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A TWISTED APPROACH TO KOSTANT'S PROBLEM

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Abstract. We use Arkhipov's twisting functors to show that the universal enveloping algebra of a semi-simple complex finite-dimensional Lie algebra surjects onto the space of ad-finite endomorphisms of the simple highest weight module $L(\lambda)$, whose highest weight is associated (in the natural way) with a subset of simple roots and a simple root in this subset. This is a new step towards a complete answer to a classical question of Kostant. We also show how one can use the twisting functors to reprove the classical results related to this question.

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1. Introduction and notation. Let $\mathfrak{g}$ be a complex semi-simple finite-dimensional Lie algebra with a fixed triangular decomposition, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and $U(\mathfrak{g})$ be its universal enveloping algebra. Then for every two $\mathfrak{g}$-modules M and N the space $\mathrm{Hom}_{\mathbb{C}}(M, N)$ can be viewed as a $U(\mathfrak{g})$-bimodule in the natural way. This bimodule then also becomes a $\mathfrak{g}$-module under the adjoint action. The bimodule $\mathrm{Hom}_{\mathbb{C}}(M, N)$ has a sub-bimodule, usually denoted by $\mathscr{L}(M, N)$ (see for example [**14**, Kapitel 6]), which consists of all elements, on which the adjoint action of $U(\mathfrak{g})$ is locally finite. Since $U(\mathfrak{g})$ itself consists of locally finite elements under the adjoint action, it naturally maps to $\mathscr{L}(M, M)$ for every $\mathfrak{g}$-module M, and the kernel of this map is the annihilator $\mathrm{Ann}(M)$ of M in $U(\mathfrak{g})$. The classical problem of Kostant (see for example [**16**]) is formulated in the following way.

For which simple $\mathfrak{g}$-modules M is the natural injection

$$U(\mathfrak{g})/\mathrm{Ann}(M) \hookrightarrow \mathscr{L}(M, M)$$

surjective?

The complete answer to this problem is not known even for simple highest weight modules. However, it is known that there are simple highest weight modules for which the answer is negative (see for example [**16**, 9.5]). There is also a classical class of simple highest weight modules, for which the answer is positive. It consists of all simple highest weight modules, whose highest weights are obtained from the antidominant one by applying the longest element of some parabolic subgroup of the Weyl group; see [**11**, **16**, **14**].

In the present paper we propose an approach to this problem that uses Arkhipov's twisting functors (see [**2**]) and is based on the properties of these functors obtained in [**3**]. In [**18**] it was shown that Arkhipov's functors are adjoint to Joseph's completion functors (see [**15**]), which suggests a close connection to Kostant's problem. We base our arguments mostly on the results of [**3**] and also use some results from [**17**, **18**, **22**].

The main properties of the twisting functors that we use are the combinatorics of their action on Verma modules and the fact that they define a self-equivalence of the bounded derived category $\mathcal{D}^b(\mathcal{O})$ of the BGG-category $\mathcal{O}$. All this can be found in [**3**].

Let R be the root system of $\mathfrak{g}$ with basis B that corresponds to the triangular decomposition above. Further let W denote the Weyl group of $\mathfrak{g}$ with identity element e. We denote by $<$ the Bruhat order on W. Then W acts on $\mathfrak{h}^*$ both in the natural way (i.e. $\lambda \mapsto w(\lambda)$ for $\lambda \in \mathfrak{h}^*$ and $w \in W$) and via the *dot action* defined as follows: $w \cdot \lambda = w(\lambda + \rho) - \rho$, $\lambda \in \mathfrak{h}^*$, $w \in W$, where ρ is the half of the sum of all positive roots. For $\alpha \in R$ denote by s_α the corresponding reflection and for a reflection $s \in W$ we let $\alpha_s \in R$ be such that $s = s_{\alpha_s}$. Fix some Weyl-Chevalley basis in $\mathfrak{g}$, say

$$\{X_\alpha \,:\, \alpha \in R\} \cup \{H_\beta \,:\, \beta \in B\},$$

and define H_α, $\alpha \in R$, in the usual way.

For $\lambda \in \mathfrak{h}^*$ the set $R_\lambda = \{\alpha \in R \,:\, \lambda(H_\alpha) \in \mathbb{Z}\}$ is a root system and the triangular decomposition of $\mathfrak{g}$ induces a uniquely defined basis B_λ of R_λ. Let W_λ be the Weyl group of R_λ. We call λ *relatively dominant* provided that λ is a dominant element in $\{w \cdot \lambda \,:\, w \in W_\lambda\}$ and *regular* provided that the stabilizer of λ in W_λ with respect to the dot action is trivial.

Throughout the paper we fix a relatively dominant and regular $\lambda \in \mathfrak{h}^*$.

For $w \in W_\lambda$ we denote by $\Delta(w)$ the Verma module with the highest weight $w \cdot \lambda$, and by $L(w)$ the unique simple quotient of $\Delta(w)$; see [**8**, Chapter 7]. For $S \subset B_\lambda$ we denote by W_λ^S the subgroup of W_λ, generated by s_α, $\alpha \in S$. Denote by w_λ^S the longest element in W_λ^S (in particular, $w_\lambda^{B_\lambda}$ is the longest element in W_λ). The main result of the present paper is the following statement.

THEOREM 1. *Let $S \subset B_\lambda$, $\alpha \in S$, and set $\mathbf{w} = s_\alpha w_\lambda^S w_\lambda^{B_\lambda}$. Then the canonical inclusion*

$$U(\mathfrak{g})/\mathrm{Ann}(L(\mathbf{w})) \hookrightarrow \mathscr{L}(L(\mathbf{w}), L(\mathbf{w}))$$

is surjective.

The paper is organized as follows. In Section 2 we collect all necessary preliminaries on the category $\mathcal{O}$ and Arkhipov's twisting functors. In Section 3 we show how one can apply the twisting functors to obtain the classical results related to Kostant's problem. In principle if one takes into account the relation between the twisting functors and Joseph's completion functors, obtained in [**18**], our approach here is rather similar to the original approach. However, here it is formulated in a shorter way. In Section 4 we prove Theorem 1 in the case $S = B_\lambda$. This is then used in Section 5 to prove Theorem 1 in the general case. In Section 6 we present an application of Theorem 1 and answer Kostant's question for some simple α-stratified modules.

