Ergod. Th. & Dynam. Sys. (1988), **8**, 331–349
Printed in Great Britain

A method for constructing attractors

MARCY BARGE

Department of Mathematical Sciences, Montana State University, Bozeman, Montana 59717, *USA*

(*Received* 15 *July* 1986; *revised* 10 *July* 1987)

Abstract. A procedure is developed for constructing C^1 diffeomorphisms of the two sphere having inverse limits of certain interval maps as attractors. The method is carried out for a particular interval map yielding a diffeomorphism with a transitive non-hyperbolic attractor.

1.

Let $f: X \to X$ be a continuous map of the compact, connected metric space X into itself. We will let (X, f) denote the inverse limit space

$$(X,f) = \{(x_0, x_1, \ldots) \mid x_n \quad \text{in } X, f(x_{n+1}) = x_n \quad \text{for } n = 0, 1, 2, \ldots\}$$

with metric

$$d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n},$$

where by $|x-y|$ we mean the distance between x and y in X. Then (X, f) is a compact, connected metric space and f induces a homeomorphism $\hat{f}: (X, f) \to (X, f)$ by

$$\hat{f}((x_0, x_1, \ldots)) = (f(x_0), x_0, x_1, \ldots).$$

In **[B–M]** the authors showed that given any continuous map $f: I \to I$ of the compact interval I, there is an embedding $i: (I, f) \to R^2$ and a homeomorphism $F: R^2 \to R^2$ such that: $F(i(I, f)) = i(I, f)$; $F \circ i = i \circ \hat{f}$; and given $z \in R^2$ there is a $y \in i(I, f)$ such that $|F^n(z) - F^n(y)| \to 0$ as $n \to \infty$. That is, $\hat{f}$ on (I, f) can be realized as the restriction of a homeomorphism of the plane to its attractor.

Here we will show that for certain maps f of the interval I, the above F can be made a C^1 diffeomorphism. The general construction will be developed in § 2. In § 3 a particular nontrivial example is worked out. In this example a C^1 diffeomorphism with a nonhyperbolic, transitive, and fairly exotic attractor is constructed. The construction can, in fact, be parametrized to demonstrate that the diffeomorphism referred to is the limit of structurally stable horseshoes.

More specifically, the example constructed is a C^1 diffeomorphism $F: S^2 \to S^2$ of the two-sphere S^2 for which there is a ball $B \subseteq S^2$ with $F(B) \subseteq B$. The attracting set $\Lambda = \bigcap_{n \geq 0} F^n(B)$ is in the interior of B, Λ is homeomorphic with the indecomposable Knaster continuum K_2 (see **[Bi]** and **[B]**), and $F|_\Lambda$ has a dense orbit. Moreover,

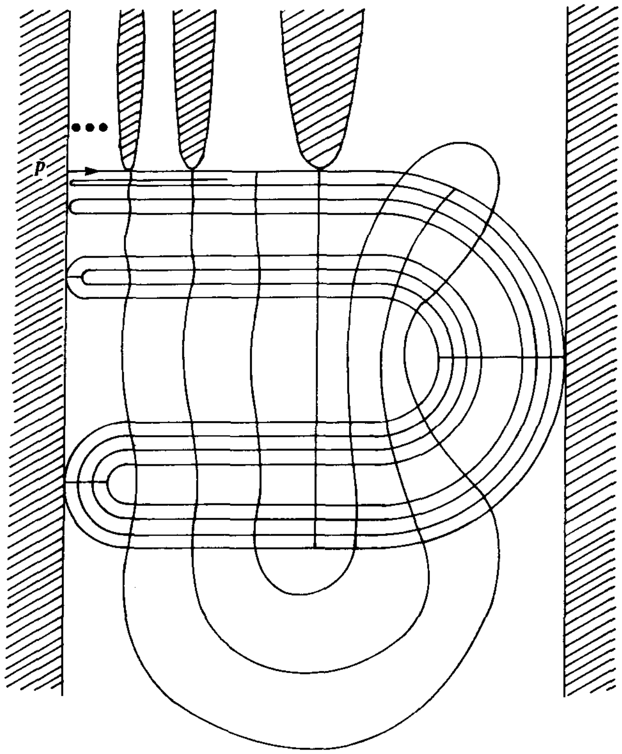

FIGURE 1. The curve emanating horizontally from p is the unstable manifold of p and its closure is the attractor Λ. The shaded regions and the bold curves sticking out of them are part of the stable set of p. The stable and unstable sets of all other points in Λ are one-dimensional manifolds.

each point of B is "in phase" with some point of Λ. That is, given $z \in B$ there is a $w \in \Lambda$ such that $|F^n(z) - F^n(w)| \to 0$ as $n \to \infty$, (see figure 1).

Misiurewicz [M] has, by a different technique, embedded our example inverse limit as an attractor for a C^∞ diffeomorphism in $\mathbf{R}^3$ and as an attractor for a homeomorphism of the plane.

2.

The construction will be carried out on the two-sphere S^2. Let B be a closed ball in S^2 and let $f: I \to I$ be a continuous map of the compact interval I. We consider maps P, G, G_1, $G_2, \ldots$ having the following properties:

(2.1) $P: B \to I$ is a continuous surjection;

(2.2) $G: S^2 \to S^2$ is a C^1 surjection, $G(B) \subseteq B$, and G is a C^1 diffeomorphism from $S^2 - B$ onto $S^2 - G(B)$;

(2.3) $P \circ G = f \circ P$;

(2.4) Given any $x \in I$ and $y, z \in P^{-1}(x)$, $|G^n(y) - G^n(z)| \to 0$ monotonically as $n \to \infty$;

(2.5) $G_n: S^2 \to S^2$ is a C^1 diffeomorphism for each $n = 1, 2, \ldots$ and there is a sequence of open sets $U_n \subseteq S^2$ such that $U_{n+1} \subseteq U_n$, $G_n = G$ off of U_n, and $G(U_1) \subseteq B$;

(2.6) $\mathit{diameter}(G^n(U_n)) \to 0$ as $n \to \infty$;

(2.7) $G^n(U_n)\cap G^k(U_k)=\varnothing$ for $n\neq k$, n, $k\geq 1$, and $G^n(U_n)\cap U_1=\varnothing$ for $n=1, 2,\ldots$;

(2.8) $\sup_{z\in G^n(U_n)}\|D(G^{n-1}\circ G_{n+1}\circ G_n^{-1}\circ G^{-(n-1)})(z)-id\|\to 0$ as $n\to\infty$, where D is the derivative and id is the identity matrix.

Assuming (2.1)-(2.8) we construct a diffeomorphism of S^2 with attractor (I,f).

Let (S^2, G) be the inverse limit space with bonding map G and define $H:(S^2, G)\to S^2$ by

$$H((z_0, z_1,\ldots))=\lim_{n\to\infty} G_1\circ\cdots\circ G_n(z_n).$$

LEMMA 2.9. *H is a homeomorphism of* (S^2, G) *onto* S^2.

Proof. Let $\underline{z}=(z_0, z_1,\ldots)\in(S^2, G)$. Suppose that $z_0\notin\bigcup_{n\geq 1} G^n(U_n)$. Then $z_n\notin U_n$ and $G_1\circ\cdots\circ G_n(z_n)=z_0$ for each $n=1, 2,\ldots$. Thus, $H(\underline{z})=z_0$.

If $z_0\in G^n(U_n)$ for some n then, by (2.7), $z_{n+k}\notin U_{n+k}$ for $k=1, 2,\ldots$. Thus $G_1\circ\cdots\circ G_n\circ G_{n+1}\circ\cdots\circ G_{n+k}(z_{n+k})=G_1\circ\cdots\circ G_n(G^k(z_{n+k}))=G_1\circ\cdots\circ G_n(z_n)$ and $H(\underline{z})=G_1\circ\cdots\circ G_n(z_n)$. We have that

$$H((z_0, z_1,\ldots))=\begin{cases} z_0, & \text{if } z_0\notin\bigcup_{n\geq 1} G^n(U_n)\\ G_1\circ\cdots\circ G_n(z_n), & \text{if } z_0\in G^n(U_n),\quad n=1,2,\ldots,\end{cases}$$

and we see that H is well defined.

Since $G_n=G$ on ∂U_n, H is continuous on

$$\left\{\underline{z}\in(S^2, G)\,|\, z_0\in\left(S^2-cl\left(\bigcup_{n\geq 1} G^n(U_n)\right)\right)\cup cl\left(\bigcup_{n=1}^{N} G^n(U_n)\right)\right\}$$

for each $N=1, 2,\ldots$. Suppose that $\underline{z}=\lim_{i\to\infty}\underline{y}^i$ where $\underline{y}^i=(y_0^i, y_1^i,\ldots)$ is such that $y_0^i\in G^{n_i}(U_{n_i})$, $n_i\to\infty$ as $i\to\infty$. Then $z_0\notin\bigcup_{n\geq 1} G^n(U_n)$ (by (2.7)) so that $H(\underline{z})=z_0$. Let

$$H(\underline{y}^i)=G_1\circ\cdots\circ G_{n_i}(y_{n_i}^i)=w^i.$$

Then $w^i\in G^{n_i}(U_{n_i})$ so that $|w^i-y_0^i|\to\infty$ as $i\to\infty$ by (2.6). Thus $H(\underline{y}^i)\to H(\underline{z})$ and H is continuous on all of (S^2, G).

Since G is one-to-one off of U_1, the G_n are one-to-one, $G^n(U_n)\cap G^k(U_k)=\varnothing$ for $n\neq k$, and $G^n(U_n)\cap U_1=\varnothing$ for $n\geq 1$, we see that H is one-to-one. Also H is a surjection since G is a surjection. Finally, (S^2, G) is compact so that H is a homeomorphism.

