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Existence of Hilbert Cusp Forms with Non-vanishing L-values

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Abstract. We develop a derivative version of the relative trace formula on PGL(2) studied in our previous work, and derive an asymptotic formula of an average of central values (derivatives) of automorphic L-functions for Hilbert cusp forms. As an application, we prove the existence of Hilbert cusp forms with non-vanishing central values (derivatives) such that the absolute degrees of their Hecke fields are arbitrarily large.

1 Introduction

Let F be a totally real number field of degree d_F, $\mathfrak{o}$ the integer ring of F, and $\mathbb{A}$ the adele ring of F. The set of non-archimedean places and the set of archimedean places of F are denoted by Σ_{fin} and Σ_∞, respectively. The completion of F at a place v is denoted by F_v. When $v \in \Sigma_{\text{fin}}$, $\mathfrak{o}_v$ denotes the maximal order of the local field F_v. Given a non-zero ideal $\mathfrak{n} \subset \mathfrak{o}$ and an even weight $l = (l_v)_{v \in \Sigma_\infty} \in (2\mathbb{N})^{\Sigma_\infty}$, let $\Pi_{\text{cus}}(l, \mathfrak{n})$ be the set of all those irreducible cuspidal automorphic representations $\pi \cong \otimes_v \pi_v$ of $\mathrm{PGL}(2, \mathbb{A})$ such that π_v is a discrete series representation of $\mathrm{PGL}(2, F_v)$ of weight l_v for all $v \in \Sigma_\infty$ and π_v has a non-zero vector invariant by the local Hecke congruence subgroup $\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v) = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathfrak{o}_v) \mid c \in \mathfrak{n}\mathfrak{o}_v\}$ for all $v \in \Sigma_{\text{fin}}$. For $\pi \in \Pi_{\text{cus}}(l, \mathfrak{n})$ and an idele class character η of $F^\times$ such that $\eta^2 = \mathbf{1}$, the standard L-function $L(s, \pi \otimes \eta)$ of $\pi \otimes \eta$ is an entire function on $\mathbb{C}$ satisfying the self-dual functional equation

$$L(s, \pi \otimes \eta) = \epsilon(s, \pi \otimes \eta) L(1 - s, \pi \otimes \eta), \tag{1.1}$$

with $\epsilon(s, \pi \otimes \eta)$ being the ϵ-factor; it is of the form

$$\epsilon(s, \pi \otimes \eta) = \pm \big(\mathrm{N}(\mathfrak{f}_\pi \mathfrak{f}_\eta^2) D_F^2\big)^{1/2 - s},$$

where D_F is the absolute discriminant of F, and $\mathfrak{f}_\eta$ and $\mathfrak{f}_\pi$ are the conductors of η and π, respectively. The number $\epsilon(1/2, \pi \otimes \eta) \in \{+1, -1\}$ is called the sign of the functional equation. The central value $L(1/2, \pi) L(1/2, \pi \otimes \eta)$ and the derivative $L(1/2, \pi) L'(1/2, \pi \otimes \eta)$ have important arithmetic meanings; there are many studies that exploit the nature of these L-values in connection with the arithmetic algebraic geometry of modular varieties ([1, 13–17]).

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1.1 Main Results

For $v \in \Sigma_{\rm fin}$, we fix a prime element ϖ_v of $\mathfrak{o}_v$ once and for all. Let $\mathfrak{p}_v = \mathfrak{o} \cap \varpi_v \mathfrak{o}_v$ be the corresponding maximal ideal of $\mathfrak{o}$ and $q_v = \#(\mathfrak{o}/\mathfrak{p}_v)$ its norm. In this article, all (fractional) ideals in F are assumed to be non-zero. For any ideal $\mathfrak{a} \subset \mathfrak{o}$, the set of places $v \in \Sigma_{\rm fin}$ such that $\mathfrak{a} \subset \mathfrak{p}_v$ is denoted by $S(\mathfrak{a})$. Let $\mathfrak{a}$ be an $\mathfrak{o}$-ideal relatively prime to $\mathfrak{f}_\eta \mathfrak{n}$ and set $S = S(\mathfrak{a})$. We write the Satake parameter of $\pi \in \Pi_{\rm cus}(l, \mathfrak{n})$ at $v \in S$ as $\operatorname{diag}({q_v}^{\nu_v(\pi)/2}, {q_v}^{-\nu_v(\pi)/2})$ with $\pm\nu_v(\pi)$ belonging to the space $\mathfrak{X}_v = \mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}$. For $\pi \in \Pi_{\rm cus}(l, \mathfrak{n})$, let $L^{S_\pi}(s, \pi; \mathrm{Ad})$ denote the adjoint L-function of π without the local v-factors for v such that $\mathfrak{f}_\pi \subset \mathfrak{p}_v^2$. Given an even holomorphic function $\alpha(\mathbf{s})$ on $\mathfrak{X}_S = \prod_{v \in S} \mathfrak{X}_v$, we are interested in the asymptotic of the average of L-values over the set $\Pi^*_{\rm cus}(l, \mathfrak{n}) = \{\pi \in \Pi_{\rm cus}(l, \mathfrak{n}) \mid \mathfrak{f}_\pi = \mathfrak{n}\}$,

$$\mathrm{AL}^*(\mathfrak{n}; \alpha) = \frac{C_l}{\mathrm{N}(\mathfrak{n})} \sum_{\pi \in \Pi^*_{\rm cus}(l, \mathfrak{n})} \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi; \mathrm{Ad})} \alpha(\nu_S(\pi)) \tag{1.2}$$

with $\nu_S(\pi) = \{\nu_v(\pi)\}_{v \in S}$ and

$$C_l = \prod_{v \in \Sigma_\infty} \frac{2\pi(l_v - 2)!}{\{(l_v/2 - 1)!\}^2}, \tag{1.3}$$

as the norm $\mathrm{N}(\mathfrak{n}) = \#(\mathfrak{o}/\mathfrak{n})$ grows under the following conditions.

(i) The number $(-1)^{\epsilon(\eta)} \widetilde{\eta}(\mathfrak{n})$, the common value of $\epsilon(s, \pi)\epsilon(s, \pi \otimes \eta)|_{s=1/2}$ for all $\pi \in \Pi^*_{\rm cus}(l, \mathfrak{n})$, equals 1, where $\epsilon(\eta)$ is the number of $v \in \Sigma_\infty$ such that $\eta_v(-1) = -1$ and $\widetilde{\eta}$ denotes the character of the group of fractional ideals relatively prime to $\mathfrak{f}_\eta$ defined by $\widetilde{\eta}(\mathfrak{p}_v) = \eta_v(\varpi_v)$ for all $v \in \Sigma_{\rm fin} - S(\mathfrak{f}_\eta)$.

(ii) $\eta_v(\varpi_v) = -1$ for all $v \in S(\mathfrak{n})$.

In our previous paper [11], we studied a weighted L-value average $\mathrm{AL}^w(\mathfrak{n}; \alpha)$ defined by (4.8), which is similar to but different from (1.2) in that the summation is taken over a larger set $\Pi_{\rm cus}(l, \mathfrak{n})$ with certain extra weighting factors $w^\eta_\mathfrak{n}(\pi)$ (see §2.3). One of our aims in this paper is to derive an asymptotic formula for (1.2) from those for $\mathrm{AL}^w(\mathfrak{m}; \alpha)$ with $\mathfrak{n} \subset \mathfrak{m}$ by a special sieving technique. Moreover, imposing the same condition (ii) as above, but the opposite sign condition $(-1)^{\epsilon(\eta)} \widetilde{\eta}(\mathfrak{n}) = -1$ to (i), we investigate the asymptotic behavior of the following average involving the central derivative of the L-function $L(s, \pi \otimes \eta)$,

$$\mathrm{ADL}^*_-(\mathfrak{n}; \alpha) = \frac{C_l}{\mathrm{N}(\mathfrak{n})} \sum_{\substack{\pi \in \Pi^*_{\rm cus}(l, \mathfrak{n}) \\ \epsilon(1/2, \pi \otimes \eta) = -1}} \frac{L(1/2, \pi) L'(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi; \mathrm{Ad})} \alpha\big(\nu_S(\pi)\big). \tag{1.4}$$

To state our main result precisely, we need further notation. Let $\mathcal{I}_{S,\eta}$ be the monoid of ideals $\mathfrak{n} \subset \mathfrak{o}$ generated by prime ideals $\mathfrak{p}_v$ with $v \notin S \cup S(\mathfrak{f}_\eta)$ such that $\widetilde{\eta}(\mathfrak{p}_v) = -1$, and $\mathcal{I}^\pm_{S,\eta} = \{\mathfrak{n} \in \mathcal{I}_{S,\eta} \mid (-1)^{\epsilon(\eta)} \widetilde{\eta}(\mathfrak{n}) = \pm 1\}$. For $n \in \mathbb{N}$, let $X_n(x)$ be the Tchebyshev polynomial $X_n(x)$ defined by the relation

$$X_n(x) = \sin((n+1)\theta)/\sin\theta \quad \text{for } x = 2\cos\theta, \tag{1.5}$$

and set

$$\alpha_\mathfrak{a}(\nu) = \prod_{v \in S} X_{n_v}(q_v^{\nu_v/2} + q_v^{-\nu_v/2}), \quad \nu = \{\nu_v\}_{v \in S} \in \mathfrak{X}_S \tag{1.6}$$

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in terms of the prime ideal decomposition $\mathfrak{a} = \prod_{v \in S(\mathfrak{a})} \mathfrak{p}_v^{n_v}$. For such $\mathfrak{a}$, define

$$\mathfrak{a}_\eta^{\pm} = \prod_{\substack{v \in S(\mathfrak{a}) \\ \widetilde{\eta}(\mathfrak{p}_v) = \pm 1}} \mathfrak{p}_v^{n_v}, \quad d_1(\mathfrak{a}) = \prod_{v \in S(\mathfrak{a})} (n_v + 1), \quad \delta_\square(\mathfrak{a}) = \prod_{v \in S(\mathfrak{a})} 2^{-1}\{1 + (-1)^{n_v}\}.$$

We have an asymptotic formula of $\mathrm{ADL}_-^*(\mathfrak{n}; \alpha_\mathfrak{a})$ with an error term whose dependence on $\mathfrak{n} \in \mathcal{I}_{S,\eta}^-$ and $\mathfrak{a}$ is made explicit. We also have a similar formula for $\mathrm{AL}^*(\mathfrak{n}; \alpha_\mathfrak{a})$ with $\mathfrak{n} \in \mathcal{I}_{S,\eta}^+$.

Theorem 1.1 *Set $\underline{l} = \min_{v \in \Sigma_\infty} l_v$ and $c = d_F^{-1}(\underline{l}/2 - 1)$, and suppose $\underline{l} \geqslant 6$. For an integral ideal $\mathfrak{n}$, set $S_k(\mathfrak{n}) = \{v \in S(\mathfrak{n}) \mid \mathfrak{f}_\pi \mathfrak{o}_v = \varpi_v^k \mathfrak{o}_v\}$ for $k \in \mathbb{N}$ and*

$$\nu(\mathfrak{n}) = \Big\{ \prod_{v \in S(\mathfrak{n}) - (S_1(\mathfrak{n}) \cup S_2(\mathfrak{n}))} (1 - q_v^{-2}) \Big\} \Big\{ \prod_{v \in S_2(\mathfrak{n})} (1 - (q_v^2 - q_v)^{-1}) \Big\}.$$

For any sufficiently small number $\epsilon > 0$, we have

(1.7)
$$\mathrm{AL}^*(\mathfrak{n}; \alpha_\mathfrak{a}) = 4 D_F^{3/2} L_{\mathrm{fin}}(1, \eta) \nu(\mathfrak{n}) \, \mathrm{N}(\mathfrak{a})^{-1/2} \delta_\square(\mathfrak{a}_\eta^-) d_1(\mathfrak{a}_\eta^+) + \mathcal{O}\big(\mathrm{N}(\mathfrak{a})^{c+2+\epsilon} \, \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}\big), \quad \mathfrak{n} \in \mathcal{I}_{S,\eta}^+,$$

(1.8)
$$\begin{aligned}
&\mathrm{ADL}_-^*(\mathfrak{n}; \alpha_\mathfrak{a}) \\
&= 4 D_F^{3/2} L_{\mathrm{fin}}(1, \eta) \nu(\mathfrak{n}) \, \mathrm{N}(\mathfrak{a})^{-1/2} d_1(\mathfrak{a}_\eta^+) \Bigg\{ \delta_\square(\mathfrak{a}_\eta^-) \Bigg(\log\big(\sqrt{\mathrm{N}(\mathfrak{n}) \, \mathrm{N}(\mathfrak{a})^{-1}} \, \mathrm{N}(\mathfrak{f}_\eta) D_F\big) \\
&\quad + \sum_{v \in S(\mathfrak{n}) - (S_1(\mathfrak{n}) \cup S_2(\mathfrak{n}))} \frac{\log q_v}{q_v^2 - 1} + \sum_{v \in S_2(\mathfrak{n})} \frac{\log q_v}{q_v^2 - q_v - 1} + \frac{L'(1, \eta)}{L(1, \eta)} + \mathfrak{C}(l) \Bigg) \\
&\quad + \sum_{v \in S(\mathfrak{a}_\eta^-)} \delta_\square(\mathfrak{a}_\eta^- \mathfrak{p}_v^{-1}) \log(q_v^{n_v + \frac{1}{2}}) \Bigg\} \\
&\quad + \mathcal{O}\big(\mathrm{N}(\mathfrak{a})^{-1/2} d_1(\mathfrak{a}_\eta^+) \delta_\square(\mathfrak{a}_\eta^-) X(\mathfrak{n}) + \mathrm{N}(\mathfrak{a})^{c+2+\epsilon} \, \mathrm{N}(\mathfrak{n})^{-\inf(1,c)+\epsilon}\big), \quad \mathfrak{n} \in \mathcal{I}_{S,\eta}^-,
\end{aligned}$$

where $L_{\mathrm{fin}}(s, \eta) = \prod_{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\eta)} (1 - \eta_v(\varpi_v) q_v^{-s})^{-1}$, $(\mathrm{Re}(s) > 1)$ is the L-function of η,

$$\mathfrak{C}(l) = \sum_{v \in \Sigma_\infty} \Big(\sum_{k=1}^{l_v/2-1} \frac{1}{k} - \frac{1}{2} \log \pi - \frac{1}{2} C_{\mathrm{Euler}} - \frac{1 - \eta_v(-1)}{2} \log 2 \Big),$$

$$X(\mathfrak{n}) = \sum_{u \in S(\mathfrak{n})} \frac{\log q_u}{q_u} + \sum_{u \in S(\mathfrak{n})} \frac{\log q_u}{(q_u - 1)^2}.$$

The constants implicit in Landau's symbols $\mathcal{O}$ in both formulas are independent of $\mathfrak{n}$ and $\mathfrak{a}$, but dependent on ϵ, l, and η.

1.2 Applications

For a positive integer N, let $J_0^{\mathrm{new}}(N)$ be the new part of the Jacobian variety of the modular curve $X_0(N)$ of level N. J.-P. Serre [7, Theorem 7] showed that the largest dimension of $\mathbb{Q}$-simple factors of $J_0^{\mathrm{new}}(N)$ tends to infinity as N grows. This result was refined in several ways by E. Royer [6], who obtained a quantitative version of

Serre's theorem giving a lower bound of the largest dimension of $\mathbb{Q}$-simple factors A of $J_0^{\rm new}(N)$ with or without rank conditions for the Mordell–Weil group of A. By the correspondence between the $\mathbb{Q}$-simple factors A of $J_0^{\rm new}(N)$ and the normalized Hecke eigen cuspidal newforms f of level $\Gamma_0(N)$ and weight 2, and by invoking the progress toward the Birch and Swinnerton-Dyer conjecture, the lower bound for the largest $\dim A$ is obtained from a lower bound of the maximum value of the absolute degree of the Hecke field $\mathbb{Q}(f)$ with or without conditions on the order of L-series $L(s,f)$ at the center of symmetry. Thus, one of Royer's results can be stated in the language of modular forms.

Theorem 1.2 *Let p be a prime. There exist constants $C_p > 0$ and $N_p > 0$ with the following properties.*

(i) *For any $N > N_p$ relatively prime to p, there exists a normalized Hecke eigen cuspidal newform f of level $\Gamma_0(N)$ and weight 2 satisfying the conditions:*
   (1) *$L(1/2, f) \neq 0$, where the functional equation of $L(s,f)$ relates the values at s and $1-s$.*
   (2) $[\mathbb{Q}(f):\mathbb{Q}] \geqslant C_p\sqrt{\log\log N}$.

(ii) *For any $N > N_p$ relatively prime to p, there exists a normalized Hecke eigen cuspidal newform f_1 of level $\Gamma_0(N)$ and weight 2 satisfying the following conditions.*
   (1) *The sign of the functional equation of $L(s, f_1)$ is -1.*
   (2) $L'(1/2, f_1) \neq 0$.
   (3) $[\mathbb{Q}(f_1):\mathbb{Q}] \geqslant C_p\sqrt{\log\log N}$.

We derive an analogue of this theorem for higher weight Hilbert modular cuspforms from Theorem 1.1. For a cuspidal representation $\pi \in \Pi_{\rm cus}^*(l, \mathfrak{n})$, the field of rationality of π (for definition, see §8.1) is denoted by $\mathbb{Q}(\pi)$.

Theorem 1.3 *Let $l = (l_v)_{v\in\Sigma_\infty}$ be a weight such that $l_v = k$ for all $v \in \Sigma_\infty$ with an even integer $k \geqslant 6$ and η a quadratic idele class character of $F^\times$. Let S be a finite subset of $\Sigma_{\rm fin} - S(\mathfrak{f}_\eta)$ and $\mathbf{J} = \{J_v\}_{v\in S}$ a family of closed subintervals of $(-2,2)$. Given a prime ideal $\mathfrak{q} = \mathfrak{p}_u$ with $u \notin S \cup S(\mathfrak{f}_\eta)$, there exist constants $C_{\mathfrak{q},l} > 0$ and $N_{\mathfrak{q},S,l,\eta,\mathbf{J}} > 0$ with the following properties. For any ideal $\mathfrak{n} \in \mathfrak{I}_{S\cup\{u\},\eta}^+$ with $\mathrm{N}(\mathfrak{n}) > N_{\mathfrak{q},S,l,\eta,\mathbf{J}}$, there exists $\pi \in \Pi_{\rm cus}^*(l,\mathfrak{n})$ such that*

(i) $L(1/2,\pi) \neq 0$ *and* $L(1/2, \pi\otimes\eta) \neq 0$,
(ii) $[\mathbb{Q}(\pi):\mathbb{Q}] \geqslant C_{\mathfrak{q},l}\sqrt{\log\log \mathrm{N}(\mathfrak{n})}$,
(iii) $q_v^{\nu_v(\pi)/2} + q_v^{-\nu_v(\pi)/2} \in J_v$ *for all* $v \in S$.

We should note that this can be regarded as a refinement of [11, Corollary 1.2]. As for derivatives, we have a conditional result.

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Theorem 1.4 *Let $l = (l_v)_{v\in\Sigma_\infty}$ and η be the same as in Theorem 1.3. Suppose that for any ideal $\mathfrak{n}$*

$$\frac{d}{ds}\Big|_{s=\frac12} L(s,\pi)L(s,\pi\otimes\eta) \geqslant 0 \quad \text{for all } \pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n}) \text{ such that } \epsilon(1/2,\pi)\epsilon(1/2,\pi\otimes\eta) = -1. \tag{1.9}$$

Let S be a finite subset of $\Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\eta)$ and $\mathbf{J} = \{J_v\}_{v\in S}$ a family of closed subintervals of $(-2,2)$. Given a prime ideal $\mathfrak{q} = \mathfrak{p}_u$ with $u \notin S\cup S(\mathfrak{f}_\eta)$ and a constant $M > 1$, there exist constants $C_{\mathfrak{q},l} > 0$ and $N_{\mathfrak{q},S,l,\eta,\mathbf{J},M} > 0$ with the following properties. For any ideal $\mathfrak{n}\in\mathcal{I}^-_{S\cup\{u\},\eta}$ with $\mathrm{N}(\mathfrak{n}) > N_{\mathfrak{q},S,l,\eta,\mathbf{J},M}$ and $\sum_{v\in S(\mathfrak{n})}\frac{\log q_v}{q_v}\leqslant M$, there exists $\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$ such that

(i) $\epsilon(1/2,\pi\otimes\eta) = -1$,
(ii) $L(1/2,\pi)\neq 0$ *and* $L'(1/2,\pi\otimes\eta)\neq 0$,
(iii) $[\mathbb{Q}(\pi):\mathbb{Q}]\geqslant C_{\mathfrak{q},l}\sqrt{\log\log \mathrm{N}(\mathfrak{n})}$,
(iv) $q_v^{\nu_v(\pi)/2} + q_v^{-\nu_v(\pi)/2}\in J_v$ *for all* $v\in S$.

We should note that the assumption (1.9) is a consequence of the Riemann hypothesis for the L-function $L(s,\pi)L(s,\pi\otimes\eta)$. Theorem 1.3 (Theorem 1.4) yields a Hilbert cuspform of arbitrarily large level with arbitrarily large degree of the field of rationality such that the central value of the L-function and the central value (derivative) of its prescribed quadratic twist are non-zero simultaneously. Although we can expect a similar result for parallel weight 2 Hilbert cuspforms, our method does not work as it is for such low weight cases. In order to treat these interesting cases, the technique of Green's function as in [10, 12] may be useful.

1.3 Framework

Let us review the proofs of Theorems 1.1, 1.3, and 1.4, explaining the organization of this paper. In our previous work [11], we constructed the renormalized smoothed automorphic Green's function $\widehat{\Psi}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha)$ as the value at $\lambda = 0$ of an entire extension of some Poincaré series $\widehat{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\alpha)$ originally defined for $\mathrm{Re}(\lambda) > 1$. Then we computed the period integral of $\widehat{\Psi}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha)$ along the diagonal split torus H adelically in a very explicit form. In the present work, instead of the period integral, we introduce a certain integral transform $\partial P^\eta_{\beta,\lambda}(\varphi)$ (see §3.2) for any cusp form φ on $\mathrm{PGL}(2,F)\backslash\mathrm{PGL}(2,\mathbb{A})$ and a quadratic idele class character η of $F^\times$, depending on a complex parameter λ and a test function β for renormalization, whose constant term at $\lambda = 0$ yields the derivative at $s = 1/2$ of the period integral of $\varphi|\det|_{\mathbb{A}}^{s-1/2}$ along H. The main step to have the formula (1.8) in Theorem 1.1 is to calculate $\partial P^\eta_{\beta,\lambda}(\widehat{\Psi}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ and its constant term at $\lambda = 0$ in two different ways; the process is completely parallel to that in [11] for period integrals. In §3, after recalling the construction of $\widehat{\Psi}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha)$, we prove a formula of $\mathrm{CT}_{\lambda=0}\,\partial P^\eta_{\beta,\lambda}(\widehat{\Psi}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ written in terms of the spectral data of cuspidal representations in $\Pi_{\mathrm{cus}}(l,\mathfrak{n})$ (Proposition 3.2). In §4, closely following [11], we compute $\partial P^\eta_{\beta,\lambda}(\widehat{\Psi}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ according to the $\big(H(F),H(F)\big)$-double coset decomposition of $\mathrm{GL}(2, F)$. By equating the two expressions of $\mathrm{CT}_{\lambda=0}\,\partial P^{\eta}_{\beta,\lambda}(\widehat{\Psi}^{l}_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ obtained in §3 and §4, we get a kind of relative trace formula, which is stated in Theorem 4.8. The formula is not for our $\mathrm{ADL}^*_-(\mathfrak{n})$, but for a similar average of L-values over all cuspidal representations $\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})$. We need to sieve out information on an average of only those $\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})$ with exact conductor $\mathfrak{n}$. For that purpose, we introduce a certain operation (see Definition 5.2), which we call the $\mathcal{N}$-transform, for any arithmetic function defined on a set of ideals. The first subsection of §5 is devoted to the study of the $\mathcal{N}$-transform. By applying the $\mathcal{N}$-transform of each term occurring in the formula (4.9), we deduce yet another formula (5.3), which relates the average $\mathrm{ADL}^*_-(\mathfrak{n})$ to the sum of the following terms: (i) the $\mathcal{N}$-transforms of $\widetilde{\mathbb{W}}^{\eta}_{\mathfrak{u}}(l, \mathfrak{n}|\alpha)$ and $\mathbb{W}^{\eta}_{\mathrm{hyp}}(l, \mathfrak{n}|\alpha)$, both of them occurring in the geometric side of (4.9), (ii) the L-value average $\mathrm{AL}^*(\mathfrak{n})$, and (iii) the $\mathcal{N}$-transform of a certain term $\mathrm{AL}^{\partial w}(\mathfrak{n})$ arising from the spectral side of (4.9). In §7, we analyze these terms separately and obtain an exact evaluation of the $\mathcal{N}$-transform of $\widetilde{\mathbb{W}}^{\eta}_{\mathfrak{u}}(l, \mathfrak{n}|\alpha)$ and estimations of the remaining terms, which lead us to the proof of (1.8). In §6, by applying the relative trace formula [11, Theorem 9.1] to the test function $\alpha_{\mathfrak{a}}$, we deduce (1.7), which is necessary to prove Theorem 1.3. In §8, we give the proof of Theorems 1.3 and 1.4. Actually, what we do there is to confirm that the argument of [6] for the classical modular forms still works with a minor modification in our setting. The analysis performed in §7 relies on explicit formulas of local orbital integrals arising from $\mathbb{W}^{\eta}_{\mathrm{hyp}}(l, \mathfrak{n}|\alpha)$ and $\widetilde{\mathbb{W}}^{\eta}_{\mathfrak{u}}(l, \mathfrak{n}|\alpha)$; the aim of §9 is to provide them. In Appendix A, we study a certain lattice sum to be used in the error term estimates in §6 and §9.

Notation and Convention

Given a condition P, $\delta(\mathrm{P})$ is 1 if P is true and 0 otherwise. For any non-negative functions $f(x)$ and $g(x)$ on a set X, we write $f(x) = \mathcal{O}(g(x))$ (or $f(x) \ll g(x)$) if there exists a constant $C > 0$ such that $f(x) \leqslant Cg(x)$ for all $x \in X$. The symbol $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $\sigma \in \mathbb{R}$, L_σ denote the vertical contour $\{\sigma + it \mid t \in \mathbb{R}\}$ directed from $\sigma - i\infty$ to $\sigma + i\infty$. Set $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. All ideals or fractional ideals appearing in this article are assumed to be non-zero.

2 Preliminary

We prepare notation recalling basic ingredients necessary in this paper.

2.1 Totally Real Number Fields

Let F be a totally real number field of absolute degree $d_F = [F{:}\mathbb{Q}]$, $\mathfrak{o}$ the integer ring of F, and $\mathbb{A}$ the ring of adeles of F. The subring of finite adeles in $\mathbb{A}$ is denoted by $\mathbb{A}_{\mathrm{fin}}$. Let $\mathfrak{d}_F$ be the different of $F/\mathbb{Q}$ and set $D_F = \#(\mathfrak{o}/\mathfrak{d}_F)$. Let Σ_∞ and Σ_{fin} be the set of archimedean places of F and the set of finite places of F, respectively. Set $\Sigma_F = \Sigma_\infty \cup \Sigma_{\mathrm{fin}}$. For $v \in \Sigma_F$, let F_v denote the completion of F at v and $|\cdot|_v$ the normalized valuation of the local field F_v. If dx is a Haar measure of the additive

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group F_v, then $d(ax) = |a|_v dx$ for all $a \in F_v^\times$. When $v \in \Sigma_{\text{fin}}$, the maximal order of the non-archimedean local field F_v is denoted by $\mathfrak{o}_v$. We fix a prime element ϖ_v of $\mathfrak{o}_v$ once and for all, and set $\mathfrak{p}_v = \mathfrak{o} \cap \varpi_v \mathfrak{o}_v$ and $q_v = \#(\mathfrak{o}/\mathfrak{p}_v)$. For any non-zero ideal $\mathfrak{m} \subset \mathfrak{o}$ and $v \in \Sigma_{\text{fin}}$, let $\text{ord}_v(\mathfrak{m})$ denote the exponent of $\mathfrak{p}_v$ in the prime factorization of $\mathfrak{m}$. Set $S(\mathfrak{m}) = \{v \in \Sigma_{\text{fin}} \mid \text{ord}_v(\mathfrak{m}) \geqslant 1\}$ and $S_k(\mathfrak{m}) = \{v \in S(\mathfrak{m}) \mid \text{ord}_v(\mathfrak{m}) = k\}$ for $k \in \mathbb{N}$. The absolute norm of an ideal $\mathfrak{m} \subset \mathfrak{o}$ is denoted by $\text{N}(\mathfrak{m})$, *i.e.*, $\text{N}(\mathfrak{m}) = \#(\mathfrak{o}/\mathfrak{m})$.

2.1.1 Real Valued Characters

For an idele class character $\eta = \prod_v \eta_v$ of $F^\times$ such that $\eta^2 = \mathbf{1}$, let $\mathfrak{f}_\eta$ be the conductor of η and set $f(\eta_v) = \text{ord}_v(\mathfrak{f}_\eta)$ for all $v \in \Sigma_{\text{fin}}$. The idele class character η gives rise to a character $\widetilde{\eta}$ of the group of fractional ideals prime to $\mathfrak{f}_\eta$ such that $\widetilde{\eta}(\mathfrak{p}_v) = \eta_v(\varpi_v)$ for all $v \in \Sigma_{\text{fin}} - S(\mathfrak{f}_\eta)$.

2.1.2 Cuspidal Representations of $\text{PGL}(2)$

Let $G = \text{GL}(2)$, viewed as an F-algebraic group, and let Z be the center of G. The adelizations of G and of Z are denoted by $G_\mathbb{A}$ and $Z_\mathbb{A}$, respectively. For $v \in \Sigma_F$, the F_v-points of G are denoted by G_v. We fix a Haar measure dx_v on the additive group F_v by requiring $\int_{\mathfrak{o}_v} dx_v = \#(\mathfrak{o}_v/\mathfrak{d}_F\mathfrak{o}_v)^{-1/2}$ if $v \in \Sigma_{\text{fin}}$ and $\int_0^1 dx_v = 1$ if $v \in \Sigma_\infty$. On the adele group $\mathbb{A}$, we take the product measure $dx = \otimes_v dx_v$. For all $v \in \Sigma_F$, we define a Haar measure on $F_v^\times$ by setting $d^\times t_v = \zeta_{F_v}(1)|t|_v^{-1} dt_v$, where $\zeta_{F_v}(s)$ is the local v-factor of the Dedekind zeta function of F. On the idele group $\mathbb{A}^\times$, we take the product measure $d^\times t = \otimes_v d^\times t_v$. For η as in §2.1.1 and a cusp form φ on $G_\mathbb{A}$, define

$$Z^*(s, \eta, \varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \varphi\left(\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\right) \eta(x_\eta^* t)|t|_\mathbb{A}^{s-1/2} d^\times t, \quad s \in \mathbb{C} \tag{2.1}$$

with $x_\eta^* = (\varpi_v^{-f(\eta_v)})_{v \in \Sigma_{\text{fin}}} \in \mathbb{A}_{\text{fin}}^\times$ and $x_\eta \in \mathbb{A}$ such that $x_{\eta,v} = 0$ for all $v \in \Sigma_\infty$ and $x_{\eta,\text{fin}} = x_\eta^*$ ([12, 2.6.2], [9, §4], [10, §2.1], [11, §6.3]). For any irreducible automorphic representation π of $G_\mathbb{A}$, we fix a family $\{\pi_v\}_{v \in \Sigma_F}$ of irreducible smooth representations of G_v having non-zero $G(\mathfrak{o}_v)$-invariant vectors for almost all v's such that π is isomorphic to the restricted tensor product $\otimes_v \pi_v$. For a non-zero ideal $\mathfrak{n} \subset \mathfrak{o}$ and $l = \{l_v\}_{v \in \Sigma_\infty} \in (2\mathbb{N})^{\Sigma_\infty}$ which we call an even weight, let $\Pi_{\text{cus}}(l, \mathfrak{n})$ be the set of all the irreducible cuspidal automorphic representations $\pi \cong \otimes_v \pi_v$ of $G_\mathbb{A}$ with trivial central character such that π_v is a discrete series representation of $\text{PGL}(2, \mathbb{R})$ of weight l_v for all $v \in \Sigma_\infty$ and π_v contains non-zero vectors invariant by the group

$$\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v) = \left\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbf{K}_v \mid c \in \mathfrak{n}\mathfrak{o}_v\right\}$$

for all $v \in \Sigma_{\text{fin}}$, where we set $\mathbf{K}_v = G(\mathfrak{o}_v)$. For $v \in \Sigma_{\text{fin}}$ and an irreducible generic smooth representation π_v of G_v with trivial central character, the conductor of π_v is defined to be the unique integer $c(\pi_v) \in \mathbb{N}_0$ such that

$$\pi_v^{\mathbf{K}_0(\varpi_v^{c(\pi_v)}\mathfrak{o}_v)}$$

is one-dimensional. Recall that the conductor $\mathfrak{f}_\pi$ of an irreducible cuspidal automorphic representation π of $Z_\mathbb{A} \backslash G_\mathbb{A} = \text{PGL}(2, \mathbb{A})$ is an ideal of $\mathfrak{o}$ such that $\text{ord}_v(\mathfrak{f}_\pi) = c(\pi_v)$ for all $v \in \Sigma_{\text{fin}}$. Set $\Pi_{\text{cus}}^*(l, \mathfrak{n}) = \{\pi \in \Pi_{\text{cus}}(l, \mathfrak{n}) \mid \mathfrak{f}_\pi = \mathfrak{n}\}$.

