}}(n)$, does not contain an open subset of a sphere, $f_2(V)$ which is affinely equivalent $F(V)$, also does not contain any open subsets of the sphere. Therefore, $\text{Im } f_1 \cap \text{Im } f_2 = \emptyset$. \square

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