Now consider the homeomorphism $F:S^2\to S^2$ given by

$$F=H\circ\hat{G}\circ H^{-1}$$

where $\hat{G}$ is the homeomorphism of (S^2, G) induced by G,

$$\hat{G}((z_0, z_1,\cdots))=(G(z_0), z_0, z_1,\ldots).$$

LEMMA 2.10.

$$F(z)=\begin{cases} G(z), & z\notin U_1\cup\left(\bigcup_{n\geq 1} G^n(U_n)\right)\\ G_1(z), & z\in U_1\\ G^n\circ G_{n+1}\circ G_n^{-1}\circ G^{-(n-1)}(z), & z\in G^n(U_n),\quad n=1,2,\ldots.\end{cases}$$

Proof. First note that if $z \notin U_n$ then $G(z) \notin G(U_n)$. For otherwise, $G_n(z) \in G_n(U_n)$. But then G_n is not one-to-one. Suppose that $z \notin U_1 \cup (\bigcup_{n\geq 1} G^n(U_n))$. Then $G(z) \notin \bigcup_{n\geq 1} G^n(U_n)$ and

$$\begin{aligned} H \circ \hat{G} \circ H^{-1}(z) &= H \circ \hat{G}((z, G^{-1}(z), G^{-2}(z), \ldots)) \\ &= H((G(z), z, G^{-1}(z), \ldots)) \\ &= G(z). \end{aligned}$$

If $z \in U_1$, then $G(z) \in G(U_1)$ so that

$$\begin{aligned} H \circ \hat{G} \circ H^{-1}(z) &= H \circ \hat{G}((z, G^{-1}(z), G^{-2}(z), \ldots)) \\ &= H((G(z), z, G^{-1}(z), \ldots)) \\ &= G_1(z). \end{aligned}$$

In case $z \in G^n(U_n)$, then $G^{n+1} \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z) \in G^{n+1}(U_n)$. If $G^{n+1} \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z) \in G^{n+1}(U_{n+1})$ then

$$\begin{aligned} H \circ \hat{G} \circ H^{-1}(z) &= H \circ \hat{G}((G^n \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z), G^{n-1} \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z), \ldots \\ &\quad \ldots, G_n^{-1} \circ \cdots \circ G_1^{-1}(z), G^{-1} \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z), \ldots) \\ &= H((G^{n+1} \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z), G^n \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z), \ldots \\ &\quad \ldots, G \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z), G_n^{-1} \circ \cdots \circ G_1^{-1}(z), \ldots) \\ &= G_1 \circ \cdots \circ G_{n+1} \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z) \\ &= G^n \circ G_{n+1} \circ G_n^{-1} \circ G^{-(n-1)}(z). \end{aligned}$$

On the other hand, if $G^{n+1} \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z) \in G^{n+1}(U_n - U_{n+1})$, then $G^{n+1} \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z) \notin \bigcup_{k\geq 1} G^k(U_k)$ so that

$$\begin{aligned} H \circ \hat{G} \circ H^{-1}(z) &= H((G^{n+1} \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z), G^n \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z), \ldots)) \\ &= G^{n+1} \circ G_n^{-1} \circ \cdots \circ G_1^{-1}(z) \\ &= G^n \circ G_{n+1} \circ G_n^{-1} \circ G^{-(n-1)}(z) \end{aligned}$$

since $G_n^{-1} \circ \cdots \circ G_1^{-1}(z) = G_n^{-1} \circ G^{-(n-1)}(z) \notin U_{n+1}$.

THEOREM 2.11. *$F: S^2 \to S^2$ is a C^1 diffeomorphism.*

Proof. In view of Lemma 2.10, (2.2) and (2.5), it suffices to show that if $z_i \in G^{n_i}(U_{n_i})$, $n_i \to \infty$, and $z_i \to z$, then $DF(z_i) \to DF(z)$. Suppose that z_i is such a sequence converging to z. Then by (2.7), $z \in S^2 - (U_1 \cup (\bigcup_{n\geq 1} G^n(U_n)))$ so that $DF(z) = DG(z)$. On the other hand,

$$\begin{aligned} DF(z_i) &= D(G^{n_i} \circ G_{n_i+1} \circ G_{n_i}^{-1} \circ G^{-(n_i-1)})(z_i) \\ &= (DG)(G^{n_i-1} \circ G_{n_i+1} \circ G_{n_i}^{-1} \circ G^{-(n_i-1)}(z_i)) \\ &\quad \circ D(G^{n_i-1} \circ G_{n_i+1} G_{n_i}^{-1} \circ G^{-(n_i-1)})(z_i). \end{aligned}$$

Now, $G^{n_i-1} \circ G_{n_i+1} \circ G_{n_i}^{-1} \circ G^{-(n_i-1)}(z_i) \in G^{n_i}(U_{n_i})$ so that $G^{n_i-1} \circ G_{n_i+1} \circ G_{n_i}^{-1} \circ G^{-(n_i-1)}(z_i) \to z$ as $i \to \infty$ by (2.6). Thus $(DG)(G^{n_i-1} \circ G_{n_i+1} \circ G_{n_i}^{-1} \circ G^{-(n_i-1)}(z_i)) \to DG(z)$ since G is C^1. Finally, assumption (2.8) insures that $D(G^{n_i-1} \circ G_{n_i+1} \circ G_{n_i}^{-1} \circ G^{-(n_i-1)})(z_i) \to id$ as $i \to \infty$ so that $DF(z_i) \to DF(z)$ as $i \to \infty$. F is a C^1 homeomorphism with DF nonsingular everywhere. Thus F is a C^1 diffeomorphism.

Now let $\Lambda=\bigcap_{n\geq 0} F^n(B)$. Then

$$H^{-1}(\Lambda)=\{(z_0, z_1, \ldots)\in(S^2, G)\,|\, z_n\in B \qquad \text{for } n=0, 1, 2, \ldots\}.$$

Let $\hat{P}: H^{-1}(\Lambda)\to(I,f)$ be given by

$$\hat{P}((z_0, z_1, \ldots))=(P(z_0), P(z_1), \ldots).$$

LEMMA 2.12. *$\hat{P}$ is a homeomorphism from $H^{-1}(\Lambda)$ onto (I,f) and $\hat{P}\circ\hat{G}=\hat{f}\circ\hat{P}$.*

Proof. That $\hat{P}$ is well defined and $\hat{P}\circ\hat{G}=\hat{f}\circ\hat{P}$ follow from (2.3). $\hat{P}$ is continuous since P is continuous. Let $\underline{z}=(z_0, z_1, \ldots)$ and $\underline{w}=(w_0, w_1, \ldots)$ be in $H^{-1}(\Lambda)$ and suppose that $\hat{P}(\underline{z})=\hat{P}(\underline{w})$. We will show that $\underline{z}=\underline{w}$. $\hat{P}(\underline{z})=\hat{P}(\underline{w})$ means that $P(z_n)=P(w_n)$ for $n=0, 1, 2, \ldots$. Let n_i be an increasing sequence of positive integers and let $z, w\in B$ be such that $z_{n_i}\to z$ and $w_{n_i}\to w$ as $i\to\infty$. Since $P(z_{n_i})=P(w_{n_i})$ we must have $P(z)=P(w)$. Let $\varepsilon>0$ and k a nonnegative integer be given. By (2.4) there is an N large enough so that $|G^N(z)-G^N(w)|<\varepsilon/3$. Since $z_{n_i}\to z$ and $w_{n_i}\to w$, we also have that $z_{n_i-N}\to G^N(z)$ and $w_{n_i-N}\to G^N(w)$. Let M be large enough so that $|z_{n_i-N}-G^N(z)|<\varepsilon/3$ and $|w_{n_i-N}-G^N(w)|<\varepsilon/3$ for all $i\geq M$. Now let $i\geq M$ be large enough so that $n_i-N\geq k$. Then, again using (2.4),

$$\begin{aligned}|w_k-z_k|&\leq|w_{n_i-N}-z_{n_i-N}|\\ &\leq|w_{n_i-N}-G^N(w)|+|G^N(w)-G^N(z)|+|z_{n_i-N}-G^N(z)|\\ &<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ and k were arbitrary, $w_k=z_k$ for all k so that $\underline{w}=\underline{z}$ and $\hat{P}$ is one-to-one.

To see that $\hat{P}$ maps $H^{-1}(\Lambda)$ onto (I,f), let $\underline{x}=(x_0, x_1, \ldots)\in(I,f)$. Then $P^{-1}(x_n)$ is a compact subset of B for each $n\geq 0$ and $G(P^{-1}(x_{n+1}))\subseteq P^{-1}(x_n)$ for each $n\geq 0$ by (2.3). It follows that $\bigcap_{k\geq 0} G^k(P^{-1}(x_{n+k}))$ is nonempty for each $n\geq 0$ and

$$G\Big(\bigcap_{k\geq 0} G^k(P^{-1}(x_{n+1+k}))\Big)=\bigcap_{k\geq 0} G^k(P^{-1}(x_{n+k})).$$

Thus, there is a $\underline{z}=(z_0, z_1, \ldots)\in(B, G)$ such that

$$z_n\in\bigcap_{k\geq 0} G^k(P^{-1}(x_{n+k}))\subseteq P^{-1}(x_n) \text{ for each } n\geq 0.$$

Then $\underline{z}\in H^{-1}(\Lambda)$ and $\hat{P}(\underline{z})=\underline{x}$. Finally, $H^{-1}(\Lambda)$ is compact so that $\hat{P}$ is a homeomorphism.