2. Preliminaries about the category $\mathcal{O}$. Let $\mathcal{O}$ denote the BGG-category $\mathcal{O}$, associated with the triangular decomposition of $\mathfrak{g}$, fixed above. See [**5**]. Let $^\star : \mathcal{O} \to \mathcal{O}$ be the classical duality on $\mathcal{O}$; that is a contravariant exact involutive equivalence, preserving the isomorphism classes of simple module; see [**13**, Section 5]. Let $\mathcal{O}_\lambda$ denote the indecomposable block of $\mathcal{O}$, whose simple modules have the form $L(w)$, $w \in W_\lambda$. Denote further by $P(w)$ the indecomposable projective cover of $L(w)$ (see [**5**]) and by θ_w the indecomposable projective functor on $\mathcal{O}_\lambda$, uniquely determined by

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the property $\theta_w \Delta(e) \cong P(w)$; see [**4**, I.3]. Then $\{\theta_w : w \in W_\lambda\}$ are exactly the direct summands of the composition of $V\otimes\_$ followed by the projection from $\mathcal{O}$ to $\mathcal{O}_\lambda$, if we let V run through all the finite-dimensional $\mathfrak{g}$-modules.

For $w \in W_\lambda$ set $\nabla(w) = \Delta(w)^\star$ and denote by $\mathcal{F}_\lambda(\Delta)$ the full subcategory of $\mathcal{O}_\lambda$, which consists of all modules, having a filtration, whose subquotients are isomorphic to Verma modules. Set $\mathcal{F}_\lambda(\nabla) = \mathcal{F}_\lambda(\Delta)^\star$.

Let $\mathcal{D}^b(\mathcal{O}_\lambda)$ denote the bounded derived category of $\mathcal{O}_\lambda$. For a right or a left exact functor F on $\mathcal{O}_\lambda$ we denote by $\mathcal{L}F$ and $\mathcal{R}F$ the corresponding left and right derived functors respectively. For $i \geq 0$ we denote by $\mathcal{L}_i F$ and $\mathcal{R}^i F$ the corresponding i-th cohomology functors. We denote by [1] the shifting functor on $\mathcal{D}^b(\mathcal{O}_\lambda)$ such that for every complex $\mathcal{X}^\bullet \in \mathcal{D}^b(\mathcal{O}_\lambda)$ and for all $i \in \mathbb{Z}$ we have $\mathcal{X}[1]^i = \mathcal{X}^{i+1}$. We consider $\mathcal{O}_\lambda$ as a subcategory of $\mathcal{D}^b(\mathcal{O}_\lambda)$ via the classical embedding in degree zero.

Via the equivalence from [**23**] for $w \in W_\lambda$ we can define on $\mathcal{O}_\lambda$ *Arkhipov's twisting functor* $\mathtt{T}_w$. See [**2**, **3**, **18**]. Denote by $\mathtt{G}_w$ its right adjoint (which is isomorphic, by [**18**, Corollary 6], to the corresponding Joseph's completion functor from [**15**], and to the functor $\star\mathtt{T}_w\star$, see [**3**, Theorem 4.1]). In this paper we shall use the following properties of $\mathtt{T}_w$. (The functor $\mathtt{G}_w$ has dual properties.)

(I) For every $w, x \in W_\lambda$ we have $\mathtt{T}_w\theta_x \cong \theta_x\mathtt{T}_w$. See [**3**, Theorem 3.2].

(II) For every $w, x \in W_\lambda$ and $i > 0$ we have $\mathcal{L}_i\mathtt{T}_w\Delta(x) = 0$. See [**3**, Theorem 2.2].

(III) For every $w \in W_\lambda$ the functor $\mathcal{L}\mathtt{T}_w$ is an autoequivalence of $\mathcal{D}^b(\mathcal{O}_\lambda)$ with inverse functor $\mathcal{R}\mathtt{G}_{w^{-1}}$. See [**3**, Corollary 4.2].

(IV) For every $w \in W_\lambda$ and every reduced decomposition, $w = s_1 \ldots s_k$, we have $\mathtt{T}_w \cong \mathtt{T}_{s_1} \cdot \cdots \cdot \mathtt{T}_{s_k}$. See [**3**, Lemma 2.1] and [**18**, Corollary 11].

(V) For every $x \in W_\lambda$ and every simple reflection $s \in W_\lambda$ such that $sx > x$ we have $\mathtt{T}_s\Delta(x) \cong \Delta(sx)$. See [**1**, Lemma 6.2].

(VI) For every $x \in W_\lambda$ and every simple reflection $s \in W_\lambda$ we have

$$\mathtt{T}_s\nabla(x) \cong \begin{cases} \nabla(x) & (x < sx), \\ \nabla(sx) & (x > sx). \end{cases}$$

See [**3**, Theorem 2.3].

(VII) For every $x \in W_\lambda$ and every simple reflection $s \in W_\lambda$ we have that $\mathtt{T}_s L(x) \neq 0$ if and only if $sx < x$. See [**3**, Section 6].

(VIII) For every simple reflection $s \in W_\lambda$ and for every $M \in \mathcal{O}_\lambda$ the module $\mathcal{L}_1\mathtt{T}_s(M)$ is the largest s-finite submodule of M. See [**22**, Theorem 1] and [**17**, Proposition 6].

3. The classical results. We start with some preparation, during which we use the twisting functors to obtain several classical results related to Kostant's problem. We base our approach on two classical statements. The first one, which can be found in [**14**, 6.8], is a very abstract property of $\mathscr{L}(M, N)$.

PROPOSITION 2. *Let M, N be $\mathfrak{g}$-modules and V be a finite-dimensional $\mathfrak{g}$-module. Then there are canonical isomorphisms*

$$\mathrm{Hom}_{\mathfrak{g}}(V, \mathscr{L}(M, N)) \cong \mathrm{Hom}_{\mathfrak{g}}(M \otimes V, N) \cong \mathrm{Hom}_{\mathfrak{g}}(M, N \otimes V^*), \tag{1}$$

where $\mathscr{L}(M, N)$ is considered as a $\mathfrak{g}$-module under the adjoint action.

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The second statement is the classical positive answer to Kostant's problem for projective Verma modules. In [**14**, 6.9] it is shown that Proposition 3 holds.

PROPOSITION 3. *For every submodule $M \subset \Delta(e)$ the canonical inclusion*

$$U(\mathfrak{g})/\mathrm{Ann}(\Delta(e)/M) \hookrightarrow \mathscr{L}(\Delta(e)/M, \Delta(e)/M)$$

is surjective, in particular, the canonical inclusion

$$U(\mathfrak{g})/\mathrm{Ann}(\Delta(e)) \hookrightarrow \mathscr{L}(\Delta(e), \Delta(e))$$

is surjective.