2.1.3 Compact Subgroups

For $v \in \Sigma_\infty$, let $\mathbf{K}_v$ denote the group of orthogonal matrices in G_v and let $\mathbf{K}_\infty = \prod_{v\in\Sigma_\infty} \mathbf{K}_v$. For a unitary representation (π, V_π) of $G_{\mathbb{A}}$, let $V_\pi[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$ be the space of vectors $\xi \in V_\pi$ such that $\pi(k_\infty k_{\mathrm{fin}})\xi = \tau_l(k_\infty)\xi$ for all $k_\infty \in \mathbf{K}_\infty^0$ and $k_{\mathrm{fin}} \in \mathbf{K}_0(\mathfrak{n})$, where τ_l is the character of $\mathbf{K}_\infty^0$ defined by $\tau_l(k_\infty) = \prod_{v\in\Sigma_\infty} e^{il_v\theta_v}$ for

$$k_\infty = \left\{ \begin{bmatrix} \cos\theta_v & -\sin\theta_v \\ \sin\theta_v & \cos\theta_v \end{bmatrix} \right\}_{v\in\Sigma_\infty},$$

and $\mathbf{K}_0(\mathfrak{n}) = \prod_{v\in\Sigma_{\mathrm{fin}}} \mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$ viewed as an open compact subgroup of $\mathbf{K}_{\mathrm{fin}} = \prod_{v\in\Sigma_{\mathrm{fin}}} \mathbf{K}_v$.

For $v \in \Sigma_F$, the group G_v is endowed with a Haar measure dg_v such that $dg_v = |t_v/t'_v|_v^{-1} dx_v d^\times t_v d^\times t'_v dk_v$ in terms of the Iwasawa decomposition $g_v = \begin{bmatrix} 1 & x_v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t_v & 0 \\ 0 & t'_v \end{bmatrix} k_v$ ($x_v \in F_v$, $t_v, t'_v \in F_v^\times$, $k_v \in \mathbf{K}_v$), where dk_v is the probability Haar measure on $\mathbf{K}_v$. We fix a Haar measure on $G_{\mathbb{A}}$ defined as the product of the Haar measures on G_v.

2.2 Local Factors

Let $v \in \Sigma_{\mathrm{fin}}$, π_v an irreducible generic smooth representation of $\mathrm{PGL}(2, F_v)$, and η_v a character of $F_v^\times$ such that $\eta_v^2 = 1$. Recall that the polynomials $Q_{j,v}^{\pi_v}(\eta_v, X)$ $(j \in \mathbb{N})$ of an indeterminate X [9, Corollary 19] and the local L-factors $L(s, \pi_v)$, $L(s, \pi_v; \mathrm{Ad})$ $(s \in \mathbb{C})$ are defined in the following manner.

- When $c(\pi_v) = 0$,

$$Q_{j,v}^{\pi_v}(\eta_v, X) = \begin{cases} \eta_v(\varpi_v)X - Q(\pi_v), & j = 1, \\ q_v^{-1}\eta_v(\varpi_v)^{j-2}X^{j-2}(a_v q_v^{1/2}\eta_v(\varpi_v)X - 1)(a_v^{-1}q_v^{1/2}\eta_v(\varpi_v)X - 1), & j \geqslant 2, \end{cases}$$

$$L(s, \pi_v) = (1 - a_v q_v^{-s})^{-1}(1 - a_v^{-1}q_v^{-s})^{-1},$$
$$L(s, \pi_v; \mathrm{Ad}) = (1 - a_v^2 q_v^{-s})^{-1}(1 - q_v^{-s})^{-1}(1 - a_v^{-2}q_v^{-s})^{-1},$$

  where $Q(\pi_v) = (a_v + a_v^{-1})/(q_v^{1/2} + q_v^{-1/2})$ with $\mathrm{diag}(a_v, a_v^{-1})$ being the Satake parameter of π_v.
- When $c(\pi_v) = 1$,

$$Q_{j,v}^{\pi_v}(\eta_v, X) = \eta_v(\varpi_v)^{j-1}X^{j-1}(\eta_v(\varpi_v)X - q_v^{-1}\chi_v(\varpi_v)^{-1}),$$

  $L(s, \pi_v) = (1 - \chi_v(\varpi_v)q_v^{-s-1/2})^{-1}$ and $L(s, \pi_v; \mathrm{Ad}) = (1 - q_v^{-(s+1)})^{-1}$, where χ_v is the unramified character of $F_v^\times$ such that $\chi_v^2 = 1$ and π_v is isomorphic to the special representation $\sigma(\chi_v|\,|_v^{1/2}, \chi_v|\,|_v^{-1/2})$ ([3, §3]).
- When $c(\pi_v) \geqslant 2$, we set $Q_{j,v}^{\pi_v}(\eta_v, X) = \eta_v(\varpi_v)^j X^j$ and $L(s, \pi_v) = 1$. (We omit the formula for $L(s, \pi_v; \mathrm{Ad})$, which is unnecessary for our purpose.)

We set $Q_{j,v}^{\pi_v}(\eta_v, X) = 1$ if $j = 0$. For $\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})$, let $\Lambda_\pi(\mathfrak{n})$ denote the set of mappings $\rho: \Sigma_{\mathrm{fin}} \to \mathbb{N}_0$ such that $0 \leqslant \rho(v) \leqslant \mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ for all $v \in \Sigma_{\mathrm{fin}}$. An explicit orthogonal basis $\{\varphi_{\pi,\rho}\}_{\rho\in\Lambda_\pi(\mathfrak{n})}$ of the finite dimensional Hilbert space $V_\pi[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$

was constructed in [9] (see also [11, §6.2]). Let $\eta = \prod_v \eta_v$ be an idele class character of $F^\times$ such that $\eta^2 = \mathbf{1}$ and $S(\mathfrak{f}_\eta) \cap S(\mathfrak{n}) = \varnothing$. For $\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})$, we set

$$Q_{\pi,\eta,\rho}(s) = \prod_{v \in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} Q^{\pi_v}_{\rho(v),v}(\eta_v, q_v^{1/2-s}), \quad \rho \in \Lambda_\pi(\mathfrak{n}),$$

and define the completed L-function $L(s,\pi)$ on $\mathrm{Re}(s) > 1$ as the absolutely convergent infinite product $L(s,\pi) = \prod_{v \in \Sigma_F} L(s,\pi_v)$, where $L(s,\pi_v) = \Gamma_{\mathbb{C}}(s + (l_v - 1)/2)$ for $v \in \Sigma_\infty$. The modified zeta integrals (2.1) for the basis $\{\varphi_{\pi,\rho}\}$ are calculated as

$$\text{(2.2)} \quad Z^*(s,\eta,\varphi_{\pi,\rho}) = D_F^{s-1/2}(-1)^{\epsilon(\eta)} \mathcal{G}(\eta) Q_{\pi,\eta,\rho}(s) L(s, \pi \otimes \eta), \quad \pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n}), \rho \in \Lambda_\pi(\mathfrak{n})$$

for $\mathrm{Re}(s) > 1$, where $\mathcal{G}(\eta)$ is the Gauss sum of η defined in [12, §2.4] and $\epsilon(\eta) = \#\{v \in \Sigma_\infty | \eta_v(-1) = -1\}$; for the proof, we refer to [9, Proposition 20] and [11, Proposition 6.1]. The function φ_{π,ρ_0} with ρ_0 such that $\rho_0(v) = 0$ for all $v \in \Sigma_{\mathrm{fin}}$ is denoted by $\varphi_\pi^{\mathrm{new}}$ and is called the new vector of π. For this particular vector, (2.2) is simplified to the well-known identity $Z^*(s,\eta,\varphi_\pi^{\mathrm{new}}) = D_F^{s-1/2}(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)L(s,\pi\otimes\eta)$, which yields a holomorphic continuation of $L(s,\pi\otimes\eta)$ to the whole complex s-plane. The L^2-norm of $\varphi_\pi^{\mathrm{new}}$ is computed in [11, Lemma 6.4] as

$$\text{(2.3)} \qquad \|\varphi_\pi^{\mathrm{new}}\|^2 = 2\Big\{\prod_{v\in\Sigma_\infty} 2^{1-l_v}\Big\} \mathrm{N}(\mathfrak{f}_\pi)[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]^{-1} L^{S_\pi}(1,\pi;\mathrm{Ad}),$$

where $L^{S_\pi}(s,\pi;\mathrm{Ad})$ is the adjoint L-function of π without the local v-factors over the set $S_\pi = \{v \in \Sigma_{\mathrm{fin}} | c(\pi_v) \geqslant 2\}$, which is defined as an analytic continuation of the Euler product $\prod_{v\in\Sigma_F - S_\pi} L(s,\pi_v;\mathrm{Ad})$ ($\mathrm{Re}(s) > 1$) with $L(s,\pi_v;\mathrm{Ad}) = \Gamma_{\mathbb{C}}(s + l_v - 1)\Gamma_{\mathbb{R}}(s+1)$ for $v \in \Sigma_\infty$.

2.3 Weight Functions

In this subsection, using a fine local structure of cuspidal representations, we define weight functions for them, which appear in the averages of L-values to be defined in §4.2. Let $\eta = \prod_v \eta_v$ be an idele class character of $F^\times$ such that $\eta^2 = \mathbf{1}$ and $S(\mathfrak{f}_\eta) \cap S(\mathfrak{n}) = \varnothing$. Let $\pi \cong \otimes_v \pi_v \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$ and $\rho \in \Lambda_\pi(\mathfrak{n})$. For a complex parameter z, we set

$$\text{(2.4)} \quad w_{\mathfrak{n}}^\eta(\pi;z) = \sum_{\rho\in\Lambda_\pi(\mathfrak{n})} \prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \overline{Q^{\pi_v}_{\rho(v),v}(\mathbf{1},1)} Q^{\pi_v}_{\rho(v),v}(\eta_v, q_v^{1/2-z}) / \tau_{\pi_v}\big(\rho(v),\rho(v)\big) = \prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} r^{(z)}(\pi_v,\eta_v)$$

with

$$r^{(z)}(\pi_v,\eta_v) = \sum_{j=0}^{\mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \overline{Q^{\pi_v}_{j,v}(\mathbf{1},1)} Q^{\pi_v}_{j,v}(\eta_v, q_v^{1/2-z}) / \tau_{\pi_v}(j,j).$$

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Here $Q_{j,v}^{\pi_v}(\eta_v, X)$ is the polynomial recalled in §2.2, and

$$\tau_{\pi_v}(j,j) = \begin{cases} 1, & j = 0 \text{ or } c(\pi_v) \geqslant 2, \\ 1 - q_v^{-2}, & c(\pi_v) = 1, j \geqslant 1, \\ 1 - Q(\pi_v)^2, & c(\pi_v) = 0, j = 1, \\ (1 - Q(\pi_v)^2)(1 - q_v^{-2}), & c(\pi_v) = 0, j \geqslant 2. \end{cases}$$

Lemma 2.1 *We have*

$$\|\varphi_{\pi,\rho}\|^2 / \|\varphi_\pi^{\mathrm{new}}\|^2 = \prod_{v \in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \tau_{\pi_v}(\rho(v), \rho(v)), \quad \rho \in \Lambda_\pi(\mathfrak{n}).$$

Proof From [9, §4] and [11, §6], $\varphi_{\pi,\rho}$ is the automorphic realization of the global Whittaker function $\otimes_{v\in\Sigma_\infty} \phi_{0,v} \otimes \otimes_{v\in\Sigma_{\mathrm{fin}}} \phi_{\rho(v),v}$, considered in the restricted tensor product of the Whittaker model $\mathcal{W}(\pi_v)$ of π_v, where $\{\phi_{v,j} | 0 \leqslant j \leqslant \mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})\}$ is the orthogonal basis of $\mathcal{W}(\pi_v)^{\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)}$ constructed in [9]. With respect to any G_v-invariant inner product of $\mathcal{W}(\pi_v)$, it is shown that the ratio $\|\phi_{j,v}\|^2 / \|\phi_{0,v}\|^2$ is $\tau_{\pi_v}(j,j)$ by [9, Proposition 10 and Corollary 12] for $c(\pi_v) = 0$, by [9, Proposition 13 and Lemma 3] for $c(\pi_v) \geqslant 2$, and by [9, Propositions 15 and Corollary 16] for $c(\pi_v) = 1$. ■

Lemma 2.2 *Let $v \in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ and set $k_v = \mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})$, $X = q_v^{1/2-z}$. If $\eta_v(\varpi_v) = -1$,*

$$r^{(z)}(\pi_v, \eta_v) = \begin{cases} \frac{1-X}{1+Q(\pi_v)} + \frac{(1+a_v q_v^{1/2} X)(1+a_v^{-1} q_v^{1/2} X)}{(q_v-1)(1+Q(\pi_v))} \frac{1-(-X)^{k_v-1}}{1+X}, & c(\pi_v) = 0, \\ 1 - \frac{X+q_v^{-1}\chi_v(\varpi_v)}{1+q_v^{-1}\chi_v(\varpi_v)} \frac{1-(-1)^{k_v} X^{k_v}}{1+X}, & c(\pi_v) = 1, \\ \frac{1+(-1)^{k_v} X^{k_v+1}}{1+X}, & c(\pi_v) \geqslant 2. \end{cases}$$

If $\eta_v(\varpi_v) = 1$,

$$r^{(z)}(\pi_v, \eta_v) = \begin{cases} \frac{1+X}{1+Q(\pi_v)} + \frac{(1-a_v q_v^{1/2} X)(1-a_v^{-1} q_v^{1/2} X)}{(q_v-1)(1+Q(\pi_v))} \sum_{j=2}^{k_v} X^{j-2}, & c(\pi_v) = 0, \\ 1 + \frac{X-q_v^{-1}\chi_v(\varpi_v)}{1+q_v^{-1}\chi_v(\varpi_v)} (\sum_{j=1}^{k_v} X^j), & c(\pi_v) = 1, \\ \sum_{j=0}^{k_v} X^j, & c(\pi_v) \geqslant 2. \end{cases}$$

Here a_v and χ_v are the same as in the definition of $Q_{j,v}^{\pi_v}(\eta_v, X)$.

Proof This is obtained by a direct computation. ■

Define $w_\mathfrak{n}^\eta(\pi) = w_\mathfrak{n}^\eta(\pi; 1/2)$, and $\partial w_\mathfrak{n}^\eta(\pi) = \frac{d}{dz} w_\mathfrak{n}^\eta(\pi; z)|_{z=1/2}$. Lemma 2.2 yields explicit formulas of $w_\mathfrak{n}^\eta(\pi)$ and $\partial w_\mathfrak{n}^\eta(\pi)$; by the Leibniz rule, the formula of $\partial w_\mathfrak{n}^\eta(\pi)$ involves $\partial r(\pi_v, \eta_v) = \frac{-1}{\log q_v} \left(\frac{d}{dz}\right)_{z=1/2} r^{(z)}(\pi_v, \eta_v)$, which is given as in the following corollary.

Corollary 2.3 *If $\eta_v(\varpi_v) = -1$,*

$$\partial r(\pi_v, \eta_v) = \begin{cases} \frac{-1}{1+Q(\pi_v)} + \frac{1+(-1)^{k_v}}{2} \frac{2q_v + (q_v+1)Q(\pi_v)}{(q_v-1)(1+Q(\pi_v))} + \frac{(-1)^{k_v}(2k_v-3)-1}{4} \frac{q_v+1}{q_v-1}, & c(\pi_v) = 0, \\ -\frac{1-(-1)^{k_v}}{2} \frac{1}{1+q_v^{-1}\chi_v(\varpi_v)} + \frac{1+(-1)^{k_v}(2k_v-1)}{4}, & c(\pi_v) = 1, \\ \frac{(-1)^{k_v}(2k_v+1)-1}{4}, & c(\pi_v) \geqslant 2. \end{cases}$$

If $\eta_v(\varpi_v) = 1$,

$$\partial r(\pi_v, \eta_v) = \begin{cases} \frac{1}{1+Q(\pi_v)} + (k_v - 1)\frac{2q_v-(q_v+1)Q(\pi_v)}{(q_v-1)(1+Q(\pi_v))} \\ \qquad + \frac{(k_v-2)(k_v-1)}{2}\frac{(q_v+1)(1-Q(\pi_v))}{(q_v-1)(1+Q(\pi_v))}, & c(\pi_v)=0, \\ \frac{k_v}{1+q_v^{-1}\chi_v(\varpi_v)} + \frac{1-q_v^{-1}\chi_v(\varpi_v)}{1+q_v^{-1}\chi_v(\varpi_v)}\frac{k_v(k_v+1)}{2}, & c(\pi_v)=1, \\ \frac{k_v(k_v+1)}{2}, & c(\pi_v)\geqslant 2. \end{cases}$$

3 Spectral Average of Derivatives of L-series: The Spectral Side

For a finite subset $S \subset \Sigma_{\rm fin}$, we set $\mathfrak{X}_S = \prod_{v\in S}(\mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z})$ and $\mathcal{A}_S = \bigotimes_{v\in S}\mathcal{A}_v$, where for $v \in \Sigma_{\rm fin}$, $\mathcal{A}_v$ denotes the space of holomorphic functions $\alpha(s)$ in

$$s \in \mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}$$

such that $\alpha(-s) = \alpha(s)$.

Throughout this paper we fix a quadratic idele class character η of $F^\times$, an ideal $\mathfrak{n} \subset \mathfrak{o}$, an even weight $l = (l_v)_{v\in\Sigma_\infty}$, and a finite subset $S \subset \Sigma_{\rm fin}$ in such a way that $S(\mathfrak{n})$, $S(\mathfrak{f}_\eta)$, and S are mutually disjoint, and $l_v \geqslant 6$ for all $v \in \Sigma_\infty$. Set $\underline{l} = \inf_{v\in\Sigma_\infty} l_v$. We also fix a function $\alpha \in \mathcal{A}_S$ until §5.

3.1 The Regularized Smoothed Kernel

Let $\mathcal{B}$ be the space of even entire functions $\beta(z)$ on $\mathbb{C}$ such that for any finite interval $I \subset \mathbb{R}$ and for any $N > 0$, $|\beta(\sigma + it)| \ll_{I,N} (1 + |t|)^{-N}$ for $\sigma + it \in I + i\mathbb{R}$. Depending on the data $(\eta, \mathfrak{n}, l, S, \alpha)$, we have constructed a cusp form $\widehat{\Psi}^l_{\rm reg}(\mathfrak{n}|\alpha)$ in [11], which plays a pivotal role in the deduction of the relative trace formula in §3 and §4. Let us review its definition briefly. For $\beta \in \mathcal{B}$ and $(\mathbf{s}, \lambda) \in \mathfrak{X}_S \times \mathbb{C}$ such that $q(\mathbf{s}) > 1$, $\mathrm{Re}(\lambda) > 1 - q(\mathbf{s})$ with $q(\mathbf{s}) = \inf_{v\in S}(\mathrm{Re}(s_v) + 1)/4$, we first set

$$\Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g) = \frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{z+\lambda}\{\Psi^{(z)}_l(\mathfrak{n}|\mathbf{s}; g) + \Psi^{(-z)}_l(\mathfrak{n}|\mathbf{s}; g)\}\,dz,$$

where the contour is taken so that $-\inf(q(\mathbf{s}) - 1, \mathrm{Re}(\lambda)) < \sigma < q(\mathbf{s}) - 1$, and $\Psi^{(z)}_l(\mathfrak{n}|\mathbf{s}; g)$ is Green's function defined as

$$\Psi^{(z)}_l(\mathfrak{n}|\mathbf{s}; g) = \prod_{v\in\Sigma_\infty}\Psi^{(z)}_v(l_v; g_v)\prod_{v\in S}\Psi^{(z)}_v(s_v; g_v)\prod_{v\in\Sigma_{\rm fin}-S}\Phi^{(z)}_{\mathfrak{n},v}(g_v),$$

for $g = (g_v)_{v\in\Sigma_F} \in G_{\mathbb{A}}$, where $\Psi^{(z)}_v(l_v; -)$ for $v \in \Sigma_\infty$ is the holomorphic Shintani function on $G_v \cong \mathrm{GL}(2, \mathbb{R})$ defined in [11, Proposition 3.1], $\Psi^{(z)}_v(s_v; -)$ for $v \in S$ is Green's function studied in [12, Chapter 5], and for any $v \in \Sigma_{\rm fin} - S$, $\Phi^{(z)}_{\mathfrak{n},v}$ is a function on G_v defined as

$$\Phi^{(z)}_{\mathfrak{n},v}(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right]k) = |t_1/t_2|_v^z\,\delta(x \in \mathfrak{o}_v)\delta(k \in \mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v))$$

for $t_1, t_2 \in F_v^\times$, $x \in F_v$, and $k \in \mathbf{K}_v$. Set

$$\widehat{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = \Big(\frac{1}{2\pi i}\Big)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g)\alpha(\mathbf{s})\,d\mu_S(\mathbf{s}),$$

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where $\mathrm{Re}(\lambda) > 0$, with $\mathbf{c} \in \mathbb{R}^S$ such that $q(\mathbf{c}) > \sup(\mathrm{Re}(\lambda)+1, 2)$, and where $\int_{\mathbb{L}_S(\mathbf{c})} f(\mathbf{s})\, d\mu_S(\mathbf{s})$ means the multidimensional contour integral along

$$\mathbb{L}_S(\mathbf{c}) = \prod_{v\in S}\{c_v + it | t \in \mathbb{R}/4\pi(\log q_v)^{-1}\mathbb{Z}\}$$

(oriented naturally) with respect to $d\mu_S(\mathbf{s}) = \prod_{v\in S} d\mu_v(s_v)$ with

$$d\mu_v(s_v) = 2^{-1}(\log q_v)(q_v^{(1+s_v)/2} - q_v^{(1-s_v)/2})ds_v.$$

The Poincaré series $\widehat{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = \sum_{\gamma\in H(F)\backslash G(F)} \widehat{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\alpha;\gamma g)$, $g \in G_{\mathbb{A}}$ is shown to be absolutely convergent on the half plane $\mathrm{Re}(\lambda) > 0$ [11, Lemma 5.5] and has a holomorphic continuation to the whole complex λ-plane defining a smooth function in $g \in G_{\mathbb{A}}$ that is a cusp form on $G_{\mathbb{A}}$ belonging to the space $L^2(Z_{\mathbb{A}}G(F)\backslash G_{\mathbb{A}})[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$ for all λ [11, §6.6]. By taking the constant term of the Taylor series of $\widehat{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ at $\lambda = 0$, we define $\widehat{\Psi}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha)$ independently of β as

$$\mathrm{CT}_{\lambda=0}\, \widehat{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = \widehat{\Psi}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha; g)\beta(0).$$

For $\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$ and $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\pi)$, the Satake parameter of π at v is denoted by $A_v(\pi) = \mathrm{diag}(q_v^{\nu_v(\pi)/2}, q_v^{-\nu_v(\pi)/2})$ with $\nu_v(\pi) \in \mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}$. From [11, §6.6], $\widehat{\Psi}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha)$ has the spectral expansion

$$\text{(3.1)}\qquad \widehat{\Psi}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha; g) = \frac{(-1)^{\#S}\{\prod_{v\in\Sigma_\infty} 2^{l_v-1}\}C_l(0)D_F^{-1/2}}{[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]} \times \sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}\sum_{\rho\in\Lambda_\pi(\mathfrak{n})} \alpha(\nu_S(\pi))\frac{\overline{Z^*(1/2,\mathbf{1},\varphi_{\pi,\rho})}}{\|\varphi_{\pi,\rho}\|^2}\varphi_{\pi,\rho}(g)$$

for all $g \in G_{\mathbb{A}}$, where $\nu_S(\pi) = \{\nu_v(\pi)\}_{v\in S} \in \mathfrak{X}_S$ and

$$C_l(0) = \prod_{v\in\Sigma_\infty} 2^{-1}\Gamma((l_v-1)/2)^2\Gamma(l_v-1)^{-1}.$$

3.2 The Periods Related to the Derivative

Given $\beta \in \mathcal{B}$, $t > 0$, and $\lambda \in \mathbb{C}$, we set $\beta^{(1)}_\lambda(t) = \frac{1}{2\pi i}\int_{L_\sigma} \frac{\beta(z)}{(z+\lambda)^2} t^z\, dz$, where $\sigma > -\mathrm{Re}(\lambda)$. The integral is independent of the choice of σ, and the resulting function $\lambda \mapsto \beta^{(1)}_\lambda(t)$ is entire on $\mathbb{C}$. We easily have

$$\text{(3.2)}\qquad \mathrm{CT}_{\lambda=0}\{\beta^{(1)}_\lambda(t) - \beta^{(1)}_\lambda(t^{-1})\} = \beta(0)\log t$$

by the residue theorem, and obtain the estimate

$$\text{(3.3)}\qquad |\beta^{(1)}_\lambda(t)| \ll_\sigma \inf\{t^\sigma, t^{-\mathrm{Re}(\lambda)}\}\log t, \quad t > 0, \sigma > -\mathrm{Re}(\lambda)$$

in the same way as [12, Lemma 7.1].

Definition 3.1 For a cusp form φ on $\mathrm{PGL}(2,\mathbb{A})$, set

$$\partial P^\eta_{\beta,\lambda}(\varphi) = \int_{F^\times\backslash\mathbb{A}^\times} \varphi\left(\left[\begin{smallmatrix} t & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\right)\eta(tx^*_\eta)\{\beta^{(1)}_\lambda(|t|_{\mathbb{A}}) - \beta^{(1)}_\lambda(|t|^{-1}_{\mathbb{A}})\}\, d^\times t, \quad \lambda \in \mathbb{C}.$$

By (3.3), the integral $\partial P^{\eta}_{\beta,\lambda}(\varphi)$ is absolutely convergent for $\lambda \in \mathbb{C}$ and the function $\lambda \mapsto \partial P^{\eta}_{\beta,\lambda}(\varphi)$ is entire on $\mathbb{C}$. Moreover, (3.2) gives us the formula

$$\mathrm{CT}_{\lambda=0}\,\partial P^{\eta}_{\beta,\lambda}(\varphi) = \int_{F^\times\backslash\mathbb{A}^\times} \varphi\left(\left[\begin{smallmatrix} t & 0 \\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1\end{smallmatrix}\right]\right)\eta(tx_\eta^*)\log|t|_{\mathbb{A}}\,d^\times t\beta(0) = \frac{d}{ds}Z^*(s,\eta,\varphi)\Big|_{s=1/2}\beta(0). \tag{3.4}$$

As we recalled in §3.1, the function $\widehat{\Psi}^{l}_{\mathrm{reg}}(\mathfrak{n}|\alpha)$ is a cusp form on $G_{\mathbb{A}}$ invariant by the center $Z_{\mathbb{A}}$. Thus we have an entire function $\lambda \mapsto \partial P^{\eta}_{\beta,\lambda}(\widehat{\Psi}^{l}_{\mathrm{reg}}(\mathfrak{n}|\alpha))$. The following is the main result of this section.

Proposition 3.2 *We have*

$$\begin{aligned}(3.5)\quad &\mathrm{CT}_{\lambda=0}\partial P^{\eta}_{\beta,\lambda}(\widehat{\Psi}^{l}_{\mathrm{reg}}(\mathfrak{n}|\alpha)) \\ &= (-1)^{\#S}\Big\{\prod_{v\in\Sigma_\infty}2^{l_v-1}\Big\}C_l(0)D_F^{-1}[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1}(-1)^{\epsilon(\eta)}\mathcal{G}(\eta) \\ &\times\Bigg[\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}(\log D_F)w^{\eta}_{\mathfrak{n}}(\pi)\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{\|\varphi^{\mathrm{new}}_\pi\|^2}\alpha(\nu_S(\pi)) \\ &+\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}\partial w^{\eta}_{\mathfrak{n}}(\pi)\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{\|\varphi^{\mathrm{new}}_\pi\|^2}\alpha(\nu_S(\pi)) \\ &+\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}w^{\eta}_{\mathfrak{n}}(\pi)\frac{L(1/2,\pi)L'(1/2,\pi\otimes\eta)}{\|\varphi^{\mathrm{new}}_\pi\|^2}\alpha(\nu_S(\pi))\Bigg]\beta(0).\end{aligned}$$

Proof Since the spectral expansion (3.1) is a finite sum, from (3.4), we have

$$\begin{aligned}\mathrm{CT}_{\lambda=0}\,\partial P^{\eta}_{\beta,\lambda}(\widehat{\Psi}^{l}_{\mathrm{reg}}(\mathfrak{n}|\alpha)) &= (-1)^{\#S}\prod 2^{l_v-1}C_l(0)D_F^{-1/2}[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1} \\ &\times\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\alpha(\nu_S(\pi))\frac{\overline{Z^*(1/2,\mathbf{1},\varphi_{\pi,\rho})}}{\|\varphi_{\pi,\rho}\|^2}\frac{d}{ds}Z^*(s,\eta,\varphi_{\pi,\rho})|_{s=1/2}\beta(0).\end{aligned}$$

From (2.2) by the Leibniz rule, the inner sum

$$\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\frac{\overline{Z^*(1/2,\mathbf{1},\varphi_{\pi,\rho})}}{\|\varphi_{\pi,\rho}\|^2}\frac{d}{ds}Z^*(s,\eta,\varphi_{\pi,\rho})|_{s=1/2}$$

is the sum of the following three quantities.

$$\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\frac{1}{\|\varphi_{\pi,\rho}\|^2}D_F^{-1/2}Q_{\pi,\mathbf{1},\rho}(1/2)L(1/2,\pi)(\log D_F)\mathcal{G}(\eta)Q_{\pi,\eta,\rho}(1/2)\times L(1/2,\pi\otimes\eta), \tag{3.6}$$

$$\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\frac{1}{\|\varphi_{\pi,\rho}\|^2}D_F^{-1/2}Q_{\pi,\mathbf{1},\rho}(1/2)L(1/2,\pi)\mathcal{G}(\eta)(Q_{\pi,\eta,\rho})'(1/2)L(1/2,\pi\otimes\eta), \tag{3.7}$$

$$\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\frac{1}{\|\varphi_{\pi,\rho}\|^2}D_F^{-1/2}Q_{\pi,\mathbf{1},\rho}(1/2)L(1/2,\pi)\mathcal{G}(\eta)Q_{\pi,\eta,\rho}(1/2)L'(1/2,\pi\otimes\eta). \tag{3.8}$$

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Since $w_{\mathfrak{n}}^{\eta}(\pi) = w_{\mathfrak{n}}^{\eta}(\pi;1/2)$ and $\partial w_{\mathfrak{n}}^{\eta}(\pi) = \frac{d}{dz} w_{\mathfrak{n}}^{\eta}(\pi;z)|_{z=1/2}$, by Lemma 2.1 and the first expression in (2.4),

$$w_{\mathfrak{n}}^{\eta}(\pi)/\|\varphi_{\pi}^{\rm new}\|^2 = \sum_{\rho\in\Lambda_\pi(\mathfrak{n})} \frac{Q_{\pi,\mathbf{1},\rho}(1/2)Q_{\pi,\eta,\rho}(1/2)}{\|\varphi_{\pi,\rho}\|^2},$$
$$\partial w_{\mathfrak{n}}^{\eta}(\pi)/\|\varphi_{\pi}^{\rm new}\|^2 = \sum_{\rho\in\Lambda_\pi(\mathfrak{n})} \frac{Q_{\pi,\mathbf{1},\rho}(1/2)Q'_{\pi,\eta,\rho}(1/2)}{\|\varphi_{\pi,\rho}\|^2}.$$

Using these relations, we easily see that (3.6), (3.7), and (3.8) equal the first, second, and third terms of (3.5), respectively. ■

4 Spectral Average of Derivatives of L-series: The Geometric Side

We continue to work with the setting of §3. From [11, §7], the function $\widehat{\Psi}_{\rm reg}^{l}(\mathfrak{n}|\alpha)$ has the expression coming from the $\big(H(F),H(F)\big)$-double coset decomposition of $G(F)$:

$$\text{(4.1)}\quad \widehat{\Psi}_{\rm reg}^{l}\Big(\mathfrak{n}|\alpha;\left[\begin{smallmatrix} t & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\Big) = \big(1+i^{\widetilde{l}}\delta(\mathfrak{n}=\mathfrak{o})\big)J_{\rm id}(\alpha;t) + J_{\rm u}(\alpha;t)+J_{\bar{\rm u}}(\alpha;t)+J_{\rm hyp}(\alpha;t),\quad t\in\mathbb{A}^\times,$$

where $\widetilde{l} = \sum_{v\in\Sigma_\infty} l_v$,

(4.2)

$$J_{\rm id}(\alpha;t) = \delta(\mathfrak{f}_\eta=\mathfrak{o})\Big(\frac{1}{2\pi i}\Big)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\prod_{v\in S}(1-q_v^{-(s_v+1)/2})^{-1}(1-q_v^{(s_v+1)/2})^{-1}\alpha(\mathbf{s})\,d\mu_S(\mathbf{s}),$$

$$\text{(4.3)}\quad J_{\rm u}(\alpha;t) = \Big(\frac{1}{2\pi i}\Big)^{\#S}\sum_{a\in F^\times}\int_{\mathbb{L}_S(\mathbf{c})} \times\Big\{\Psi_l^{(0)}\Big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix} 1 & at^{-1}\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\Big)+\Psi_l^{(0)}\Big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix} 1 & 0\\ at^{-1} & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0\\ -x_\eta & 1\end{smallmatrix}\right]w_0\Big)\Big\}\alpha(\mathbf{s})\,d\mu_S(\mathbf{s}),$$

$$\text{(4.4)}\quad J_{\bar{\rm u}}(\alpha;t) = \Big(\frac{1}{2\pi i}\Big)^{\#S}\sum_{a\in F^\times}\int_{\mathbb{L}_S(\mathbf{c})} \times\Big\{\Psi_l^{(0)}\Big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix} 1 & 0\\ at & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\Big)+\Psi_l^{(0)}\Big(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix} 1 & at\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0\\ -x_\eta & 1\end{smallmatrix}\right]w_0\Big)\Big\}\alpha(\mathbf{s})\,d\mu_S(\mathbf{s})$$

with $w_0 = \left[\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right]$ and x_η is the adele defined in §2.1.2, and

$$J_{\rm hyp}(\alpha;t) = \sum_{b\in F^\times-\{-1\}}\sum_{a\in F^\times}\widehat{\Psi}_l^{(0)}\Big(\mathfrak{n}|\alpha;\left[\begin{smallmatrix} 1+b^{-1} & 1\\ 1 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} at & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\Big)$$

with

$$\text{(4.5)}\qquad \widehat{\Psi}_l^{(0)}(\mathfrak{n}|\alpha;g) = \Big(\frac{1}{2\pi i}\Big)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\Psi_l^{(0)}(\mathfrak{n}|\mathbf{s};g)\alpha(\mathbf{s})\,d\mu_S(\mathbf{s}).$$

The convergence of the integrals and series was fully discussed in [11, §7].