LEMMA 2.13. *Given $z\in B$ there is a $y\in\Lambda$ such that $|F^n(z)-F^n(y)|\to 0$ as $n\to\infty$.*

Proof. Let $\underline{z}=(z_0, z_1, \ldots)=H^{-1}(z)$. Then $z_0\in B$. Let $\underline{x}=(x_0, x_1, \ldots)\in(I,f)$ be such that $P(z_0)=x_0$. Then $\underline{y}=(y_0, y_1, \ldots)=\hat{P}^{-1}(\underline{x})$ is in $H^{-1}(\Lambda)$ and $P(\pi_0(\underline{z}))=P(\pi_0(\underline{y}))$ where $\pi_k:(S^2, G)\to S^2$ is given by $\pi_k((z_0, z_1, \ldots, z_k, \ldots))=z_k$. But then

$$P(\pi_k(\hat{G}^n(\underline{z})))=P(\pi_k(\hat{G}^n(\underline{y})))$$

for all k, $0\leq k\leq n$. It follows from (2.4) that, for fixed k,

$$|\pi_k\hat{G}^n(\underline{z})-\pi_k\hat{G}^n(\underline{y})|\to 0 \text{ as } n\to\infty.$$

This means that $d(\hat{G}^n(\underline{z}), \hat{G}^n(\underline{y}))\to 0$ as $n\to\infty$. Now let $y=H(\underline{y})$. Then $y\in\Lambda$, $F^n(z)=H\circ\hat{G}^n\circ H^{-1}(z)=H\circ\hat{G}^n(\underline{z})$ and $F^n(y)=H\circ\hat{G}^n\circ H^{-1}(y)=H\circ\hat{G}^n(\underline{y})$. Thus, $|F^n(z)-F^n(y)|\to 0$ as $n\to\infty$. □

https://doi.org/10.1017/S0143385700004491 Published online by Cambridge University Press

Assuming (2.1)–(2.8) we have obtained the following:

COROLLARY 2.14. *There is a C^1 diffeomorphism $F: S^2 \to S^2$ with invariant attracting set $\Lambda \subseteq B \subseteq S^2$ such that $F|_\Lambda$ is topologically conjugate to $\hat{f}: (I, f) \to (I, f)$. Moreover, given $z \in B$ there is a $y \in \Lambda$ such that $|f^n(\check{z}) - F^n(y)| \to 0$ as $n \to \infty$.*

3.

We explicitly construct the maps G and G_n on a ball in the two-sphere S^2.

Let $B \subseteq S^2$ have coordinates x and y with

$$B = \{(x, y) \mid -\tfrac{1}{4} \le x \le \tfrac{5}{4}, -\tfrac{1}{2} \le y \le \tfrac{1}{2}\}.$$

The map G on B will have the form $G(x, y) = (f(x), k(x, y))$. We begin with the construction of f.

LEMMA 3.1. *Let $\{y_n\}_{n=1}^{\infty}$ be a sequence, $0 < \cdots < y_{n+1} < y_n < \cdots < y_1 < \frac{1}{3}$, with $y_n \to 0$ as $n \to \infty$. Then there is a map $f_1: [0, 1] \to [0, 1]$ with the properties:*

(i) $f_1(y) = 2y$ *for* $\frac{1}{3} \le y \le \frac{1}{2}$;

(ii) $f_1(\frac{1}{2} + y) = f_1(\frac{1}{2} - y)$ *for* $0 \le y \le \frac{1}{2}$;

(iii) $f_1(0) = 0$;

(iv) f_1 *is* C^1 *on* $[0, 1] - \{\frac{1}{2}\}$;

(v) $f_1'(0) = 1$ *and* $f_1'(y) > 1$ *for* $0 < y < \frac{1}{2}$;

(vi) $f_1^n(y_n) \to 0$ *as* $n \to \infty$; *and*

(vii) $\sup_{0 \le y \le y_n} (f_1^n)'(y) \to 1$ *as* $n \to \infty$.

Proof. The construction is straightforward and is omitted.

LEMMA 3.2. *Let $I = [0, 1]$ and let $f: I \to I$ be such that:*

(i) $f(0) = 0, f(\frac{1}{2}) = 1, f(1) = 0$;

(ii) f *is continuous and* C^1 *on* $[0, 1] - \{\frac{1}{2}\}$; *and*

(iii) $|f'(x)| > 1$ *for* $x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

Then f is topologically conjugate to $g: I \to I$ where g is given by

$$g(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \le x \le 1. \end{cases}$$

Proof. Let U be an open interval (nonempty) in I. We claim that $\frac{1}{2} \in f^n(U)$ for some $n \ge 0$. To see this, let C be a connected component of $\bigcup_{n \ge 0} f^n(U)$ of maximal length. Then $\frac{1}{2} \in C$ for otherwise the length of $f(C)$ is greater than the length of C by (i) and (iii) but this is impossible since $f(C)$ is a connected subset of $\bigcup_{n \ge 0} f^n(U)$. Thus $\frac{1}{2} \in f^n(U)$ for some $n \ge 0$ and we have established that $\bigcup_{n \ge 0} f^{-n}(\frac{1}{2})$ is dense in I. Similarly, $\bigcup_{n \ge 0} g^{-n}(\frac{1}{2})$ is dense in I.

We now define $h: \bigcup_{n \ge 0} f^{-n}(\frac{1}{2}) \to \bigcup_{n \ge 0} g^{-n}(\frac{1}{2})$ recursively. Let $h(\frac{1}{2}) = \frac{1}{2}$ and suppose that we have defined h of $f^{-n}(\frac{1}{2})$. Let $x \in f^{-n}(\frac{1}{2})$. Then there are precisely two inverse images of x under f: denote by $f_l^{-1}(x)$ the preimage of x in $(0, \frac{1}{2})$ and denote by $f_r^{-1}(x)$ the preimage of x in $(\frac{1}{2}, 1)$. Similarly, let $g_l^{-1}(h(x))$ and $g_r^{-1}(h(x))$ be the preimages of $h(x)$ under g in $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ respectively. Set $h(f_l^{-1}(x)) = g_l^{-1}(h(x))$ and $h(f_r^{-1}(x)) = g_r^{-1}(h(x))$.

In this way h is defined on all of $\bigcup_{n \ge 0} f^{-n}(\frac{1}{2})$ and it is clear that h is one-to-one, maps $\bigcup_{n \ge 0} f^{-n}(\frac{1}{2})$ onto $\bigcup_{n \ge 0} g^{-n}(\frac{1}{2})$, and $h \circ f = g \circ h$ on $\bigcup_{n \ge 0} f^{-n}(\frac{1}{2})$. A simple induction shows that h is order preserving and hence uniformly continuous. Thus, h extends to a continuous, order preserving map of I onto I (since $\bigcup_{n\geq 0} f^{-n}(\frac{1}{2})$ and $\bigcup_{n\geq 0} g^{-n}(\frac{1}{2})$ are dense in I) and we see, in fact, that h is a homeomorphism with $h \circ f = g \circ h$ on I.

It follows from the above that the map f_1 in Lemma 3.1 is topologically conjugate to

$$g(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2-2x, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

COROLLARY 3.3. *Given a sequence* $\{x_n\}_{n=1}^{\infty}$, $0 < \cdots < x_{n+1} < x_n < \cdots < x_1 < \frac{1}{4}$, *there is a* C^1 *map* $f: [0, 1] \to [0, 1]$ *satisfying*:

(i) f *is topologically conjugate to*

$$g(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2-2x, & \frac{1}{2} \leq x \leq 1; \end{cases}$$

(ii) $f(x) = 4x(1-x)$ *for* $\frac{1}{4} \leq x \leq \frac{3}{4}$;
(iii) $f(\frac{1}{2}+x) = f(\frac{1}{2}-x)$ *for* $0 \leq x \leq \frac{1}{2}$;
(iv) $f^n(x_n) \leq x_{n-1} \to 0$ *as* $n \to \infty$; *and*
(v) $\sup_{0\leq x\leq x_n} (f^n)'(x) \to 1$ *as* $n \to \infty$.

Proof. Let $h(x) = (2/\pi) \arcsin \sqrt{x}$. Then h is a homeomorphism of $[0, 1]$ and h and h^{-1} are C^1 on $(0, 1)$. It is straightforward that

$$h^{-1} \circ g \circ h(x) = 4x(1-x) \qquad \text{for all } x \in I.$$

Let $y_n = h(x_n)$. Then $0 < \cdots < y_{n+1} < y_n < \cdots < y_1 < \frac{1}{3}$. Now let f_1 be as in Lemma 3.1 for this sequence $\{y_n\}$. Let $f(x) = h^{-1} \circ f_1 \circ h(x)$. We will show that f satisfies (i)-(v) of this corollary.

Property (i) is clear. Indeed, f is conjugate to f_1 and f_1 is conjugate to $g(x)$ by Lemma 3.2. Since

$$f_1(x) = \begin{cases} 2x, & \frac{1}{3} \leq x \leq \frac{1}{2} \\ 2-2x, & \frac{1}{2} \leq x \leq \frac{2}{3}, \end{cases}$$

property (ii) is immediate. Also, the symmetry of f_1 and h guarantee (iii). Property (iv) follows from $f^n(x_n) = h^{-1} \circ f_1^n \circ h(x_n) = h^{-1}(f_1^n(y_n))$ and $f_1^n(y_n) \to 0$ as $n \to \infty$.