Using the twisting functors we obtain the following result.

COROLLARY 4. ([**16**, Corollary 6.4], [**14**, 7.25]) *For every $w \in W_\lambda$ the canonical inclusion*

$$U(\mathfrak{g})/\mathrm{Ann}(\Delta(w)) \hookrightarrow \mathscr{L}(\Delta(w), \Delta(w))$$

is surjective.

Proof. We have the obvious map $\mathscr{L}(\Delta(w), \Delta(w)) \to \mathscr{L}(\Delta(e), \Delta(e))$ induced by the inclusion $\Delta(w) \subset \Delta(e)$. Since $\mathrm{Ann}(\Delta(w)) = \mathrm{Ann}(\Delta(e))$ by [**8**, Theorem 8.4.4], it is enough to show that for every simple finite-dimensional $\mathfrak{g}$-module V we have the equality

$$\dim \mathrm{Hom}_{\mathfrak{g}}(V, \mathscr{L}(\Delta(w), \Delta(w))) = \dim \mathrm{Hom}_{\mathfrak{g}}(V, \mathscr{L}(\Delta(e), \Delta(e))).$$

For this we compute

$$\begin{array}{ll}
\mathrm{Hom}_{\mathfrak{g}}(V, \mathscr{L}(\Delta(w), \Delta(w))) & = (1) \\
\mathrm{Hom}_{\mathfrak{g}}(\Delta(w), \Delta(w) \otimes V^*) & = \text{(IV) and (V)} \\
\mathrm{Hom}_{\mathfrak{g}}(\mathrm{T}_w \Delta(e), \mathrm{T}_w(\Delta(e)) \otimes V^*) & = \text{(I)} \\
\mathrm{Hom}_{\mathfrak{g}}(\mathrm{T}_w \Delta(e), \mathrm{T}_w(\Delta(e) \otimes V^*)) & = \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_\lambda)}(\mathrm{T}_w \Delta(e), \mathrm{T}_w(\Delta(e) \otimes V^*)) & = \text{(II)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_\lambda)}(\mathcal{L}\mathrm{T}_w \Delta(e), \mathcal{L}\mathrm{T}_w(\Delta(e) \otimes V^*)) & = \text{(III)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O}_\lambda)}(\Delta(e), \Delta(e) \otimes V^*) & = \\
\mathrm{Hom}_{\mathfrak{g}}(\Delta(e), \Delta(e) \otimes V^*) & = (1) \\
\mathrm{Hom}_{\mathfrak{g}}(V, \mathscr{L}(\Delta(e), \Delta(e))). &
\end{array}$$

This completes the proof. □

PROPOSITION 5. *Let $w \in W_\lambda$ and*

$$0 \to X \to \Delta(w) \to Y \to 0 \tag{2}$$

be a short exact sequence such that for every finite-dimensional $\mathfrak{g}$-module V we have

$$\mathrm{Ext}^1_{\mathcal{O}}(\Delta(w), X \otimes V) = 0. \tag{3}$$

Then the canonical inclusion

$$U(\mathfrak{g})/\mathrm{Ann}(Y) \hookrightarrow \mathscr{L}(Y, Y)$$

is surjective.

Proof. Applying $\mathrm{Hom}_{\mathfrak{g}}(\Delta(w), \_ \otimes V)$ to (2) and using (3) yields the short exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(\Delta(w), X \otimes V) \to \mathrm{Hom}_{\mathfrak{g}}(\Delta(w), \Delta(w) \otimes V) \to \\ \to \mathrm{Hom}_{\mathfrak{g}}(\Delta(w), Y \otimes V) \to 0,$$

which implies that $\mathscr{L}(\Delta(w), \Delta(w))$ surjects onto $\mathscr{L}(\Delta(w), Y)$, where the vector space $\mathscr{L}(Y, Y)$ is a subspace. Since $U(\mathfrak{g})$ surjects onto $\mathscr{L}(\Delta(w), \Delta(w))$, by Corollary 4, the statement follows. $\square$

Now we can prove the classical result of Gabber and Joseph.

THEOREM 6. ([**11**, Theorem 4.4], [**14**, 7.32]) *Let $S \subset B_\lambda$ and $\overline{\mathbf{w}} = w_\lambda^S w_\lambda^{B_\lambda}$. Then the canonical inclusion*

$$U(\mathfrak{g})/\mathrm{Ann}(L(\overline{\mathbf{w}})) \hookrightarrow \mathscr{L}(L(\overline{\mathbf{w}}), L(\overline{\mathbf{w}}))$$

is surjective.

Proof. Let V be a finite-dimensional $\mathfrak{g}$-module. Consider the short exact sequence

$$0 \to K(\overline{\mathbf{w}}) \to \Delta(\overline{\mathbf{w}}) \to L(\overline{\mathbf{w}}) \to 0, \tag{4}$$

where $K(\overline{\mathbf{w}})$ is just the kernel of the canonical projection from $\Delta(\overline{\mathbf{w}})$ to $L(\overline{\mathbf{w}})$. Then we have

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathcal{O}}(\Delta(\overline{\mathbf{w}}), K(\overline{\mathbf{w}}) \otimes V) && = \\
&\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\Delta(\overline{\mathbf{w}}), K(\overline{\mathbf{w}}) \otimes V[1]) && = (\text{duals of (IV) and (V)}) \\
&\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(\mathtt{G}_{w_\lambda^S}\Delta\big(w_\lambda^{B_\lambda}\big), K(\overline{\mathbf{w}}) \otimes V[1]\big) && = (_ \otimes V \text{ is exact} \\
& && \quad\ \text{and preserves projectives}) \\
&\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(V^* \otimes \mathtt{G}_{w_\lambda^S}\Delta\big(w_\lambda^{B_\lambda}\big), K(\overline{\mathbf{w}})[1]\big) && = (\text{dual of (I)}) \\
&\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(\mathtt{G}_{w_\lambda^S}\big(V^* \otimes \Delta\big(w_\lambda^{B_\lambda}\big)\big), K(\overline{\mathbf{w}})[1]\big) && = (\Delta(w_\lambda^{B_\lambda}) = \nabla(w_\lambda^{B_\lambda})) \\
&\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(\mathtt{G}_{w_\lambda^S}\big(V^* \otimes \nabla\big(w_\lambda^{B_\lambda}\big)\big), K(\overline{\mathbf{w}})[1]\big) && = (\text{dual of (II)}) \\
&\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(\mathcal{R}\mathtt{G}_{w_\lambda^S}\big(V^* \otimes \nabla\big(w_\lambda^{B_\lambda}\big)\big), K(\overline{\mathbf{w}})[1]\big) && = (\text{III}) \\
&\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(V^* \otimes \nabla\big(w_\lambda^{B_\lambda}\big), \mathcal{L}\mathtt{T}_{w_\lambda^S}K(\overline{\mathbf{w}})[1]\big).
\end{aligned}$$