4.1 Orbital Integrals

For $\bullet \in \{\mathrm{id}, \mathrm{u}, \bar{\mathrm{u}}, \mathrm{hyp}\}$, set

$$\mathbb{W}^{\eta}_{\bullet}(\beta,\lambda;\alpha) = \int_{F^\times\backslash\mathbb{A}^\times} J_{\bullet}(\alpha;t)\{\beta^{(1)}_{\lambda}(|t|_{\mathbb{A}}) - \beta^{(1)}_{\lambda}(|t|^{-1}_{\mathbb{A}})\}\eta(tx^*_\eta)\,d^\times t, \quad \mathrm{Re}(\lambda) > 1.$$

In this subsection, we shall show that these integrals converge absolutely when $\mathrm{Re}(\lambda) > 1$ and admit an analytic continuation in a neighborhood of $\lambda = 0$.

Lemma 4.1 *The integral $\mathbb{W}^{\eta}_{\mathrm{id}}(\beta,\lambda;\alpha)$ converges absolutely and $\mathbb{W}^{\eta}_{\mathrm{id}}(\beta,\lambda;\alpha) = 0$ for* $\mathrm{Re}(\lambda) > 0$.

Proof Let λ and w be complex numbers such that $\mathrm{Re}(w) < \mathrm{Re}(\lambda)$, and ξ an idele class character of $F^\times$. Then in the same way as [12, Lemma 7.6], we have

$$\int_{F^\times\backslash\mathbb{A}^\times} \beta^{(1)}_{\lambda}(|t|_{\mathbb{A}})\xi(t)|t|^{w}_{\mathbb{A}}d^\times t = \delta_{\xi,\mathbf{1}}\,\mathrm{vol}(F^\times\backslash\mathbb{A}^1)\frac{\beta(-w)}{(\lambda-w)^2}.$$

From this combined with (4.2), we have the conclusion because our η is non-trivial. ■

Set

$$\underline{l} = \inf_{v\in\Sigma_\infty} l_v, \qquad \widetilde{l} = \sum_{v\in\Sigma_\infty} l_v, \qquad q(\mathbf{s}) = \inf_{v\in S}\frac{\mathrm{Re}(s_v)+1}{4} \qquad (\mathbf{s} = (s_v)_{v\in S} \in \mathfrak{X}_S)$$

and $\epsilon_v = 2^{-1}(1-\eta_v(-1))$ for all $v \in \Sigma_\infty$. For $\mathbf{s} \in \mathfrak{X}_S$ and $z \in \mathbb{C}$, set

$$\Upsilon^{\eta}_{S}(z;\mathbf{s}) = \prod_{v\in S}(1-\eta_v(\varpi_v)q_v^{-(z+(s_v+1)/2)})^{-1}(1-q_v^{(s_v+1)/2})^{-1},$$

$$\Upsilon^{\eta}_{S,l}(z;\mathbf{s}) = D_F^{-1/2}\{\#(\mathfrak{o}/\mathfrak{f}_\eta)^\times\}^{-1} \times\Big\{\prod_{v\in\Sigma_\infty}\frac{2\Gamma(-z)\Gamma(l_v/2+z)}{\Gamma_{\mathbb{R}}(-z+\epsilon_v)\Gamma(l_v/2)}i^{\epsilon_v}\cos\Big(\frac{\pi}{2}(-z+\epsilon_v)\Big)\Big\}\Upsilon^{\eta}_{S}(z;\mathbf{s}).$$

Lemma 4.2 *For $\mathbf{s} \in \mathfrak{X}_S$ and $\lambda \in \mathbb{C}$ such that $q(\mathrm{Re}(\mathbf{s})) > \mathrm{Re}(\lambda) > \sigma > 1$ and $1 < \sigma < \underline{l}/2$, the integrals*

$$V^{\pm}_{0,\eta}(\lambda;\mathbf{s}) = \int_{\mathbb{A}^\times}\Big(\frac{1}{2\pi i}\int_{L_{\mp\sigma}}\frac{\beta(z)}{(z+\lambda)^2}|t|^{\pm z}_{\mathbb{A}}\,dz\Big)\Psi^{(0)}_{l}\big(\mathfrak{n}|\mathbf{s};\big[\begin{smallmatrix}1 & t^{-1}\\ 0 & 1\end{smallmatrix}\big]\big[\begin{smallmatrix}1 & x_\eta\\ 0 & 1\end{smallmatrix}\big]\big)\eta(tx^*_\eta)d^\times t,$$

$$V^{\pm}_{1,\eta}(\lambda;\mathbf{s}) = \int_{\mathbb{A}^\times}\Big(\frac{1}{2\pi i}\int_{L_{\mp\sigma}}\frac{\beta(z)}{(z+\lambda)^2}|t|^{\pm z}_{\mathbb{A}}\,dz\Big) \times\Psi^{(0)}_{l}\big(\mathfrak{n}|\mathbf{s};\big[\begin{smallmatrix}1 & 0\\ t^{-1} & 1\end{smallmatrix}\big]\big[\begin{smallmatrix}1 & 0\\ -x_\eta & 1\end{smallmatrix}\big]w_0\big)\eta(tx^*_\eta)d^\times t$$

converge absolutely as double integrals and

$$V^{\pm}_{0,\eta}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\mathrm{N}(\mathfrak{f}_\eta)^{\mp z}L(\mp z,\eta)(-1)^{\epsilon(\eta)}\Upsilon^{\eta}_{S,l}(\pm z;\mathbf{s})\,dz,$$

$$V^{\pm}_{1,\eta}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\mathrm{N}(\mathfrak{f}_\eta)^{\mp z}\mathrm{N}(\mathfrak{n})^{\pm z}\widetilde{\eta}(\mathfrak{n})\delta(\mathfrak{n}=\mathfrak{o})L(\mp z,\eta)i^{\widetilde{l}}\Upsilon^{\eta}_{S,l}(\pm z;\mathbf{s})\,dz.$$

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Proof As in [11, Lemma 8.2], we change the order of integrals and compute the t-integrals first. Since $\eta \neq 1$, the integrands in the remaining contour integrals in z are holomorphic on $|\operatorname{Re}(z)| < \sigma$; thus we can shift the contour $L_{-\sigma}$ to L_σ for $V_{0,\eta}^+$ and $V_{1,\eta}^+$. ∎

Lemma 4.3 *The integral $\mathbb{W}_{\mathfrak{u}}^{\eta}(\beta,\lambda;\alpha)$ has an analytic continuation to the region $\operatorname{Re}(\lambda) > -\underline{l}/2$ as a function in λ. The constant term of $\mathbb{W}_{\mathfrak{u}}^{\eta}(\beta,\lambda;\alpha)$ at $\lambda = 0$ equals $\mathbb{W}_{\mathfrak{u}}^{\eta}(l,\mathfrak{n}|\alpha)\beta(0)$ with*

$$\mathbb{W}_{\mathfrak{u}}^{\eta}(l,\mathfrak{n}|\alpha) = (-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\big(1+(-1)^{\epsilon(\eta)}\widetilde{\eta}(\mathfrak{n})i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})\big)\Big(\frac{1}{2\pi i}\Big)^{\#S} \times \int_{\mathbb{L}_S(\mathbf{c})} \mathfrak{W}_{S,\mathfrak{u}}^{\eta}(l,\mathfrak{n}|\mathbf{s})\alpha(\mathbf{s})\,d\mu_S(\mathbf{s}),$$

where $\Upsilon_S^{\eta}(\mathbf{s}) = \Upsilon_S^{\eta}(0;\mathbf{s})$ and

$$\mathfrak{W}_{S,\mathfrak{u}}^{\eta}(l,\mathfrak{n}|\mathbf{s}) = \pi^{\epsilon(\eta)}\Upsilon_S^{\eta}(\mathbf{s})L(1,\eta)\Big\{\log D_F + \frac{L'(1,\eta)}{L(1,\eta)} + \sum_{v\in\Sigma_\infty}\Big(\sum_{k=1}^{l_v/2-1}\frac{1}{k} - \frac{1}{2}\log\pi - \frac{1}{2}C_{\mathrm{Euler}} - \delta_{\epsilon_v,1}\log 2\Big) + \sum_{v\in S}\frac{\log q_v}{1-\eta_v(\varpi_v)q_v^{(s_v+1)/2}}\Big\}.$$

Proof From (4.3) and Lemma 4.2, we have

$$\begin{aligned}
&\mathbb{W}_{\mathfrak{u}}^{\eta}(\beta,\lambda;\alpha)\\
&= \Big(\frac{1}{2\pi i}\Big)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\{V_{0,\eta}^+(\lambda;\mathbf{s}) - V_{0,\eta}^-(\lambda;\mathbf{s}) + V_{1,\eta}^+(\lambda;\mathbf{s}) - V_{1,\eta}^-(\lambda;\mathbf{s})\}\alpha(\mathbf{s})d\mu_S(\mathbf{s})\\
&= \big((-1)^{\epsilon(\eta)} + i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})\big)\Big(\frac{1}{2\pi i}\Big)^{\#S}\\
&\qquad\times\int_{\mathbb{L}_S(\mathbf{c})}\frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\{\mathrm{N}(\mathfrak{f}_\eta)^{-z}L(-z,\eta)\Upsilon_{S,l}^{\eta}(z;\mathbf{s})\\
&\quad - \mathrm{N}(\mathfrak{f}_\eta)^{z}L(z,\eta)\Upsilon_{S,l}^{\eta}(-z;\mathbf{s})\}dz\alpha(\mathbf{s})d\mu_S(\mathbf{s}),
\end{aligned}$$

which is holomorphic on $\operatorname{Re}(\lambda) > -\sigma$. Since $1 < \sigma < \underline{l}/2$ is arbitrary, this gives an analytic continuation of $\mathbb{W}_{\mathfrak{u}}^{\eta}(\beta,\lambda;\alpha)$ to the region $\operatorname{Re}(\lambda) > -\underline{l}/2$ and yields the first equality of

$$\begin{aligned}
&\mathrm{CT}_{\lambda=0}\,\mathbb{W}_{\mathfrak{u}}^{\eta}(\beta,\lambda;\alpha)\\
&\quad= \Big(\frac{1}{2\pi i}\Big)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\Big(\frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{z^2}\{f_{\mathfrak{u}}(z) - f_{\mathfrak{u}}(-z)\}\,dz\Big)\alpha(\mathbf{s})d\mu_S(\mathbf{s})\\
&\quad= \big((-1)^{\epsilon(\eta)} + i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})\big)\operatorname{Res}_{z=0}\Big(\frac{\beta(z)}{z^2}f_{\mathfrak{u}}(z)\Big)\\
&\quad= \big((-1)^{\epsilon(\eta)} + i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})\big)\{\mathrm{CT}_{z=0}\frac{f_{\mathfrak{u}}(z)}{z}\beta(0) + \frac{1}{2}\operatorname{Res}_{z=0}f_{\mathfrak{u}}(z)\beta''(0)\},
\end{aligned}$$

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where $f_{\mathrm{u}}(z) = \mathrm{N}(\mathfrak{f}_\eta)^{-z}L(-z,\eta)\Upsilon_{S,l}^{\eta}(z;\mathbf{s})$. Since η is non-trivial, by the functional equation $L(s,\eta) = i^{\epsilon(\eta)}D_F^{1-s}\mathrm{N}(\mathfrak{f}_\eta)^{-s}\#((\mathfrak{o}/\mathfrak{f}_\eta)^\times)\mathcal{G}(\eta)L(1-s,\eta)$, $f_{\mathrm{u}}(z)$ is holomorphic at $z=0$. Thus, $\mathrm{Res}_{z=0}f_{\mathrm{u}}(z) = 0$, and $\mathrm{CT}_{z=0}\frac{f_{\mathrm{u}}(z)}{z} = f_{\mathrm{u}}'(0)$; the derivative $f_{\mathrm{u}}'(0)$ is computed as

$$\begin{aligned}
-(\log \mathrm{N}(\mathfrak{f}_\eta))L(0,\eta)\Upsilon_{S,l}^{\eta}(0;\mathbf{s}) - L'(0,\eta)\Upsilon_{S,l}^{\eta}(0;\mathbf{s}) + L(0,\eta)(\Upsilon_{S,l}^{\eta})'(0;\mathbf{s})\\
&= i^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\widetilde{\Upsilon}_{S,l}^{\eta}(0;\mathbf{s})\{-L(1,\eta)\log\mathrm{N}(\mathfrak{f}_\eta)\\
&\qquad + L(1,\eta)\log(D_F\mathrm{N}(\mathfrak{f}_\eta))L'(1,\eta) + L(1,\eta)\tfrac{d}{dz}\log\widetilde{\Upsilon}_{S,l}(z;\mathbf{s})|_{z=0}\}\\
&= \mathcal{G}(\eta)D_F^{1/2}\pi^{\epsilon(\eta)}\Upsilon_S^{\eta}(\mathbf{s})\{L(1,\eta)\log D_F + L'(1,\eta)\\
&\qquad + L(1,\eta)\frac{d}{dz}\log\widetilde{\Upsilon}_{S,l}(z;\mathbf{s})|_{z=0}\},
\end{aligned}$$

where $\widetilde{\Upsilon}_{S,l}^{\eta}(z;\mathbf{s}) = D_F^{1/2}\#((\mathfrak{o}/\mathfrak{f}_\eta)^\times)\Upsilon_{S,l}^{\eta}(z;\mathbf{s})$, whose logarithmic derivative at $z=0$ is computed as

$$\sum_{v\in\Sigma_\infty}\Big(\psi(\tfrac{l_v}{2}) - \frac{1}{2}\log\pi + \frac{1}{2}\psi\Big(\frac{-z+\epsilon_v}{2}\Big) - \psi(-z) + \frac{\pi}{2}\tan\frac{\pi}{2}(-z+\epsilon_v)\Big)\Big|_{z=0} + \sum_{v\in S}\frac{\log q_v}{1-\eta_v(\varpi_v)q_v^{(s_v+1)/2}}.$$

Here, by $\psi(1) = -C_{\mathrm{Euler}}$, $\psi(1/2) = -C_{\mathrm{Euler}} - 2\log 2$, and $\frac{d}{dt}(t\cot t)|_{t=0} = 0$, we have

$$\frac{1}{2}\psi\Big(\frac{-z+\epsilon_v}{2}\Big) - \psi(-z) + \frac{\pi}{2}\tan\frac{\pi}{2}(-z+\epsilon_v)\Big|_{z=0} = \begin{cases}\frac{1}{2}C_{\mathrm{Euler}} & \epsilon_v = 0,\\ \frac{1}{2}\psi(\frac{1}{2}) - \psi(1) = \frac{1}{2}C_{\mathrm{Euler}} - \log 2 & \epsilon_v = 1.\end{cases}$$

■

In the same way as Lemma 4.2, we obtain the following.

Lemma 4.4 *For $\mathbf{s}\in\mathfrak{X}_S$ and $\lambda\in\mathbb{C}$ such that $q(\mathrm{Re}(\mathbf{s})) > \mathrm{Re}(\lambda) > \sigma > 1$ and $1 < \sigma < \underline{l}/2$, the integrals*

$$\widetilde{V}_{1,\eta}^{\pm}(\lambda;\mathbf{s}) = \int_{\mathbb{A}^\times}\Big(\frac{1}{2\pi i}\int_{L_{\pm\sigma}}\frac{\beta(z)}{(z+\lambda)^2}|t|_{\mathbb{A}}^{\pm z}\,dz\Big)\Psi_l^{(0)}(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right])\eta(tx_\eta^*)\,d^\times t,$$

$$\widetilde{V}_{0,\eta}^{\pm}(\lambda;\mathbf{s}) = \int_{\mathbb{A}^\times}\Big(\frac{1}{2\pi i}\int_{L_{\pm\sigma}}\frac{\beta(z)}{(z+\lambda)^2}|t|_{\mathbb{A}}^{\pm z}\,dz\Big)\Psi_l^{(0)}(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\-x_\eta&1\end{smallmatrix}\right]w_0)\eta(tx_\eta^*)\,d^\times t$$

converge absolutely as double integrals, and

$$\widetilde{V}_{1,\eta}^{\pm}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\mathrm{N}(\mathfrak{f}_\eta)^{\mp z}\mathrm{N}(\mathfrak{n})^{\mp z}\widetilde{\eta}(\mathfrak{n})L(\pm z,\eta)\Upsilon_{S,l}^{\eta}(\mp z;\mathbf{s})\,dz,$$

$$\widetilde{V}_{0,\eta}^{\pm}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\mathrm{N}(\mathfrak{f}_\eta)^{\mp z}\delta(\mathfrak{n}=\mathfrak{o})L(\pm z,\eta)(-1)^{\epsilon(\eta)}i^{\tilde{l}}\Upsilon_{S,l}^{\eta}(\mp z;\mathbf{s})\,dz.$$

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Lemma 4.5 *The integral* $\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha)$ *converges absolutely on* $\mathrm{Re}(\lambda)>1$ *and has an analytic continuation to the region* $\mathrm{Re}(\lambda)>-\underline{l}/2$ *as a function in* λ. *The constant term of* $\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha)$ *at* $\lambda=0$ *equals* $\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(l,\mathfrak{n}|\alpha)\beta(0)$ *with*

$$\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(l,\mathfrak{n}|\alpha)=(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}((-1)^{\epsilon(\eta)}\widetilde{\eta}(\mathfrak{n})+i^{\widetilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\Big(\frac{1}{2\pi i}\Big)^{\#S}\times\int_{\mathbb{L}_S(\mathbf{c})}\mathfrak{W}^{\eta}_{S,\bar{\mathrm{u}}}(l,\mathfrak{n}|\mathbf{s})\alpha(\mathbf{s})\,d\mu_S(\mathbf{s}),$$

where

$$\mathfrak{W}^{\eta}_{S,\bar{\mathrm{u}}}(l,\mathfrak{n}|\mathbf{s})=-\pi^{\epsilon(\eta)}\Upsilon^{\eta}_S(\mathbf{s})L(1,\eta)\log(\mathrm{N}(\mathfrak{n})\mathrm{N}(\mathfrak{f}_\eta)^2)-\mathfrak{W}^{\eta}_{S,\mathrm{u}}(l,\mathfrak{n}|\mathbf{s}).$$

Proof By (4.4) and Lemma 4.4, we see that $\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha)$ equals

$$\begin{aligned}&\Big(\frac{1}{2\pi i}\Big)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\{\widetilde{V}^{+}_{0,\eta}(\lambda;\mathbf{s})-\widetilde{V}^{-}_{0,\eta}(\lambda;\mathbf{s})+\widetilde{V}^{+}_{1,\eta}(\lambda;\mathbf{s})-\widetilde{V}^{-}_{1,\eta}(\lambda;\mathbf{s})\}\alpha(\mathbf{s})d\mu_S(\mathbf{s})\\&=(\widetilde{\eta}(\mathfrak{n})+(-1)^{\epsilon(\eta)}i^{\widetilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\Big(\frac{1}{2\pi i}\Big)^{\#S}\\&\qquad\times\int_{\mathbb{L}_S(\mathbf{c})}\frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\{\mathrm{N}(\mathfrak{f}_\eta)^{-z}\mathrm{N}(\mathfrak{n})^{-z}L(z,\eta)\Upsilon^{\eta}_{S,l}(-z;\mathbf{s})\\&\quad-\mathrm{N}(\mathfrak{f}_\eta)^{z}\mathrm{N}(\mathfrak{n})^{z}L(-z,\eta)\Upsilon^{\eta}_{S,l}(z;\mathbf{s})\}\,dz\alpha(\mathbf{s})d\mu_S(\mathbf{s}),\end{aligned}$$

which gives an analytic continuation of $\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha)$ to the region $\mathrm{Re}(\lambda)>-\underline{l}/2$. We set $f_{\bar{\mathrm{u}}}(z)=-\mathrm{N}(\mathfrak{f}_\eta)^{2z}\mathrm{N}(\mathfrak{n})^{z}f_{\mathrm{u}}(z)$. Since $f_{\mathrm{u}}(z)$ is holomorphic at $z=0$, so is $f_{\bar{\mathrm{u}}}(z)$. Thus

$$\begin{aligned}\mathrm{CT}_{\lambda=0}\,\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha)&=\big(\widetilde{\eta}(\mathfrak{n})+(-1)^{\epsilon(\eta)}i^{\widetilde{l}}\delta(\mathfrak{n}=\mathfrak{o})\big)\\&\qquad\times\Big(CT_{z=0}\frac{f_{\bar{\mathrm{u}}}(z)}{z}\beta(0)+\frac{1}{2}\mathrm{Res}_{z=0}f_{\bar{\mathrm{u}}}(z)\beta''(0)\Big)\\&=(\widetilde{\eta}(\mathfrak{n})+(-1)^{\epsilon(\eta)}i^{\widetilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))(f'_{\bar{\mathrm{u}}}(0)\beta(0)).\end{aligned}$$

The derivative $f'_{\bar{\mathrm{u}}}(0)$ is computed as $-\log(\mathrm{N}(\mathfrak{n})\mathrm{N}(\mathfrak{f}_\eta^2))f_{\mathrm{u}}(0)-f'_{\mathrm{u}}(0)$, which becomes

$$\mathcal{G}(\eta)D_F^{1/2}\{-\pi^{\epsilon(\eta)}\Upsilon^{\eta}_S(\mathbf{s})L(1,\eta)\log(\mathrm{N}(\mathfrak{n})\mathrm{N}(\mathfrak{f}_\eta^2))-\mathfrak{W}^{\eta}_{S,\mathrm{u}}(l,\mathfrak{n}|\mathbf{s})\}.\qquad\blacksquare$$

Lemma 4.6 *The integral* $\mathbb{W}^{\eta}_{\mathrm{hyp}}(\beta,\lambda;\alpha)$ *converges absolutely for* $\mathrm{Re}(\lambda)>1$ *and has an analytic continuation to the region* $\mathrm{Re}(\lambda)>-\epsilon$ *for some* $\epsilon>0$. *The constant term of* $\mathbb{W}^{\eta}_{\mathrm{hyp}}(\beta,\lambda;\alpha)$ *at* $\lambda=0$ *equals* $\mathbb{W}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)\beta(0)$. *Here*

$$\mathbb{W}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)=\Big(\frac{1}{2\pi i}\Big)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\mathfrak{L}_\eta(l,\mathfrak{n}|\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s})$$

with $\mathbf{c}=(c)_{v\in\Sigma_\infty}\in\mathbb{R}^S$ *such that* $1<c<\underline{l}/2-1$ *and*

$$\mathfrak{L}_\eta(l,\mathfrak{n}|\mathbf{s})=\sum_{b\in F-\{0,-1\}}\int_{\mathbb{A}^\times}\Psi^{(0)}_l(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1+b^{-1}&1\\1&1\end{smallmatrix}\right]\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right])\eta(tx_\eta^*)\log|t|_{\mathbb{A}}\,d^\times t.$$

Proof In the same way as [11, Lemma 8.5], we see that there exists $\epsilon > 0$ such that, for $0 < |\rho| < \epsilon$ the integral

$$\int_{\mathbb{L}_S(\mathbf{c})} |\alpha(\mathbf{s})|\,|d\mu_S(\mathbf{s})| \int_{L_\rho} \frac{|\beta(z)|}{|z+\lambda|^2} \sum_{b\in F^\times-\{-1\}} \int_{\mathbb{A}^\times} \left|\Psi_l^{(0)}(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1+b^{-1} & 1\\ 1 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}t & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & x_\eta\\ 0 & 1\end{smallmatrix}\right])\right| \times \{|t|_{\mathbb{A}}^{\rho}+|t|_{\mathbb{A}}^{-\rho}\}\, d^\times t|dz|$$

is convergent. The analytic continuation of $\mathbb{W}_{\rm hyp}^{\eta}(\beta,\lambda;\alpha)$ to $\mathrm{Re}(\lambda) > -\epsilon$ is obtained from this. The absolute convergence of $\mathbb{W}_{\rm hyp}^{\eta}(l,\mathfrak{n}|\alpha)$ follows by the majorization $|\log x| \ll x^{\rho}+x^{-\rho}$ $(x>0)$. We obtain the last assertion with the aid of (3.2). ∎

From (4.1) combined with Lemmas 4.1, 4.3, 4.5, and 4.6, we get the formula

$$\partial P_{\beta,\lambda}^{\eta}(\widehat{\Psi}_{\rm reg}^{l}(\mathfrak{n}|\alpha)) = \mathbb{W}_{\rm u}^{\eta}(\beta,\lambda;\alpha)+\mathbb{W}_{\check{\rm u}}^{\eta}(\beta,\lambda;\alpha)+\mathbb{W}_{\rm hyp}^{\eta}(\beta,\lambda;\alpha), \tag{4.6}$$

which is valid on a half plane $\mathrm{Re}(\lambda) > -\epsilon$ containing $\lambda = 0$.

4.2 The Relative Trace Formula

For any ideal $\mathfrak{m} \subset \mathfrak{o}$, set

$$\iota(\mathfrak{m}) = [\mathbf{K}_{\rm fin}:\mathbf{K}_0(\mathfrak{m})] = \prod_{v\in S(\mathfrak{m})}(1+q_v)q_v^{\mathrm{ord}_v(\mathfrak{m})-1}. \tag{4.7}$$

Let $\mathcal{J}_{S,\eta}$ be the monoid of ideals generated by prime ideals $\mathfrak{p}_v$ $(v \in \Sigma_{\rm fin} - S \cup S(\mathfrak{f}_\eta))$. We shall introduce several functionals in $\alpha \in \mathcal{A}_S$ depending on $\mathfrak{m} \in \mathcal{J}_{S,\eta}$:

$$\mathrm{AL}^{w}(\mathfrak{m};\alpha) = C_l \sum_{\pi\in\Pi_{\rm cus}(l,\mathfrak{m})} \frac{w_{\mathfrak{m}}^{\eta}(\pi)\iota(\mathfrak{f}_\pi)}{\mathrm{N}(\mathfrak{f}_\pi)\iota(\mathfrak{m})} \frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi;\mathrm{Ad})}\alpha(\nu_S(\pi)), \tag{4.8}$$

$$\mathrm{AL}^{\partial w}(\mathfrak{m};\alpha) = C_l \sum_{\pi\in\Pi_{\rm cus}(l,\mathfrak{m})} \frac{\partial w_{\mathfrak{m}}^{\eta}(\pi)\iota(\mathfrak{f}_\pi)}{\mathrm{N}(\mathfrak{f}_\pi)\iota(\mathfrak{m})} \frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi;\mathrm{Ad})}\alpha(\nu_S(\pi)),$$

$$\mathrm{ADL}_{\pm}^{w}(\mathfrak{m};\alpha) = C_l \sum_{\substack{\pi\in\Pi_{\rm cus}(l,\mathfrak{m})\\ \epsilon(1/2,\pi\otimes\eta)=\pm1}} \frac{w_{\mathfrak{m}}^{\eta}(\pi)\iota(\mathfrak{f}_\pi)}{\mathrm{N}(\mathfrak{f}_\pi)\iota(\mathfrak{m})} \frac{L(1/2,\pi)L'(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi;\mathrm{Ad})}\alpha(\nu_S(\pi)),$$

where C_l is defined as (1.3), $w_{\mathfrak{m}}^{\eta}(\pi)$ and $\partial w_{\mathfrak{m}}^{\eta}(\pi)$ are the weight functions defined in §2.3, and $L^{S_\pi}(s,\pi;\mathrm{Ad})$ is the partial adjoint L-function of π (see the sentence below (2.3)). The derivative of L-functions in $\mathrm{ADL}_{+}^{w}(\mathfrak{m};\alpha)$ is eliminated by the functional equation (1.1).

Proposition 4.7 *We have*

$$\mathrm{ADL}_{+}^{w}(\mathfrak{m};\alpha) = C_l \sum_{\pi\in\Pi_{\rm cus}(l,\mathfrak{m})} \log\{\mathrm{N}(\mathfrak{f}_\pi\mathfrak{f}_\eta^2)D_F^2\}^{-1/2}\frac{w_{\mathfrak{m}}^{\eta}(\pi)\iota(\mathfrak{f}_\pi)}{\mathrm{N}(\mathfrak{f}_\pi)\iota(\mathfrak{m})} \times \frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi;\mathrm{Ad})}\alpha(\nu_S(\pi)).$$

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Proof By differentiating (1.1), $L'(1/2,\pi\otimes\eta)=\frac{\epsilon'(1/2,\pi\otimes\eta)}{2}L(1/2,\pi\otimes\eta)$ if $\epsilon(1/2,\pi\otimes\eta)=1$. Then from $\epsilon(s,\pi\otimes\eta)=\epsilon(1/2,\pi\otimes\eta)\{\mathrm{N}(\mathfrak{f}_\pi)\mathrm{N}(\mathfrak{f}_\eta)^2D_F^2\}^{1/2-s}$, we obtain the assertion immediately. ∎

The following is the main result of §3 and §4.