To verify (v), we note that there is a sequence $\{s_n\}$ such that $f^n(y) \leq s_n \cdot y$ for $0 \leq y \leq y_n$ and $s_n \to 1$ as $n \to \infty$. Thus, for $0 < x \leq x_n$,

$$\begin{aligned} (f^n)'(x) &= (h^{-1})'(f_1^n(h(x))) \cdot (f_1^n)'(h(x)) \cdot h'(x) \\ &\leq (h^{-1})'(s_n \cdot h(x)) \cdot (f_1^n)'(h(x)) \cdot h'(x). \end{aligned}$$

Now $(h^{-1})'(f_1^n(h(x)) \cdot h'(x) > 1$ and $(f_1^n)'(h(x)) > 1$ so that $(f^n)'(x) > 1$. Since $(f_1^n)'(h(x)) \to 1$ as $n \to \infty$ for $0 \leq x \leq x_n$, we only need to show that $(h^{-1})'(s_n \cdot h(x)) \cdot h'(x) \to 1$ as $n \to \infty$ to conclude that $\sup_{0\leq x\leq x_n} (f^n)'(x) \to 1$ as $n \to \infty$.

To this end, we calculate:

$$h'(x) = \frac{1}{\pi\sqrt{x - x^2}}$$

and

$$(h^{-1})'(s_n \cdot h(x)) = \frac{1}{h'(h^{-1}(s_n \cdot h(x)))}$$
$$= \frac{1}{\pi(h^{-1}(s_n \cdot h(x)) - (h^{-1}(s_n \cdot h(x)))^2)^{1/2}}.$$

Replacing s_n by $1+\varepsilon_n$ we obtain:

$$\left[\frac{h'(x)}{h'(h^{-1}(1+\varepsilon_n)h(x)))}\right]^2 = \frac{1}{x(1-x)} \cdot [x \cos^2(\varepsilon_n \arcsin\sqrt{x})$$
$$+\sqrt{x-x^2}\sin(2\varepsilon_n \arcsin\sqrt{x})$$
$$+(1-x)\sin^2(\varepsilon_n \arcsin\sqrt{x})]$$
$$\cdot[1-(x\cos^2(\varepsilon_n \arcsin\sqrt{x})+\sqrt{x-x^2}\sin(2\varepsilon_n \arcsin\sqrt{x})$$
$$+(1-x)\sin^2(\varepsilon_n \arcsin\sqrt{x}))].$$

Now let δ_n be such that:

$$\sin(2\varepsilon_n \arcsin\sqrt{x}) \le \delta_n x;$$

and

$$\sin^2(\varepsilon_n \arcsin\sqrt{x}) \le \delta_n x \qquad \text{for } x \le x_n,$$

where $\delta_n \to 0$ as $n \to \infty$. Then, continuing from above, we have

$$\left[\frac{h'(x)}{h'(h^{-1}(s_n \cdot h(x)))}\right]^2 \le \frac{[x+(\sqrt{x-x^2})\delta_n\sqrt{x}+(1-x)\delta_n x]\cdot[1-x]}{x(1-x)}$$
$$\le 1+(\sqrt{1-x})\delta_n+(1-x)\delta_n$$
$$\le 1+2\delta_n.$$

Thus $(f^n)'(x) \to 1$ uniformly as $n \to \infty$ for $x \in [0, x_n]$.

LEMMA 3.4. *Given a sequence* $\{z_n\}_{n=1}^{\infty}$, $-\frac{1}{4} < z_1 < \cdots < z_n < z_{n+1} < \cdots < 0$, *there is a* C^1 *map* $f: [-\frac{1}{4}, 0] \to [-\frac{1}{4}, 0]$ *satisfying*:

(i) $f(x) > x$ *for* $x \in [-\frac{1}{4}, 0)$ *and* $f(0) = 0$;
(ii) $0 < f'(x) \le 1$ *for* $x \in [-\frac{1}{4}, 0]$ *and* $f'(0) = 1$;
(iii) $f^n(z_n) \le z_{n-1} \to 0$ *as* $n \to \infty$; *and*
(iv) $\sup_{z_n \le x \le 0}(f^n)'(x) \to 1$ *as* $n \to \infty$.

Proof. The construction of such an f is much like the construction for Lemma 3.1 and is also omitted.

Now, given sequences $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ as in Corollary 3.3 and Lemma 3.4 respectively, let $f: [-\frac{1}{4}, \frac{5}{4}] \to [-\frac{1}{4}, \frac{5}{4}]$ be given by

$$f(x) = \begin{cases} f(x) \text{ as in Lemma 3.4,} & \text{for } -\frac{1}{4} \le x \le 0 \\ f(x) \text{ as in Corollary 3.3,} & \text{for } 0 \le x \le 1 \\ f(x-1) \text{ as in Lemma 3.4,} & \text{for } 1 \le x \le \frac{5}{4}. \end{cases}$$

Then f has the properties:

$$\left.\begin{array}{l} \text{(i) } f:[-\tfrac{1}{4},\tfrac{5}{4}]\to[-\tfrac{1}{4},\tfrac{5}{4}] \text{ is } C^1; \\ \text{(ii) } f|_{[0,1]} \text{ is topologically conjugate to} \\ \qquad g(x)=\begin{cases} 2x, & 0\le x\le \tfrac{1}{2} \\ 2-2x, & \tfrac{1}{2}\le x\le 1; \end{cases} \\ \text{(iii) } f^n(x_n)\le x_{n-1}\to 0 \text{ and } f^n(z_n)\ge z_{n-1}\to 0 \text{ as } n\to\infty; \text{ and} \\ \text{(iv) } \sup_{z_n\le x\le z_n}(f^n)'(x)\to 1 \text{ as } n\to\infty. \end{array}\right\}$$

The function $G: S^2\to S^2$ that we wish to construct will be of the form

$$G(x,y)=(f(x),k(x,y)) \quad \text{on} \quad B=\{(x,y)\,|\,-\tfrac{1}{4}\le x\le \tfrac{5}{4}, |y|\le\tfrac{1}{2}\},$$

where f is as in (3.5) for sequences $\{x_n\}$ and $\{z_n\}$ to be determined.

We now construct the function k. The map k will have the form $k(x,y)=k_0(y)$ for $-\frac{1}{4}\le x\le\frac{1}{8}$.

LEMMA 3.6. *Given a sequence of intervals, $J_n=[l_n,r_n]$, $n=1,2,\ldots$ such that*

$$J_{n+1}\subseteq \text{interior}(J_n), \qquad l_1=-\tfrac{1}{2}, \qquad \frac{\text{length}(J_n)}{\text{length}(J_{n+1})}\to 1 \quad \text{as } n\to\infty,$$

and length $(J_1)+2\sum_{n=1}^{\infty}$ length $(J_n)=\frac{3}{4}$, *there is a C^1 map $k_0:[-\frac{1}{2},\frac{1}{2}]\to[-\frac{1}{2},\frac{1}{2}]$ such that*:

(i) $k_0(\frac{1}{4})=\frac{1}{4}$ *and* $k_0'(\frac{1}{4})=1$;
(ii) $|k_0(y)-\frac{1}{4}|<|y-\frac{1}{4}|$ *for* $y\in[-\frac{1}{2},\frac{1}{2}]-\{\frac{1}{4}\}$;
(iii) $k_0'(y)\le 1$ *for all* $y\in[-\frac{1}{2},\frac{1}{2}]$; *and*
(iv) $(k_0^n)'|_{J_n}\equiv 1$.

Proof. We construct a $\tilde{k}_0$ on $[-\frac{1}{4},\frac{3}{4}]$. Let $\tilde{l}_n=-(r_n-\frac{1}{4})=\frac{1}{4}-r_n$, $\tilde{r}_n=-(l_n-\frac{1}{4})=\frac{1}{4}-l_n$, and $\tilde{J}_n=[\tilde{l}_n,\tilde{r}_n]=-J_n+\frac{1}{4}$. Let $a_n=\text{length}(\tilde{J}_n)=\text{length}(J_n)$. We have: $a_1+2\sum_{n=1}^{\infty}a_n=\frac{3}{4}$ and $\tilde{r}_1=1$.

Let $b_n=\tilde{l}_n-\tilde{l}_{n-1}$ for $n\ge 2$ and let $b_1=0$. Also, let $S_n=a_1+\sum_{i=1}^{n}2a_i$ for $n\ge 1$ and let $S_0=a_1$. We define $\tilde{k}_0^{-1}:[0,\frac{3}{4}-2a_1]\to[0,\frac{3}{4}]$ by

$$\tilde{k}_0^{-1}(x)=\begin{cases} x+2a_n+b_n, & \text{for } x\in[\frac{3}{4}-S_n,\frac{3}{4}-S_n+a_n],\ n\ge 1 \\ p_n(x), & \text{for } x\in[\frac{3}{4}-S_{n+1}+a_{n+1},\frac{3}{4}-S_n],\ n\ge 1 \\ 0, & \text{for } x=0. \end{cases}$$

The above p_n is the cubic polynomial satisfying, for $n\ge 1$:

$$p_n(\tfrac{3}{4}-S_{n+1}+a_{n+1})=\tfrac{3}{4}-S_n+a_{n+1}+b_{n+1}; \qquad p_n(\tfrac{3}{4}-S_n)=\tfrac{3}{4}-S_{n-1}+b_n;$$
$$p_n'(\tfrac{3}{4}-S_{n+1}+a_{n+1})=1; \qquad p_n'(\tfrac{3}{4}-S_n)=1.$$