Let us calculate $\mathcal{L}\mathtt{T}_{w_\lambda^S}K(\overline{\mathbf{w}})$. Because of our choice of w_λ^S we can use Proposition 11, which will be proved in Section 5 (alternatively one can use [**11**, Section 2]), and [**6**] to get that the module $K(\overline{\mathbf{w}})$ admits a BGG-type resolution, which has the following form:

$$0 \to X^k \to X^{k-1} \to \cdots \to X^1 \to X^0 \to K(\overline{\mathbf{w}}) \to 0,$$

where every X^i is a direct sum of some $\Delta(xw_\lambda^{B_\lambda})$ with $x \in W_\lambda^S$, $x \neq w_\lambda^S$. Let $\mathcal{X}^\bullet$ denote the corresponding complex in $\mathcal{D}^b(\mathcal{O}_\lambda)$. Using (II) we have

$$\mathcal{L}\mathtt{T}_{w_\lambda^S}K(\overline{\mathbf{w}}) = \mathcal{L}\mathtt{T}_{w_\lambda^S}\mathcal{X}^\bullet = \mathtt{T}_{w_\lambda^S}\mathcal{X}^\bullet.$$

Now for every $x \in W_\lambda^S$, $x \neq w_\lambda^S$, let $y \in W_\lambda^S$ be such that $yx^{-1} = w_\lambda^S$. Then, using (IV), (V) and (VI), we have

$$\mathtt{T}_{w_\lambda^S}\Delta\big(xw_\lambda^{B_\lambda}\big) = \mathtt{T}_y\mathtt{T}_{x^{-1}}\Delta\big(xw_\lambda^{B_\lambda}\big) = \mathtt{T}_y\Delta\big(w_\lambda^{B_\lambda}\big) = \mathtt{T}_y\nabla\big(w_\lambda^{B_\lambda}\big) = \nabla\big(yw_\lambda^{B_\lambda}\big).$$

This implies that $\mathtt{T}_{w_\lambda^S}\mathcal{X}^\bullet$ is a complex of dual Verma modules in $\mathcal{O}_\lambda$. At the same time the module $\nabla(w_\lambda^{B_\lambda}) \otimes V^* \cong \Delta(w_\lambda^{B_\lambda}) \otimes V^*$ is a tilting module in $\mathcal{O}_\lambda$. Hence, by [**12**, Chap. III, Lemma 2.1], the space

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(\nabla\big(w_\lambda^{B_\lambda}\big) \otimes V^*, \mathcal{L}\mathtt{T}_{w_\lambda^S}K(\overline{\mathbf{w}})[1]\big)$$

can be computed already in the homotopy category, where it is obviously zero, since the only non-zero component of the first complex is in degree zero and the above computation shows that the zero component of the second complex is zero. Hence we obtain

$$\mathrm{Ext}^1_{\mathcal{O}}(\Delta(\overline{\mathbf{w}}), K(\overline{\mathbf{w}}) \otimes V) = 0. \tag{5}$$

The statement of our theorem now follows by applying Proposition 5 to the short exact sequence (4). □

4. Proof of Theorem 1: the case $S = B_\lambda$. In this section we prove Theorem 1 in the case $S = B_\lambda$. Throughout the section we fix $\alpha \in B_\lambda$ and set $s = s_\alpha$.

PROPOSITION 7. *The canonical inclusion $U(\mathfrak{g})/\mathrm{Ann}(L(s)) \hookrightarrow \mathscr{L}(L(s), L(s))$ is surjective.*

To prove this statement we shall need several lemmas.

LEMMA 8. *Let $w \in W_\lambda$ be such that* $\mathrm{Hom}_{\mathfrak{g}}(L(s), \theta_w L(s)) \neq 0$. *Then $w = s$ or $w = e$.*

Proof. Assume that $w \neq e, s$. Let

$$0 \to L(s) \to X \to L(e) \to 0 \tag{6}$$

be a non-split short exact sequence, which exists because of the Kazhdan-Lusztig theorem. (See for example [**19**, Theorem 1].) Then $\theta_w L(e) = 0$, since $w \neq e$, and hence $\theta_w X = \theta_w L(s)$. However, since (6) is non-split, X is a homomorphic image of $\Delta(e)$, and hence $\theta_w X$ is a homomorphic image of $\theta_w\Delta(e) = P(w)$. In particular, $\theta_w X$ is either zero or has simple top $L(w)$. On the other hand $\theta_w L(s)$ is self-dual and thus $\theta_w X = \theta_w L(s)$ is either zero or has simple socle $L(w)$. In each of these two cases we have the equality $\mathrm{Hom}_{\mathfrak{g}}(L(s), \theta_w L(s)) = 0$ since $w \neq s$. This completes the proof. □

The above result naturally motivates the following question.

QUESTION. *Let $S \subset B_\lambda$ and $w \in W_\lambda$ be such that the vector space* $\mathrm{Hom}_{\mathfrak{g}}\big(L\big(w_\lambda^S\big), \theta_w L\big(w_\lambda^S\big)\big)$ *is non-zero. Does this imply that $w \in W_\lambda^S$?*

Recall that a $\mathfrak{g}$-module, M, is called *s-finite* provided that it is locally finite over the $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$, which corresponds to s. The module $L(w)$ is s-finite if and only if w is the minimal coset representative of some coset from $\{e, s\}\backslash W_\lambda$, that is if and only if $sw > w$.

Define $F(s)$ as the minimal submodule of the radical $\mathrm{Rad}(\Delta(s))$ of $\Delta(s)$ such that the quotient $\mathrm{Rad}(\Delta(s))/F(s)$ is s-finite and consider the short exact sequence

$$0 \to F(s) \to \Delta(s) \to N(s) \to 0, \tag{7}$$

where $N(s)$ is the cokernel. Our next step is to prove the following result.