Theorem 4.8 *Let η be a quadratic idele class character of $F^\times$ and S a finite subset of $\Sigma_{\mathrm{fin}}-S(\mathfrak{f}_\eta)$. Let $l=(l_v)_{v\in\Sigma_\infty}$ be an even weight such that $l_v\geqslant 6$ for all $v\in\Sigma_\infty$. Set $\epsilon(\eta)=\#\{v\in\Sigma_\infty|\eta_v(-1)=-1\}$ and $\tilde{l}=\sum_{v\in\Sigma_\infty}l_v$. For any ideal $\mathfrak{n}\in\mathcal{J}_{S,\eta}$ and for any $\alpha\in\mathcal{A}_S$,*

$$(4.9)\quad 2^{-1}(-1)^{\#S+\epsilon(\eta)}\mathcal{G}(\eta)D_F^{-1}\times\{\mathrm{ADL}_-^w(\mathfrak{n};\alpha)+\mathrm{ADL}_+^w(\mathfrak{n};\alpha)+(\log D_F)\,\mathrm{AL}^w(\mathfrak{n};\alpha)+\mathrm{AL}^{\partial w}(\mathfrak{n};\alpha)\}=\widetilde{\mathbb{W}}_{\mathrm{u}}^\eta(l,\mathfrak{n}|\alpha)+\mathbb{W}_{\mathrm{hyp}}^\eta(l,\mathfrak{n}|\alpha).$$

Here $\mathbb{W}_{\mathrm{hyp}}^\eta(l,\mathfrak{n}|\alpha)$ is defined in Lemma 4.6,

$$(4.10)\quad \widetilde{\mathbb{W}}_{\mathrm{u}}^\eta(l,\mathfrak{n}|\alpha)=(1-(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n}))(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\{1+(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n})i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})\}\times\Big(\frac{1}{2\pi i}\Big)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\widetilde{\mathfrak{W}}_S^\eta(l,\mathfrak{n}|\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s})$$

with $d\mu_S(\mathbf{s})=\prod_{v\in S}2^{-1}\log q_v(q_v^{(1+s_v)/2}-q_v^{(1-s_v)/2})ds_v$ and $\mathbb{L}_S(\mathbf{c})$ being the multidimensional contour $\prod_{v\in S}\{c_v+it\mid t\in\mathbb{R}/4\pi(\log q_v)^{-1}\mathbb{Z}\}$ directed as usual, and

$$(4.11)\quad \widetilde{\mathfrak{W}}_S^\eta(l,\mathfrak{n}|\mathbf{s})=\pi^{\epsilon(\eta)}\Upsilon_S^\eta(\mathbf{s})L(1,\eta)\Big\{\log\big(\sqrt{\mathrm{N}(\mathfrak{n})}D_F\mathrm{N}(\mathfrak{f}_\eta)\big)+\frac{L'(1,\eta)}{L(1,\eta)}+\mathfrak{C}(l)+\sum_{v\in S}\frac{\log q_v}{1-\eta_v(\varpi_v)q_v^{(s_v+1)/2}}\Big\},$$

$$\Upsilon_S^\eta(\mathbf{s})=\prod_{v\in S}(1-\eta_v(\varpi_v)q_v^{-(1+s_v)/2})^{-1}(1-q_v^{(1+s_v)/2})^{-1},$$

$$\mathfrak{C}(l)=\sum_{v\in\Sigma_\infty}\Big(\sum_{k=1}^{l_v/2-1}\frac{1}{k}-\frac{1}{2}\log\pi-\frac{1}{2}C_{\mathrm{Euler}}-\delta_{\epsilon_v,1}\log 2\Big).$$

Proof From Proposition 3.2 together with (2.3),

$$\mathrm{CT}_{\lambda=0}\,\partial P_{\beta,\lambda}^\eta(\widehat{\Psi}_{\mathrm{reg}}^l(\mathfrak{n}|\alpha))=2^{-1}(-1)^{\#S+\epsilon(\eta)}\mathcal{G}(\eta)D_F^{-1}\times\big\{\mathrm{ADL}_-^w(\mathfrak{n};\alpha)+\mathrm{ADL}_+^w(\mathfrak{n};\alpha)+(\log D_F)\,\mathrm{AL}^w(\mathfrak{n};\alpha)+\mathrm{AL}^{\partial w}(\mathfrak{n};\alpha)\big\}.$$

On the other hand, from the formula (4.6), the same $\mathrm{CT}_{\lambda=0}\,\partial P_{\beta,\lambda}^\eta(\widehat{\Psi}_{\mathrm{reg}}^l(\mathfrak{n}|\alpha))$ is computed by Lemmas 4.3, 4.5, and 4.6. ∎

5 Extraction of the New Part: The Totally Inert Case

Let $\mathcal{I}_{S,\eta}$ be the monoid of ideals generated by prime ideals $\mathfrak{p}_v$ such that $v \in \Sigma_{\mathrm{fin}} - S \cup S(\mathfrak{f}_\eta)$ and $\widetilde{\eta}(\mathfrak{p}_v) = -1$. Note that $\mathcal{I}_{S,\eta}$ is a submonoid of $\mathcal{J}_{S,\eta}$ defined in §4.2.

In this section, we separate the contribution of those π with $\mathfrak{f}_\pi = \mathfrak{n}$ from the total average $\mathrm{ADL}_-^w(\mathfrak{n};\alpha)$ under the condition $\mathfrak{n} \in \mathcal{I}_{S,\eta}$. For that purpose, we introduce the notion of $\mathcal{N}$-transform for arithmetic functions on a set of ideals and study its properties in the first place.

5.1 The $\mathcal{N}$-transform

For any ideal $\mathfrak{c}$ and a place $v \in \Sigma_{\mathrm{fin}}$, set $\omega_v(\mathfrak{c}) = 1$ if $v \in S(\mathfrak{c})$ and $\omega_v(\mathfrak{c}) = \frac{q_v+1}{q_v-1}$ if $v \notin S(\mathfrak{c})$. For any pair of integral ideals $\mathfrak{m}$ and $\mathfrak{b}$, define

$$\omega(\mathfrak{m},\mathfrak{b}) = \delta(\mathfrak{m} \subset \mathfrak{b}) \prod_{v \in S(\mathfrak{b})} \omega_v(\mathfrak{m}\mathfrak{b}^{-1}).$$

Given an ideal $\mathfrak{n}$, let $\mathfrak{n}_0$ denote the smallest square free integral ideal such that $\mathfrak{n} \subset \mathfrak{n}_0$; thus, there exists the unique integral ideal $\mathfrak{n}_1$ such that $\mathfrak{n} = \mathfrak{n}_0\mathfrak{n}_1^2$. Let $\mathcal{I}$ be a set of integral ideals such that if $\mathfrak{n} \in \mathcal{I}$, then all ideals $\mathfrak{m} \subset \mathfrak{o}$ dividing $\mathfrak{n}$ are elements of $\mathcal{I}$.

Proposition 5.1 *Let $B(\mathfrak{m})$ and $A(\mathfrak{m})$ be two arithmetic functions defined for ideals $\mathfrak{m} \in \mathcal{I}$. Then, the following two conditions are equivalent:*

(i) *For any $\mathfrak{n} \in \mathcal{I}$, $B(\mathfrak{n}) = \sum_{\mathfrak{b}|\mathfrak{n}_1} \omega(\mathfrak{n},\mathfrak{b}^2)A(\mathfrak{n}\mathfrak{b}^{-2})$.*

(ii) *For any $\mathfrak{n} \in \mathcal{I}$, $A(\mathfrak{n}) = \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\#I}\{\prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0)\}B(\mathfrak{n}\prod_{v\in I}\mathfrak{p}_v^{-2})$.*

Proof We show that (i) implies (ii). By substituting (i), the right-hand side of (ii) becomes

$$\begin{aligned}
&\sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\#I}\Big\{\prod_{v \in I\cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0)\Big\}\Big\{\sum_{\mathfrak{b}|\mathfrak{n}_1\prod_{v\in I}\mathfrak{p}_v^{-1}} \omega(\mathfrak{n}\prod_{v\in I}\mathfrak{p}_v^{-2},\mathfrak{b}^2)A(\mathfrak{n}\mathfrak{b}^{-2}\prod_{v\in I}\mathfrak{p}_v^{-2})\Big\}\\
&= \sum_{\mathfrak{b}_1|\mathfrak{n}_1}\Big\{\sum_{I\subset S(\mathfrak{n}_1\mathfrak{b}_1^{-1})} (-1)^{\#I}\omega\Big(\mathfrak{n}\prod_{v\in I}\mathfrak{p}_v^{-2},\mathfrak{n}_1^2\mathfrak{b}_1^{-2}\prod_{v\in I}\mathfrak{p}_v^{-2}\Big)\prod_{v\in I\cap S_1(\mathfrak{n}_1)\cap S(\mathfrak{n}_1\mathfrak{b}_1^{-1})}\omega_v(\mathfrak{n}_0)\Big\}\\
&\qquad\times A(\mathfrak{n}_0\mathfrak{b}_1^2).
\end{aligned}$$

To get the equality here, we made the substitution $\mathfrak{b}_1 = \mathfrak{n}_1\mathfrak{b}^{-1}\prod_{v\in I}\mathfrak{p}_v^{-1}$. If $\mathfrak{b}_1 = \mathfrak{n}_1$, the term inside the bracket is 1 obviously; otherwise it equals

$$\begin{aligned}
&\sum_{I\subset S(\mathfrak{n}_1\mathfrak{b}_1^{-1})} (-1)^{\#I} \prod_{v\in S(\mathfrak{n}_1\mathfrak{b}_1^{-1}\prod_{v\in I}\mathfrak{p}_v^{-1})-S(\mathfrak{n}_0\mathfrak{b}_1^2)} \frac{q_v+1}{q_v-1} \prod_{v\in I\cap S(\mathfrak{n}_1\mathfrak{b}_1^{-1})\cap S_1(\mathfrak{n}_1)-S(\mathfrak{n}_0)} \frac{q_v+1}{q_v-1}\\
&= \sum_{I\subset S(\mathfrak{n}_1\mathfrak{b}_1^{-1})} (-1)^{\#I} \prod_{v\in[(I-S_1(\mathfrak{n}_1\mathfrak{b}_1^{-1}))\cup(S(\mathfrak{n}_1\mathfrak{b}_1^{-1})-I)]-S(\mathfrak{n}_0\mathfrak{b}_1^2)} \frac{q_v+1}{q_v-1} \prod_{v\in I\cap S_1(\mathfrak{n}_1\mathfrak{b}_1^{-1})-S(\mathfrak{n}_0\mathfrak{b}_1^2)} \frac{q_v+1}{q_v-1}\\
&= \sum_{I\subset S(\mathfrak{n}_1\mathfrak{b}_1^{-1})} (-1)^{\#I} \prod_{v\in(S(\mathfrak{n}_1\mathfrak{b}_1^{-1})-I)-S(\mathfrak{n}_0\mathfrak{b}_1^2)} \frac{q_v+1}{q_v-1} \prod_{v\in I-S(\mathfrak{n}_0\mathfrak{b}_1^2)} \frac{q_v+1}{q_v-1}\\
&= \prod_{v\in S(\mathfrak{n}_1\mathfrak{b}_1^{-1})} (\omega_v(\mathfrak{n}_0\mathfrak{b}_1^2) - \omega_v(\mathfrak{n}_0\mathfrak{b}_1^2)),
\end{aligned}$$

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which is zero by $S(\mathfrak{n}_1\mathfrak{b}_1^{-1}) \neq \varnothing$. The assertion that (ii) implies (i) is proved similarly. ■

Definition 5.2 For an arithmetic function $B: \mathcal{I} \to \mathbb{C}$, we define its $\mathcal{N}$-transform $\mathcal{N}[B]: \mathcal{I} \to \mathbb{C}$ by the formula

$$\mathcal{N}[B](\mathfrak{n}) = \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\# I} \Big\{ \prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0) \Big\} \frac{\iota(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})} B\Big(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2}\Big).$$

Lemma 5.3 *For $t \in \mathbb{C}$, let N^t be the arithmetic function $\mathfrak{n} \mapsto \mathrm{N}(\mathfrak{n})^t$ on $\mathcal{I}$. For any ideal $\mathfrak{n}$, we have*

$$\mathcal{N}[\mathrm{N}^t](\mathfrak{n}) = \mathrm{N}(\mathfrak{n})^t \Big\{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 - q_v^{-2(1+t)}) \Big\} \Big\{ \prod_{v \in S_2(\mathfrak{n})} (1 - (1 - q_v^{-1})^{-1} q_v^{-2(1+t)}) \Big\}.$$

Proof By (4.7), we have $\frac{\iota(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})} = \prod_{v \in I} q_v^{-2} \prod_{v \in I \cap S_2(\mathfrak{n})} (1 + q_v^{-1})^{-1}$ for any $I \subset S(\mathfrak{n})$. Therefore,

$$\begin{aligned}
&\sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\# I} \Big\{ \prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0) \Big\} \frac{\iota(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})} \mathrm{N}(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})^t \\
&\quad = \mathrm{N}(\mathfrak{n})^t \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\# I} \Big\{ \prod_{v \in I \cap S_2(\mathfrak{n})} \frac{q_v + 1}{q_v - 1} \Big\} \prod_{v \in I \cap S_2(\mathfrak{n})} (1 + q_v^{-1})^{-1} \Big\{ \prod_{v \in I} q_v^{-2(1+t)} \Big\} \\
&\quad = \mathrm{N}(\mathfrak{n})^t \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\# I} \prod_{v \in I \cap S_2(\mathfrak{n})} (1 - q_v^{-1})^{-1} \Big\{ \prod_{v \in I} q_v^{-2t} \Big\} \\
&\quad = \mathrm{N}(\mathfrak{n})^t \Big\{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 - q_v^{-2(1+t)}) \Big\} \Big\{ \prod_{v \in S_2(\mathfrak{n})} (1 - (1 - q_v^{-1})^{-1} q_v^{-2(1+t)}) \Big\}. \quad ■
\end{aligned}$$

Corollary 5.4 *The $\mathcal{N}$-transform of $\mathfrak{n} \mapsto \log \mathrm{N}(\mathfrak{n})$ on $\mathcal{I}$ is given by*

$$\begin{aligned}
\mathcal{N}[\log \mathrm{N}](\mathfrak{n}) = &\prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 - q_v^{-2}) \prod_{v \in S_2(\mathfrak{n})} (1 - (q_v^2 - q_v)^{-1}) \\
&\times \Big(\log \mathrm{N}(\mathfrak{n}) + \sum_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} \frac{2 \log q_v}{q_v^2 - 1} + \sum_{v \in S_2(\mathfrak{n})} \frac{2 \log q_v}{q_v^2 - q_v - 1} \Big).
\end{aligned}$$

Proof Take the derivative at $t = 0$ of the formula in Lemma 5.3. ■

For any arithmetic function $B: \mathcal{I} \to \mathbb{C}$, we define another function $\mathcal{N}^+[B]$ by setting

$$\mathcal{N}^+[B](\mathfrak{n}) = \sum_{I \subset S(\mathfrak{n}_1)} \Big\{ \prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0) \Big\} \frac{\iota(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})} B\Big(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2}\Big)$$

for $\mathfrak{n} = \mathfrak{n}_0 \mathfrak{n}_1^2 \in \mathcal{I}$. In a similar way to Lemma 5.3, we have

$$\mathcal{N}^+[\mathrm{N}^t] = \mathrm{N}(\mathfrak{n})^t \Big\{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 + q_v^{-2(t+1)}) \Big\} \Big\{ \prod_{v \in S_2(\mathfrak{n})} (1 + (1 - q_v^{-1})^{-1} q_v^{-2(t+1)}) \Big\} \tag{5.1}$$

for any $t \in \mathbb{C}$.

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Lemma 5.5 *Let $c > 0$. Then for any sufficiently small $\epsilon > 0$, we have*

$$\mathcal{N}^+[\mathrm{N}^{-c+\epsilon}](\mathfrak{n}) \ll_\epsilon \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+3\epsilon}, \quad \mathfrak{n} \in \mathcal{I}.$$

Proof From $\mathrm{N}(\mathfrak{n})^{-c+\epsilon} \leqslant \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}$, we have

$$\mathcal{N}^+[\mathrm{N}^{-c+\epsilon}](\mathfrak{n}) \leqslant \mathcal{N}^+[\mathrm{N}^{-\inf(c,1)+\epsilon}](\mathfrak{n})$$

obviously. Let us set $t = -\inf(c,1) + \epsilon$ and examine the right-hand side of formula (5.1). We note that $t+1 = 1-\inf(c,1)+\epsilon \geqslant \epsilon > 0$. The set $P(\epsilon) = \{v \in \Sigma_{\mathrm{fin}} | 1-q_v^{-1} < q_v^{-\epsilon}\}$ is a finite set. For $v \in S_2(\mathfrak{n}) - P(\epsilon)$, we have $(1-q_v^{-1})^{-1} \leqslant q_v^{\epsilon}$ and $q_v^{-2(t+1)} \leqslant q_v^{-2\epsilon}$; by these, the factor $1 + (1-q_v^{-1})q_v^{-2(t+1)}$ is bounded by $1 + q_v^{-\epsilon}$. For $v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})$ or $v \in S_2(\mathfrak{n}) \cap P(\epsilon)$, we simply apply $q_v^{-2(t+1)} \leqslant q_v^{-2\epsilon}$. Thus,

$$\text{(5.2)} \qquad \mathcal{N}^+[\mathrm{N}^t](\mathfrak{n}) \leqslant \mathrm{N}(\mathfrak{n})^t \Big\{ \prod_{v \in P(\epsilon)} (1 + (1-q_v^{-1})^{-1} q_v^{-2\epsilon}) \Big\} \times \Big\{ \prod_{v \in S_2(\mathfrak{n}) - P(\epsilon)} (1 + q_v^{-\epsilon}) \Big\} \Big\{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 + q_v^{-2\epsilon}) \Big\}.$$

In the right-hand side, the second factor is independent of $\mathfrak{n}$. The last two factors combined are majorized by $\{\prod_{v \in S(\mathfrak{n})} (1 + q_v^{-\epsilon})\}^2 \ll_\epsilon \{\prod_{v \in S(\mathfrak{n})} q_v^{\epsilon}\}^2 \leqslant \mathrm{N}(\mathfrak{n})^{2\epsilon}$. Hence there exists a constant $C(\epsilon) > 0$ such that (5.2) is less than $C(\epsilon)\,\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+3\epsilon}$ for any $\mathfrak{n} \in \mathcal{I}$. ∎

5.2 The Totally Inert Case Over $\mathfrak{n}$

Set $\mathcal{I} = \mathcal{I}_{S,\eta}$. Fixing a test function $\alpha \in \mathcal{A}_S$ for a while, we study the arithmetic functions $\mathrm{AL}^*: \mathcal{I} \to \mathbb{C}$ and $\mathrm{ADL}_-^*: \mathcal{I} \to \mathbb{C}$ defined by (1.2) and (1.4), respectively. We relate these functions to the $\mathcal{N}$-transforms of arithmetic functions AL^w, $\mathrm{ADL}_\pm^w$ defined in §4.2.

Any ideal $\mathfrak{n} \in \mathcal{I}$ satisfies the condition $\eta_v(\varpi_v) = -1$ for all $v \in S(\mathfrak{n})$, under which the quantities $w_\mathfrak{n}^\eta(\pi)$ and $\partial w_\mathfrak{n}^\eta(\pi)$ turn out to be written explicitly in terms of $\omega(\mathfrak{m}, \mathfrak{b})$.

Lemma 5.6 *Let $\mathfrak{n} \in \mathcal{I}$. Then for any $\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})$, we have $w_\mathfrak{n}^\eta(\pi) = 0$ unless $\mathfrak{n}\mathfrak{f}_\pi^{-1} = \mathfrak{b}^2$ for some integral ideal $\mathfrak{b}$, in which case $w_\mathfrak{n}^\eta(\pi) = \omega(\mathfrak{n}, \mathfrak{n}\mathfrak{f}_\pi^{-1})$.*

Proof From (2.4), $w_\mathfrak{n}^\eta(\pi)$ is the product of $r^{(1/2)}(\pi_v, \eta_v)$ over $v \in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})$. By Lemma 2.2, we see that $r^{(1/2)}(\pi_v, \eta_v)$ is zero unless $\mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ is even, in which case it equals $\omega_v(\mathfrak{f}_\pi)$. ∎

Lemma 5.7 *Let $\mathfrak{n} \in \mathcal{I}$. For any $\pi \cong \otimes_v \pi_v \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})$, we have the following.*

(i) *If $\mathfrak{n}\mathfrak{f}_\pi^{-1} = \mathfrak{b}^2$ with an integral ideal $\mathfrak{b}$, then*

$$\partial w_\mathfrak{n}^\eta(\pi) = \omega(\mathfrak{n}, \mathfrak{n}\mathfrak{f}_\pi^{-1}) \sum_{v \in S(\mathfrak{b})} (-\log q_v)\, \mathrm{ord}_v(\mathfrak{b}).$$

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(ii) *If* $\mathfrak{n}\mathfrak{f}_\pi^{-1} = \mathfrak{b}^2\mathfrak{p}_u$ *with an integral ideal* $\mathfrak{b}$ *and a place* $u \in S(\mathfrak{n})$, *then*

$$\partial w_{\mathfrak{n}}^{\eta}(\pi) = \omega(\mathfrak{n}, \mathfrak{n}\mathfrak{f}_\pi^{-1})(\log q_u)\begin{cases} \mathrm{ord}_u(\mathfrak{b}) + \frac{q_u - 1}{(1+a_u q_u^{1/2})(1+a_u^{-1}q_u^{1/2})}, & c(\pi_u) = 0,\\ \mathrm{ord}_u(\mathfrak{b}) + \frac{1}{1+q_u^{-1}\chi_u(\varpi_u)}, & c(\pi_u) = 1,\\ \mathrm{ord}_u(\mathfrak{b}) + 1, & c(\pi_u) \geqslant 2,\end{cases}$$

where $\mathrm{diag}(a_u, a_u^{-1})$ *is the Satake parameter of* π_u *if* $c(\pi_u) = 0$ *and* χ_u *denotes the unramified character of* $F_u^\times$ *such that* $\pi_u \cong \sigma(\chi_u|\,|_u^{1/2}, \chi_u|\,|_u^{-1/2})$ *if* $c(\pi_u) = 1$.

Except for the two cases (i) *and* (ii), *we have* $\partial w_{\mathfrak{n}}^{\eta}(\pi) = 0$.

Lemma 5.8 *For any* $\mathfrak{n} \in \mathcal{I}$,

$$\mathrm{AL}^{w}(\mathfrak{n};\alpha) = \sum_{\mathfrak{b}} \omega(\mathfrak{n},\mathfrak{b}^2)\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2})}{\iota(\mathfrak{n})}\,\mathrm{AL}^{*}(\mathfrak{n}\mathfrak{b}^{-2};\alpha),$$

$$\mathrm{ADL}_{-}^{w}(\mathfrak{n};\alpha) = \sum_{\mathfrak{b}} \omega(\mathfrak{n},\mathfrak{b}^2)\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2})}{\iota(\mathfrak{n})}\,\mathrm{ADL}_{-}^{*}(\mathfrak{n}\mathfrak{b}^{-2};\alpha),$$

$$\mathrm{ADL}_{+}^{w}(\mathfrak{n};\alpha) = \sum_{\mathfrak{b}} \omega(\mathfrak{n},\mathfrak{b}^2)\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2})}{\iota(\mathfrak{n})}\log(\mathrm{N}(\mathfrak{n}\mathfrak{b}^{-2})^{-1/2}\,\mathrm{N}(\mathfrak{f}_\eta)^{-1}D_F^{-1})\,\mathrm{AL}^{*}(\mathfrak{n}\mathfrak{b}^{-2};\alpha),$$

where $\mathfrak{b}$ *runs through all the integral ideals such that* $\mathfrak{n} \subset \mathfrak{b}^2$.

Proof This follows immediately from Lemma 5.6. To have the last formula, we also need Proposition 4.7. ■

Lemma 5.9 *For any* $\mathfrak{n} \in \mathcal{I}$,

$$\mathrm{AL}^{*}(\mathfrak{n};\alpha) = \mathcal{N}[\mathrm{AL}^{w}](\mathfrak{n}),$$
$$\mathrm{ADL}_{-}^{*}(\mathfrak{n};\alpha) = \mathcal{N}[\mathrm{ADL}_{-}^{w}](\mathfrak{n}),$$
$$-\log(\sqrt{\mathrm{N}(\mathfrak{n})}\,\mathrm{N}(\mathfrak{f}_\eta)D_F)\,\mathrm{AL}^{*}(\mathfrak{n};\alpha) = \mathcal{N}[\mathrm{ADL}_{+}^{w}](\mathfrak{n}).$$

Proof By Lemma 5.8, we obtain the first formula by applying Proposition 5.1 with $B(\mathfrak{m}) = \iota(\mathfrak{m})\,\mathrm{AL}^{w}(\mathfrak{m};\alpha)$ and $A(\mathfrak{m}) = \iota(\mathfrak{m})\,\mathrm{AL}^{*}(\mathfrak{m};\alpha)$ both defined for $\mathfrak{m} \in \mathcal{I}$. The remaining two formulas are proved in the same way. ■

Formula (4.9) can be applied to an arbitrary ideal $\mathfrak{m} \in \mathcal{I}$. In the right-hand side of the formula, we have two terms $\widetilde{\mathbb{W}}_{\mathfrak{u}}^{\eta}(l,\mathfrak{m}|\alpha)$ and $\mathbb{W}_{\mathrm{hyp}}^{\eta}(l,\mathfrak{m}|\alpha)$, which we regard as arithmetic functions in $\mathfrak{m}$ for a while and consider their $\mathcal{N}$-transforms $\mathcal{N}[\widetilde{\mathbb{W}}_{\mathfrak{u}}^{\eta}]$ and $\mathcal{N}[\mathbb{W}_{\mathrm{hyp}}^{\eta}]$. The following is the main result of this section.

Proposition 5.10 *For any* $\mathfrak{n} \in \mathcal{I}$, *we have the identity among linear functionals in* $\alpha \in \mathcal{A}_S$:

$$\mathrm{ADL}_{-}^{*}(\mathfrak{n};\alpha) = 2(-1)^{\#S+\epsilon(\eta)}\mathcal{G}(\eta)^{-1}D_F\{\mathcal{N}[\widetilde{\mathbb{W}}_{\mathfrak{u}}^{\eta}](\mathfrak{n}) + \mathcal{N}[\mathbb{W}_{\mathrm{hyp}}^{\eta}](\mathfrak{n})\} + \log(\mathrm{N}(\mathfrak{n})^{1/2}\,\mathrm{N}(\mathfrak{f}_\eta))\mathrm{AL}^{*}(\mathfrak{n};\alpha) - \mathcal{N}[\mathrm{AL}^{\partial w}](\mathfrak{n}). \tag{5.3}$$

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Proof We take the $\mathcal{N}$-transform of both sides of the formula (4.9), regarding it as an identity among arithmetic functions on $\mathcal{I}$. Then apply Lemma 5.9. ■

Set $\mathcal{I}_{S,\eta}^{\pm} = \{\mathfrak{n} \in \mathcal{I} | (-1)^{\epsilon(\eta)}\widetilde{\eta}(\mathfrak{n}) = \pm 1\}$.

Lemma 5.11 *We have* $\mathrm{AL}^*(\mathfrak{n};\alpha) = 0$ *for all* $\mathfrak{n} \in \mathcal{I}_{S,\eta}^-$, *and* $\mathrm{ADL}_-^*(\mathfrak{n};\alpha) = 0$ *for all* $\mathfrak{n} \in \mathcal{I}_{S,\eta}^+$.

Proof By the sign of the functional equation, $L(1/2,\pi)L(1/2,\pi\otimes\eta) = 0$ for $\pi \in \Pi_{\mathrm{cus}}^*(l,\mathfrak{n})$ with $\mathfrak{n} \in \mathcal{I}_{S,\eta}^-$. This shows the first assertion. To prove the second claim, let $\mathfrak{n} \in \mathcal{I}_{S,\eta}^+$. Then $\epsilon(1/2,\pi)\epsilon(1/2,\pi\otimes\eta) = +1$ for all $\pi \in \Pi_{\mathrm{cus}}^*(l,\mathfrak{n})$, which means $\epsilon(1/2,\pi) = -1$ and hence $L(1/2,\pi) = 0$ for all π occurring in the sum $\mathrm{ADL}_-^*(\mathfrak{n};\alpha)$. ■

6 An Error Term Estimate for Averaged L-values

In this section we shall prove (1.7) in Theorem 1.1, starting with the following asymptotic formula of $\mathrm{AL}^w(\mathfrak{m};\alpha_{\mathfrak{a}})$.

Proposition 6.1 *Suppose* $\underline{l} = \inf_{v\in\Sigma_\infty} l_v \geqslant 6$. *For any ideal* $\mathfrak{a} \subset \mathfrak{o}$ *prime to* $\mathfrak{f}_\eta$ *and for any* $\mathfrak{m} \in \mathcal{I}_{S(\mathfrak{a}),\eta}^+$, *we have*

$$\mathrm{AL}^w(\mathfrak{m};\alpha_{\mathfrak{a}}) = 4D_F^{3/2}\{1 + D(\mathfrak{m})\}L_{\mathrm{fin}}(1,\eta)\,\mathrm{N}(\mathfrak{a})^{-1/2}\delta_{\square}(\mathfrak{a}_\eta^-)d_1(\mathfrak{a}_\eta^+) + \mathcal{O}_{\epsilon,l,\eta}(\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{m})^{-c+\epsilon}), \tag{6.1}$$

where we set $c = d_F^{-1}(\underline{l}/2 - 1)$ *and* $D(\mathfrak{m}) = i^{\tilde{l}}\delta(\mathfrak{m} = \mathfrak{o})$ *with* $\tilde{l} = \sum_{v\in\Sigma_\infty} l_v$.

To derive (1.7) from this, we apply the first formula of Lemma 5.9 substituting (6.1). The main term of (1.7) is computed by Lemma 5.3. The error term of (1.7) stems from the remainder term of (6.1) whose $\mathcal{N}$-transform is estimated by Lemma 5.5, and from the terms $D(\mathfrak{m})$ whose $\mathcal{N}$-transform amounts at most to $\mathcal{O}(\mathrm{N}(\mathfrak{n})^{-1+\epsilon})$ as seen by the formula

$$\mathcal{N}[D](\mathfrak{n}) = \delta(S(\mathfrak{n}) = S_2(\mathfrak{n}))\Big\{\prod_{v\in S(\mathfrak{n})}\frac{q_v+1}{q_v-1}\Big\}\frac{i^{\tilde{l}}(-1)^{\#S(\mathfrak{n})}}{\iota(\mathfrak{n})} \tag{6.2}$$

combined with $\iota(\mathfrak{n})^{-1} \leqslant \mathrm{N}(\mathfrak{n})^{-1}$. The rest of this section is devoted to proving Proposition 6.1. Set $S = S(\mathfrak{a})$. Let $\alpha = \otimes_{v\in S}\alpha_v \in \mathcal{A}_S$ and $\mathfrak{m} \in \mathcal{I}_{S,\eta}^+$. From (4.5), the function $\widehat{\Psi}_l^{(0)}(\mathfrak{m}|\alpha, g)$ in adele points $g = \{g_v\}$ is a product of functions $\Psi_v(g_v)$ on groups G_v such that $\Psi_v(g_v) = \Psi_v^{(0)}(l_v; g_v)$ for $v \in \Sigma_\infty$,

$$\Psi_v(g_v) = \frac{1}{2\pi i}\int_{L_v(\sigma)}\Psi_v^{(0)}(s_v; g_v)\alpha_v(s_v)\,d\mu_v(s_v)$$

for $v \in S$, where $L_v(\sigma)$ denotes the contour $\{\sigma + it \mid t \in \mathbb{R}/4\pi(\log q_v)^{-1}\mathbb{Z}\}$ with usual direction, and $\Psi_v(g_v) = \Phi_{\mathfrak{m},v}^{(0)}(g_v)$ for $v \in \Sigma_{\mathrm{fin}} - S$. We apply the relative trace formula [11, Theorem 9.1], which asserts

$$\mathrm{AL}^w(\mathfrak{m};\alpha) = 2(-1)^{\#S+\epsilon(\eta)}\mathcal{G}(\eta)^{-1}\{\widetilde{\mathbb{J}}_{\mathrm{u}}^\eta(l,\mathfrak{m}|\alpha) + \mathbb{J}_{\mathrm{hyp}}^\eta(l,\mathfrak{m}|\alpha)\},$$

where

$$\widetilde{\mathbb{J}}^{\eta}_{\mathfrak{u}}(l,\mathfrak{m}|\alpha) = 2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}(1+D(\mathfrak{m}))L_{\mathrm{fin}}(1,\eta)\prod_{v\in S(\mathfrak{a})}U_v^{\eta_v}(\alpha_v),$$

$$\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{m}|\alpha) = \sum_{b\in F-\{0,-1\}}\prod_{v\in\Sigma_F}J_v(b),$$

(6.3) $$U_v^{\eta_v}(\alpha_v) = \frac{1}{2\pi i}\int_{L_v(\sigma)}\frac{1}{(1-\eta_v(\varpi_v)q_v^{-(s+1)/2})(1-q_v^{(s+1)/2})}\alpha_v(s)\,d\mu_v(s),$$

(6.4) $$J_v(b) = \int_{F_v^\times}\Psi_v\left(\left[\begin{smallmatrix}1+b^{-1} & 1\\ 1 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}t_v & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & x_{\eta,v}\\ 0 & 1\end{smallmatrix}\right]\right)\eta_v(t_v x^*_{\eta,v})\,d^\times t_v.$$

We apply this formula to the function $\alpha_{\mathfrak{a}} = \otimes_{v\in S(\mathfrak{a})}\alpha_{\mathfrak{p}_v^{n_v}}$, noting the relation

(6.5) $$\alpha_{\mathfrak{p}_v^n}(\nu) = \frac{z^{n+1}-z^{-(n+1)}}{z-z^{-1}} = \sum_{m=0}^{[n/2]}\alpha_v^{(n-2m)}(\nu) - \delta(n\in 2\mathbb{N}_0),$$

with $\alpha_v^{(m)}(\nu) = z^m + z^{-m}$, $z = q_v^{\nu/2}$, which is proved by (1.5).