Explicitly,

$$\begin{aligned} p_n(x)=&\frac{-2[2(a_n-a_{n+1})+(b_n-b_{n+1})]}{(a_{n+1})^3}(x-(\tfrac{3}{4}-S_{n+1}+a_{n+1}))^3 \\ &+\frac{3[2(a_n-a_{n+1})+(b_n-b_{n+1})]}{(a_{n+1})^2}(x-(\tfrac{3}{4}-S_{n+1}+a_{n+1}))^2 \\ &+(x-(\tfrac{3}{4}-S_{n+1}+a_{n+1})) \\ &+(\tfrac{3}{4}-S_{n-1}+a_n+b_n), \qquad \text{for } n\ge 1. \end{aligned}$$

Then $\tilde{k}_0^{-1}$ is C^1 on $(0, \frac{3}{4}-2a_1]$. Furthermore,

$$1 \le p_n'(x) \le 1 + \frac{3}{2}\left[\frac{2(a_n - a_{n+1}) + (b_n - b_{n+1})}{a_{n+1}}\right]$$

for $x \in [\frac{3}{4} - S_{n+1} + a_{n+1}, \frac{3}{4} - S_n]$. Now, since $(a_n/a_{n+1}) = (\text{length}(J_n)/\text{length}(J_{n+1})) \to 1$ as $n \to \infty$ and

$$\frac{a_{n+1}-a_n}{a_{n+1}} \le \frac{-b_{n+1}}{a_{n+1}} \le \frac{b_n - b_{n+1}}{a_{n+1}} \le \frac{b_n}{a_{n+1}} \le \frac{a_{n-1}-a_n}{a_{n+1}},$$

we see that $p_n'(x) \to 1$ uniformly for $x \in [\frac{3}{4} - S_{n+1} + a_{n+1}, \frac{3}{4} - S_n]$, as $n \to \infty$. Thus $\tilde{k}_0^{-1}$ is C^1 on $[0, \frac{3}{4} - 2a_1]$. The above calculation also shows that $(\tilde{k}_0^{-1})'(x) \ge 1$ for all $x \in [0, \frac{3}{4} - 2a_1]$ so that $\tilde{k}_0 : [0, \frac{3}{4}] \to [0, \frac{3}{4} - 2a_1]$ is C^1 and $\tilde{k}_0'(y) \le 1$ for all $y \in [0, \frac{3}{4}]$ and $\tilde{k}_0'(0) = 1$.

Finally, $\tilde{k}_0^n(y) = y - 2\sum_{i=1}^n a_i - \sum_{i=1}^n b_i$ for $y \in \tilde{J}_n$ so that $(\tilde{k}_0^n)'(y) = 1$ for $y \in \tilde{J}_n$ and $n = 1, 2, \ldots$. Now let $k_0(y) = \frac{1}{4} - \tilde{k}_0(\frac{1}{4} - y)$ for $y \in [-\frac{1}{2}, \frac{1}{4}]$ and extended k_0 to $[-\frac{1}{2}, \frac{1}{2}]$ by $k_0(x) = \frac{1}{4} + \sin(x - \frac{1}{4})$ for $x \in [\frac{1}{4}, \frac{1}{2}]$. Then k_0 is C^1 and has properties (i)-(iv).

We proceed to the explicit construction of G. Let $G : [\frac{1}{4}, \frac{3}{4}] \times [-1, 1] \to [\frac{1}{4}, \frac{5}{4}] \times [-\frac{1}{2}, \frac{1}{2}] = B$ by

$$\left.\begin{aligned} &G(x,y) = (1 - 4(x-\tfrac{1}{2})^2 + \alpha(y), (\tfrac{1}{2}-x)(\tfrac{1}{2}y+1)) \\ &\text{where} \\ &\alpha(y) = \begin{cases} (y-\frac{1}{2})^2, & \frac{1}{2} \le y \le 1 \\ 0, & |y| \le \frac{1}{2} \\ -(y+\frac{1}{2})^2, & -1 \le y \le -\frac{1}{2}. \end{cases} \end{aligned}\right\} \tag{3.7}$$

Then G is C^1 and G collapses the interval $J = \{(\frac{1}{2}, y) \mid -\frac{1}{2} \le y \le \frac{1}{2}\}$ to the point $(1, 0)$. It is straightforward to check that G is a C^1 diffeomorphism from $[\frac{1}{4}, \frac{3}{4}] \times [-1, 1] - J$ onto its image.

Now let $K_n : [\frac{1}{4}, \frac{3}{4}] \times [-1, 1] \to B$, $n = 2, 3, \ldots$ by

$$K_n(x,y) = \begin{cases} G(x,y), & \text{for } |x-\frac{1}{2}| \ge \dfrac{1}{(2n)^2} \text{ or } |y| \ge \dfrac{1}{2} + \dfrac{1}{n^2}; \\ (\alpha_n(y)(x-\frac{1}{2})^4 + \beta_n(y)(x-\frac{1}{2})^2 + \gamma_n(y), (\frac{1}{2}-x)(\frac{1}{2}y+1)), & \\ \qquad \text{for } |x-\frac{1}{2}| \le \dfrac{1}{(2n)^3} \text{ and } \frac{1}{2} \le y \le \frac{1}{2} + \dfrac{1}{n^3}; \\ ((\frac{1}{2}-y)f_n(x) + (\frac{1}{2}+y)f(x), (\frac{1}{2}-x)(\frac{1}{2}y+1)), & \\ \qquad \text{for } |x-\frac{1}{2}| \le \dfrac{1}{(2n)^3} \text{ and } |y| \le \frac{1}{2}; \\ (a_n(y)(x-\frac{1}{2})^4 + b_n(y)(x-\frac{1}{2})^2 + c_n(y), (\frac{1}{2}-x)(\frac{1}{2}y+1)), & \\ \qquad \text{for } |x-\frac{1}{2}| \le \dfrac{1}{(2n)^3} \text{ and } -\frac{1}{2} - \dfrac{1}{n^3} \le y \le -\frac{1}{2}. \end{cases}$$

The functions appearing in the above definition have the following formulas:

$$f(x) = 1 - 4(x-\tfrac{1}{2})^2 = 4x(1-x);$$

$$f_n(x) = -2^5 n^4 (x-\tfrac{1}{2})^4 + \left(1 - \frac{1}{8n^4}\right);$$

https://doi.org/10.1017/S0143385700004491 Published online by Cambridge University Press

$$\gamma_n(y)=\tfrac{1}{8}(y-\tfrac{1}{2})^3+\left(1-\frac{1}{4n^3}\right)(y-\tfrac{1}{2})^2+\frac{1}{8n^4}(y-\tfrac{1}{2})+1;$$

$$\alpha_n(y)=(2n)^8[-1-(y-\tfrac{1}{2})^2+\gamma_n(y)];$$

$$\beta_n(y)=2(2n)^4[1+(y-\tfrac{1}{2})^2-\gamma_n(y)]-4;$$

$$c_n(y)=\left(\frac{n^2}{4}+\tfrac{1}{8}\right)(y+\tfrac{1}{2})^3+\left(\frac{1}{4n^3}-\tfrac{5}{8}\right)(y+\tfrac{1}{2})^2+\frac{1}{8n^4}(y+\tfrac{1}{2})+\left(1-\frac{1}{8n^4}\right);$$

$$a_n(y)=(2n)^8[-1+(y+\tfrac{1}{2})^2+c_n(y)];$$

and

$$b_n(y)=2(2n)^4[1-(y+\tfrac{1}{2})^2-c_n(y)]-4.$$

By construction the K_n are continuously differentiable and $K_n=G$ off of

$$V_n=\left[\tfrac{1}{2}-\frac{1}{(2n)^3},\tfrac{1}{2}+\frac{1}{(2n)^3}\right]\times\left[-\tfrac{1}{2}-\frac{1}{n^3},\tfrac{1}{2}+\frac{1}{n^3}\right].$$

Moreover, we will establish the following proposition.

PROPOSITION 3.8. *There is an N such that for all $n\geq N$, K_n is a C^1 diffeomorphism onto its image and*

$$\left\|D(K_{n+1}\circ K_n^{-1})(z)-\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\|\to 0$$

uniformly for

$$z\in G([\tfrac{1}{4},\tfrac{1}{2}]\times[-1,1])\text{ as }n\to\infty.$$

Temporarily assuming the validity of the above proposition, we complete the construction.

Let $N\geq 2$ be as in Proposition 3.8 and define G_n on $[\tfrac{1}{4},\tfrac{3}{4}]\times[-1,1]$ by $G_n=K_{n+N}$ for $n=1,2,\ldots.$ Also, let $U_n=V_{n+N}$. Note that

$$G(U_n)\subseteq\left[1-\frac{5}{4(n+N)^4},1+\frac{1}{(n+N)^4}\right]\times\left[-\left(\frac{5(n+N)^2+2}{8(n+N)^4}\right),\left(\frac{5(n+N)^2+2}{8(n+N)^4}\right)\right].$$

Let L be such that

$$L\cdot\left(3+2\sum_{n=4}^{\infty}\left[\frac{5(n+N)^2+2}{4(n+N)^4}\right]\right)=\tfrac{3}{4}.$$

Then $0<L<1$. Define $k_1:[-\tfrac{1}{2},\tfrac{1}{2}]\to[-\tfrac{1}{2},\tfrac{1}{2}]$ by $k_1(y)=-Ly+(L-1)/2$, let $J_1=k_1([-\tfrac{1}{2},\tfrac{1}{2}])=[-\tfrac{1}{2},-\tfrac{1}{2}+L]$, and let

$$J_n=k_1\left(\left[-\left(\frac{5(n+2+N)^2+2}{8(n+2+N)^4}\right),\left(\frac{5(n+2+N)^2+2}{8(n+2+N)^4}\right)\right]\right)\qquad\text{for } n=2,3,\ldots.$$

Then the intervals J_n satisfy the conditions of Lemma 3.6. Let $k_0:[-\tfrac{1}{2},\tfrac{1}{2}]\to[-\tfrac{1}{2},\tfrac{1}{2}]$ be the map determined by Lemma 3.6 and these J_n.