LEMMA 9. *The canonical inclusion* $U(\mathfrak{g})/\mathrm{Ann}(N(s)) \hookrightarrow \mathscr{L}(N(s), N(s))$ *is surjective.*

Proof. For every $w \in W_\lambda$ we have

$$\begin{array}{ll}
\mathrm{Ext}^1_{\mathcal{O}}(\Delta(s), \theta_w F(s)) & = \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\Delta(s), \theta_w F(s)[1]) & = \text{(V)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathtt{T}_s\Delta(e), \theta_w F(s)[1]) & = \text{(properties of } \theta_w) \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(\theta_{w^{-1}}\mathtt{T}_s\Delta(e), F(s)[1]\big) & = \text{(I)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}\big(\mathtt{T}_s\theta_{w^{-1}}\Delta(e), F(s)[1]\big) & = \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathtt{T}_s P(w^{-1}), F(s)[1]) & = \text{(II)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{L}\mathtt{T}_s P(w^{-1}), F(s)[1]) & = \text{(III)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(P(w^{-1}), \mathcal{R}\mathtt{G}_s F(s)[1]) & = (P(w^{-1}) \text{ is projective}) \\
\mathrm{Hom}_{\mathfrak{g}}(P(w^{-1}), \mathcal{R}^1\mathtt{G}_s F(s)) & = \text{(dual of (VIII).} \\
0. &
\end{array}$$

The statement now follows from Corollary 4 and Proposition 5. □

Consider now the short exact sequence

$$0 \to X(s) \to N(s) \xrightarrow{p'} L(s) \to 0. \tag{8}$$

LEMMA 10. *For every finite-dimensional* $\mathfrak{g}$*-module* V *the sequence* (8) *induces the following isomorphism:*

$$\mathrm{Hom}_{\mathfrak{g}}(N(s), N(s) \otimes V) \cong \mathrm{Hom}_{\mathfrak{g}}(L(s), L(s) \otimes V).$$

Proof. Let $0 \neq f \in \mathrm{Hom}_{\mathfrak{g}}(N(s), N(s) \otimes V)$. Since $N(s)$ has simple top, f cannot annihilate it. Consider the map $(p' \otimes \mathrm{id}) \circ f \in \mathrm{Hom}_{\mathfrak{g}}(N(s), L(s) \otimes V)$. Since the kernel of the projection $N(s) \otimes V \overset{p'\otimes \mathrm{id}}{\twoheadrightarrow} L(s) \otimes V$ is s-finite and the top of $N(s)$ is not, we have $(p' \otimes \mathrm{id}) \circ f \neq 0$. On the other hand, since the socle of $L(s) \otimes V$ consists exclusively of s-infinite modules, the map $(p' \otimes \mathrm{id}) \circ f$ must annihilate $X(s)$ and hence it factors through $L(s)$. This implies that (8) induces the following inclusion:

$$\mathrm{Hom}_{\mathfrak{g}}(N(s), N(s) \otimes V) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(L(s), L(s) \otimes V).$$

To complete the proof we now have to compare the dimensions. Thus it is enough to show that, for every $w \in W_\lambda$, we have

$$\dim \mathrm{Hom}_{\mathfrak{g}}(N(s), \theta_w N(s)) = \dim \mathrm{Hom}_{\mathfrak{g}}(L(s), \theta_w L(s)).$$

This is obvious for $w = e$ since both spaces are one-dimensional in this case. For $w = s$ we have non-zero adjunction morphisms in both spaces; moreover, the module $\theta_s L(s)$ has simple socle. This implies that

$$\dim \mathrm{Hom}_{\mathfrak{g}}(N(s), \theta_s N(s)) = \dim \mathrm{Hom}_{\mathfrak{g}}(L(s), \theta_s L(s)) = 1.$$

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For $w \neq s, e$, Lemma 8 implies that $\dim \mathrm{Hom}_{\mathfrak{g}}(L(s), \theta_w L(s)) = 0$. The statement follows. □

Now we are ready to prove Proposition 7.

Proof of Proposition 7. Since $X(s)$ is s-finite and $L(s)$ is simple and s-infinite, we have $\mathscr{L}(X(s), L(s)) = 0$ by Proposition 2, which implies that (8) induces the isomorphism $\mathscr{L}(L(s), L(s)) \cong \mathscr{L}(N(s), L(s))$.

Since $X(s)$ is s-finite and the top of $N(s)$ is simple and s-infinite, we have $\mathscr{L}(N(s), X(s)) = 0$ by Proposition 2, which implies that (8) induces the inclusion $\mathscr{L}(N(s), N(s)) \hookrightarrow \mathscr{L}(N(s), L(s))$. However, Lemma 10 and Proposition 2 show that this inclusion is in fact an isomorphism. Since $U(\mathfrak{g})$ surjects onto $\mathscr{L}(N(s), N(s))$, by Corollary 4, it follows that (8) induces a surjection of $U(\mathfrak{g})$ onto $\mathscr{L}(L(s), L(s))$. This completes the proof. □

5. Proof of Theorem 1: the general case. In this Section we prove Theorem 1 in the general case. Our approach is similar to the one we use in Section 4. However, it requires more delicate arguments in several places. Moreover, in some places we shall use the reduction to the case considered in Section 4. Set $s = s_\alpha$, where $\alpha \in S$, and recall the notation $\overline{\mathbf{w}} = w_\lambda^S w_\lambda^{B_\lambda}$, $\mathbf{w} = s\overline{\mathbf{w}}$.

Using the equivalence from **[23]** we can assume that λ is integral. Let $\mathfrak{a} = \mathfrak{a}(S)$ denote the semi-simple Lie subalgebra of $\mathfrak{g}$, generated by $X_{\pm\alpha}$, $\alpha \in S$. If M is a weight $\mathfrak{g}$-module with the weight-space decomposition $M = \oplus_{\mu\in\mathfrak{h}^*} M_\mu$, and $\nu \in \mathfrak{h}^*$, then the subspace

$$M_{\mathfrak{a}}^{\nu} = \oplus_{\mu\in\nu+\mathbb{Z}S} M_\mu$$

is stable under the action of $\mathfrak{a}$ and hence is an $\mathfrak{a}$-submodule of M. This induces a functor, which we shall denote by R^ν, from the category of all weight $\mathfrak{g}$-modules to the category of all weight $\mathfrak{a}$-modules, that sends M to $M_{\mathfrak{a}}^{\nu}$. Let $\mathcal{O}^{\mathfrak{a}}$ denote the category $\mathcal{O}$ for the algebra $\mathfrak{a}$. From the PBW theorem it follows that for every $w \in W_\lambda$ and every $\nu \in \mathfrak{h}^*$ the module $\mathrm{R}^\nu\Delta(w)$ has a finite Verma flag as an $\mathfrak{a}$-module. In particular, $\mathrm{R}^\nu\Delta(w) \in \mathcal{O}^{\mathfrak{a}}$. From this one easily deduces that R^ν maps $\mathcal{O}$ to $\mathcal{O}^{\mathfrak{a}}$.