Lemma 6.2 *Set* $\Upsilon_v^{\eta_v}(s) = (1-\eta_v(\varpi_v)q_v^{-(1+s)/2})^{-1}(1-q_v^{(1+s)/2})^{-1}$. *For* $n\in\mathbb{N}_0$,

$$\frac{1}{2\pi i}\int_{L_v(\sigma)}\Upsilon_v^{\eta_v}(s)\alpha_{\mathfrak{p}_v^n}(s)\,d\mu_v(s) = -q_v^{-n/2}\begin{cases}\delta(n\in 2\mathbb{N}_0), & \eta_v(\varpi_v)=-1,\\ n+1, & \eta_v(\varpi_v)=+1,\end{cases}$$

$$\frac{\log q_v}{2\pi i}\int_{L_v(\sigma)}\frac{\Upsilon_v^{\eta_v}(s)\alpha_{\mathfrak{p}_v^n}(s)}{1-\eta_v(\varpi_v)q_v^{(s+1)/2}}\,d\mu_v(s)$$
$$= q_v^{-n/2}\log q_v\begin{cases}(-1)^n\left[\frac{n+1}{2}\right], & \eta_v(\varpi_v)=-1,\\ \frac{n(n+1)}{2}, & \eta_v(\varpi_v)=+1.\end{cases}$$

Proof The second formula, whose left-hand side is $\widetilde{U}_v^{\eta_v}(\alpha_{\mathfrak{p}_v^n})$ defined by (9.8), follows from Lemma 9.13 by (6.5). The first formula, whose left-hand side is $U_v^{\eta_v}(\alpha_{\mathfrak{p}_v^n})$ defined by (6.3), is shown similarly. ∎

By the first formula of Lemma 6.2, we get

$$\widetilde{\mathbb{J}}^{\eta}_{\mathfrak{u}}(l,\mathfrak{m}|\alpha_{\mathfrak{a}}) = 2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}(1+i^{\tilde{l}}\delta(\mathfrak{m}=\mathfrak{o}))$$
$$\times L_{\mathrm{fin}}(1,\eta)(-1)^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a})^{-1/2}\delta_{\square}(\mathfrak{a}_\eta^-)d_1(\mathfrak{a}_\eta^+).$$

This completes the evaluation of $\widetilde{\mathbb{J}}^{\eta}_{\mathfrak{u}}(l,\mathfrak{m}|\alpha_{\mathfrak{a}})$. To estimate $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{m}|\alpha_{\mathfrak{a}})$, let us recall results on local orbital integrals from [11]. For any place $v\in\Sigma_{\mathrm{fin}}$, we define a function $\Lambda_v\colon F_v-\{0,-1\}\to\mathbb{Z}$ by setting $\Lambda_v(b) = \delta(b\in\mathfrak{o}_v)\{\mathrm{ord}_v(b(b+1))+1\}$. Let $w_2\in S(\mathfrak{m})$, $w_3\in S(\mathfrak{f}_\eta)$, and $w_4\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{f}_\eta)$. Then

(6.6) $$|J_{w_2}(b)| \leqslant \delta(b\in\mathfrak{m}\mathfrak{o}_{w_2})\Lambda_{w_2}(b),$$

(6.7) $$|J_{w_3}(b)| \leqslant 4\delta(b\in\mathfrak{f}_\eta^{-1}\mathfrak{o}_{w_3}), \qquad |J_{w_4}(b)| \leqslant \delta(b\in\mathfrak{o}_{w_4})\Lambda_{w_4}(b)$$

for any $b \in F - \{0, -1\}$. Indeed, the estimate (6.6) and both estimates of (6.7) follow immediately from [11, Lemmas 10.5, 10.10, and 10.4]. For an integer $k \geqslant 4$ and a real valued character ε of $\mathbb{R}^\times$, define $J^\varepsilon(k;b)$ in $b \in \mathbb{R} - \{0, -1\}$ as

$$J^{\mathbf{1}}(k;b) = \begin{cases} (1+b)^{-k/2} \frac{2\Gamma(k/2)^2}{\Gamma(k)} {}_2F_1(k/2, k/2; k; (b+1)^{-1}), & b(b+1) > 0, \\ 2\log|(b+1)/b| P_{k/2-1}(2b+1) & \\ \qquad - \sum_{m=1}^{[k/4]} \frac{8(k-4m+1)}{(2m-1)(k-2m)} P_{k/2-2m}(2b+1), & b(b+1) < 0, \end{cases}$$

$$J^{\mathrm{sgn}}(k;b) = \begin{cases} 0, & b(b+1) > 0, \\ 2\pi i P_{k/2-1}(2b+1), & b(b+1) < 0, \end{cases}$$

where $P_n(x)$ is the Legendre polynomial of degree n and we put $\mathrm{sgn}(\tau) = \tau/|\tau|$ for $\tau \in \mathbb{R}^\times$.

Lemma 6.3 *For any $\epsilon' > 0$, we have the estimate*

$$|b(b+1)|^{\epsilon'} |J^\varepsilon(k;b)| \ll_{\epsilon',k} (1+|b|)^{-k/2+2\epsilon'}, \quad b \in \mathbb{R} - \{0, -1\}.$$

Proof For $J^{\mathrm{sgn}}(k;b)$ the estimate is obvious. As for $J^{\mathbf{1}}(k;b)$, the estimate for $-1 < b < 0$ is also obvious. For $b > 0$, the estimate follows from ${}_2F_1(k/2, k/2; k; (b+1)^{-1}) = \mathcal{O}(|\log b|)$, $b \to +0$ by [4, p. 49]. For $b < -1$, we only have to consider the estimate as above and the functional equation $J^{\mathbf{1}}(k;b) = (-1)^{k/2} J^{\mathbf{1}}(k; -b-1)$ $(b < -1)$, which is easily confirmed by the formula ${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1(a, c-b; c; \frac{z}{z-1})$ ([24, p. 47]). ■

Given relatively prime $\mathfrak{o}$-ideals $\mathfrak{n}$ and $\mathfrak{b}$ and for $\epsilon \geqslant 0$, we set

$$\mathfrak{I}_\epsilon^\eta(l, \mathfrak{n}, \mathfrak{b}) = \sum_{b \in \mathfrak{n}\mathfrak{b}^{-1} - \{0,-1\}} \tau^{S(\mathfrak{b})}(b)^2 |\mathrm{N}(b(b+1))|^\epsilon \prod_{v \in \Sigma_\infty} |J^{\eta_v}(l_v; b)|,$$

where

$$\tau^{S(\mathfrak{b})}(b) = \Big\{ \prod_{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{b})} \Lambda_v(b) \Big\} \prod_{v \in S(\mathfrak{b})} \delta(b \in \mathfrak{b}^{-1}\mathfrak{o}_v), \quad b \in F - \{0, -1\}.$$

Proposition 6.4 *Suppose $\underline{l} \geqslant 6$ and set $\underline{l} = \inf_{v \in \Sigma_\infty} l_v$. Let $\mathfrak{b}$ and $\mathfrak{n}$ be relatively prime ideals. For any $\epsilon \geqslant 0$ and $\epsilon' > 0$ such that $\underline{l}/4 - 1 \geqslant \epsilon + 2\epsilon'$, we have*

$$\mathfrak{I}_\epsilon^\eta(l, \mathfrak{n}, \mathfrak{b}) \ll_{\epsilon, \epsilon', l} \mathrm{N}(\mathfrak{b})^{1+c+\epsilon'} \mathrm{N}(\mathfrak{n})^{-c+2\epsilon+\epsilon'}$$

with the implied constant independent of $\mathfrak{b}$ and $\mathfrak{n}$.

Proof Let $\epsilon \geqslant 0$ and $\epsilon' > 0$. By [11, Lemma 12.3] and Lemma 6.3, we have

$$\mathfrak{I}_\epsilon^\eta(l, \mathfrak{n}, \mathfrak{b}) \ll_{\epsilon, \epsilon', l} \mathrm{N}(\mathfrak{b})^{4\epsilon'} \sum_{b \in \mathfrak{n}\mathfrak{b}^{-1} - \{0\}} \prod_{v \in \Sigma_\infty} (1+|b_v|)^{-l_v/2+2\epsilon+4\epsilon'} = \mathrm{N}(\mathfrak{b})^{4\epsilon'} \theta(\mathfrak{n}\mathfrak{b}^{-1}),$$

where we regard the fractional ideal $\Lambda = \mathfrak{n}\mathfrak{b}^{-1}$ as a $\mathbb{Z}$-lattice in the Euclidean space $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ and $\theta(\Lambda) = \sum_{b \in \Lambda - \{0\}} f(b)$ with $f(x) = \prod_{v \in \Sigma_\infty} (1+|x_v|)^{-(l_v - 4\epsilon - 8\epsilon')/2}$ (see Appendix A). If $\underline{l}/4 - 1 \geqslant \epsilon + 2\epsilon'$, then $l' = \{l_v - 4\epsilon - 8\epsilon'\}_{v \in \Sigma_\infty}$ satisfies $\underline{l'} \geqslant 4$. The desired estimate follows if we apply Theorem A.1 with $\Lambda = \mathfrak{n}\mathfrak{b}^{-1}$ and $\Lambda_0 = \mathfrak{b}^{-1}$ noting $D(\mathfrak{n}\mathfrak{b}^{-1}) = \mathrm{N}(\mathfrak{n})\,\mathrm{N}(\mathfrak{b})^{-1}$, $D(\mathfrak{b}^{-1}) = \mathrm{N}(\mathfrak{b})^{-1}$, and $r(\mathfrak{b}^{-1}) \leqslant r(\mathfrak{o})$. ■

Proposition 6.5 *Suppose $\underline{l} \geqslant 6$. Given $\mathfrak{o}$-ideals $\mathfrak{n}$ and $\mathfrak{a} = \prod_{v \in S(\mathfrak{a})} \mathfrak{p}_v^{n_v}$ relatively prime to each other, for any $\epsilon > 0$, we have $|\mathbb{J}_{\mathrm{hyp}}^{\eta}(l, \mathfrak{n}|\alpha_{\mathfrak{a}})| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon}$ with the implied constant independent of $\mathfrak{a}$ and $\mathfrak{n}$.*

Proof Let $v \in S(\mathfrak{a})$ and $n \in \mathbb{N}_0$. By [11, Lemma 10.3], we have

$$|J_v^{\eta_v}(b, \alpha_v^{(m)})| \ll (1+m)^2 \delta(|b|_v \leqslant q_v^m) q_v^{\delta(m>0)-m/2}\{1 + \Lambda_v(b)\}, \quad b \in F^\times - \{-1\}$$

with the implied constant independent of $m \in \mathbb{N}_0$ and v. Let $n > 0$. From (6.5),

$$\begin{aligned}|J_v^{\eta_v}(b, \alpha_{\mathfrak{p}_v^n})| &\ll \delta(|b|_v \leqslant q_v^n)\Big\{\sum_{m=0}^{n}(1+m)^2 q_v^{1-m/2}\Big\}\{1 + \Lambda_v(b)\} \\ &\leqslant \delta(|b|_v \leqslant q_v^n) q_v \Big(\sum_{m=0}^{\infty}(1+m)^2 2^{-m/2}\Big)\{1 + \Lambda_v(b)\}.\end{aligned}$$

Thus we have a constant C independent of $v \in S(\mathfrak{a})$ and $n \in \mathbb{N}_0$ such that

$$(6.8) \qquad |J_v^{\eta_v}(b, \alpha_{\mathfrak{p}_v^n})| \leqslant C q_v^{\delta(n>0)} \delta(|b|_v \leqslant q_v^n)\{1 + \Lambda_v(b)\}, \quad b \in F^\times - \{0, -1\}.$$

Let $w_5 \in \Sigma_\infty$. Then from [11, Lemma 10.15], we have the equality $J_{w_5}(b) = J^{\eta_{w_5}}(l_{w_5}; b)$. Combining (6.8) with Lemma 6.3, Proposition 6.4, (6.6), and (6.7), we obtain

$$\begin{aligned}|\mathbb{J}_{\mathrm{hyp}}^{\eta}(l, \mathfrak{n}|\alpha_{\mathfrak{a}})| &\leqslant C^{\#S(\mathfrak{a})}\Big\{\prod_{v \in S(\mathfrak{a})} q_v^{\delta(n_v > 0)}\Big\} \\ &\quad \times \sum_{I \subset S(\mathfrak{a})} \sum_{b \in \mathfrak{n}(\prod_{v \in I}\mathfrak{p}_v^{n_v})^{-1}\mathfrak{f}_\eta^{-1}} \tau^{S(\prod_{v\in I}\mathfrak{p}_v^{n_v})}(b) \prod_{w_5 \in \Sigma_\infty} |J_{w_5}(b)| \\ &\leqslant C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a}) \sum_{I \subset S(\mathfrak{a})} \mathfrak{I}_0^{\eta}(l, \mathfrak{n}, \mathfrak{f}_\eta \prod_{v \in I}\mathfrak{p}_v^{n_v}) \\ &\ll_{\epsilon,l} C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a}) \sum_{I \subset S(\mathfrak{a})} \mathrm{N}(\mathfrak{f}_\eta \prod_{v \in I}\mathfrak{p}_v^{n_v})^{1+c+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon} \\ &\ll_{\epsilon,l,\eta} C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a}) \times 2^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a})^{1+c+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon}.\end{aligned}$$

By the estimate $(2C)^{\#S(\mathfrak{a})} \ll_\epsilon \mathrm{N}(\mathfrak{a})^\epsilon$, we are done. ∎

7 An Error Term Estimate for Averaged Derivative of L-values

The aim of this section is to prove the formula (1.8) in Theorem 1.1. Starting from the formula (5.3) with α specialized to $\alpha_{\mathfrak{a}}$ for $\mathfrak{a} = \prod_{v \in S(\mathfrak{a})} \mathfrak{p}_v^{n_v}$, we examine the four terms in the right-hand side separately. Here are the highlights in the analysis for each term.

(i) We compute the term $\mathcal{N}[\widetilde{\mathbb{W}}_{\mathrm{u}}^{\eta}](\mathfrak{n})$ explicitly by using Lemma 9.13, Lemma 5.3 and Corollary 5.4, which yields the main term of the formula (modulo a part of the error term); see §7.1 for detail.

(ii) We prove $\mathcal{N}[\mathbb{W}_{\mathrm{hyp}}^{\eta}](\mathfrak{n}) = \mathcal{O}_{\epsilon,l,\eta}(\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-\inf(1,c)+\epsilon})$ by using the explicit formula of local terms given in §9; see §7.2 for detail.

(iii) Since $\mathfrak{n} \in \mathcal{I}_{S(\mathfrak{a}),\eta}^{-}$, the term $\mathrm{AL}^*(\mathfrak{n})$ vanishes by the reason of the sign of the functional equations.

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(iv) We prove $\mathcal{N}[\mathrm{AL}^{\partial w}](\mathfrak{n}) = \mathcal{O}_{\epsilon,l,\eta}(\mathrm{N}(\mathfrak{a})^{-1/2+\epsilon}X(\mathfrak{n}) + \mathrm{N}(\mathfrak{a})^{c+2}\,\mathrm{N}(\mathfrak{n})^{-\inf(1,c)+\epsilon})$. This part is most subtle and the term $X(\mathfrak{n})$ arises from this stage; see §7.3 for detail.

Combining these, we obtain (1.8) immediately.

7.1 Computation of $\mathcal{N}[\widetilde{\mathbb{W}}^{\eta}_{\mathfrak{u}}](\mathfrak{n})$

Let us describe the procedure (i). We take α to be the function $\alpha_{\mathfrak{a}}$. Set $S = S(\mathfrak{a})$. From (4.10), noting $(-1)^{\epsilon(\eta)}\widetilde{\eta}(\mathfrak{n}) = -1$, we have that $\mathcal{N}[\widetilde{\mathbb{W}}^{\eta}_{\mathfrak{u}}](\mathfrak{n})$ is the sum of the following two integrals:

$$2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\Big(\frac{1}{2\pi i}\Big)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\mathcal{N}[\widetilde{\mathfrak{W}}^{\eta}_{S}(l,\bullet|\mathbf{s})](\mathfrak{n})\alpha_{\mathfrak{a}}(\mathbf{s})\,d\mu_S(\mathbf{s}), \tag{7.1}$$

$$-2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\Big(\frac{1}{2\pi i}\Big)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\mathcal{N}[D\widetilde{\mathfrak{W}}^{\eta}_{S}(l,\bullet|\mathbf{s})](\mathfrak{n})\alpha_{\mathfrak{a}}(\mathbf{s})\,d\mu_S(\mathbf{s}), \tag{7.2}$$

where $\widetilde{\mathfrak{W}}^{\eta}_{S}(l,\bullet|\mathbf{s})$ is the quantity (4.11) viewed as an arithmetic function in $\mathfrak{n} \in \mathcal{I}^{-}_{S,\eta}$, and D is an arithmetic function defined as $D(\mathfrak{n}) = i^{\widetilde{l}}\delta(\mathfrak{n} = \mathfrak{o})$ (cf. §6). By formula (4.11),

$$\mathcal{N}[\widetilde{\mathfrak{W}}^{\eta}_{S}(l,\bullet|\mathbf{s})](\mathfrak{n}) = \pi^{\epsilon(\eta)}\Upsilon^{\eta}_{S}(\mathbf{s})L(1,\eta)\Big\{2^{-1}\mathcal{N}[\log\mathrm{N}](\mathfrak{n}) + \Big(\log(D_F\,\mathrm{N}(\mathfrak{f}_\eta)) + \frac{L'(1,\eta)}{L(1,\eta)} + \mathfrak{C}(l) + \sum_{v\in S}\frac{\log q_v}{1-\eta_v(\varpi_v)q_v^{(s_v+1)/2}}\Big)\mathcal{N}[1](\mathfrak{n})\Big\}.$$

By Lemma 5.3 and Corollary 5.4, we have formulas of $\mathcal{N}[\log\mathrm{N}](\mathfrak{n})$ and of $\mathcal{N}[1](\mathfrak{n})$; substituting these, and by using Lemma 6.2, we complete the evaluation of the integral (7.1).

The evaluation of the integral (7.2) is similar; instead of $\mathcal{N}[\log\mathrm{N}]$ and $\mathcal{N}[1]$, we need $\mathcal{N}[D\log\mathrm{N}]$ and $\mathcal{N}[D]$. The former one is 0 because $D\log\mathrm{N} = 0$; as in §6, the latter one is given by (6.2) and estimated as $|\mathcal{N}[D](\mathfrak{n})| \ll_\epsilon \mathrm{N}(\mathfrak{n})^{-1+\epsilon}$. Hence the integral (7.2) amounts at most to $\mathrm{N}(\mathfrak{n})^{-1+\epsilon}\,\mathrm{N}(\mathfrak{a})^{-1/2+\epsilon}$.

7.2 Estimation of the Term $\mathcal{N}[\mathbb{W}^{\eta}_{\mathrm{hyp}}](\mathfrak{n})$

Let us describe the procedure (ii). We need the following estimation, which we prove in §9.4.

Proposition 7.1 *For any small $\epsilon > 0$,*

$$|\mathbb{W}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha_{\mathfrak{a}})| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon}, \quad \mathfrak{n}\in\mathcal{I}^{-}_{S(\mathfrak{a}),\eta},$$

where the implied constant is independent of the ideal $\mathfrak{a}$.

By this proposition and Lemma 5.5,

$$|\mathcal{N}[\mathbb{W}^{\eta}_{\mathrm{hyp}}](\mathfrak{n})| \leqslant \mathcal{N}^{+}[|\mathbb{W}^{\eta}_{\mathrm{hyp}}|](\mathfrak{n}) \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\mathcal{N}^{+}[\mathrm{N}^{-c+\epsilon}](\mathfrak{n}) \ll_\epsilon \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+3\epsilon}.$$

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7.3 Estimation of the Term $\mathcal{N}[\mathrm{AL}^{\partial w}](\mathfrak{n})$

Let us describe the procedure (iv).

Lemma 7.2 *Let $\alpha \in \mathcal{A}_S$. Then for any $\mathfrak{n} \in \mathcal{I}^-_{S(\mathfrak{a}),\eta}$, we have the inequality*

$$|\mathrm{AL}^{\partial w}(\mathfrak{n};\alpha)| \leqslant \sum_{(\mathfrak{b},u)} D(\mathfrak{n};\mathfrak{b},u)\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})}{\iota(\mathfrak{n})}|\mathrm{AL}^*(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1};\alpha)|,$$

where $(\mathfrak{b},u)$ runs through all the pairs of an integral ideal $\mathfrak{b}$ and a place u such that $\mathfrak{n}\subset\mathfrak{b}^2\mathfrak{p}_u$. For such $(\mathfrak{b},u)$, we set

$$D(\mathfrak{n};\mathfrak{b},u) = \omega(\mathfrak{n},\mathfrak{b}^2\mathfrak{p}_u)(\log q_u)\Big(\mathrm{ord}_u(\mathfrak{b}) + \tfrac{q_u^{1/2}+1}{q_u^{1/2}-1}\Big).$$

Proof By Lemma 5.7, the π-summand of $\mathrm{AL}^{\partial w}(\mathfrak{n};\alpha)$ vanishes unless $\mathfrak{f}_\pi$ satisfies either (i) $\mathfrak{n}\mathfrak{f}_\pi^{-1} = \mathfrak{b}^2$ with some $\mathfrak{n}\subset\mathfrak{b}$, or (ii) $\mathfrak{n}\mathfrak{f}_\pi^{-1} = \mathfrak{b}^2\mathfrak{p}_u$ with some $\mathfrak{n}\subset\mathfrak{b}$ and $u\in S(\mathfrak{n})$. In the case (i), the π-summand vanishes. Indeed, from $\mathfrak{f}_\pi\in\mathcal{I}^-_{S(\mathfrak{a}),\eta}$, it turns out $L(1/2,\pi)L(1/2,\pi\otimes\eta) = 0$ by the functional equation. In the case (ii), from Lemma 5.7, noting the Ramanujan bound $|a_v| = 1$ and the obvious relation $|\chi_v(\varpi_v)| = 1$, we have

$$|\partial w_{\mathfrak{n}}^{\eta}(\pi)| \leqslant \omega(\mathfrak{n},\mathfrak{b}^2\mathfrak{p}_u)(\log q_u)\begin{cases}\mathrm{ord}_u(\mathfrak{b}) + \frac{q_u-1}{(1-q_u^{1/2})^2}, & c(\pi_u)=0,\\ \mathrm{ord}_u(\mathfrak{b}) + \frac{1}{1-q_u^{-1}}, & c(\pi_u)=1,\\ \mathrm{ord}_u(\mathfrak{b})+1, & c(\pi_u)\geqslant 2\end{cases}$$

$$\leqslant \omega(\mathfrak{n},\mathfrak{b}^2\mathfrak{p}_u)(\log q_u)\Big(\tfrac{q_u^{1/2}+1}{q_u^{1/2}-1} + \mathrm{ord}_v(\mathfrak{b})\Big) = D(\mathfrak{n};\mathfrak{b},u).$$

Here, we used $\frac{1}{1-q_u^{-1}} < \frac{q_u-1}{(1-q_u^{1/2})^2} = \frac{q_u^{1/2}+1}{q_u^{1/2}-1}$ to have the second inequality. ∎

Lemma 7.3 *For any $\epsilon\in(0,1)$, we have*

$$\sum_{(\mathfrak{b},u)} \mathrm{N}(\mathfrak{b}^2\mathfrak{p}_u)^\epsilon\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})}{\iota(\mathfrak{n})}\mathrm{N}(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})^{-\inf(c,1)+\epsilon} \ll_\epsilon \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+2\epsilon}, \tag{7.3}$$

$$\sum_{(\mathfrak{b},u)} \mathrm{N}(\mathfrak{b})^\epsilon\Big(\frac{q_u+1}{q_u-1}\Big)^2(\log q_u)\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})}{\iota(\mathfrak{n})} \ll_\epsilon X(\mathfrak{n}), \tag{7.4}$$

where $(\mathfrak{b},u)$ runs through the same range as in Lemma 7.2.

Proof Let us show (7.4). By the inequality $\iota(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})/\iota(\mathfrak{n}) \leqslant \mathrm{N}(\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})$,

$$\begin{aligned}&\sum_{(\mathfrak{b},u)} \mathrm{N}(\mathfrak{b})^\epsilon\Big(\frac{q_u+1}{q_u-1}\Big)^2(\log q_u)\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})}{\iota(\mathfrak{n})}\\ &\qquad\leqslant \sum_{(\mathfrak{b},u)} \mathrm{N}(\mathfrak{b})^{-2+\epsilon}\Big(\frac{q_u+1}{q_u-1}\Big)^2\frac{\log q_u}{q_u}\\ &\qquad\leqslant \Big\{\sum_{\mathfrak{b}\subset\mathfrak{o}}\mathrm{N}(\mathfrak{b})^{-2+\epsilon}\Big\}\Big\{\sum_{u\in S(\mathfrak{n})}\Big(\frac{q_u+1}{q_u-1}\Big)^2\frac{\log q_u}{q_u}\Big\}\end{aligned}$$

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$$= \zeta_{F,\mathrm{fin}}(2-\epsilon)\Big\{\sum_{u\in S(\mathfrak{n})}\frac{\log q_u}{q_u}+\sum_{u\in S(\mathfrak{n})}\frac{4\log q_u}{(q_u-1)^2}\Big\} \ll_\epsilon X(\mathfrak{n}).$$

The first estimate (7.3) is proved similarly. ■

Proposition 7.4 *For any $\epsilon>0$, we have*

$$|\,\mathrm{AL}^{\partial w}(\mathfrak{n};\alpha_{\mathfrak{a}})| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{-1/2}d_1(\mathfrak{a}_\eta^+)\delta_\square(\mathfrak{a}_\eta^-)X(\mathfrak{n})+\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon},$$

where $\mathfrak{n}\in\mathcal{I}_{S(\mathfrak{a}),\eta}^-$.

Proof Let $\epsilon>0$. From $\frac{x+1}{x-1}\ll_\epsilon x^\epsilon$ for $x\geqslant 2$, we first have

$$\omega(\mathfrak{n},\mathfrak{b}^2\mathfrak{p}_u)\leqslant\Big(\prod_{v\in S(\mathfrak{b})}\frac{q_v+1}{q_v-1}\Big)\frac{q_u+1}{q_u-1}\ll_\epsilon \mathrm{N}(\mathfrak{b})^\epsilon\frac{q_u+1}{q_u-1},$$

and then $D(\mathfrak{n};\mathfrak{b},u)\ll_\epsilon \mathrm{N}(\mathfrak{b})^\epsilon(\log q_u)\big(\frac{q_u+1}{q_u-1}\big)^2\ll_\epsilon \mathrm{N}(\mathfrak{b}^2\mathfrak{p})^\epsilon$ with the implied constant independent of $\mathfrak{n}$ and $(\mathfrak{b},u)$. Using this, we have the desired bound with the aid of (1.7) and Lemmas 7.2 and 7.3. ■

Proposition 7.5 *For any $\epsilon>0$, we have*

$$|\mathcal{N}[\mathrm{AL}^{\partial w}](\mathfrak{n};\alpha_{\mathfrak{a}})| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{-1/2}d_1(\mathfrak{a}_\eta^+)\delta_\square(\mathfrak{a}_\eta^-)X(\mathfrak{n})+\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-\inf(1,c)+\epsilon},$$

where $\mathfrak{n}\in\mathcal{I}_{S(\mathfrak{a}),\eta}^-$.

Proof From Proposition 7.4, we have

$$\begin{aligned}|\mathcal{N}[\mathrm{AL}^{\partial w}](\mathfrak{n};\alpha_{\mathfrak{a}})| \ll_{\epsilon,l,\eta} &\ \mathrm{N}(\mathfrak{a})^{-1/2}d_1(\mathfrak{a}_\eta^+)\delta_\square(\mathfrak{a}_\eta^-)\mathcal{N}^+[X](\mathfrak{n})\\ &+\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\mathcal{N}^+[\mathrm{N}^{-\inf(1,c)+\epsilon}](\mathfrak{n})\end{aligned}$$

for all $\mathfrak{n}\in\mathcal{I}_{S(\mathfrak{a}),\eta}^-$. Since $X(\mathfrak{m})\leqslant X(\mathfrak{n})$ if $\mathfrak{n}\subset\mathfrak{m}\subset\mathfrak{o}$, we have

$$\begin{aligned}\mathcal{N}^+[X](\mathfrak{n})&\leqslant X(\mathfrak{n})\mathcal{N}^+[1](\mathfrak{n})\\ &=X(\mathfrak{n})\Big\{\prod_{v\in S(\mathfrak{n}_1)-S_2(\mathfrak{n})}(1+q_v^{-2})\Big\}\Big\{\prod_{v\in S_2(\mathfrak{n})}(1+(1-q_v^{-1})^{-1}q_v^{-2})\Big\}\\ &\leqslant X(\mathfrak{n})\Big\{\prod_{v\in\Sigma_{\mathrm{fin}}}(1+q_v^{-2})\Big\}\Big\{\prod_{v\in\Sigma_{\mathrm{fin}}}(1+(1-q_v^{-1})^{-1}q_v^{-2})\Big\}.\end{aligned}$$

By the convergence of the Euler products, this yields $\mathcal{N}^+[X](\mathfrak{n})\ll X(\mathfrak{n})$. Therefore, with the aid of Lemma 5.5, we obtain the estimate

$$|\mathcal{N}[\mathrm{AL}^{\partial w}](\mathfrak{n};\alpha_{\mathfrak{a}})| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{-1/2}d_1(\mathfrak{a}_\eta^+)\delta_\square(\mathfrak{a}_\eta^-)X(\mathfrak{n})+\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-\inf(1,c)+3\epsilon}$$

for any sufficiently small $\epsilon\in(0,1)$ with the implied constant independent of $\mathfrak{n}$ and $\mathfrak{a}$. ■

8 An Estimation of Number of Cusp Forms

Recall $c = d_F^{-1}(\underline{l}/2-1)$. Suppose that for each ideal $\mathfrak{a} \subset \mathfrak{o}$, we are given a set $\mathcal{J}_{\mathfrak{a}}$ of ideals prime to $\mathfrak{f}_\eta \mathfrak{a}$ in such a way that $\mathcal{J}_{\mathfrak{a}} \subset \mathcal{J}_{\mathfrak{a}'}$ for any $\mathfrak{a} \subset \mathfrak{a}'$, and a family of real numbers $\{\omega_{\mathfrak{n}}(\pi) | \pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})\}$ for each $\mathfrak{n} \in \mathcal{J}_{\mathfrak{a}}$ which satisfies the following estimate for any $\epsilon > 0$:

$$(8.1)\quad \Big| \sum_{\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})} \omega_{\mathfrak{n}}(\pi) \prod_{v \in S(\mathfrak{a})} X_{n_v}(\lambda_v(\pi)) - \prod_{v \in S(\mathfrak{a})} \mu_{v,\eta_v}(X_{n_v}) \Big| \ll_{\epsilon,l,\eta} \frac{\mathrm{N}(\mathfrak{a})^{-1/2+\epsilon}}{\log \mathrm{N}(\mathfrak{n})} + \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon},$$

with the implied constant independent of $\mathfrak{a}$ and $\mathfrak{n} \in \mathcal{J}_{\mathfrak{a}}$, where $\lambda_v(\pi)$ is the trace of the Satake parameter of π at v. Moreover we assume the non-negativity condition

$$(8.2)\qquad \omega_{\mathfrak{n}}(\pi) \geqslant 0 \quad \text{for all } \pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n}) \text{ and } \mathfrak{n} \in \mathcal{J}_{\mathfrak{a}}.$$

Let $\mathfrak{q}$ be a prime ideal relatively prime to $\mathfrak{f}_\eta$. In what follows, we abuse the symbol $\mathfrak{q}$ to denote the corresponding place $v_{\mathfrak{q}}$ of F; for example, we write $\eta_{\mathfrak{q}}$, $\lambda_{\mathfrak{q}}(\pi)$ in place of $\eta_{v_{\mathfrak{q}}}$, $\lambda_{v_{\mathfrak{q}}}(\pi)$, etc. Let $S = \{v_1,\dots,v_r\}$ be a finite subset of $\Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\eta \mathfrak{q})$ and set $\mathfrak{a}_S = \prod_{v \in S} \mathfrak{p}_v$. Let $\mathbf{J} = \{J_j\}_{j=1}^r$ a family of closed subintervals of $(-2,2)$. For each J_j, we choose an open interval J_j' such that $\overline{J_j'} \subset J_j^\circ$ together with a C^∞-function $\chi_j\colon \mathbb{R} \to [0,\infty)$ such that $\chi_j(x) \neq 0$ for all $x \in J_j'$, $\mathrm{supp}(\chi_j) \subset J_j$ and $\int_{-2}^2 \chi_j(x)\,d\mu_{v,\eta_v}(x) = 1$, where

$$d\mu_{v,\eta_v}(x) = \begin{cases} \dfrac{q_v - 1}{(q_v^{1/2} + q_v^{-1/2} - x)^2} d\mu^{\mathrm{ST}}(x), & \eta_v(\varpi_v) = +1, \\[2ex] \dfrac{q_v + 1}{(q_v^{1/2} + q_v^{-1/2})^2 - x^2}\, d\mu^{\mathrm{ST}}(x), & \eta_v(\varpi_v) = -1. \end{cases}$$

Here $d\mu^{\mathrm{ST}}(x) = (2\pi)^{-1}\sqrt{4-x^2}dx$. Fixing such a family of functions $\{\chi_j\}$, we set $\Omega_{\mathfrak{n}}(\pi) = \omega_{\mathfrak{n}}(\pi)\prod_{j=1}^r \chi_j(\lambda_{v_j}(\pi))$, for any $\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$ and $\mathfrak{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}$.