Next, set

$$x_n=1-\left(1-\frac{5}{4(n+N)^4}\right)=\frac{5}{4(n+N)^4}$$

and

$$z_n = 1 - \left(1 + \frac{1}{(n+N)^4}\right) = -\frac{1}{(n+N)^4}$$

for $n = 1, 2, \ldots$, and let f be as in (3.5) for these sequences $\{x_n\}$ and $\{z_n\}$.

We now define $G: B \cup ([\frac{1}{4}, \frac{3}{4}] \times [-1, 1]) \to B$ by:

$$G(x, y) = \begin{cases} (f(x), k_0(y)), & \text{for } -\frac{1}{4} \le x \le \frac{1}{8}, -\frac{1}{2} \le y \le \frac{1}{2}, \\ (f(x), k(x, y)), & \text{for } \frac{1}{8} \le x \le \frac{1}{4}, -\frac{1}{2} \le y \le \frac{1}{2}, \\ \text{as in (3.7)}, & \text{for } \frac{1}{4} \le x \le \frac{3}{4}, -1 \le y \le 1, \\ (f(x), k(x, y)), & \text{for } \frac{1}{4} \le x \le \frac{7}{8}, -\frac{1}{2} \le y \le \frac{1}{2}, \\ (f(x), k_1(y)), & \text{for } \frac{7}{8} \le x \le \frac{5}{4}, -\frac{1}{2} \le y \le \frac{1}{2}. \end{cases}$$

In the above definition of G, $k(x, y)$ is a C^1 function that smoothly interpolates the other values of G and is such that

$$-\tfrac{1}{2} \le k(x, y) \le \tfrac{1}{2} \qquad \text{and, for } x > \tfrac{1}{8}, 0 < \left|\frac{\partial k}{\partial y}(x, y)\right| \le 1.$$

Then G restricted to $B \cup ([\frac{1}{4}, \frac{3}{4}] \times [-1, 1]) - (\{\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}])$ is a C^1 diffeomorphism into B and $G(\{\frac{1}{2}\} \times [-\frac{1}{2}, \frac{1}{2}]) = \{(1, 0)\}$. Moreover, f and k_0 have been constructed in such a way that $G^n(U_n) \cap G^k(U_k) = \emptyset$ for $n \ne k$, $n, k \ge 1$ and $G^n(U_n) \cap U_1 = \emptyset$ for $n \ge 1$.

Now it is clear that G can be extended to S^2 in such a way that

$$G: S^2 - (\{\tfrac{1}{2}\} \times [-\tfrac{1}{2}, \tfrac{1}{2}]) \to S^2 - \{(1, 0)\}$$

is a C^1 diffeomorphism. We then define $G_n: S^2 \to S^2$ to agree with G off of U_n. Then G_n is a C^1 diffeomorphism of S^2 for each $n = 1, 2, \ldots$.

Let $P: B \to I = [0, 1]$ be given by

$$P(x, y) = \begin{cases} 0, & \text{for } -\frac{1}{4} \le x \le 0, \\ x, & \text{for } 0 \le x \le 1, \\ 1, & \text{for } 1 \le x \le \frac{5}{4}. \end{cases}$$

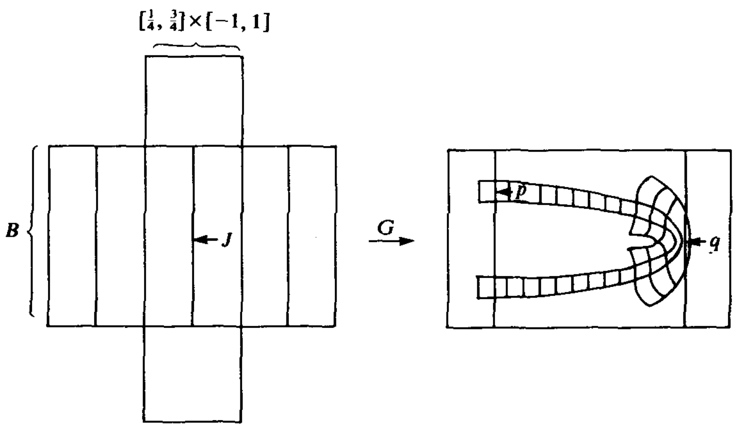

FIGURE 2. G takes vertical line segments in B into vertical line segments, $G(J) = \{q\}$, and $G|_{(B \cup ([\frac{1}{4}, \frac{3}{4}] \times [-1,1])) - J}$ is a C^1 diffeomorphism onto its image.

https://doi.org/10.1017/S0143385700004491 Published online by Cambridge University Press

We now have that (2.1)-(2.5) and (2.7) are satisfied by G, G_n, P, and $f|_I$. That the diameter of $G^n(U_n)$ goes to zero follows from diameter $(f^n([z_n, x_n]))\to 0$ and diameter $(k_0^n(J_n))\to 0$. Thus (2.6) is satisfied.

That $(f^n)'|_{[z_n,x_n]}\to 1$ uniformly as $n\to\infty$ and $(k_0^n)'|_{J_n}\equiv 1$ implies

$$DG^{-(n-2)}(z)\to\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and

$$DG^{(n-2)}(w)\to\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

uniformly as $n\to\infty$ for $z\in G^n(U_n)$ and $w\in G^2(U_n)$.

Assuming Proposition 3.8 and letting $z\in G^n(U_n)$ we have:

$$\begin{aligned} D(G^{n-1}\circ G_{n+1}\circ G_n^{-1}\circ G^{-(n-1)})(z) &= DG^{n-2}(G\circ G_{n+1}\circ G_n^{-1}\circ G^{-(n-1)}(z))\\ &\quad\cdot DG(G_{n+1}\circ G_n^{-1}\circ G^{-(n-1)}(z))\\ &\quad\cdot D(G_{n+1}\circ G_n^{-1})(G^{-(n-1)}(z))\\ &\quad\cdot DG^{-1}(G^{-(n-2)}(z))\cdot DG^{-(n-2)}(z).\end{aligned}$$

Now $z\in G^n(U_n)$ so that $G\circ G_{n+1}\circ G_n^{-1}\circ G^{-(n-1)}(z)\in G^2(U_n)$. Thus

$$DG^{n-2}(G\circ G_{n+1}\circ G_n^{-1}\circ G^{-(n-1)}(z))\to\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and

$$DG^{-(n-2)}(z)\to\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

uniformly for $z\in G^n(U_n)$. By Proposition 3.8,

$$D(G_{n+1}\circ G_n^{-1}(G^{-(n-1)}(z))\to\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

uniformly for $z\in G^n(U_n)$ and, since $G_{n+1}\circ G_n^{-1}$ goes to the identity uniformly,

$$DG(G_{n+1}\circ G_n^{-1}\circ G^{-(n-1)}(z))\cdot DG^{-1}(G^{-(n-2)}(z))\to\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

uniformly. Thus,

$$D(G^{n-1}\circ G_{n+1}\circ G_n^{-1}\circ G^{-(n-1)})(z)\to\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

uniformly for $z\in G^n(U_n)$ as $n\to\infty$ and 2.8 is satisfied.

Corollary 2.14 now supplies the example diffeomorphism $F:S^2\to S^2$ promised in § 1. The diffeomorphism F has an attracting set $\Lambda\subseteq$ interior (B) and $F|_\Lambda$ is topologically conjugate to the homeomorphism $\hat{f}:(I,f)\to(I,f)$. By Corollary 3.3, $f|_I$ is topologically conjugate to

$$g(x)=\begin{cases}2x, & 0\le x\le\frac{1}{2}\\ 2-2x, & \frac{1}{2}\le x\le 1.\end{cases}$$

Thus $\hat{f}:(I,f)\to(I,f)$ is topologically conjugate to $\hat{g}:(I,g)\to(I,g)$. The continuum (I,g) is the indecomposable Knaster continuum and $\hat{g}:(I,g)\to(I,g)$ is transitive (i.e., has a dense orbit). Thus, the attractor Λ for F is the indecomposable Knaster continuum and $F|_\Lambda$ is transitive. The construction is complete with the proof of Proposition 3.8.

Proof of Proposition 3.8. The determinant of $DK_n(x, y)$, $d_n(x, y)$, is:

$$d_n(x,y)=\begin{cases} 4(x-\frac{1}{2})^2+2(y-\frac{1}{2})(\frac{1}{2}y+1), \\ \qquad \text{for } |x-\frac{1}{2}|\geq \frac{1}{(2n)^2}, \frac{1}{2}+\frac{1}{n^2}\leq y\leq 1; \\ 4(x-\frac{1}{2})^2-2(y+\frac{1}{2})(\frac{1}{2}y+1), \\ \qquad \text{for } |x-\frac{1}{2}|\geq \frac{1}{(2n)^2}, -1\leq y\leq -\frac{1}{2}-\frac{1}{n^2}; \\ -(x-\frac{1}{2})^2[2\alpha_n(y)(x-\frac{1}{2})^2+\beta_n(y)]+(\frac{1}{2}y+1) \\ \qquad\qquad [\alpha_n'(y)(x-\frac{1}{2})^4+\beta_n'(y)(x-\frac{1}{2})^2+\gamma_n'(y)], \\ \qquad \text{for } |x-\frac{1}{2}|\leq \frac{1}{(2n)^2}, \frac{1}{2}\leq y\leq \frac{1}{2}+\frac{1}{n^3}; \\ \frac{1}{2}(\frac{1}{2}-x)[(\frac{1}{2}-y)f_n'(x)+(\frac{1}{2}+y)f'(x)]+(\frac{1}{2}y+1)[f(x)-f_n(x)], \\ \qquad \text{for } |x-\frac{1}{2}|\leq \frac{1}{(2n)^2}, |y|\leq \frac{1}{2}; \\ -(x-\frac{1}{2})^2[2a_n(y)(x-\frac{1}{2})^2+b_n(y)]+(\frac{1}{2}y+1) \\ \qquad\qquad [a_n'(y)(x-\frac{1}{2})^4+b_n'(y)(x-\frac{1}{2})^2+c_n'(y)], \\ \qquad \text{for } |x-\frac{1}{2}|\leq \frac{1}{(2n)^2}, -\frac{1}{2}-\frac{1}{n^2}\leq y\leq -\frac{1}{2}. \end{cases}$$

We wish to show that, for n sufficiently large, $d_n(x, y) > 0$.