Let $\mathfrak{h}^\perp$ be the orthogonal complement to $\mathfrak{a}\cap\mathfrak{h}$ in $\mathfrak{h}$ with respect to the Killing form. Let ξ be the restriction of $\overline{\mathbf{w}}\cdot\lambda$ to $\mathfrak{h}^\perp$. Define the *parabolic induction functor* $\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{g}}$ in the following way: for $M \in \mathcal{O}^{\mathfrak{a}}$ let $\mathfrak{h}^\perp$ act on M via ξ, and let $X_\alpha M = 0$ for all positive roots $\alpha \in R$ such that $X_\alpha \notin \mathfrak{a}$. In this way we can regard M as a module over the parabolic subalgebra $\mathfrak{p} = \mathfrak{a} + \mathfrak{h} + \mathfrak{n}_+$ of $\mathfrak{g}$. We set

$$\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{g}}(M) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M,$$

which obviously defines a functor from $\mathcal{O}^{\mathfrak{a}}$ to $\mathcal{O}$. From the PBW theorem it follows that this functor sends Verma modules to Verma modules. Let ζ be the restriction of $\overline{\mathbf{w}}\cdot\lambda$ to $\mathfrak{a}\cap\mathfrak{h}$. Note that ζ is regular and dominant for $\mathfrak{a}$.

Finally, denote by $\mathcal{C}$ the full subcategory of $\mathcal{O}^\lambda$, that consists of all modules M, whose composition factors all have the form $L(y)$, $y \in W_\lambda^S w_\lambda^{B_\lambda}$.

PROPOSITION 11. $\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{g}}$ *and* $\mathrm{R}^{\overline{\mathbf{w}}\cdot\lambda}$ *induce mutually inverse equivalences between* $\mathcal{O}_\zeta^{\mathfrak{a}}$ *and* $\mathcal{C}$.

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Proof. The classical adjunction between the restriction and induction implies that $(\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{g}}, \mathtt{R}^{\overline{\mathbf{w}}\cdot\lambda})$ is an adjoint pair of functors, which gives us the natural maps

$$\mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{g}} \mathtt{R}^{\overline{\mathbf{w}}\cdot\lambda} \to \mathrm{Id}_{\mathcal{C}}, \quad \text{and} \quad \mathrm{Id}_{\mathcal{O}_{\zeta}^{\mathfrak{a}}} \to \mathtt{R}^{\overline{\mathbf{w}}\cdot\lambda} \mathrm{Ind}_{\mathfrak{a}}^{\mathfrak{g}}.$$

These maps are obviously isomorphisms on Verma modules, and then by induction one shows that they are isomorphisms on simple modules. The statement follows. □

As in Section 4 we define $F(\mathbf{w})$ as the minimal submodule of the radical $\mathrm{Rad}(\Delta(\mathbf{w}))$ of $\Delta(\mathbf{w})$ such that the quotient $\mathrm{Rad}(\Delta(\mathbf{w}))/F(\mathbf{w})$ is s-finite and consider the short exact sequence

$$0 \to F(\mathbf{w}) \to \Delta(\mathbf{w}) \to N(\mathbf{w}) \to 0, \tag{9}$$

where $N(\mathbf{w})$ is the cokernel.

Proposition 12. *The canonical inclusion*

$$U(\mathfrak{g})/\mathrm{Ann}(N(\mathbf{w})) \hookrightarrow \mathscr{L}(N(\mathbf{w}), N(\mathbf{w}))$$

is surjective.

Proof. Let $w \in W_\lambda$. Using the same arguments as in the proof of Lemma 9 we obtain

$$\mathrm{Ext}^1_{\mathcal{O}}(\Delta(\mathbf{w}), \theta_w F(\mathbf{w})) = \mathrm{Ext}^1_{\mathcal{O}}(\theta_{w^{-1}} \Delta(\overline{\mathbf{w}}), \mathtt{G}_s F(\mathbf{w})).$$

Let us prove that the last space is zero. For this we shall need the following statement.

Lemma 13. *All simple subquotients of* $\mathtt{G}_s F(\mathbf{w})$ *are of the form* $L(x)$, $x \in W_\lambda^S w_\lambda^{B_\lambda}$.

Proof. From the definition of W_λ^S and $w_\lambda^{B_\lambda}$ for $z \in W$ we have $z > w_\lambda^{B_\lambda}$ implies that $z \in W_\lambda^S w_\lambda^{B_\lambda}$. In particular, from the BGG Theorem ([**8**, Theorem 7.6.23]) it follows that all composition subquotients of $\Delta(z)$, $z \in W_\lambda^S w_\lambda^{B_\lambda}$, are of the form $L(y)$, $y \in W_\lambda^S w_\lambda^{B_\lambda}$. Therefore, all composition subquotients of $F(\mathbf{w})$ are of this form since $\mathbf{w} \in W_\lambda^S w_\lambda^{B_\lambda}$ and $F(\mathbf{w}) \subset \Delta(\mathbf{w})$.

Using the left exactness of $\mathtt{G}_s$ it is enough to prove that for every $y \in W_\lambda^S w_\lambda^{B_\lambda}$ all simple subquotients of $\mathtt{G}_s L(y)$ are of the form $L(x)$, $x \in W_\lambda^S w_\lambda^{B_\lambda}$. By (VII), we can even assume $sy < y$. Note that we automatically have $sy \in W_\lambda^S w_\lambda^{B_\lambda}$ provided that $y \in W_\lambda^S w_\lambda^{B_\lambda}$ since $\alpha \in S$. Applying $\mathtt{G}_s$ to the short exact sequence

$$0 \to K(y) \to \Delta(y) \to L(y) \to 0$$

and using the dual of (VI) we obtain the following exact sequence:

$$0 \to \mathtt{G}_s K(y) \to \mathtt{G}_s \Delta(y)\,(\cong \Delta(sy)) \to \mathtt{G}_s L(y) \to \mathcal{L}_1 \mathtt{G}_s K(y).$$