Lemma 8.1 *For any sufficiently small $\epsilon > 0$, there exists $N_{\epsilon,S,l} > 0$ such that*

$$(8.3)\quad \Big| \sum_{\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})} \Omega_{\mathfrak{n}}(\pi) X_n(\lambda_{\mathfrak{q}}(\pi)) - \mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_n) \Big| \ll_{\epsilon,l,\eta,S,\mathbf{J}} \frac{n+1}{(\log \mathrm{N}(\mathfrak{n}))^3} + \frac{\mathrm{N}(\mathfrak{q}^n)^{-1/2+\epsilon}}{\log \mathrm{N}(\mathfrak{n})} + \mathrm{N}(\mathfrak{q}^n)^{2+c+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}$$

for $n \in \mathbb{N}_0$ and $\mathfrak{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}$ with $\mathrm{N}(\mathfrak{n}) > N_{\epsilon,S,l}$. Here the implied constant is independent of n and $\mathfrak{n}$. Moreover, $\Omega_{\mathfrak{n}}(\pi) \geqslant 0$ for all $\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$ and $\mathfrak{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}$.

Proof Given an integer $M > 1$, define $\chi_j^M(x) = \sum_{n=0}^M \widehat{\chi}_j(n) X_n(x)$ for $x \in [-2,2]$ with $\widehat{\chi}_j(n) = \int_{-2}^2 \chi_j(x) X_n(x)\,d\mu^{\mathrm{ST}}(x)$ and set

$$\chi(\mathbf{x}) = \prod_{j=1}^r \chi_j(x_j), \qquad \chi^M(\mathbf{x}) = \prod_{j=1}^r \chi_j^M(x_j)$$

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for $\mathbf{x} = \{x_j\}_{j=1}^r$ in the product space $[-2,2]^r$. Let $\mathfrak{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}$. By the triangle inequality, the left-hand side of (8.3) is no greater than the sum of the following three terms :

$$(8.4)\qquad \Big|\sum_{\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{n})} \omega_{\mathfrak{n}}(\pi)X_n(\lambda_{\mathfrak{q}}(\pi))\big\{\chi(\lambda_S(\pi)) - \chi^M\big(\lambda_S(\pi)\big)\big\}\Big|,$$

$$(8.5)\qquad \Big|\sum_{\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{n})} \omega_{\mathfrak{n}}(\pi)X_n(\lambda_{\mathfrak{q}}(\pi))\chi^M(\lambda_S(\pi)) - \mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_n)\mu_{S,\eta}(\chi^M)\Big|,$$

$$(8.6)\qquad |\{\mu_{S,\eta}(\chi^M) - \mu_{S,\eta}(\chi)\}\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_n)|,$$

where $\lambda_S(\pi) = (\lambda_v(\pi))_{v\in S}$ and $\mu_{S,\eta} = \otimes_{v\in S}\mu_{v,\eta_v}$. Note $\mu_{S,\eta}(\chi) = 1$. We shall estimate these quantities. Since $|\widehat{\chi}_j(n)| \ll_{\chi_j} n^{-5}$ for any $n > 0$ by integration by parts and by $\max_{[-2,2]}|X_n| \ll n+1$, we have

$$|\chi_j^M(x)| \leqslant \sum_{n\leqslant M}|\widehat{\chi}_j(n)||X_n(x)| \ll_{\chi_j} \sum_{n\leqslant M} n^{-4} \leqslant \zeta(4)$$

and

$$\max_{x\in[-2,2]}|\chi_j(x) - \chi_j^M(x)| \leqslant \sum_{n>M}|\widehat{\chi}_j(n)|\max_{[-2,2]}|X_n| \ll_{\chi_j} \sum_{n>M} n^{-4} \ll M^{-3}.$$

By these,

$$(8.7)\quad \max_{[-2,2]^r}|\chi(\mathbf{x})-\chi^M(\mathbf{x})| \leqslant \max_{[-2,2]^r}\Big(\sum_{j=1}^r\Big|\prod_{h=1}^{j-1}\chi_h^M(x_h)\Big||\chi_j(x_j)-\chi_j^M(x_j)|\Big) \ll_{S,\chi} M^{-3}.$$

From (8.1) for $\mathfrak{a} = \mathfrak{o}$, noting $\mathfrak{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S} \subset \mathcal{J}_{\mathfrak{o}}$, we have the estimate

$$\Big|\sum_{\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{n})}\omega_{\mathfrak{n}}(\pi) - 1\Big| \ll_{\epsilon,l,\eta} (\log \mathrm{N}(\mathfrak{n}))^{-1} + \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}.$$

Hence (8.4) is majorized by

$$\Big\{\max_{[-2,2]}|X_n|\Big\}\Big\{\max_{\mathbf{x}\in[-2,2]^r}|\chi(\mathbf{x}) - \chi^M(\mathbf{x})|\Big\}\sum_{\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{n})}\omega_{\mathfrak{n}}(\pi)$$
$$\ll_{\epsilon,l,\eta,S,\chi} (n+1)M^{-3}(1+\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}).$$

By (8.7), the quantity (8.6) is majorized by $\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_n)M^{-3}$, which amounts at most to $(n+1)M^{-3}$. Let us estimate (8.5). By expanding the product, $\chi^M(\mathbf{x})$ is expressed as a sum of the terms $\prod_{j=1}^r\widehat{\chi}_j(n_j)\prod_{j=1}^r X_{n_j}(x_j)$ over all $\mathbf{n} = (n_j)_{j=1}^r \in \{0,\dots,M\}^r$. Hence by using (8.1), we can majorize (8.5) from above by

$$\sum_{\mathbf{n}\in\{0,\cdots,M\}^r}\Big|\sum_{\pi\in\Pi_{\mathrm{cus}}^*(l,\mathfrak{n})}\omega_{\mathfrak{n}}(\pi)\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(\lambda_{\mathfrak{q}}(\pi))\prod_{j=1}^r X_{n_j}(\lambda_{v_j}(\pi)) - \mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_n)\mu_{S,\eta}\Big(\prod_{j=1}^r X_{n_j}\Big)\Big|$$
$$\ll_{\epsilon,l,\eta,S,\chi} \frac{\mathrm{N}(\mathfrak{a}_S^M\mathfrak{q}^n)^{-1/2+\epsilon}}{\log\mathrm{N}(\mathfrak{n})} + \mathrm{N}(\mathfrak{a}_S^M\mathfrak{q}^n)^{2+c+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}.$$

Combining the estimations made so far, we have that the left-hand side of (8.3) is majorized by

$$(8.8)\quad (n+1)M^{-3}(1+\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}) + \frac{\mathrm{N}(\mathfrak{a}_S^M\mathfrak{q}^n)^{-1/2+\epsilon}}{\log \mathrm{N}(\mathfrak{n})} + \mathrm{N}(\mathfrak{a}_S^M\mathfrak{q}^n)^{2+c+\epsilon}\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}.$$

Now take

$$M = \Big[\frac{\epsilon}{2+c+\epsilon}\frac{\log \mathrm{N}(\mathfrak{n})}{\log \mathrm{N}(\mathfrak{a}_S)}\Big].$$

Then $\mathrm{N}(\mathfrak{a}_S)^{M(2+c+\epsilon)} \leqslant \mathrm{N}(\mathfrak{n})^{\epsilon}$, and also $\mathrm{N}(\mathfrak{a}_S)^{M(-1/2+\epsilon)} \leqslant 1$ evidently. By these, (8.8) is majorized by

$$\begin{aligned}&(n+1)(\log \mathrm{N}(\mathfrak{n}))^{-3}\log \mathrm{N}(\mathfrak{a}_S)^3(1+\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon})\\ &\qquad + \frac{\mathrm{N}(\mathfrak{q}^n)^{-1/2+\epsilon}}{\log \mathrm{N}(\mathfrak{n})} + \mathrm{N}(\mathfrak{q}^n)^{2+c+\epsilon}\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+2\epsilon}\\ &\ll_{\epsilon,S} (n+1)(\log \mathrm{N}(\mathfrak{n}))^{-3} + \frac{\mathrm{N}(\mathfrak{q}^n)^{-1/2+\epsilon}}{\log \mathrm{N}(\mathfrak{n})} + \mathrm{N}(\mathfrak{q}^n)^{2+c+\epsilon}\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+2\epsilon}.\end{aligned}$$

■

Lemma 8.2 *Let $I \subset [-2,2]$ be an open interval disjoint from the set*

$$\{\lambda_{\mathfrak{q}}(\pi) | \pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n}), \quad \Omega_{\mathfrak{n}}(\pi) \neq 0\}.$$

Then for any small $\epsilon > 0$, there exists a constant $N_{\epsilon,l,\eta,S,\mathfrak{q}} > 0$ such that for any ideal $\mathfrak{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}$ with $\mathrm{N}(\mathfrak{n}) > N_{\epsilon,l,\eta,S,\mathfrak{q}}$, $\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(I) \ll_{\epsilon,l,\eta,S,\mathrm{J}} \mathrm{N}(\mathfrak{q})^{\epsilon}(\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon}$ with the implied constant independent of I, $\mathfrak{n}$, and $\mathfrak{q}$.

Proof The proof of [6, Proposition 5.1 and Lemma 5.2] goes through as it is with a small modification. We reproduce the argument for convenience.

Let $\Delta > 0$ be a parameter to be specified below and K a closed subinterval of I such that

(i) $\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(I-K) \leqslant \Delta$.

Depending on Δ and K, we choose a C^{∞}-function f on $\mathbb{R}$ such that

(ii) $\mathrm{supp}(f) \subset \bar{I}$,
(iii) $f(x) = 1$ if $x \in K$ and $0 \leqslant f(x) \leqslant 1$ for $x \in \mathbb{R}$,
(iv) $|f^{(k)}(x)| \ll_k \Delta^{-k}$ for $k \in \mathbb{N}_0$.

Since I does not contain the relevant $\lambda_{\mathfrak{q}}(\pi)$'s, from (ii) we have $\Omega_{\mathfrak{n}}(\pi)f(\lambda_{\mathfrak{q}}(\pi)) = 0$ for all $\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$. Using this, from (i) and (iii), we have the inequalities

$$\begin{aligned}(8.9)\quad \mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(I) &\leqslant \mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(K) + \Delta \leqslant \int_{-2}^{2} f\, d\mu_{\mathfrak{q},\eta_{\mathfrak{q}}} + \Delta\\ &\leqslant \Big|\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})} \Omega_{\mathfrak{n}}(\pi)f(\lambda_{\mathfrak{q}}(\pi)) - \int_{-2}^{2} f\, d\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}\Big| + \Delta.\end{aligned}$$

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If we set $f_M(x) = \sum_{n=0}^{M} \widehat{f}(n) X_n(x)$, then the first term of (8.9) is bounded by the sum of the following three terms

$$\Big(\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})} |\Omega_{\mathfrak{n}}(\pi)|\Big)\cdot \max_{[-2,2]} |f - f_M|, \tag{8.10}$$

$$\int_{-2}^{2} \max_{[-2,2]} |f - f_M|\, d\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}, \tag{8.11}$$

$$\Big|\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})} \Omega_{\mathfrak{n}}(\pi) f_M(\lambda_{\mathfrak{q}}(\pi)) - \int_{-2}^{2} f_M\, d\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}\Big|. \tag{8.12}$$

We remark that by the non-negativity of $\Omega_{\mathfrak{n}}(\pi)$, the absolute value in (8.10) can be deleted. Then by the estimate $|\widehat{f}(n)| \ll_k n^{-k}\Delta^{-k}$ which follows from (iv) by integration by parts, and by $\max_{[-2,2]} |X_n| \ll n+1$, we have

$$\max_{[-2,2]} |f - f_M| \leqslant \sum_{n>M} |\widehat{f}(n)| \max_{[-2,2]} |X_n| \ll_k \sum_{n>M} n^{-k}\Delta^{-k} n \ll M^{2-k}\Delta^{-k}$$

with $k \geqslant 3$. From (8.3) applied with $n = 0$, noting $\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_0) = 1$, we have the estimate $|\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})} \Omega_{\mathfrak{n}}(\pi) - 1| \ll_{\epsilon,l,\eta,S,\mathbf{J}} (\log \mathrm{N}(\mathfrak{n}))^{-1} + \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}$. Hence the sum of (8.10) and (8.11) is majorized by

$$\Delta^{-k} M^{2-k}(1 + (\log \mathrm{N}(\mathfrak{n}))^{-1} + \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}) \ll \Delta^{-k} M^{2-k}$$

with the implied constant independent of Δ, M, $\mathfrak{q}$ and $\mathfrak{n}$. By (8.3) and by $|\widehat{f}(n)| \ll 1$, the term (8.12) is majorized by

$$\begin{aligned}
&\sum_{n=0}^{M} |\widehat{f}(n)| \Big|\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})} \Omega_{\mathfrak{n}}(\pi) X_n(\lambda_{\mathfrak{q}}(\pi)) - \mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_n)\Big| \\
&\quad \ll_{\epsilon,l,\eta,S,\mathbf{J}} \sum_{n=0}^{M} \Big(\frac{n+1}{(\log \mathrm{N}(\mathfrak{n}))^3} + \frac{\mathrm{N}(\mathfrak{q}^n)^{-1/2+\epsilon}}{\log \mathrm{N}(\mathfrak{n})} + \mathrm{N}(\mathfrak{q}^n)^{2+c+\epsilon}\, \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon} \Big) \\
&\quad \ll_{\epsilon} \frac{M^2}{(\log \mathrm{N}(\mathfrak{n}))^3} + \frac{1}{\log \mathrm{N}(\mathfrak{n})} + \mathrm{N}(\mathfrak{q})^{c'M}\, \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon},
\end{aligned}$$

where $c' = 2 + c + \epsilon$. Putting all relevant estimations together, we obtain

$$\begin{aligned}
\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(I) \ll_{k,\epsilon,l,\eta,S,\mathbf{J}} &\ \Delta + \Delta^{-k} M^{2-k} \\
&+ \frac{1}{\log \mathrm{N}(\mathfrak{n})} + \frac{M^2}{(\log \mathrm{N}(\mathfrak{n}))^3} + \mathrm{N}(\mathfrak{q})^{c'M}\, \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}
\end{aligned}$$

with the implied constant independent of I, Δ, M, $\mathfrak{q}$, and $\mathfrak{n}$. By setting

$$M = \Big[\frac{\inf(c,1)}{2c'} \frac{\log \mathrm{N}(\mathfrak{n})}{\log \mathrm{N}(\mathfrak{q})}\Big],$$

this yields the estimate

$$\begin{aligned}
\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(I) \ll_{k,\epsilon,l,\eta,S,\mathbf{J}} &\ \Delta + \Delta^{-k} (\log \mathrm{N}(\mathfrak{q}))^{k-2} (\log \mathrm{N}(\mathfrak{n}))^{2-k} \\
&+ (\log \mathrm{N}(\mathfrak{n}))^{-1} + \mathrm{N}(\mathfrak{n})^{-\inf(c,1)/2+\epsilon}.
\end{aligned}$$

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Let $\epsilon > 0$ and we let Δ vary so that it satisfies $\Delta^{-k}(\log \mathrm{N}(\mathfrak{n}))^{2-k} \asymp_k (\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon}$, or equivalently $\Delta \asymp_k (\log \mathrm{N}(\mathfrak{n}))^{-1+(3-\epsilon)/k}$. By taking $k = [3/\epsilon] + 1$, we have

$$(\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon/2} \ll_\epsilon \Delta \ll_\epsilon (\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon}.$$

Hence,

$$\begin{aligned}\mu_{\mathfrak{q},\eta_\mathfrak{q}}(I) \ll_{\epsilon,l,\eta,S,\mathbf{J}} &(\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon} + (\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon}(\log \mathrm{N}(\mathfrak{q}))^{k-2} \\ &+ (\log \mathrm{N}(\mathfrak{n}))^{-1} + \mathrm{N}(\mathfrak{n})^{-\inf(c,1)/2+\epsilon} \\ \ll_\epsilon &\mathrm{N}(\mathfrak{q})^\epsilon (\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon}.\end{aligned}$$

This completes the proof. ∎

Lemma 8.3 *Given $\epsilon > 0$, there exists a positive number $N_{\epsilon,l,\eta,S,\mathfrak{q},\mathbf{J}}$ such that for any ideal $\mathfrak{n} \in \mathfrak{J}_{\mathfrak{q}\mathfrak{a}_S}$ with $\mathrm{N}(\mathfrak{n}) > N_{\epsilon,l,\eta,S,\mathfrak{q},\mathbf{J}}$, we have the inequality*

$$\#\{\lambda_\mathfrak{q}(\pi) | \pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n}), \Omega_\mathfrak{n}(\pi) \neq 0\} \geqslant \mathrm{N}(\mathfrak{q})^{-\epsilon}(\log \mathrm{N}(\mathfrak{n}))^{1-\epsilon}.$$

Proof It follows in the same way as [6, Lemma 5.3]. ∎

8.1 Hecke Fields

Let $\Gamma = \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. We let the group Γ act on the set $(2\mathbb{N})^{\Sigma_\infty}$ of even weights by the rule ${}^\sigma l = (l_{\sigma^{-1}\circ v})_{v\in\Sigma_\infty}$ for $l = (l_v)_{v\in\Sigma_\infty}$ and $\sigma \in \Gamma$, regarding $\Sigma_\infty = \mathrm{Hom}(F,\mathbb{C})$. Let $\mathbb{Q}(l)$ be the fixed field of $\mathrm{Stab}_\Gamma(l)$, which is a finite extension of $\mathbb{Q}$. From [8] (see [5] also), the Satake parameter $A_v(\pi)$ belongs to $\mathrm{GL}(2,\bar{\mathbb{Q}})$ for any $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n})$ and the set $\Pi_{\mathrm{cus}}(l,\mathfrak{n})$ has a natural action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(l))$ in such a way that $({}^\sigma\pi)_v \cong \pi_{\sigma^{-1}\circ v}$ for all $v \in \Sigma_\infty$ and

$$q_v^{1/2} A_v({}^\sigma\pi) = \sigma(q_v^{1/2} A_v(\pi)) \quad \text{for all } v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n}). \tag{8.13}$$

The field of rationality of $\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$, to be denoted by $\mathbb{Q}(\pi)$, is defined as the fixed field of the group $\{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(l)) | {}^\sigma\pi = \pi\}$. From (8.13), by the strong multiplicity one theorem for GL(2), we have $\mathbb{Q}(\pi) = \mathbb{Q}(l)\big(q_v^{1/2}\lambda_v(\pi) | v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n})\big)$.

Proposition 8.4 *Assume that l is a parallel weight,* i.e., *there exists $k \in 2\mathbb{N}$ such that $l_v = k$ for all $v \in \Sigma_\infty$. Let S be a finite subset of $\Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\eta)$ and $\mathbf{J} = \{J_v\}_{v\in S}$ a family of closed subintervals of $(-2,2)$. Given a sufficiently small $\epsilon > 0$ and a prime ideal $\mathfrak{q}$ prime to $S \cup S(\mathfrak{f}_\eta)$, there exists a positive integer $N_{\epsilon,l,\eta,S,\mathfrak{q},\mathbf{J}}$ such that for any $\mathfrak{n} \in \mathfrak{J}_{\mathfrak{q}\mathfrak{a}_S}$ with $\mathrm{N}(\mathfrak{n}) > N_{\epsilon,l,\eta,S,\mathfrak{q},\mathbf{J}}$, there exists $\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$ such that $\omega_\mathfrak{n}(\pi) \neq 0$, $\lambda_v(\pi) \in J_v$ for all $v \in S$, and*

$$[\mathbb{Q}(\pi):\mathbb{Q}] \geqslant \sqrt{\max\Big\{\frac{(1-\epsilon)\log\log \mathrm{N}(\mathfrak{n})}{\log(16\,\mathrm{N}(\mathfrak{q})^{(k-1)/2})} - 2\epsilon, 0\Big\}}.$$

Proof Let $\Omega_\mathfrak{n}(\pi)$ be as above. We follow the proof of [6, Proposition 7.3]. Let $d(\Omega,\mathfrak{n})$ denote the maximal degree of algebraic numbers $\lambda_\mathfrak{q}(\pi)$ for all $\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$

such that $\Omega_{\mathfrak{n}}(\pi) \neq 0$. Then

$$d(\Omega,\mathfrak{n}) \leqslant \max\{[\mathbb{Q}(\pi):\mathbb{Q}] \mid \pi \in \Pi^*_{\rm cus}(l,\mathfrak{n}), \Omega_{\mathfrak{n}}(\pi) \neq 0\}$$
$$\leqslant \max\{[\mathbb{Q}(\pi):\mathbb{Q}] \mid \pi \in \Pi^*_{\rm cus}(l,\mathfrak{n}), \omega_{\mathfrak{n}}(\pi) \neq 0, \lambda_v(\pi) \in J_v (\forall v \in S)\}.$$

Let $\mathcal{E}(M,d)$ denote the set of algebraic integers which, together with its conjugates, have the absolute values at most M and the absolute degrees at most d. From the parallel weight assumption, the Hecke eigenvalues $\mathrm{N}(\mathfrak{q})^{(k-1)/2}\lambda_{\mathfrak{q}}(\pi)$ are known to be algebraic integers [8, Proposition 2.2]. Since

$$\sigma(\mathrm{N}(\mathfrak{q})^{(k-1)/2}\lambda_{\mathfrak{q}}(\pi)) = \mathrm{N}(\mathfrak{q})^{(k-1)/2}\lambda_{\mathfrak{q}}({}^{\sigma}\pi)$$

from (8.13), by the Ramanujan bound, we have

$$\mathrm{N}(\mathfrak{q})^{(k-1)/2}\lambda_{\mathfrak{q}}(\pi) \in \mathcal{E}\big(2\,\mathrm{N}(\mathfrak{q})^{(k-1)/2}, d(\Omega,\mathfrak{n})\big).$$

Then the cardinality of the set $\{\mathrm{N}(\mathfrak{q})^{(k-1)/2}\lambda_{\mathfrak{q}}(\pi) | \pi \in \Pi^*_{\rm cus}(l,\mathfrak{n}), \Omega_{\mathfrak{n}}(\pi) \neq 0\}$ is bounded from above by $\#\mathcal{E}(2\,\mathrm{N}(\mathfrak{q})^{(k-1)/2}, d(\Omega,\mathfrak{n}))$ which in turn is no greater than $(16\,\mathrm{N}(\mathfrak{q})^{(k-1)/2})^{d(\Omega,\mathfrak{n})^2}$ by [6, Lemma 6.2]. Combining this with the lower bound provided by Lemma 8.3, we have

$$\mathrm{N}(\mathfrak{q})^{-\epsilon}(\log \mathrm{N}(\mathfrak{n}))^{1-\epsilon} \leqslant (16\,\mathrm{N}(\mathfrak{q})^{(k-1)/2})^{d(\Omega,\mathfrak{n})^2}.$$

By taking the logarithm, we are done. ∎

Remark 8.5 The parallel weight assumption can be removed if the integrality of the Hecke eigenvalues $q_v^{(k-1)/2}\lambda_v(\pi)$ for all $v \in \Sigma_{\rm fin} - S(\mathfrak{f}_\pi)$ is known in a broader generality, where $k = \max_{v\in\Sigma_\infty} l_v$.

8.2 The Proof of Theorem 1.3

Theorem 1.1 means that the numbers

$$\omega_{\mathfrak{n}}(\pi) = \frac{C_l}{4D_F^{3/2}L_{\rm fin}(1,\eta)\nu(\mathfrak{n})}\frac{1}{\mathrm{N}(\mathfrak{n})}\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi;\mathrm{Ad})}$$

for $\pi \in \Pi^*_{\rm cus}(l,\mathfrak{n})$ and $\mathfrak{n} \in \mathcal{I}^+_{S\cup S(\mathfrak{q}),\eta}$ satisfy our first assumption (8.1). The second assumption (8.2) follows from [2]. Thus Theorem 1.3 is a corollary of Proposition 8.4 with this particular $\{\omega_{\mathfrak{n}}(\pi)\}$.

8.3 The Proof of Theorem 1.4

For any $M > 1$, let $\mathcal{I}^-_{S\cup S(\mathfrak{q}),\eta}[M]$ be the set of $\mathfrak{n} \in \mathcal{I}^-_{S\cup S(\mathfrak{q}),\eta}$ such that $\sum_{v\in S(\mathfrak{n})}\frac{\log q_v}{q_v} \leqslant M$. Theorem 1.1 means that

$$\omega_{\mathfrak{n}}(\pi) = \frac{C_l}{4D_F^{3/2}L_{\rm fin}(1,\eta)\nu(\mathfrak{n})\log\sqrt{\mathrm{N}(\mathfrak{n})}}\frac{1}{\mathrm{N}(\mathfrak{n})}\frac{L(1/2,\pi)L'(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi;\mathrm{Ad})}$$

for $\pi \in \Pi^*_{\rm cus}(l,\mathfrak{n})$ and $\mathfrak{n} \in \mathcal{I}^-_{S\cup S(\mathfrak{q}),\eta}[M]$ satisfy our first assumption (8.1). By our non-negativity assumption (1.9), the second assumption (8.2) is also met. Thus Theorem 1.4 follows from Proposition 8.4.

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Remark 8.6 In the parallel weight two case (*i.e.*, $l_v = 2$ for all $v \in \Sigma_\infty$) with totally imaginary condition on η (*i.e.*, $\eta_v(-1) = -1$ for all $v \in \Sigma_\infty$), the assumption (1.9) follows from [17, Theorem 6.1] due to the non-negativity of the Neron–Tate height pairing. Similar results may be expected in the parallel higher weight case [15].

9 Computations of Local Terms

In this section, we shall give the postponed proof of Proposition 7.1. Let $\alpha = \otimes_{v\in S}\alpha_v \in \mathcal{A}_S$ be such that $\alpha_v = \alpha_v^{(m_v)}$ for some $(m_v)_{v\in S} \in \mathbb{N}_0^S$, where $\alpha_v^{(m)}(s) = q_v^{ms/2} + q_v^{-ms/2}$ for $v \in S$ and $m \in \mathbb{N}_0$. We examine the term $\mathbb{W}^\eta_{\rm hyp}(l, \mathfrak{n}|\alpha)$ appearing in formula (4.9). Set $\delta_b = \left[\begin{smallmatrix} 1+b^{-1} & 1 \\ 1 & 1\end{smallmatrix}\right]$ for $b \in F^\times$. From Lemma 4.6, by changing the order of integrals, we have the first equality of the formula

$$\text{(9.1)}\quad \mathbb{W}^\eta_{\rm hyp}(l, \mathfrak{n}|\alpha) = \sum_{b\in F-\{0,-1\}} \int_{\mathbb{A}^\times} \widehat{\Psi}^{(0)}_l(\mathfrak{n}|\alpha, \delta_b\left[\begin{smallmatrix} t & 0 \\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1\end{smallmatrix}\right])\eta(tx^*_\eta)\log|t|_{\mathbb{A}}\, d^\times t$$
$$= \sum_{b\in F-\{0,-1\}} \sum_{w\in\Sigma_F} \Big\{ \prod_{v\in\Sigma_F-\{w\}} J_v(b)\Big\} W_w(b),$$

where $J_v(b)$ is defined by (6.4),

$$W_w(b) = \int_{F_w^\times} \Psi_w(\delta_b\left[\begin{smallmatrix} t_w & 0 \\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_{\eta,w} \\ 0 & 1\end{smallmatrix}\right])\eta_w(t_w x^*_{\eta,w})\log|t_w|_w\, d^\times t_w$$

for $b \in F_v - \{0, -1\}$, and $\Psi_w(g_w)$ denotes the w-th factor of $\widehat{\Psi}^{(0)}_l(\mathfrak{n}|\alpha; g)$ as in §6. The second equality of (9.1) is justified by $\sum_b \sum_w \{\prod_{v\neq w} |J_v(b)|\}|W_w(b)| < \infty$, which results from the analysis to be made in §9.4. The integrals $J_v(b)$ are studied and their explicit evaluations are obtained in [11, §10]. In what follows, we examine the integral $W_w(b)$ separating cases $w \in S$, $w \in \Sigma_{\rm fin} - S$, and $w \in \Sigma_\infty$. We remark that $\mathrm{vol}(\mathfrak{o}_v^\times) = \#(\mathfrak{o}_v/\mathfrak{d}_F\mathfrak{o}_v)^{-1/2}$ in the computations below.

9.1 Hyperbolic Non-Archimedean Terms for S

Let $v \in S$. Then the integral $W_v(b)$ depends on the test function $\alpha_v \in \mathcal{A}_v$ and the character η_v of $F_v^\times$; we write $W_v^{\eta_v}(b; \alpha_v)$ in place of $W_v(b)$ in this subsection. We have

$$W_v^{\eta_v}(b, \alpha_v) = \frac{1}{2\pi i}\int_{L_v(c)} \Big\{ \int_{F_v^\times} \Psi_v^{(0)}(s_v; \delta_b\left[\begin{smallmatrix} t & 0 \\ 0 & 1\end{smallmatrix}\right])\eta_v(t)\log|t|_v d^\times t\Big\}\, \alpha_v(s_v)\, d\mu_v(s_v),$$

where $L_v(c) = \{c + it \mid t \in \mathbb{R}/4\pi(\log q_v)^{-1}\mathbb{Z}\}$ (see §6).