Case 1.

$$|x-\tfrac{1}{2}|\geq \frac{1}{(2n)^2}, \tfrac{1}{2}+\frac{1}{n^3}\leq y\leq 1.$$

We get

$$d_n(x,y)\geq \frac{5}{2n^2}\left(\frac{1}{2n^2}+1\right)>0 \qquad \text{for all } n.$$

Case 2.

$$|x-\tfrac{1}{2}|\geq \frac{1}{(2n)^2}, -1\leq y\leq -\tfrac{1}{2}-\frac{1}{n^3}.$$

In this case

$$d_n(x,y)\geq \frac{1}{n^2}\left(\frac{1}{4n^2}+1\right)>0 \qquad \text{for all } n.$$

Case 3.

$$|x-\tfrac{1}{2}|\leq \frac{1}{(2n)^2}, \quad \tfrac{1}{2}\leq y\leq \tfrac{1}{2}+\frac{1}{n^2}.$$

Let $x=\frac{1}{2}+s/(2n)^2$, $y=\frac{1}{2}+t/n^3$, $-1\le s\le 1$, and $0\le t\le 1$. Then

$$d_n(x,y)=\left(\frac{1}{n^4}\right)\left[\tfrac{1}{8}\left(\tfrac{5}{4}+\frac{t}{2n^2}\right)(3t-1)(t-1)(s^2-1)^2\right]$$
$$-\left(\frac{1}{n^6}\right)[\tfrac{1}{4}(t-1)^2(t)(s^4)]+\left(\frac{1}{n^4}\right)[\tfrac{1}{4}s^2]+\left(\frac{1}{n^2}\right)\left[\left(\tfrac{5}{2}+\frac{t}{n^2}\right)(t)\right].$$

For $t\ge\frac{1}{3}$, the last term dominates and $d_n(x,y)>0$ for n sufficiently large. For $t<\frac{1}{3}$ the first, third, and fourth terms are non-negative and the fourth term dominates the second term unless $t=0$. If $t=0$ the sum of the first and third terms is positive. In any case, $d_n(x,y)>0$ for all (x,y) and n sufficiently large.

Case 4.

$$|x-\tfrac{1}{2}|\le\frac{1}{(2n)^3},\quad |y|\le\tfrac{1}{2}.$$

Let $x-\frac{1}{2}=t/(2n)^2$, $-1\le t\le 1$. We then get

$$d_n(x,y)=\frac{1}{8n^4}[(\tfrac{3}{2}+3y)t^4-(1+3y)t^2+(\tfrac{1}{2}y+1)].$$

The quantity in the brackets is zero when

$$t^2=\frac{(1+3y)\pm\sqrt{(1+3y)^2-4(\frac{3}{2}+3y)(\frac{1}{2}y+1)}}{2(\frac{3}{2}+3y)}.$$

The polynomial in y inside the radical is negative for $y\ge-\frac{1}{3}$ so that, for real zeros, y must be less than $-\frac{1}{3}$. But then, $1+3y<0$ and, if $y>-\frac{1}{2}$, we have $t^2<0$. Therefore the quantity $(\frac{3}{2}+3y)t^4-(1+3y)t^2+(\frac{1}{2}y+1)$ is bounded above zero for $-1\le t\le 1$ and $-\frac{1}{2}\le y\le\frac{1}{2}$ and we have $d_n(x,y)\ge k/n^4$ for some positive constant k and all $n>0$.

Case 5. $|x-\frac{1}{2}|\le 1/(2n)^3$ and $-\frac{1}{2}\le 1/n^3\le y\le-\frac{1}{2}$. Let $x=\frac{1}{2}+s/(2n)^2$ and $y=-\frac{1}{2}-t/n^2$, $-1\le s\le 1$, $0\le t\le 1$. Then:

$$d_n(x,y)=\frac{1}{n^2}\left(\tfrac{3}{4}-\frac{t}{2n^2}\right)(s^2-1)^2\left[\left(\tfrac{3}{4}+\frac{3}{8n^2}\right)t^2-\frac{1}{2n^2}t\right]$$
$$+\frac{1}{n^2}\left(\tfrac{3}{4}-\frac{t}{2n^2}\right)[\tfrac{5}{4}+\tfrac{3}{4}(2s^2-s^4)]t$$
$$+\frac{1}{n^4}\left[\left(\tfrac{3}{4}-\frac{t}{2n^2}\right)(s^2-1)^2(\tfrac{1}{8})+\frac{s^4}{4}\right]$$
$$+\frac{1}{n^4}(s^4-s^2)\left[\left(\tfrac{1}{2}+\frac{1}{4n^2}\right)t^3-\left(\frac{1}{2n^2}+\tfrac{3}{4}\right)t^2+\frac{1}{4n^2}t\right].$$

First consider $t\ge 1/n$. Then the first term above is positive, the second term is bigger than k/n^2 for some positive constant k, and the last two terms are smaller in absolute value than l/n^4 for some l. Thus, for $t\ge 1/n$, $d_n(x,y)\ge k/n^2$ for some positive k and sufficiently large n.

Now if $t\le 1/n$, the second term is larger than $k(1/n^3)$ for some positive constant k and the other three terms are in absolute value less than $l(1/n^4)$ for some l. Thus, $d_n(x,y)\ge k(1/n^3)$ for some positive constant k and sufficiently large n. In any case, $d_n(x,y)$ is positive for sufficiently large n.

We have established the existence of N such that K_n is a diffeomorphism for $n\geq N$. Our final task is to show that $D(K_{n+1}\circ K_n^{-1})$ goes to the identity as $n\to\infty$ $(n\geq N)$.

Let $n\geq N$ and $(x,y)\in[\frac{1}{4},\frac{3}{4}]\times[-1,1]$, let $\mathbf{I}=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ be the identity matrix, and $0=\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ the zero matrix. The expression $(DK_{n+1}(x,y))(DK_n(x,y))^{-1}-\mathbf{I}$ takes on one of seven forms depending on (x,y).

Case (*i*). $|x-\frac{1}{2}|\geq 1/(2n)^3$ or $|y|\geq\frac{1}{2}+1/n^3$. In this case, $(DK_{n+1}(x,y))\cdot(DK_n(x,y))^{-1}-\mathbf{I}=0$.

Case (*ii*). $|x-\frac{1}{2}|\leq 1/(2n)^2$, $\frac{1}{2}\leq y\leq\frac{1}{2}+1/n^2$ and either $|x-\frac{1}{2}|\geq 1/(2(n+1))^2$ or $y\geq\frac{1}{2}+1/(n+1)^3$. Let

$$(DK_{n+1}(x,y))\cdot(DK_n(x,y))^{-1}-\mathbf{I}$$

be denoted by

$$\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}.$$

Replacing x by $\frac{1}{2}+s/(2n)^3$, $-1\leq s\leq 1$, and y by $\frac{1}{2}+t/n^3$, $0\leq t\leq 1$, we get:

$$A_{11}=\frac{\left(\frac{1}{n^6}\right)[\frac{1}{4}(t-1)^2(t)(s^4)]-\left(\frac{1}{n^4}\right)\left[(\frac{1}{8})\left(\frac{5}{4}+\frac{t}{2n^3}\right)(3t-1)(t-1)(s^2-1)^2\right]}{\left(\frac{1}{n^6}\right)[(-\frac{1}{4})(t-1)^2(t)(s^4)]+\left(\frac{1}{n^4}\right)\left[(\frac{1}{8})\left(\frac{5}{4}+\frac{t}{2n^3}\right)(3t-1)(t-1)(s^2-1)^2\right]+\left(\frac{1}{n^4}\right)(\frac{1}{4}s^2)+\left(\frac{1}{n^2}\right)\left(\frac{5}{4}+\frac{t}{n^2}\right)(t)}.$$

If $t\geq 1/n$ then the numerator of A_{11} clearly goes to zero faster than the last term in the denominator so that $A_{11}\to 0$ as $n\to\infty$. On the other hand, if $t<1/n$ then $y<\frac{1}{2}+1/(n+1)^2$ so that $|x-\frac{1}{2}|\geq 1/(2(n+1))^3$, then $s^2\geq n^4/(n+1)^4$. We see that the numerator then goes to 0 with the $1/n^6$ while the denominator is greater than k/n^4 for some positive k. Thus, in any event, $A_{11}\to 0$.

A_{21} isn't quite as messy: $A_{21}=0$. For A_{12} we have:

$$A_{12}=\frac{\left(\frac{1}{n^6}\right)[(\frac{1}{4})(3t-1)(t-1)(s^2-1)^2(s)+4(s)(s^2-1)(t-1)^2t^2]}{d_n},$$

where d_n is the same denominator as in A_{11}. We saw before that $d_n\geq k/n^4$ for some positive constant k and $n\geq N$. Thus, $A_{12}\to 0$ as $n\to\infty$.