From the first paragraph of the proof we have that all simple subquotients of $\Delta(sy)$ and $\Delta(y)$ have the necessary form as $sy, y \in W_\lambda^S w_\lambda^{B_\lambda}$. From the dual of (VIII) it follows that all simple subquotients of $\mathcal{L}_1 \mathtt{G}_s K(y)$ have the necessary form as well. The statement follows. □

We have $\theta_{w^{-1}} \Delta(\overline{\mathbf{w}}) \in \mathcal{F}_\lambda(\Delta)$. Let $Q_1 = \oplus_{x \in W_\lambda \setminus W_\lambda^S w_\lambda^{B_\lambda}} P(x)$ and consider the trace Z (i.e. the sum of the images of all homomorphism) of Q_1 in $\theta_{w^{-1}} \Delta(\overline{\mathbf{w}})$. The module

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$\theta_{w^{-1}}\Delta(\overline{\mathbf{w}})$ has a Verma flag, say

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = \theta_{w^{-1}}\Delta(\overline{\mathbf{w}}). \tag{10}$$

Assume that for $i \in \{1, \dots, l\}$ we have $M_i/M_{i-1} \cong \Delta(y)$ and $M_{i+1}/M_i \cong \Delta(x)$ for some $y \in W_\lambda^S w_\lambda^{B_\lambda}$ and $x \notin W_\lambda^S w_\lambda^{B_\lambda}$. Then $y \not\leq x$, and hence

$$\mathrm{Ext}^1_{\mathcal{O}}(\Delta(x), \Delta(y)) = 0.$$

This means that we can substitute M_i in (10) by some other submodule M_i' of M_{i+1}, which contains M_{i-1}, and such that $M_i'/M_{i-1} \cong \Delta(x)$ and $M_{i+1}/M_i' \cong \Delta(y)$. Applying this procedure inductively, we can rearrange (10) such that the following condition is satisfied. There exists $l' \in \{1, \dots, l\}$ such that for every $i \leq l'$ there exists $x \notin W_\lambda^S w_\lambda^{B_\lambda}$ such that $M_i/M_{i-1} \cong \Delta(x)$, and for every $i > l'$ there exists $y \in W_\lambda^S w_\lambda^{B_\lambda}$ such that $M_i/M_{i-1} \cong \Delta(y)$. From the definition of the trace it follows immediately that $Z = M_{l'}$. In particular, all modules in the short exact sequence

$$0 \to Z \to \theta_{w^{-1}}\Delta(\overline{\mathbf{w}}) \to \mathrm{Coker} \to 0$$

have Verma flags. Since for every $x \in W_\lambda \backslash W_\lambda^S w_\lambda^{B_\lambda}$ and $y \in W_\lambda^S w_\lambda^{B_\lambda}$ we have $y \not\leq x$, for all such x and y we obtain

$$\mathrm{Ext}^1_{\mathcal{O}}(\Delta(x), L(y)) = 0,$$

which, because of Lemma 13, yields

$$\mathrm{Ext}^1_{\mathcal{O}}(Z, \mathtt{G}_s F(\mathbf{w})) = 0.$$

Now let us consider the module Coker $\in \mathcal{C}$. We claim that $\mathtt{R}^{\overline{\mathbf{w}}\cdot\lambda}$Coker is a projective module in the category $\mathcal{O}_\zeta^{\mathfrak{a}}$. Indeed, the module $\Delta(\overline{\mathbf{w}})$ is obtained by the parabolic induction from some projective Verma $\mathfrak{a}$-module. Since the adjoint action of $\mathfrak{a}$ on $U(\mathfrak{g})$ is locally finite, it follows that $\mathtt{R}^{\nu}(\Delta(\overline{\mathbf{w}}))$ is projective in $\mathcal{O}_\zeta^{\mathfrak{a}}$ for every $\nu \in \mathfrak{h}^*$. Further, for every finite-dimensional $\mathfrak{g}$-module V we have

$$\mathtt{R}^{\overline{\mathbf{w}}\cdot\lambda}(V \otimes \Delta(\overline{\mathbf{w}})) = \oplus_{(\nu_1,\nu_2)}\mathtt{R}^{\nu_1}V \otimes \mathtt{R}^{\nu_2}\Delta(\overline{\mathbf{w}}),$$

where the sum is taken over all pairs $(\nu_1, \nu_2) \in \mathfrak{h}^* \times \mathfrak{h}^*$ with different $\mathfrak{h}^{\perp}$-restrictions of ν_1 such that $\nu_1 + \nu_2 = \overline{\mathbf{w}} \cdot \lambda$. In particular, $\mathtt{R}^{\overline{\mathbf{w}}\cdot\lambda}(V \otimes \Delta(\overline{\mathbf{w}}))$ is projective in $\mathcal{O}_\zeta^{\mathfrak{a}}$. The inductive construction of the Verma flag in [**5**] implies that

$$\mathtt{R}^{\overline{\mathbf{w}}\cdot\lambda}(\mathrm{Coker}) = \mathtt{R}^0 V \otimes \mathtt{R}^{\overline{\mathbf{w}}\cdot\lambda}\Delta(\overline{\mathbf{w}}),$$

which is also projective in $\mathcal{O}_\zeta^{\mathfrak{a}}$. In particular, the first extension between $\mathtt{R}^{\overline{\mathbf{w}}\cdot\lambda}$(Coker) and all simple $\mathfrak{a}$-modules in $\mathcal{O}_\zeta^{\mathfrak{a}}$ vanishes and hence from Proposition 11 we derive

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathrm{Coker}, L(y)) = 0,$$

for all $y \in W_\lambda^S w_\lambda^{B_\lambda}$. Therefore, using Lemma 13 we get

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathrm{Coker}, \mathtt{G}_s F(\mathbf{w})) = 0.$$

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Thus

$$\mathrm{Ext}^1_{\mathcal{O}}\big(\theta_{w^{-1}}\Delta(\overline{\mathbf{w}}), \mathtt{G}_s F(\mathbf{w})\big) = 0$$

and the statement of the proposition follows from Corollary 4 and Proposition 5. $\square$

Consider now the short exact sequence

$$0 \to X(\mathbf{w}) \to N(\mathbf{w}) \xrightarrow{\hat{p}} L(\mathbf{w}) \to 0. \tag{11}$$

LEMMA 14. *For every finite-dimensional* $\mathfrak{g}$*-module* V *the sequence* (11) *induces the isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(N(\mathbf{w}), N(\mathbf{w}) \otimes V) \cong \mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), L(\mathbf{w}) \otimes V).$$

Proof. The same arguments as in Lemma 10 show that (11) induces the inclusion

$$\mathrm{Hom}_{\mathfrak{g}}(N(\mathbf{w}), N(\mathbf{w}) \otimes V) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), L(\mathbf{w}) \otimes V).$$

Let $f \in \mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), L(\mathbf{w}) \otimes V)$. We should like to lift f to an element in the space $\mathrm{Hom}_{\mathfrak{g}}(N(\mathbf{w}), N(\mathbf{w}) \otimes V)$. For this we consider the auxiliary module $\mathtt{T}_s L(\mathbf{w})$.