Lemma 9.1 *Let $v \in S$. Then for any $m > 0$ and any $b \in F_v - \{0, -1\}$,*

$$W_v^{\eta_v}(b; \alpha_v^{(m)}) = \widetilde{I}_v^+(m; b) + \eta_v(\varpi_v)\{(\log q_v)I_v^+(m; \varpi_v^{-1}(b+1)) - \widetilde{I}_v^+(m; \varpi_v^{-1}(b+1))\}$$

with

$$I_v^+(m;b) = \mathrm{vol}(\mathfrak{o}_v^\times)2^{\delta(m=0)} \times \Big(-q_v^{\frac{-m}{2}}\delta_m^{\eta_v}(b) + \sum_{l=o(b)}^{m-1}\{(m-l-1)q_v^{\frac{2-m}{2}} - (m-l+1)q_v^{\frac{-m}{2}}\}\delta_l^{\eta_v}(b)\Big),$$

$$\widetilde{I}_v^+(m;b) = \mathrm{vol}(\mathfrak{o}_v^\times)(\log q_v)2^{\delta(m=0)} \times \Big(-q_v^{\frac{-m}{2}}\widetilde{\delta}_m^{\eta_v}(b) + \sum_{l=o_1(b)}^{m-1}\{(m-l-1)q_v^{\frac{2-m}{2}} - (m-l+1)q_v^{\frac{-m}{2}}\}\widetilde{\delta}_l^{\eta_v}(b)\Big),$$

where we set $o(b) = \sup(0, -\mathrm{ord}_v(b))$, $o_1(b) = \sup(0, 1-\mathrm{ord}_v(b))$, *and*

$$\delta_n^{\eta_v}(b) = \delta(|b|_v \leqslant q_v^n)\eta_v(\varpi_v^n)\begin{cases}(\mathrm{ord}_v(b)+1)^{\delta(n=0)}, & \eta_v(\varpi_v)=1,\\ (2^{-1}(\eta_v(b)+1))^{\delta(n=0)}, & \eta_v(\varpi_v)=-1,\end{cases}$$

$$\widetilde{\delta}_n^{\eta_v}(b) = \delta(|b|_v < q_v^n)\eta_v(\varpi_v^n) \times \begin{cases}\eta_v(b)(-n-\mathrm{ord}_v(b)), & n>0,\\ -2^{-1}\mathrm{ord}_v(b)(\mathrm{ord}_v(b)+1), & n=0, \eta_v(\varpi_v)=1,\\ 4^{-1}(\eta_v(b)-1)+2^{-1}\mathrm{ord}_v(b)\eta_v(b), & n=0, \eta_v(\varpi_v)=-1.\end{cases}$$

When $m=0$,

$$W_v^{\eta_v}(b;\alpha_v^{(0)}) = -2\,\mathrm{vol}(\mathfrak{o}_v^\times)(\log q_v)\big(\widetilde{\delta}_0^{\eta_v}(b) + \eta_v(\varpi_v)\delta_0^{\eta_v}(\varpi_v^{-1}(b+1)) - \eta_v(\varpi_v)\widetilde{\delta}_0^{\eta_v}(\varpi_v^{-1}(b+1))\big).$$

Proof We decompose the integral into the sum $W_v(b;\alpha_v^{(m)}) = \widetilde{I}_v^+(m;b) + \widetilde{I}_v^-(m;b)$, where

$$\widetilde{I}_v^+(m;b) = \int_{t\in F_v^\times, |t|\leqslant 1}\widehat{\Phi}_{vm}(\delta_b\left[\begin{smallmatrix} t & 0\\ 0 & 1\end{smallmatrix}\right])\eta_v(t)\log|t|_v\,d^\times t,$$

$$\widetilde{I}_v^-(m;b) = \int_{|t|_v>1}\widehat{\Phi}_{vm}(\delta_b\left[\begin{smallmatrix} t & 0\\ 0 & 1\end{smallmatrix}\right])\eta_v(t)\log|t|_v\,d^\times t$$

with $\widehat{\Phi}_{vm}(g_v) = \frac{1}{2\pi i}\int_{L_v(c)}\Psi_v^{(0)}(s_v;g_v)\alpha_v^{(m)}(s_v)\,d\mu_v(s_v)$. We consider the case $m>0$. By the evaluation of $\widehat{\Phi}_{vm}(g_v)$ made in [11, Lemma 10.1],

$$\widetilde{I}_v^+(m;b) = \int_{|t|\leqslant 1, \sup(1,|t|_v^{-1}|b|_v)=q_v^m}(-q_v^{-m/2})\eta_v(t)\log|t|_v\,d^\times t + \sum_{l=0}^{m-1}\int_{|t|\leqslant 1,\sup(1,|t|_v^{-1}|b|_v)=q_v^l}\{(m-l-1)q_v^{1-m/2} - (m-l+1)q_v^{-m/2}\} \times \eta_v(t)\log|t|_v\,d^\times t.$$

We have the following three equalities.

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- If $l = 0$ and $\eta_v(\varpi_v) = 1$,

$$\int_{|t|\leqslant 1, \sup(1,|t|_v^{-1}|b|_v)=q_v^l} \eta_v(t)\log|t|_v\, d^\times t = \delta(|b|_v < 1)\operatorname{vol}(\mathfrak{o}_v^\times)\log q_v \frac{-\operatorname{ord}_v(b)(\operatorname{ord}_v(b)+1)}{2}.$$

- If $l = 0$ and $\eta_v(\varpi_v) = -1$,

$$\int_{|t|\leqslant 1, \sup(1,|t|_v^{-1}|b|_v)=q_v^l} \eta_v(t)\log|t|_v\, d^\times t = \delta(|b|_v < 1)\operatorname{vol}(\mathfrak{o}_v^\times)\log q_v \Big(\frac{\eta_v(b)-1}{4} + \frac{\operatorname{ord}_v(b)\eta_v(b)}{2}\Big).$$

- If $l > 0$,

$$\begin{aligned}\int_{|t|\leqslant 1, \sup(1,|t|_v^{-1}|b|_v)=q_v^l} &\eta_v(t)\log|t|_v\, d^\times t \\ &= \delta(|b| < q_v^l)\int_{|t|_v = q_v^{-l}|b|_v} \eta_v(t)\log|t|_v\, d^\times t \\ &= -\delta(|b|_v < q_v^l)\operatorname{vol}(\mathfrak{o}_v^\times)(\log q_v)\eta_v(\varpi_v^l b)(l + \operatorname{ord}_v(b)).\end{aligned}$$

By the variable change $y = \varpi_v^{-1}t^{-1}$, $\widetilde{I}_v^-(m;b)$ becomes

$$\begin{aligned}\int_{|y|_v<1} &\widehat{\Phi}_{vm}\big(\delta_b\big[\begin{smallmatrix}\varpi_v^{-1}y^{-1} & 0\\ 0 & 1\end{smallmatrix}\big]\big)\eta_v(\varpi_v^{-1}y^{-1})\log|\varpi_v^{-1}y^{-1}|_v\, d^\times y \\ &= \eta_v(\varpi_v^{-1})\int_{|y|_v\leqslant 1}\widehat{\Phi}_{vm}\big(\delta_b\big[\begin{smallmatrix}\varpi_v^{-1}y^{-1} & 0\\ 0 & 1\end{smallmatrix}\big]\big)\eta_v(y)(\log q_v - \log|y|_v)\, d^\times y \\ &= \eta_v(\varpi_v)\{(\log q_v)I_v^+(m;\varpi_v^{-1}(b+1)) - \widetilde{I}_v^+(m;\varpi_v^{-1}(b+1))\},\end{aligned}$$

where the integral $I_v^+(m;b)$ is evaluated in the proof of [11, Lemma 10.2]. From the results above, we have the lemma for $m > 0$. The case $m = 0$ is similar. ∎

Lemma 9.2 *For $m \in \mathbb{N}$,*

$$|W_v^{\eta_v}(b;\alpha_v^{(m)})| \ll (\log q_v)\delta(|b|_v \leqslant q_v^{m-1})q_v^{1-m/2}m(2m + \operatorname{ord}_v(b(b+1)))^2,$$

where $b \in F_v^\times - \{-1\}$. When $m = 0$,

$$|W_v^{\eta_v}(b;\alpha_v^{(0)})| \ll (\log q_v)\delta(|b|_v \leqslant 1)\big(\operatorname{ord}_v(b(b+1)) + 1\big)^2, \quad b \in F_v^\times - \{-1\}.$$

Here the implied constants are independent of v, m, and b. Moreover, for $n \in \mathbb{N}_0$,

$$|W_v^{\eta_v}(b;\alpha_{\mathfrak{p}_v^n})| \ll (\log q_v)q_v\delta(|b|_v \leqslant q_v^n)\{\operatorname{ord}_v\big(b(b+1)\big) + 2n + 1\}^2, \quad b \in F_v^\times - \{-1\}$$

with the implied constant independent of v, n and b.

Proof Noting (6.5), from the first and the second estimates in the lemma, we derive the last one in the same way as we did to have (6.8). Let us prove the first estimate. The second one is easier. Suppose $m \geqslant 1$. From Lemma 9.1, it suffices to estimate $I_v^+(m;\varpi_v^{-1}(b+1))$, $\widetilde{I}_v^+(m;b)$, and $\widetilde{I}_v^+(m;\varpi_v^{-1}(b+1))$. By the formula in Lemma 9.1,

$$|I_v^+(m,\varpi_v^{-1}(b+1))| \ll \delta(|b|_v \leqslant q_v^{m-1})(m+1)^2 q_v^{1-m/2}.$$

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Next we examine $\widetilde{I}_v^+(m;b)$. From the definition of $\widetilde{\delta}_m^{\eta_v}$ in Lemma 9.1, we have

$$|\widetilde{\delta}_0^{\eta_v}(b)| \leqslant \delta(|b|_v < 1)2^{-1}(\mathrm{ord}_v(b)+1)^2.$$

By using this,

$$\begin{aligned}
&\sum_{l=o_1(b)}^{m-1}(m-l-1)q_v^{1-m/2}|\widetilde{\delta}_l^{\eta_v}(b)| \\
&\leqslant \delta(m\geqslant 1, |b|_v \leqslant q_v^{m-2})q_v^{1-m/2}\Big\{\sum_{l=1}^{m-1}(m-l-1)|\widetilde{\delta}_l^{\eta_v}(b)| + (m-1)|\widetilde{\delta}_0^{\eta_v}(b)|\Big\} \\
&\leqslant \delta(m\geqslant 1, |b|_v \leqslant q_v^{m-2})q_v^{1-m/2} \\
&\qquad\qquad \times \Big\{\sum_{l=1}^{m-1}(m-l-1)(l+\mathrm{ord}_v(b)) + (m-1)|\widetilde{\delta}_0^{\eta_v}(b)|\Big\} \\
&= \delta(m\geqslant 1, |b|_v \leqslant q_v^{m-2})q_v^{1-m/2}(m-1) \\
&\qquad \times \Big\{6^{-1}(m-2)m + 2^{-1}(m-2)\,\mathrm{ord}_v(b) + |\widetilde{\delta}_0^{\eta_v}(b)|\Big\} \\
&\ll \delta(m\geqslant 2, |b|_v \leqslant q_v^{m-2})q_v^{1-m/2}m\Big(m^2 + m\,\mathrm{ord}_v(b) + (\mathrm{ord}_v(b)+1)^2\Big) \\
&\ll \delta(m\geqslant 2, |b|_v \leqslant q_v^{m-2})q_v^{1-m/2}m\Big(m + \mathrm{ord}_v(b)\Big)^2.
\end{aligned}$$

Similarly,

$$\sum_{l=o_1(b)}^{m-1}(m-l+1)q_v^{-m/2}|\widetilde{\delta}_l^{\eta_v}(b)| \ll \delta(m\geqslant 1, |b|_v \leqslant q_v^{m-2})q_v^{-m/2}m(m+\mathrm{ord}_v(b)+1)^2.$$

Hence, we obtain

$$|\widetilde{I}_v^+(m;b)| \ll (\log q_v)\delta(|b|_v \leqslant q_v^{m-1})q_v^{1-m/2}m(m+\mathrm{ord}_v(b))^2,$$

where $m\in\mathbb{N},\ b\in F_v^\times - \{-1\}$, which also yields

$$|\widetilde{I}_v^+(m;\varpi_v^{-1}(b+1))| \ll (\log q_v)\delta(|b+1|_v \leqslant q_v^{m-1})q_v^{1-m/2}m(m+\mathrm{ord}_v(b+1))^2,$$

where $m\in\mathbb{N},\ b\in F_v^\times - \{-1\}$. ∎

9.2 Hyperbolic Non-Archimedean Terms Outside S

Let $v\in\Sigma_{\mathrm{fin}} - S$. There are three cases to be considered: $v\in\Sigma_{\mathrm{fin}} - S(\mathfrak{n}\mathfrak{f}_\eta)$, $v\in S(\mathfrak{n})$, and $v\in S(\mathfrak{f}_\eta)$.

Lemma 9.3 *Let $v\in\Sigma_{\mathrm{fin}} - (S\cup S(\mathfrak{n}\mathfrak{f}_\eta))$. For $b\in F_v^\times - \{-1\}$, we have*

$$W_v(b) = \int_{F_v^\times}\Phi_{v,0}^{(0)}(\delta_b[\begin{smallmatrix} t & 0\\ 0 & 1\end{smallmatrix}])\eta_v(t)\log|t|_v\,d^\times t = \mathrm{vol}(\mathfrak{o}_v^\times)(\log q_v)\{\widetilde{\delta}_0^{\eta_v}(b) - \widetilde{\delta}_0^{\eta_v}(b+1)\}.$$

In particular, $|W_v(b)| \ll (\log q_v)\delta(|b(b+1)|_v<1)(\mathrm{ord}_v(b(b+1))+1)^2,\ b\in F_v^\times - \{-1\}$.

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Proof By [12, Lemma 11.4],

$$\int_{F_v^\times} \Phi_{v,0}^{(0)}(\delta_b[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}])\eta_v(t)\log|t|_v\, d^\times t$$
$$= \int_{|b|_v\leqslant|t|_v<1} \eta_v(t)\log|t|_v\, d^\times t + \int_{|b+1|_v^{-1}\geqslant|t|_v>1} \eta_v(t)\log|t|_v\, d^\times t$$
$$= \mathrm{vol}(\mathfrak{o}_v^\times)(\log q_v)\{\widetilde{\delta}_0^{\eta_v}(b) - \widetilde{\delta}_0^{\eta_v}(b+1)\}. \qquad \blacksquare$$

Lemma 9.4 *Let $v \in S(\mathfrak{n})$. Then*

$$W_v(b) = 2^{-1}\,\mathrm{vol}(\mathfrak{o}_v^\times)(-\log q_v)\delta(b\in \mathfrak{n}\mathfrak{o}_v)$$
$$\times \begin{cases} (\mathrm{ord}_v(b)+\mathrm{ord}_v(\mathfrak{n}))(\mathrm{ord}_v(b)-\mathrm{ord}_v(\mathfrak{n})+1), & \eta_v(\varpi_v)=1, \\ [\,\mathrm{ord}_v(\mathfrak{n})\eta_v(\varpi_v^{\mathrm{ord}_v(\mathfrak{n})}) + \mathrm{ord}_v(b)\eta_v(b) & \\ \qquad +2^{-1}\{\eta_v(b) - \eta_v(\varpi_v^{\mathrm{ord}_v(\mathfrak{n})})\}], & \eta_v(\varpi_v)=-1. \end{cases}$$

In particular,

$$|W_v(b)| \ll \delta(b\in\mathfrak{n}\mathfrak{o}_v)(\log q_v)(\mathrm{ord}_v(b)+\mathrm{ord}_v(\mathfrak{n})+1)^2, \quad b\in F_v^\times - \{-1\}.$$

Proof By [12, Lemma 11.4],

$$\int_{F_v^\times} \Phi_{v,\mathfrak{n}}^{(0)}(\delta_b[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}])\eta_v(t)\log|t|_v\, d^\times t$$
$$= \int_{|b|_v\leqslant|t|_v<1} \delta(t\in\mathfrak{n}\mathfrak{o}_v)\eta_v(t)\log|t|_v\, d^\times t$$
$$= \delta(b\in\mathfrak{n}\mathfrak{o}_v)\sum_{n=\mathrm{ord}_v(\mathfrak{n})}^{\mathrm{ord}_v(b)} \int_{\mathfrak{o}_v^\times} \eta_v(\varpi_v^n u)\log|\varpi_v^n u|_v\, d^\times u$$
$$= \delta(b\in\mathfrak{n}\mathfrak{o}_v)\,\mathrm{vol}(\mathfrak{o}_v^\times)(-\log q_v)\sum_{n=\mathrm{ord}_v(\mathfrak{n})}^{\mathrm{ord}_v(b)} \eta_v(\varpi_v^n)n. \qquad \blacksquare$$

Lemma 9.5 *Let $v\in S(\mathfrak{f}_\eta)$ and put $f = f(\eta_v)\in\mathbb{N}$. For $b\in F_v^\times-\{-1\}$,*

$$W_v(b) = \mathrm{vol}(\mathfrak{o}_v^\times)\delta(b\in\mathfrak{p}_v^{-f})\eta_v(-1)(1-q_v^{-1})^{-1}q_v^{-f}(\log q_v)$$
$$\times\Big\{-f+\eta_v(b(b+1))\Big(\delta(b\in\mathfrak{p}_v)(-f-\mathrm{ord}_v(b))$$
$$+\delta(b\in\mathfrak{o}_v^\times)(-f+\mathrm{ord}_v(b+1)) + \delta(b\notin\mathfrak{o}_v)(-f)q_v^{\mathrm{ord}_v(b)}\Big)\Big\}.$$

In particular,

$$|W_v(b)| \ll (\log q_v)q_v^{-f}\delta(|b|_v\leqslant q_v^f)\big\{f+\delta(|b|_v\leqslant 1)\,\mathrm{ord}_v(b(b+1))\big\},$$

where $b\in F_v^\times-\{-1\}$.

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Proof By [12, Lemma 11.4], we have $W_v(b) = \delta(b \in \mathfrak{p}_v^{-f})(W_{v,1}(b) + W_{v,2}(b))$ with

$$W_{v,1}(b) = \int_{-t \in \varpi_v^f U_v(f) |t|_v |b+1|_v \leqslant 1} \eta_v(t\varpi_v^{-f}) \log|t|_v \, d^\times t$$
$$= \mathrm{vol}(\mathfrak{o}_v^\times)\eta_v(-1)(-f\log q_v) q_v^{-f}(1-q_v^{-1})^{-1},$$
$$W_{v,2}(b) = \int_{-t \in F_v^\times - \varpi_v^f U_v(f)} |1 + t\varpi_v^{-f}|_v |b + t\varpi_v^{-f}(b+1)|_v \leqslant |t|_v \eta_v(t\varpi_v^{-f}) \log|t|_v \, d^\times t.$$

The integration domain of $W_{v,2}(b)$ is a disjoint union of the sets $D_l(b)$ ($l \in \mathbb{Z}$) defined in [11, §10.2]. By [11, Lemmas 10.6, 10.7, and 10.8],

$$\begin{aligned}
&\mathrm{vol}(\mathfrak{o}_v^\times)^{-1} W_{v,2}(b) = \mathrm{vol}(\mathfrak{o}_v^\times)^{-1} \sum_{l \in \mathbb{Z}} (-l \log q_v) \int_{D_l(b)} \eta_v(t\varpi_v^{-f}) \, d^\times t \\
&= \delta(|b|_v < 1 = |b+1|_v)\{(-f + \mathrm{ord}_v(b+1) - \mathrm{ord}_v(b))\log q_v\} \\
&\qquad \times \eta_v\Big(\frac{-b}{b+1}\Big)(1-q_v^{-1})^{-1} q_v^{-f} \\
&\quad + \delta(|b|_v = |b+1_v| \geqslant 1)(-f\log q_v) \\
&\qquad \times \eta_v\Big(\frac{-b}{b+1}\Big)(1-q_v^{-1})^{-1} q_v^{-f+\mathrm{ord}_v(b)} \\
&\quad + \delta(|b+1|_v < 1 = |b|_v)\{(-f + \mathrm{ord}_v(b+1) - \mathrm{ord}_v(b))\log q_v\} \\
&\qquad \times \eta_v\Big(\frac{-b}{b+1}\Big)(1-q_v^{-1})^{-1} q_v^{-f} \\
&= \eta_v\Big(\frac{-b}{b+1}\Big)(1-q_v^{-1})^{-1} q_v^{-f} (\log q_v)\Big\{\delta(|b|_v < 1 = |b+1|_v)\big(-f - \mathrm{ord}_v(b)\big) \\
&\qquad + \delta(|b|_v = |b+1|_v \geqslant 1)(-f) q_v^{\mathrm{ord}_v(b)} \\
&\qquad + \delta(|b+1|_v < 1 = |b|_v)(-f + \mathrm{ord}_v(b+1))\Big\} \\
&= \eta_v\Big(\frac{-b}{b+1}\Big)(1-q_v^{-1})^{-1} q_v^{-f} (\log q_v)\Big\{\delta(b \in \mathfrak{p}_v)\big(-f - \mathrm{ord}_v(b)\big) \\
&\qquad + \delta(b \in \mathfrak{o}_v^\times)(-f + \mathrm{ord}_v(b+1)) + \delta(b \notin \mathfrak{o}_v)(-f) q_v^{\mathrm{ord}_v(b)}\Big\}.
\end{aligned}$$

This completes the proof. ∎

9.3 Hyperbolic Archimedean Terms

Let $v \in \Sigma_\infty$ and fix an identification $F_v \cong \mathbb{R}$. In this paragraph, we abbreviate l_v to l, omitting the subscript v. From the proof of [11, Lemma 10.12], we have

$$\begin{aligned}
W_v(b) &= \int_{\mathbb{R}^\times} \Big(\frac{1+it}{\sqrt{t^2+1}}\Big)^l \{1 + i(bt^{-1} + t(b+1))\}^{-l/2} \eta_v(t) \log|t|_v \, d^\times t \\
&= \int_{\mathbb{R}^\times} (1-it)^{-l/2}(1+b+t^{-1}bi)^{-l/2} \eta_v(t) \log|t|_v \, d^\times t \\
&= W_+(b) + \eta_v(-1)\overline{W_+(b)},
\end{aligned}$$

where we set $W_+(b) = i^{l/2}(1+b)^{-l/2}\int_0^\infty (t+i)^{-l/2}\big(t + \frac{bi}{b+1}\big)^{-l/2} t^{l/2-1} \log t \, dt$.

Here is an explicit formula of $W_+(b)$.

Lemma 9.6 *Suppose $l \geqslant 4$. Then for $b \in \mathbb{R}^\times - \{-1\}$, we have*

$$W_+(b) = -\pi i J_+(l;b) - A(b) - iB(b)$$

with

$$\begin{aligned} J_+(l;b) = &\Big\{ \log\Big|\frac{b+1}{b}\Big| + \delta(b(b+1)<0)\pi i\Big\} \\ &\times P_{l/2-1}(2b+1) - \sum_{m=1}^{[l/4]} \frac{4(l-4m+1)}{(2m-1)(l-2m)} P_{l/2-2m}(2b+1), \\ A(b) = &\sum_{k=0}^{l/2-1} \binom{l/2+k-1}{k}\binom{l/2-1}{k} \\ &\times \Big\{ \frac{b^k}{2}\Big(\log\Big|\frac{b}{b+1}\Big|\Big)^2 - \frac{\theta(b)^2}{2} b^k - \frac{9\pi^2}{8}(-1)^{k+l/2}(b+1)^k\Big\} \\ &+ \sum_{k=0}^{l/2-1}\binom{l/2+k-1}{k}\sum_{j=1}^{l/2-k-1}\binom{l/2-1}{k+j}\frac{(-1)^j}{j} \\ &\times \Big(\sum_{m=1}^{j-1}\frac{1}{m}\{b^k + (-1)^{k+l/2}(b+1)^k\} - b^k \log\Big|\frac{b}{b+1}\Big|\Big), \\ B(b) = &\sum_{k=0}^{l/2-1}\binom{l/2+k-1}{k}\binom{l/2-1}{k} b^k \log\Big|\frac{b}{b+1}\Big|\theta(b) \\ &- \sum_{k=0}^{l/2-1}\binom{l/2+k-1}{k}\sum_{j=1}^{l/2-k-1}\binom{l/2-1}{k+j}\frac{(-1)^j}{j} \\ &\times \Big\{\frac{3\pi}{2}(-1)^{k+l/2}(b+1)^k + b^k\theta(b)\Big\}, \end{aligned}$$

where $\theta(b) = \pi/2$ if $b(b+1)<0$, and $\theta(b) = 3\pi/2$ if $b(b+1)>0$.

Proof For $b \in \mathbb{R}^\times - \{-1\}$, put

$$g(z) = i^{l/2}(1+b)^{-l/2}(z+i)^{-l/2}\Big(z + \frac{bi}{b+1}\Big)^{-l/2} z^{l/2-1}(\log z)^2,$$

where $\log z = \log|z| + i\arg(z)$ with $\arg(z) \in [0, 2\pi)$. Then $g(z)$ is meromorphic on $\mathbb{C} - \mathbb{R}_{\geqslant 0}$ and holomorphic except for poles at $z = -i, \frac{-bi}{b+1}$. We note that $\frac{-bi}{b+1} \in i\mathbb{R} - \{0, -i\}$. By the residue theorem, we have

$$\text{(9.2)}\qquad \begin{aligned} &2\pi i(\mathrm{Res}_{z=-i} + \mathrm{Res}_{z=\frac{-bi}{b+1}})g(z) \\ &= \int_\epsilon^R g(t)\,dt - \int_\epsilon^R g(te^{2\pi i}) + \oint_{|z|=R} g(z)\,dz - \oint_{|z|=\epsilon} g(z)\,dz \end{aligned}$$

with R sufficiently large and $\epsilon > 0$ sufficiently small. By letting $R \to +\infty$, $\epsilon \to +0$, noting the relation $(\log t + 2\pi i)^2 = (\log t)^2 + 4\pi i \log t - 4\pi^2$, we have

$$2\pi i(\mathrm{Res}_{z=-i} + \mathrm{Res}_{z=\frac{-bi}{b+1}})g(z) = -4\pi i W_+(b) + 4\pi^2 J_+(l;b),$$

because the last two terms on the right-hand side of (9.2) vanish in the limit, where $J_+(l;b) := i^{l/2}(1+b)^{-l/2}\int_0^\infty (t+i)^{-l/2}\big(t + \frac{bi}{b+1}\big)^{-l/2} t^{l/2-1}\,dt$ is evaluated in the proof of [11, Lemma 10.13]. Hence, we obtain

$$W_+(b) = -\frac{1}{2}\Big\{\mathrm{Res}_{z=-i} + \mathrm{Res}_{z=\frac{-bi}{b+1}}\Big\}\, g(z) - \pi i J_+(l;b).$$

A direct computation reveals $\frac{1}{2}\big\{\mathrm{Res}_{z=-i} + \mathrm{Res}_{z=\frac{-bi}{b+1}}\big\}\, g(z) = A(b) + iB(b)$. This completes the proof. ∎

Lemma 9.7 *Suppose $l > 4$. For any $\epsilon > 0$, we have*

$$|b(b+1)|^{\epsilon}|W_v(b)| \ll_{\epsilon,l} (1+|b|)^{-l/2+2\epsilon}, \qquad b \in \mathbb{R} - \{0,-1\}.$$

Proof From Lemmas 6.3 and 9.6, for any $\epsilon > 0$, $|b(b+1)|^{\epsilon} W_+(b)$ is locally bounded around the points $b = 0, -1$. For b away from the set $\{0,-1\}$, we have

$$|W_+(b)| \leqslant |2b(b+1)|^{-l/4} \int_0^{\infty} t^{l/4}(t^2+1)^{-l/4}|\log t|\, \frac{dt}{t}$$

by $t^2(b+1)^2 + b^2 \geqslant 2|t|\,|b(b+1)|$. Since $l > 4$, the last integral is convergent; hence the above inequality gives us $|b(b+1)|^{\epsilon}|W_+(b)| \ll_{\epsilon,l} (1+|b|)^{-l/2+2\epsilon}$ for large $|b|$. ∎

9.4 The Proof of Proposition 7.1

We start from the formula (9.1) taking α to be $\alpha_{\mathfrak{a}} \in \mathcal{A}_{S(\mathfrak{a})}$ defined by (1.6). If we set

$$\mathbb{W}(T) = \sum_{b\in F-\{0,-1\}} \sum_{w\in T} \Big\{ \prod_{v\in\Sigma_F-\{w\}} J_v(b) \Big\}\, W_w(b) \tag{9.3}$$

for any subset $T \subset \Sigma_F$, then (9.1) can be written in the form

$$\mathbb{W}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha_{\mathfrak{a}}) = \mathbb{W}(\Sigma_\infty) + \mathbb{W}(S(\mathfrak{a})) + \mathbb{W}(S(\mathfrak{n})) + \mathbb{W}(S(\mathfrak{f}_\eta)) + \mathbb{W}(\Sigma_{\mathrm{fin}} - S(\mathfrak{n}\mathfrak{a}\mathfrak{f}_\eta)).$$

We shall estimate each term in the right-hand side of this equality, explicating the dependence on $\mathfrak{n}$ and $\mathfrak{a} = \prod_{v\in S(\mathfrak{a})} \mathfrak{p}_v^{n_v}$. Set $c = (\underline{l}/2-1)/d_F$. In the remaining part of this section, all the constants implied by the Vinogradov symbols are independent of $\mathfrak{n}$ and $\mathfrak{a}$ but may depend on l, η and a given small number $\epsilon > 0$. For convenience, we collect here all the estimates used below (other than these, we also need (6.6), (6.7), (6.8), and Lemma 9.7). Let $w_1 \in S(\mathfrak{a})$, $w_2 \in S(\mathfrak{n})$, $w_3 \in S(\mathfrak{f}_\eta)$, $w_4 \in \Sigma_{\mathrm{fin}} - S(\mathfrak{a}\mathfrak{f}_\eta)$. Then

$$|W_{w_1}(b)| \ll (\log q_{w_1}) q_{w_1} \delta(b \in \mathfrak{a}^{-1}\mathfrak{o}_{w_1})\{2n_{w_1} + \mathrm{ord}_{w_1}(b(b+1)) + 1\}^2, \tag{9.4}$$

$$|W_{w_2}(b)| \ll (\log q_{w_2}) \delta(b \in \mathfrak{n}\mathfrak{o}_{w_2})\{\mathrm{ord}_{w_2}(b) + \mathrm{ord}_{w_2}(\mathfrak{n}) + 1\}^2, \tag{9.5}$$

$$|W_{w_3}(b)| \ll (\log q_{w_3}) \delta(b \in \mathfrak{f}_\eta^{-1}\mathfrak{o}_{w_3})\{2f(\eta_{w_3}) + \mathrm{ord}_{w_3}(b(b+1)) + 1\}, \tag{9.6}$$

$$|W_{w_4}(b)| \ll (\log q_{w_4}) \delta(|b(b+1)|_{w_4} < 1)\Lambda_{w_4}(b)^2 \tag{9.7}$$

for $b \in F^\times - \{0,-1\}$, where all the constants implied by the Vinogradov symbol are independent of b, $\mathfrak{n}$, $\mathfrak{a}$, and the places w_i $(1 \leqslant i \leqslant 4)$. Indeed, the estimate (9.4) follows from Lemma 9.2, (9.5) is from Lemma 9.4, (9.6) is from Lemma 9.5, and (9.7) is from Lemma 9.3.

Proposition 7.1 follows from Lemmas 9.8, 9.9, 9.10, 9.11 and 9.12 to be shown below.

Lemma 9.8 *We have $|\mathbb{W}(\Sigma_\infty)| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon}$.*

Proof Similarly to the proof of Proposition 6.5, by Lemma 9.7 and Proposition 6.4, we have $|\mathbb{W}(\Sigma_\infty)| \ll_{\epsilon,l,\eta} C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a}) \sum_{I\subset S(\mathfrak{a})} \mathrm{N}(\mathfrak{f}_\eta \prod_{v\in I} \mathfrak{p}_v^{n_v})^{1+c+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon}$. Thus, we are done. ∎

Lemma 9.9 *We have* $|\mathbb{W}(S(\mathfrak{a}))| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon}$.