Finally, $A_{22}=0$ for all (x,y) and case (ii) is finished.

Case (*iii*). $|x-\frac{1}{2}|\leq 1/(2(n+1))^2$ and $\frac{1}{2}\leq y\leq\frac{1}{2}+1/(n+1)^3$. Again, let

$$(DK_{n+1}(x,y))\cdot(DK_n(x,y))^{-1}-\mathbf{I}=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}.$$

We have:

$$\begin{aligned}A_{11}=\{&[2(\alpha_n(y)-\alpha_{n+1}(y))+(\tfrac{1}{2}y+1)(\alpha'_{n+1}(y)-\alpha'_n(y))](x-\tfrac{1}{2})^4\\&+[(\beta_n(y)-\beta_{n+1}(y))+(\tfrac{1}{2}y+1)(\beta'_{n+1}(y)-\beta'_n(y))](x-\tfrac{1}{2})^2\\&+[(\tfrac{1}{2}y+1)(\gamma'_{n+1}(y)-\gamma'_n(y))]\}\cdot\frac{1}{d_n(x,y)}\end{aligned}$$

https://doi.org/10.1017/S0143385700004491 Published online by Cambridge University Press

where $d_n(x, y)$ is as in case (ii). We calculate:

$$\alpha_n(y)-\alpha_{n+1}(y)=(2)^5[(n^8-(n+1)^8)(y-\tfrac{1}{2})^3-2(n^6-(n+1)^6)(y-\tfrac{1}{2})^2 + (n^4-(n+1)^4)(y-\tfrac{1}{2})].$$

For $0\le y-\frac{1}{2}\le 1/(n+1)^2$ (as in this case) we see that

$$|\alpha_n(y)-\alpha_{n+1}(y)|\le n\cdot k$$

for some positive constant k. Similarly:

$$|\alpha'_{n+1}(y)|-\alpha'_n(y)|\le n^3\cdot k;$$

$$|\beta_n(y)-\beta_{n+1}(y)|\le\left(\frac{1}{n^3}\right)\cdot k;$$

$$|\beta'_{n+1}(y)-\beta'_n(y)|\le\left(\frac{1}{n}\right)\cdot k;$$

and

$$|\gamma'_{n+1}(y)-\gamma'_n(y)|\le\left(\frac{1}{n^5}\right)\cdot k.$$

Thus, if $|x-\frac{1}{2}|\le 1/(2(n+1))^2$, the numerator of A_{11} is smaller in absolute value than $(1/n^5)k$ for some positive k. In case (ii) we determined that $d_n(x, y)\ge(1/n^4)k$ for some $k\ge 0$. Thus $A_{11}\to 0$ as $n\to\infty$. In this case also, $A_{21}=0$.

A_{12} is given by:

$$A_{12}=\{-[4\alpha_{n+1}(y)(x-\tfrac{1}{2})^3+2\beta_{n+1}(y)(x-\tfrac{1}{2})][\alpha'_n(y)(x-\tfrac{1}{2})^4+\beta'_n(y)(x-\tfrac{1}{2})^2+\gamma'_n(y)] + [\alpha'_{n+1}(y)(x-\tfrac{1}{2})^4+\beta'_{n+1}(y)(x-\tfrac{1}{2})^2+\gamma'_{n+1}(y)] \cdot[4\alpha_n(y)(x-\tfrac{1}{2})^3+2\beta_n(y)(x-\tfrac{1}{2})]\} \cdot(1/d_n(x, y)).$$

For $|x-\frac{1}{2}|\le 1/(2(n+1))^2$ and $\frac{1}{2}\le y\le\frac{1}{2}+1/(n+1)^2$ one finds that the numerator of A_{12} is in absolute value smaller than $(1/n^8)k$ for some $k>0$. Since $d_n(x, y)\ge(1/n^4)k$, $A_{12}\to 0$ as $n\to\infty$.

$A_{22}=0$ for all n so that in case (iii) we have

$$(DK_{n+1}(x, y))(DK_n(x, y))^{-1}-\mathbf{I}\to 0 \qquad \text{as } n\to\infty.$$

Case (iv). $1/(2(n+1))^3\le|x-\frac{1}{2}|\le 1/(2n)^3$ and $|y|\le\frac{1}{2}$. Again letting

$$(DK_{n+1}(x, y))(DK_n(x, y))^{-1}-\mathbf{I}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$$

and $|x-\frac{1}{2}|=s/(2n)^3$ we have:

$$A_{11}=\frac{\frac{1}{(2n)^4}[(3s^4-2s^2-1)y+(-4s^4+6s^2-2)]}{d_n(x, y)}.$$

Here

$$d_n(x, y)=\frac{1}{8n^4}[(\tfrac{3}{2}+3y)s^4-(1+3y)s^2+(\tfrac{1}{2}y+1)].$$

It was demonstrated in the previous Case 4 that $d_n(x, y)\ge k(1/n^4)$ for some $k>0$.

Since $s=(2n)^2|x-\frac{1}{2}|$ and

$$\frac{1}{(2(n+1))^2}\le|x-\tfrac{1}{2}|\le\frac{1}{(2n)^3},$$

$$\left(\frac{n}{n+1}\right)^2\le s\le 1.$$

Thus, $s\to 1$ as $n\to\infty$ and $A_{11}\to 0$ as $n\to\infty$.

A_{21} is identically zero.

$$A_{12}=\frac{\left(\frac{1}{n^6}\right)[\frac{1}{4}t^5-\frac{1}{2}t^3+\frac{1}{4}]}{d_n(x,y)}\to 0$$

as $n\to\infty$ since $d_n(x,y)\ge k(1/n^4)$.

$A_{22}=0$ for all (x,y). Thus, in Case (iv),

$$(DK_{n+1}(x,y))(DK_n(x,y))^{-1}-\mathbf{I}\to 0 \qquad \text{as } n\to\infty.$$

Case (*v*). $|x-\frac{1}{2}|\le 1/(2(n+1))^2$, $|y|\le\frac{1}{2}$. With the notation as above, we have

$$A_{11}=\frac{(2)^4[4-3y][(n+1)^4-n^4](x-\frac{1}{2})^4+\frac{1}{8}(\frac{1}{2}y+1)\left[\frac{1}{(n+1)^4}-\frac{1}{(n)^4}\right]}{d_n(x,y)}.$$

We see that the numerator of A_{11} is in absolute value less than $k(1/n^5)$ for some $k>0$ ($|x-\frac{1}{2}|\le 1/(2(n+1))^2$). The $d_n(x,y)$ is as in Case (iv) and is larger that $k(1/n^4)$ for some $k>0$. Thus $A_{11}\to 0$ as $n\to\infty$.

$A_{21}=0$ for all (x,y).

$$\begin{aligned}A_{12}=\Bigg\{&\left[-(2)^5n^4(x-\tfrac{1}{2})^4+4(x-\tfrac{1}{2})^2-\frac{1}{8n^4}\right]\\ &\cdot[(\tfrac{1}{2}-y)(-(2)^7)(n+1)^4(x-\tfrac{1}{2})^3-8(\tfrac{1}{2}+y)(x-\tfrac{1}{2})]\\ &+\left[2^5(n+1)^4(x-\tfrac{1}{2})^4-4(x-\tfrac{1}{2})^2+\frac{1}{8(n+1)^4}\right]\\ &\cdot[(\tfrac{1}{2}-y)(-(2)^7)n^4(x-\tfrac{1}{2})^3-8(\tfrac{1}{3}+y)(x-\tfrac{1}{2})]\Bigg\}\\ &\cdot\frac{1}{d_n(x,y)}.\end{aligned}$$

One sees that, for $|x|\le 1/(2(n+1))^2$, the numerator is in absolute value $\le k(1/n^6)$ for some $k>0$. Since $d_n(x,y)\ge k(1/n^4)$, $A_{12}\to 0$ as $n\to\infty$. $A_{22}=0$ for all (x,y). Thus, in Case (v),

$$(DK_{n+1}(x,y))(DK_n(x,y))^{-1}-\mathbf{I}\to 0 \qquad \text{as } n\to\infty.$$

There are two remaining cases to be considered. One of these is: $|x-\frac{1}{2}|\le 1/(2(n+1))^2$, $-\frac{1}{2}-1/(n+1)^2\le y\le-\frac{1}{2}$. The analysis of this case proceeds almost exactly as in Case (iii). The other remaining case is: $|x-\frac{1}{2}|\le 1/(2n)^2$, $-\frac{1}{2}-1/n^3\le y\le -\frac{1}{2}$, and either $y\le -1/(n+1)^3$ or $|x-\frac{1}{2}|\ge 1/(2(n+1))^2$. This case is very much like Case (ii). We trust the reader to check these cases. Our construction is complete.

REFERENCES

[**B**] M. Barge. Horseshoe maps and inverse limits. To appear.

[**Bi**] R. H. Bing. Snake-like continua. *Duke Math. J.* (3) **18** (1951), 653–663.

[**Br**] M. Brown. Some applications of an approximation theorem for inverse limits. *Proc. Amer. Math. Soc.* **11** (1960), 478–483.

[**B–M**] M. Barge & J. Martin. The construction of global attractors, preprint available.

[**M**] M. Misiurewicz. Embedding inverse limits of interval maps as attractors. *Fund. Math.* (1) **125** (1985), 23–40.

[**S**] S. Smale. Diffeomorphisms with many periodic points. In *Differential and Combinatorial Topology—A Symposium in Honor of Marston Mores,* (ed. Stewart S. Cairns), pp. 63–80. Princeton University Press: Princeton, New Jersey, 1965.