LEMMA 15. $\mathrm{Ext}^1_{\mathcal{O}}(\mathtt{T}_s L(\mathbf{w}), L(x)) = 0$ *for each* s*-finite* $L(x)$.

Proof. We have

$$\begin{array}{ll}
\mathrm{Ext}^1_{\mathcal{O}}(\mathtt{T}_s L(\mathbf{w}), L(x)) & = \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathtt{T}_s L(\mathbf{w}), L(x)[1]) & = \text{(VIII)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{L}\mathtt{T}_s L(\mathbf{w}), L(x)[1]) & = \text{(III)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(L(\mathbf{w}), \mathcal{R}\mathtt{G}_s L(x)[1]) & = \text{(VII), (VIII)} \\
\mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(L(\mathbf{w}), L(x)) & = \\
\mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), L(x)) & = (\mathbf{w} \neq x) \\
0. &
\end{array}$$

$\square$

First of all we claim that there is a non-zero map from $\mathtt{T}_s L(\mathbf{w})$ to $N(\mathbf{w})$. Indeed, applying $\mathrm{Hom}_{\mathfrak{g}}(\mathtt{T}_s L(\mathbf{w}), \_)$ to (11) and using Lemma 15 we obtain the surjection

$$\mathrm{Hom}_{\mathfrak{g}}(\mathtt{T}_s L(\mathbf{w}), N(\mathbf{w})) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{g}}(\mathtt{T}_s L(\mathbf{w}), L(\mathbf{w})).$$

Using this surjection we can lift the canonical projection $\tilde{p} : \mathtt{T}_s L(\mathbf{w}) \twoheadrightarrow L(\mathbf{w})$ to obtain the short exact sequence

$$0 \to \mathrm{Ker} \to \mathtt{T}_s L(\mathbf{w}) \xrightarrow{q} N(\mathbf{w}) \to 0. \tag{12}$$

Applying $\mathrm{Hom}_{\mathfrak{g}}(\mathtt{T}_s L(\mathbf{w}), \_ \otimes V)$ to (11) and using Lemma 15 we obtain the surjection

$$\mathrm{Hom}_{\mathfrak{g}}(\mathtt{T}_s L(\mathbf{w}), N(\mathbf{w}) \otimes V) \twoheadrightarrow \mathrm{Hom}_{\mathfrak{g}}(\mathtt{T}_s L(\mathbf{w}), L(\mathbf{w}) \otimes V).$$

In particular, we can lift the map $f \circ \hat{p} \circ q \in \mathrm{Hom}_{\mathfrak{g}}(\mathtt{T}_s L(\mathbf{w}), L(\mathbf{w}) \otimes V)$ to some map $\overline{f} \in \mathrm{Hom}_{\mathfrak{g}}(\mathtt{T}_s L(\mathbf{w}), N(\mathbf{w}) \otimes V)$.

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Now recall that $\mathrm{T}_s L(\mathbf{w}) \in \mathcal{C}$ by Lemma 13 (and the fact that $\mathrm{T}_s \cong \star \mathrm{G}_s \star$). Applying $\mathrm{R}^{\overline{\mathbf{w}}\cdot\lambda}$ and using Proposition 11 and Lemma 10 we obtain that $\mathrm{R}^{\overline{\mathbf{w}}\cdot\lambda}(\overline{f})$ annihilates the module $\mathrm{R}^{\overline{\mathbf{w}}\cdot\lambda}(\mathrm{Ker})$, which implies that $\overline{f}$ annihilates Ker by Proposition 11. In particular, $\overline{f}$ factors through $N(\mathbf{w})$. Since all the modules $L(\mathbf{w})$, $N(\mathbf{w})$, and $\mathrm{T}_s L(\mathbf{w})$, have the same simple top, it follows that $f \neq 0$ if and only if $\overline{f} \neq 0$. This gives us the injection

$$\mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), L(\mathbf{w}) \otimes V) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(N(\mathbf{w}), N(\mathbf{w}) \otimes V),$$

and the statement follows. □

Now we have the same amount of information as at the end of Section 4 and hence the proof of Theorem 1 can be easily completed in the same way as the proof of Proposition 7.

Proof of Theorem 1. Mutatis mutandis the proof of Proposition 7. □

6. Application to α-stratified simple modules. For $c \in \mathbb{C}$ denote by $\Theta_s^c = \Theta_{\alpha_s}^c$ *Mathieu's* twisting functor from [**20**, 4.3].

Corollary 16. *Under the assumptions of Theorem* 1 *we have that the canonical injection*

$$U(\mathfrak{g})/\mathrm{Ann}\big(\Theta_s^c L(\mathbf{w})\big) \hookrightarrow \mathscr{L}\big(\Theta_s^c L(\mathbf{w}), \Theta_s^c L(\mathbf{w})\big)$$

is surjective. Moreover, for every $w \in W_\lambda$ the canonical injection

$$U(\mathfrak{g})/\mathrm{Ann}\big(\Theta_s^c \Delta(w)\big) \hookrightarrow \mathscr{L}\big(\Theta_s^c \Delta(w), \Theta_s^c \Delta(w)\big)$$

is surjective.

Proof. From the definition of Θ_s^c it follows that Θ_s^c preserves $\mathrm{Ann}(M)$ and induces an isomorphism between $\mathscr{L}(M, M)$ and $\mathscr{L}(\Theta_s^c M, \Theta_s^c M)$ for any $M \in \mathcal{O}$ on which $X_{-\alpha}$ acts injectively. The first statement now follows from Theorem 1 and the second one from Corollary 4. □

When the modules $\Theta_s^c L(\mathbf{w})$ are simple, they are simple α_s-stratified modules considered in [**7**, **9**]. The modules $\Theta_s^c \Delta(w)$ are proper standard objects in the parabolic generalization of $\mathcal{O}$ studied in [**10**]. (See also [**21**].)

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