Proof By the estimates recalled above, the range of b in the summation (9.3) with $T = S(\mathfrak{a})$ can be restricted to $\mathfrak{n}\mathfrak{a}^{-1}\mathfrak{f}_\eta^{-1} - \{0,-1\}$. If $b \in \mathfrak{n}\mathfrak{a}^{-1}\mathfrak{f}_\eta^{-1}$, then $b(b+1)\mathfrak{a}^2\mathfrak{f}_\eta^2$ is an ideal of $\mathfrak{o}$. From this, noting that η is unramified over $S(\mathfrak{a})$, we have the equality $\mathrm{ord}_w(b(b+1)\mathfrak{a}^2\mathfrak{f}_\eta^2) = 2n_w + \mathrm{ord}_w(b(b+1))$ for any $w \in S(\mathfrak{a})$; by taking summation over $w \in S(\mathfrak{a})$,

$$\sum_{w\in S(\mathfrak{a})} \{2n_w + \mathrm{ord}_w(b(b+1)) + 1\} \log q_w \leqslant \log \mathrm{N}\big(b(b+1)\mathfrak{a}^2\mathfrak{f}_\eta^2\big) + \log \mathrm{N}(\mathfrak{a})$$
$$\ll_{\epsilon,\eta} |\mathrm{N}(b(b+1))|^{\epsilon/2}\,\mathrm{N}(\mathfrak{a})^{\epsilon}.$$

Using this, from (9.4), (6.6), (6.7), and (6.8), we obtain

$$\begin{aligned}
|\mathbb{W}(S(\mathfrak{a}))| &\ll \sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} \sum_{w_1\in S(\mathfrak{a})} \Big\{ \prod_{v\in\Sigma_F-\{w_1\}} |J_v(b)| \Big\} \\
&\qquad\times (\log q_{w_1}) q_{w_1} \{\mathrm{ord}_{w_1}(b(b+1)) + 2n_{w_1} + 1\}^2 \\
&\ll_{\epsilon,\eta} C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a})^{1+2\epsilon} \sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} |\mathrm{N}(b(b+1))|^{\epsilon} \prod_{v\in\Sigma_\infty} |J_v(b)| \\
&\qquad\times \prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{f}_\eta)} \Lambda_v(b) \prod_{v\in S(\mathfrak{a})} \{1+\Lambda_v(b)\} \\
&\leqslant C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a})^{1+2\epsilon} \sum_{I\subset S(\mathfrak{a})} \mathfrak{I}_\epsilon^\eta(l,\mathfrak{n},\mathfrak{f}_\eta \prod_{v\in I} \mathfrak{p}_v^{n_v}),
\end{aligned}$$

where C is the implied constant in (6.8) and (9.4). ∎

Lemma 9.10 *We have* $|\mathbb{W}(S(\mathfrak{n}))| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon}$.

Proof From the estimations recalled above,

$$\begin{aligned}
&|\mathbb{W}(S(\mathfrak{n}))| \ll_{\epsilon,\eta} C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a}) \\
&\times \sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} \prod_{v\in\Sigma_\infty} |J_v(b)| \prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)} \Lambda_v(b) \prod_{v\in S(\mathfrak{a})} \{1+\Lambda_v(b)\} \sum_{w_2\in S(\mathfrak{n})} |W_{w_2}(b)|,
\end{aligned}$$

where C is the constant in (6.8). By (9.5),

$$\begin{aligned}
\sum_{w_2\in S(\mathfrak{n})} |W_{w_2}(b)| &\ll \sum_{w_2\in S(\mathfrak{n})} (\log q_{w_2})(\mathrm{ord}_{w_2}(\mathfrak{n}) + \mathrm{ord}_{w_2}(b) + 1)^2 \\
&\ll \sum_{w_2\in S(\mathfrak{n})} \mathrm{ord}_{w_2}(\mathfrak{n})^2 (\log q_{w_2}) + \sum_{w_2\in S(\mathfrak{n})} (\log q_{w_2}) \Lambda_{w_2}(b)^2 \\
&\ll_\epsilon \mathrm{N}(\mathfrak{n})^\epsilon \prod_{v\in S(\mathfrak{n})} \Lambda_v(b)^2
\end{aligned}$$

for $b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}$. From this,

$$\begin{aligned} |\mathbb{W}(S(\mathfrak{n}))| &\ll_{\epsilon,\eta} C^{\# S(\mathfrak{a})}\, \mathrm{N}(\mathfrak{a})\, \mathrm{N}(\mathfrak{n})^{\epsilon} \\ &\quad \times \sum_{b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} \prod_{v \in \Sigma_\infty} |J_v(b)| \prod_{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{a}\mathfrak{f}_\eta)} \Lambda_v(b)^2 \prod_{v \in S(\mathfrak{a})} \{1 + \Lambda_v(b)\} \\ &\leqslant C^{\# S(\mathfrak{a})}\, \mathrm{N}(\mathfrak{a})\, \mathrm{N}(\mathfrak{n})^{\epsilon} \sum_{I \subset S(\mathfrak{a})} \mathfrak{I}_0^{\eta}(l, \mathfrak{n}, \mathfrak{f}_\eta \prod_{v \in I} \mathfrak{p}_v^{n_v}). \end{aligned}$$

Then the desired estimate is obtained by Proposition 6.4. ∎

Lemma 9.11 *We have* $|\mathbb{W}(S(\mathfrak{f}_\eta))| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\, \mathrm{N}(\mathfrak{n})^{-c+\epsilon}$.

Proof By the same argument as in the proof of Lemma 9.9, we have

$$\begin{aligned} \sum_{w \in S(\mathfrak{f}_\eta)} \{2f(\eta_w) + \mathrm{ord}_w(b(b+1)) + 1\} \log q_w &\leqslant \log \mathrm{N}\big(b(b+1)\mathfrak{a}^2\mathfrak{f}_\eta^2\big) + \log \mathrm{N}(\mathfrak{f}_\eta) \\ &\ll_{\epsilon,\eta} |\mathrm{N}(b(b+1))|^{\epsilon}\, \mathrm{N}(\mathfrak{a})^{2\epsilon} \end{aligned}$$

for $b \in \mathfrak{n}\mathfrak{a}^{-1}\mathfrak{f}_\eta^{-1} - \{0,-1\}$. From the estimations recalled as above,

$$\begin{aligned} |\mathbb{W}(S(\mathfrak{f}_\eta))| &\leqslant \sum_{b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} \sum_{w_3 \in S(\mathfrak{f}_\eta)} \{ \prod_{v \in \Sigma_F - \{w_3\}} |J_v(b)| \} \\ &\qquad \times (\log q_{w_3})\{2f(\eta_{w_3}) + \mathrm{ord}_{w_3}(b(b+1)) + 1\} \\ &\ll_{\epsilon,l,\eta} C^{\# S(\mathfrak{a})}\, \mathrm{N}(\mathfrak{a}) \sum_{b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} |\mathrm{N}(b(b+1))|^{\epsilon}\, \mathrm{N}(\mathfrak{a})^{2\epsilon} \\ &\qquad \times \prod_{v \in \Sigma_\infty} |J_v(b)| \prod_{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{a}\mathfrak{f}_\eta)} \Lambda_v(b) \prod_{v \in S(\mathfrak{a})} \{1 + \Lambda_v(b)\} \\ &\ll_{\epsilon,l,\eta} C^{\# S(\mathfrak{a})}\, \mathrm{N}(\mathfrak{a})^{1+2\epsilon} \sum_{I \subset S(\mathfrak{a})} \mathfrak{I}_\epsilon^{\eta}(l, \mathfrak{n}, \mathfrak{f}_\eta \prod_{v \in I} \mathfrak{p}_v^{n_v}). \end{aligned}$$

Then the desired estimate is obtained by Proposition 6.4. ∎

Lemma 9.12 *We have* $|\mathbb{W}(\Sigma_{\mathrm{fin}} - S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta))| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\, \mathrm{N}(\mathfrak{n})^{-c+\epsilon}$.

Proof In the summation on the left-hand side of (9.3) with $T = \Sigma_{\mathrm{fin}} - S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)$, the range of (b, w) is restricted to $b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1} - \{0,-1\}$ and $w \in S(b(b+1)\mathfrak{o} \cap \mathfrak{o}) \cap T$, due to the estimations recalled above. Thus,

$$\begin{aligned} &|\mathbb{W}(\Sigma_{\mathrm{fin}} - S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta))| \\ &\quad \leqslant \sum_{b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} \sum_{w_4 \in S(b(b+1)\mathfrak{o}\cap\mathfrak{o}) - S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)} \Big\{ \prod_{v \in \Sigma_F - \{w_4\}} |J_v(b)| \Big\} |W_{w_4}(b)| \\ &\quad \ll_{\epsilon,\eta} C^{\# S(\mathfrak{a})}\, \mathrm{N}(\mathfrak{a}) \sum_{b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} \prod_{v \in \Sigma_\infty} |J_v(b)| \prod_{v \in S(\mathfrak{a})} \big\{1 + \Lambda_v(b)\big\} \\ &\qquad \times \sum_{w_4 \in S(b(b+1)\mathfrak{o}\cap\mathfrak{o}) - S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)} \Big\{ \prod_{\substack{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{a}\mathfrak{f}_\eta) \\ v \neq w_4}} \Lambda_v(b) \Big\} (\log q_{w_4}) \Lambda_{w_4}(b)^2 \end{aligned}$$

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$$
\begin{aligned}
&\ll_{\epsilon,\eta} C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a}) \sum_{b\in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} \prod_{v\in\Sigma_\infty} |J_v(b)| \prod_{v\in S(\mathfrak{a})} \{1+\Lambda_v(b)\}\\
&\quad\times\Big\{\sum_{w_4\in S(b(b+1)\mathfrak{o}\cap\mathfrak{o})-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)} \log q_{w_4}\Big\} \prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{f}_\eta)} \Lambda_v(b)^2\\
&\ll_{\epsilon,\eta} C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a}) \sum_{b\in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} \prod_{v\in\Sigma_\infty} |J_v(b)| \prod_{v\in S(\mathfrak{a})} \{1+\Lambda_v(b)\}\\
&\quad\times \prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{f}_\eta)} \Lambda_v(b)^2\times \mathrm{N}(\mathfrak{a})^{2\epsilon}|\mathrm{N}(b(b+1))|^{\epsilon}\\
&= C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a})^{1+2\epsilon} \sum_{I\subset S(\mathfrak{a})} \mathfrak{I}_\epsilon^\eta(l,\mathfrak{n},\mathfrak{f}_\eta\prod_{v\in I}\mathfrak{p}_v^{n_v}).
\end{aligned}
$$

Here we note

$$
\sum_{w_4\in S(b(b+1)\mathfrak{o}\cap\mathfrak{o})-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)} \log q_{w_4} \ll_{\epsilon,\eta} \mathrm{N}(\mathfrak{a})^{2\epsilon}|\mathrm{N}(b(b+1))|^{\epsilon},\quad b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}.
$$

Indeed, if $b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}$, we have $b(b+1)\mathfrak{f}_\eta^2\mathfrak{a}^2\subset\mathfrak{o}$ and

$$
S(b(b+1)\mathfrak{o}\cap\mathfrak{o})-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)\subset S(b(b+1)\mathfrak{f}_\eta^2\mathfrak{a}^2).
$$

Hence,

$$
\begin{aligned}
\sum_{w_4\in S(b(b+1)\mathfrak{o}\cap\mathfrak{o})-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)} \log q_{w_4} &\leqslant \sum_{w_4\in S(b(b+1)\mathfrak{f}_\eta^2\mathfrak{a}^2)} \log q_{w_4}\leqslant \log\mathrm{N}(b(b+1)\mathfrak{f}_\eta^2\mathfrak{a}^2)\\
&\ll_\epsilon |\mathrm{N}(b(b+1))\,\mathrm{N}(\mathfrak{f}_\eta)^2\,\mathrm{N}(\mathfrak{a})^2|^{\epsilon},
\end{aligned}
$$

for $b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}$. Therefore, the assertion follows from Proposition 6.4 and from $C^{\#S(\mathfrak{a})}\ll_\epsilon \mathrm{N}(\mathfrak{a})^{\epsilon}$. ■

9.5 Unipotent Terms

It is seen from (4.10) and (4.11) that $\widetilde{\mathbb{W}}_{\mathfrak{u}}^{\eta}(l,\mathfrak{n}|\alpha)$ is a linear combination of

$$
\prod_{v\in S} U_v^{\eta_v}(\alpha_v)\quad\text{and}\quad \widetilde{U}_w^{\eta_w}(\alpha_w)\prod_{v\in S-\{w\}} U_v^{\eta_v}(\alpha_v)\quad (w\in S),
$$

where $U_v^{\eta_v}(\alpha_v)$ is defined in (6.3) and

(9.8) $$\widetilde{U}_v^{\eta_v}(\alpha_v)=\frac{1}{2\pi i}\int_{\sigma-2\pi i(\log q_v)^{-1}}^{\sigma+2\pi i(\log q_v)^{-1}} \frac{\eta_v(\varpi_v)\log q_v}{(1-\eta_v(\varpi_v)q_v^{-(s+1)/2})^2(1-q_v^{-(s+1)/2})q_v^{s+1}}\,\alpha_v(s)\,d\mu_v(s)$$

with $d\mu_v(s)=2^{-1}(\log q_v)(q_v^{(1+s)/2}-q_v^{(1-s)/2})ds$ and $\sigma>0$.

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Lemma 9.13 *For any $m \in \mathbb{N}_0$, we have*

$$U_v^{\eta_v}(\alpha_v^{(m)}) = -2\,\delta(m=0) + q_v^{1-m/2} \times \begin{cases} \delta(m \in 2\mathbb{N})(1-q_v^{-1}), & \eta_v(\varpi_v) = -1, \\ \delta(m>0)\{(m-1)-(m+1)q_v^{-1}\}, & \eta_v(\varpi_v) = 1, \end{cases}$$

$$\widetilde{U}_v^{\eta_v}(\alpha_v^{(m)}) = -\delta(m>0)q_v^{-m/2}(\log q_v) \times \begin{cases} \left\{\frac{q_v-1}{2}m(-1)^m - \frac{3q_v+1}{4}(-1)^m + \frac{1-q_v}{4}\right\}, & \eta_v(\varpi_v) = -1, \\ \left\{\frac{(m-1)(m-2)}{2}q_v - \frac{m(m+1)}{2}\right\}, & \eta_v(\varpi_v) = +1. \end{cases}$$

Proof The first formula is proved in [11, Proposition 11.1]. The second formula is shown similarly. ∎

Appendix A An Estimation of a Certain Lattice Sum

Let $d \geqslant 1$ be an integer. We fix $l = (l_j)_{1 \leqslant j \leqslant d} \in \mathbb{R}^d$ such that $l_d \geqslant \cdots \geqslant l_1 \geqslant 4$, and consider a positive function $f(x)$ on $\mathbb{R}^d$ defined by

$$f(x) = \prod_{j=1}^{d}(1+|x_j|)^{-l_j/2}, \quad x = (x_j)_{1\leqslant j\leqslant d} \in \mathbb{R}^d.$$

Given a $\mathbb{Z}$-lattice $\Lambda \subset \mathbb{R}^d$ (of full rank), we define

$$\theta(\Lambda) = \sum_{b \in \Lambda - \{0\}} f(b), \qquad r(\Lambda) = \frac{1}{2}\min_{b\in\Lambda-\{0\}} \|b\|.$$

Viewing $\theta(\Lambda)$ as a function in Λ, we need to compare its asymptotic size with a certain power of $D(\Lambda)$, the Euclidean volume of a fundamental domain of $\mathbb{R}^d/\Lambda$. The following is the main result of this section.

Theorem A.1 *Let F be a totally real number field of degree d. Let Λ_0 and Λ be fractional ideals such that $\Lambda \subset \Lambda_0$; we regard them as $\mathbb{Z}$-lattices in $\mathbb{R}^d$ by the embedding $F \to \mathbb{R}^{\mathrm{Hom}(F,\mathbb{R})} \cong \mathbb{R}^d$. Then, $\theta(\Lambda) \ll \{1 + r(\Lambda_0)\}^{dl_d/2} D(\Lambda_0)^{-1} D(\Lambda)^{(1-l_1/2)/d}$ with the implied constant independent of Λ and Λ_0.*

A.1 Proof of Theorem A.1

Let $d\mu(\omega)$ denote the Euclidean measure on the sphere $\mathbb{S}^{d-1} = \{x = (x_j)_{1\leqslant j \leqslant d} \in \mathbb{R}^d \mid \sum_{j=1}^d x_j^2 = 1\}$.

Lemma A.2 *For any $\lambda = (\lambda_j) \in \mathbb{C}^d$ such that $\mathrm{Re}(\lambda_j) < 1$, we have*

$$I(\lambda) = \int_{\mathbb{S}^{d-1}} \prod_{j=1}^{d} |\omega_j|^{-\lambda_j} d\mu(\omega) = 2\Gamma\Big(\sum_{j=1}^{d} \tfrac{1-\lambda_j}{2}\Big)^{-1} \prod_{j=1}^{d} \Gamma\Big(\frac{1-\lambda_j}{2}\Big).$$

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Proof The formula is obtained by computing the integral

$$\int_{\mathbb{R}^d} \exp(-\epsilon\|x\|^2) \prod_{j=1}^{d} |x_j|^{-\lambda_j}\, dx \tag{A.1}$$

in two different ways, where $\epsilon > 0$ and $\mathrm{Re}(\lambda_j) < 1$. By expressing (A.1) as an iterating integral, we compute it as

$$\prod_{j=1}^{d} \int_{\mathbb{R}} e^{-\epsilon x_j^2} |x_j|^{-\lambda_j}\, dx_j = \prod_{j=1}^{d} \epsilon^{(\lambda_j-1)/2} \Gamma\Big(\frac{1-\lambda_j}{2}\Big) = \epsilon^{(\sum_{j=1}^{d}\lambda_j - d)/2} \prod_{j=1}^{d} \Gamma\Big(\frac{1-\lambda_j}{2}\Big),$$

on one hand. On the other hand, by the polar decomposition, (A.1) becomes

$$\begin{aligned}
&\int_0^{\infty} \int_{\mathbb{S}^{d-1}} e^{-\epsilon\rho^2} \prod_{j=1}^{d} |\rho\omega_j|^{-\lambda_j} \rho^{d-1} d\rho\, d\mu(\omega) \\
&= \Big(\int_{\mathbb{S}^{d-1}} \prod_{j=1}^{d} |\omega_j|^{-\lambda_j}\, d\mu(\omega)\Big)\Big(\int_0^{\infty} e^{-\epsilon\rho^2} \rho^{-\sum_{j=1}^{d}\lambda_j + d-1}\, d\rho\Big) \\
&= I(\lambda) 2^{-1} \epsilon^{(\sum_{j=1}^{d}\lambda_j - d)/2} \Gamma\Big(\sum_{j=1}^{d} \frac{1-\lambda_j}{2}\Big).
\end{aligned}$$

■

Lemma A.3 *For $t = (t_j)_{1\leqslant j\leqslant d} \in [1,\infty)^d$, set*

$$\varphi(t_1,\dots,t_d) = \int_{\mathbb{S}^{d-1}} f(t_1\omega_1,\dots,t_d\omega_d)\, d\mu(\omega).$$

For $t > 1$, let $\underline{t}$ denote the diagonal element (t_j) defined by $t_j = t$ $(1 \leqslant j \leqslant d)$. Then

$$\varphi(\underline{t}) = \mathcal{O}(t^{1-d-l_1/2}), \qquad t \in [1,\infty).$$

Proof For $\lambda = (\lambda_j) \in \mathbb{C}^d$ such that $0 < \mathrm{Re}(\lambda_j) < 1$, we compute the multiple Mellin transform $\widetilde{\varphi}(\lambda) = \int_0^{\infty} \cdots \int_0^{\infty} \varphi(t_1,\dots,t_d) \prod_{j=1}^{d} t_j^{\lambda_j} \frac{dt_j}{t_j}$. By Lemma A.2,

$$\begin{aligned}
\widetilde{\varphi}(\lambda) &= \int_{\mathbb{S}^{d-1}} \Big\{\prod_{j=1}^{d} \int_0^{\infty} (1 + t_j|\omega_j|)^{-l_j/2} t_j^{\lambda_j-1}\, dt_j\Big\}\, d\mu(\omega) \\
&= \int_{\mathbb{S}^{d-1}} \Big\{\prod_{j=1}^{d} |\omega_j|^{-\lambda_j} \int_0^{\infty} (1+t_j)^{-l_j/2} t_j^{\lambda_j-1}\, dt_j\Big\}\, d\mu(\omega) \\
&= \Big(\int_{\mathbb{S}^{d-1}} \prod_{j=1}^{d} |\omega_j|^{-\lambda_j}\, d\mu(\omega)\Big)\Big(\prod_{j=1}^{d} \int_0^{\infty} (1+t_j)^{-l_j/2} t_j^{\lambda_j-1} dt_j\Big)
\end{aligned}$$

$$= I(\lambda)\Big\{\prod_{j=1}^{d}\Gamma(l_j/2)^{-1}\Gamma(l_j/2-\lambda_j)\Gamma(\lambda_j)\Big\}$$

$$= 2\Gamma\Big(\sum_{j=1}^{d}\frac{1-\lambda_j}{2}\Big)^{-1}\Big\{\prod_{j=1}^{d}\Gamma(l_j/2)^{-1}\Gamma((1-\lambda_j)/2)\Gamma(l_j/2-\lambda_j)\Gamma(\lambda_j)\Big\}.$$

By Stirling's formula, this is bounded by a constant multiple of

$$P(\operatorname{Im}\lambda)\exp(-\pi\sum_{j=1}^{d}|\operatorname{Im}(\lambda_j)|)$$

with some polynomial $P(x_1,\dots,x_d)$ which can be taken uniformly with $\operatorname{Re}(\lambda)$ varied compactly. Thus, by a successive application of the Mellin inversion formula, we obtain

$$\varphi(\underline{t}) =$$

$$\Big(\frac{1}{2\pi i}\Big)^d\int_{L_{\sigma_1}}\cdots\int_{L_{\sigma_d}} 2\Big\{\prod_{j=1}^{d}\frac{\Gamma\big(\frac{1-\lambda_j}{2}\big)\Gamma\big(\frac{l_j}{2}-\lambda_j\big)\Gamma(\lambda_j)}{\Gamma(l_j/2)}\Big\}\frac{t^{-\sum_{j=1}^{d}\lambda_j}}{\Gamma\big(\sum_{j=1}^{d}\frac{1-\lambda_j}{2}\big)}\prod_{j=1}^{d}d\lambda_j,$$

where $0<\sigma_j<1$ for all $1\leqslant j\leqslant d$. We shift the contours $L_{\sigma_j}=\{\operatorname{Re}(\lambda)=\sigma_j\}$ in some order far to the right. The residues arise when the moving contour L_{σ_j} passes the points in $(1+2\mathbb{Z}_{\geqslant 0})\cup(l_j/2+\mathbb{Z}_{\geqslant 0})$. Among those residues, the one with the smallest possible power of t^{-1} comes from the pole at $\lambda_1=l_1/2$, $\lambda_j=1(2\leqslant j\leqslant d)$ if $l_2>l_1$, which we assume for simplicity in the rest of the proof of this lemma. (When $l_2=l_1$, there are several terms giving the same power in t^{-1}.) The residue term is $\mathcal{O}(t^{-(d-1+l_2/2)})$, by which the contribution from the remaining terms are majorized. This completes the proof. ■

Lemma A.4

(i) $f(x+y)\geqslant f(x)f(y)$ *for all* $x,y\in\mathbb{R}^d$.
(ii) $\operatorname{vol}(\mathbb{S}^{d-1})(1+\rho)^{-dl_d/2}\leqslant\int_{\mathbb{S}^{d-1}}f(\rho\omega)\,d\mu(\omega)\ll(1+\rho)^{1-d-l_1/2}$, $\rho>0$, *with the implied constant depending on* l *and* d.

Proof (i) is immediate from the inequality $1+|x_j+y_j|\leqslant(1+|x_j|)(1+|y_j|)$. As for (ii), we first note the inequality $0\leqslant|\omega_j|\leqslant 1$ for $\omega\in\mathbb{S}^{d-1}$. Using this, we have $\prod_{j=1}^{d}(1+|\rho\omega_j|)\leqslant(1+\rho)^d$. By this,

$$f(\rho\omega)\geqslant\Big\{\prod_{j=1}^{d}(1+|\rho\omega_j|)\Big\}^{-l_d/2}\geqslant(1+\rho)^{-dl_d/2}.$$

Taking the integral in ω, we have the estimation from below as desired. The upper bound is provided by Lemma A.3. ■

We compare $\theta(\Lambda)$ with the integral of $f(x)$ on the ball $B_\Lambda=\{x\in\mathbb{R}^d\,|\,\|x\|<r(\Lambda)\}$. For convenience, we set $I(D)=\int_D f(x)\,dx$ for any Borel set D in $\mathbb{R}^d$.

Lemma A.5 *Let* Λ_0 *and* Λ *be* $\mathbb{Z}$*-lattices such that* $\Lambda\subset\Lambda_0$. *Then we have the inequality* $\theta(\Lambda)\leqslant I(B_{\Lambda_0})^{-1}I(\mathbb{R}^d-B_\Lambda)$.

Proof Lemma A.4 (i) gives us $I(B_\Lambda)\theta(\Lambda) \leqslant \sum_{b\in\Lambda-\{0\}} \int_{B_\Lambda} f(b+x)\,dx$. Since $\Lambda \subset \Lambda_0$, we have $B_{\Lambda_0} \subset B_\Lambda$, from which $I(B_{\Lambda_0}) \leqslant I(B_\Lambda)$ is obtained by the non-negativity of $f(x)$. Since $(B_\Lambda + B_\Lambda) \cap \Lambda = \{0\}$, the translated sets $B_\Lambda + b (b \in \Lambda - \{0\})$ are mutually disjoint. From this remark

$$\sum_{b\in\Lambda-\{0\}} \int_{B_\Lambda} f(b+x)\,dx \leqslant \int_{\mathbb{R}^d - B_\Lambda} f(x)\,dx = I(\mathbb{R}^d - B_\Lambda). \qquad \blacksquare$$

Lemma A.6 *Let Λ be a $\mathbb{Z}$-lattice. Then $I(B_\Lambda) \geqslant \mathrm{vol}(\mathbb{S}^{d-1})(1+r(\Lambda))^{-dl_d/2} r(\Lambda)^d/d$. We also have $I(\mathbb{R}^d - B_\Lambda) \ll r(\Lambda)^{1-l_1/2}$ with the implied constant independent of Λ.*

Proof By Lemma A.4 (ii),

$$\begin{aligned}
\mathrm{I}(B_\Lambda) &= \int_0^{r(\Lambda)} \int_{\mathbb{S}^{d-1}} f(\rho\omega)\,d\omega \rho^{d-1}\,d\rho \\
&\geqslant \mathrm{vol}(\mathbb{S}^{d-1}) \int_0^{r(\Lambda)} (1+\rho)^{-dl_d/2} \rho^{d-1}\,d\rho \\
&\geqslant \mathrm{vol}(\mathbb{S}^{d-1})(1+r(\Lambda))^{-dl_d/2} \int_0^{r(\Lambda)} \rho^{d-1}\,d\rho \\
&= \mathrm{vol}(\mathbb{S}^{d-1})(1+r(\Lambda))^{-dl_d/2} r(\Lambda)^d/d, \\
\mathrm{I}(\mathbb{R}^d - B_\Lambda) &= \int_{r(\Lambda)}^\infty \int_{\mathbb{S}^{d-1}} f(\rho\omega) d\omega \rho^{d-1} d\rho \\
&\ll \int_{r(\Lambda)}^\infty (1+\rho)^{1-d-l_1/2} \rho^{d-1}\,d\rho \\
&\leqslant \int_{r(\Lambda)}^\infty \rho^{-l_1/2}\,d\rho = (l_1/2-1)^{-1} r(\Lambda)^{1-l_1/2}. \qquad \blacksquare
\end{aligned}$$

Lemma A.7 *Let F be a totally real number field of degree d. There exist positive constants C_d and C'_d such that $C_d r(\Lambda)^d \leqslant D(\Lambda) \leqslant C'_d r(\Lambda)^d$ for any fractional ideal Λ.*

Proof The first inequality follows from Minkowski's convex body theorem. The second inequality is proved as follows. For any $b \in \Lambda - \{0\}$, there exists an ideal $\mathfrak{a} \subset \mathfrak{o}$ such that $(b) = \mathfrak{a}\Lambda$; hence $|\mathrm{N}(b)| = \mathrm{N}(\Lambda)\,\mathrm{N}(\mathfrak{a}) \geqslant \mathrm{N}(\Lambda)$. Thus, by the arithmetic-geometric mean inequality,

$$D(\Lambda)^{1/d} = \mathrm{N}(\Lambda)^{1/d} \leqslant \Big\{ \prod_{j=1}^d |b_j|^2 \Big\}^{1/(2d)} \leqslant \Big\{ \sum_{j=1}^d |b_j|^2/d \Big\}^{1/2} = d^{-1/2}\|b\|.$$

Hence, $D(\Lambda)^{1/d} \leqslant 2d^{-1/2} r(\Lambda)$. This shows $D(\Lambda) \leqslant C'_d\, r(\Lambda)^d$ with $C'_d = (2d^{-1/2})^d$. ■

Theorem A.1 follows from Lemmas A.5, A.6 and A.7 immediately.

References

[1] B. H. Gross, and D. B. Zagier, *Heegner points and derivatives of L-series*. Inventiones Math. **84**(1986), 225–320. http://dx.doi.org/10.1007/BF01388809

[2] H. Jacquet, and N. Chen, *Positivity of quadratic base change L-functions*. Bull. Soc. Math. France **129**(2001), no. 1, 33–90.

[3] H. Jacquet, and L. P. Langlands, *Automorphic forms on* $\mathrm{GL}(2)$. Lecture Notes in Mathematics, 114, Springer-Verlag, Berlin, 1970.

[4] W. Magnus, F. Oberhettinger, and R. Soni, *Formulas and theorems for the special functions of mathematical physics*. In: Grundlehren der mathematischen Wissenschafter, 52, 3rd edition. Springer-Verlag, NewYork, 1966.

[5] A. Raghuram and N. Tanabe, *Notes of the arithmetic of Hilbert modular forms*. J. Ramanujan Math. Soc. **26**(2011), no. 3, 261–319.

[6] E. Royer, *Facteurs* $\mathbb{Q}$*-simples de* $J_0(N)$ *de grande dimension et de grand rang*. Bull. Soc. Math. France **128**(2000), no. 2, 219–248.

[7] J.P. Serre, *Répartition asymptotique des valeurs propres de l'operateur de Hecke* T_p. J. Amer. Math. Soc. **10**(1997), no. 1, 75–102. http://dx.doi.org/10.1090/S0894-0347-97-00220-8

[8] G. Shimura, *The special values of the zeta functions associated with Hilbert modular forms*. Duke Math. J. **45**(1978), no.3, 637–679. http://dx.doi.org/10.1215/S0012-7094-78-04529-5

[9] S. Sugiyama, *Regularized periods of automorphic forms on GL(2)*. Tohoku Math. J. **65**(2013), no. 3, 73–409. http://dx.doi.org/10.2748/tmj/1378991022

[10] ———, *Asymptotic behaviors of means of central values of automorphic L-functions for GL(2)*. J. Number Theory, **156**(2015), 195–246. http://dx.doi.org/10.1016/j.jnt.2015.04.003

[11] S. Sugiyama, and M. Tsuzuki, *Relative trace formulas and subconvexity estimates of L-functions for Hilbert modular forms*. (preprint 2013). arxiv:http://arxiv.org/abs/1305.2261

[12] M. Tsuzuki, *Spectral means of central values of automorphic L-functions for* $GL(2)$. Mem. Amer. Math. Soc. **235**(1110), 2015. http://dx.doi.org/10.1090/memo/1110

[13] J. L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symmétrie*. Compositio Math. **54**(1985), 173–242.

[14] X. Yuan, S. W. Zhang, and W. Zhang, *The Gross-Zagier formula on Shimura curves*. Annals of Mathematics Studies, 184, Princeton University Press, Princeton, NJ, 2013.

[15] S. W. Zhang, *Heights of Heegner cycles and derivatives of L-series*. Invent. Math. **130**(1997), 99–152. http://dx.doi.org/10.1007/s002220050179

[16] ———, *Gross-Zagier formula for* $\mathrm{GL}(2)$. Asian J. Math. **5**(2001), 183–290. http://dx.doi.org/10.4310/AJM.2001.v5.n2.a1

[17] ———, *Gross-Zagier formula for* $\mathrm{GL}(2)$. *II*. In: Heegner points and Rankin L-series. Math. Sci. Res. Inst. Publ. 49, Cambridge University Press, Cambridge, 2004, pp. 191–214. http://dx.doi.org/10.1017/CBO9780511756375.008

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