REMARKS ON THE COMMUTATIVITY OF THE RADICALS OF GROUP ALGEBRAS

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(Received 15 August, 1977)

Let K be an arbitrary field with characteristic $p>0$, G a finite group of order $p^a g'$ with $(p, g')=1$, P a p-Sylow subgroup of G and G' the commutator subgroup of G. For a ring R denote by $J(R)$ the Jacobson radical of R and by $Z(R)$ the centre of R. We write KG for the group algebra of G over K.

On the commutativity of $J(KG)$ there are works of D. A. R. Wallace [**11**] and W. Hamernik [**3**]. The first aim of this paper is to investigate the structure of G when $J(KG)$ is commutative. Our result can be stated as follows: if $J(KG)$ is commutative and $J(KG)^2 \neq 0$, then $N_G(P)=C_G(P)$ and $N_G(P)$ is abelian, where $N_G(P)$ and $C_G(P)$ are the normalizer of P in G and the centralizer of P in G, respectively.

D. A. R. Wallace [**11**] and W. Hamernik [**3**] obtained a necessary and sufficient condition on G for $J(KG)$ to be commutative when p is odd. Indeed, they proved that when p is odd and when G is a nonabelian group of order divisible by p, $J(KG)$ is commutative if and only if $G'P$ is a Frobenius group with complement P with kernel G'. So in the present paper we shall obtain a necessary and sufficient condition of G for $J(KG)$ to be commutative for any prime number p not necessarily odd. That is to say, we shall prove that $J(KG)$ is commutative if and only if G is a group of the following two types: (*i*) $|G|$ is not divisible by 2^2 when $p=2$, and $|G|$ is not divisible by p when p is odd; (*ii*) G is a p-nilpotent group with an abelian p-Sylow subgroup P, $b_0=|O_{p'}(G):G'|$, $b_1=\ldots=b_{a-2}=0$, and if p is odd, $b_{a-1}=0$, where b_k is the number of p-regular conjugate classes K_j of G such that the number of elements of K_j is divisible by p^k and not by p^{k+1} for $k=0,\ldots,a$. By [**11**, Theorem 1] and [**3**, Corollary 5.2], when p is odd, $J(KG)$ is commutative if and only if $J(KG)\subseteq Z(KG)$. But when $p=2$, this does not hold in general.

Throughout this paper we shall use the following notations. Denote by $[V:K]$ the K-dimension of a K-vector space V. If S is a subset of G, $|S|$ will denote the number of elements of S, $N_G(S)$ and $C_G(S)$ will denote the normalizer of S in G and the centralizer of S in G, respectively, and let $\hat{S}=\sum_{s\in S} s$ in KG when $S\neq\varnothing$ and let $\hat{S}=0$ in KG when $S=\varnothing$. For a positive integer t and a ring R we write R_t for the ring of all $t\times t$ matrices with entries in R.

To begin with we shall study G when $J(KG)$ is commutative.

THEOREM 1. *Suppose that $|G|$ is divisible by 2^2 when $p=2$ and that $|G|$ is divisible by p when p is odd. If $J(KG)$ is commutative, then $N_G(P)=C_G(P)$ and $N_G(P)$ is abelian, where P is a p-Sylow subgroup of G.*

Proof. Since $KG/J(KG)$ is a separable K-algebra (cf. [**4**, Proposition 12.11]), $J(EG)=E\otimes_K J(KG)$ for any extension field E of K. So we may assume that K is algebraically

https://doi.org/10.1017/S0017089500003736 Published online by Cambridge University Press

closed. Let $1=\sum_{i=1}^{n}\sum_{j=1}^{f_i} e_{ij}$ be a decomposition of the unit element of KG into a sum of mutually orthogonal primitive idempotents of KG such that $KGe_{ij}\cong KGe_{i'j'}$ if and only if $i=i'$. Set $e_i=e_{i1}$, $U_i=KGe_i$, $F_i=U_i/J(KG)U_i$ and $u_i=[U_i:K]$, and hence $f_i=[F_i:K]$ for each i. Let F_1 be the trivial KG-module, and so $f_1=1$.

By [**11**, Theorem 2], G is p-nilpotent and P is abelian. So KG is primary decomposable from [**6**, Theorem 1]. Thus each block of KG contains, up to isomorphism, only one irreducible KG-module. Put $B_i=\sum_{j=1}^{f_i}\oplus KGe_{ij}$ for each i. $B_1,\ldots,B_n$ are all blocks of KG. Set $H=O_{p'}(G)$, the largest normal subgroup of G of order prime to p. Since G is p-nilpotent, it follows from [**5**, Theorems 2, 7] that

$$B_i\cong KHe'_{i1}\otimes_K K^cP_i\otimes_K K_{t_i}, \text{ as } K\text{-algebras},$$

where e'_{i1} is a centrally primitive idempotent of KH, $G_i=\{x\in G\mid x^{-1}e'_{i1}x=e'_{i1}\}$, $t_i=|G:G_i|$, P_i is a p-Sylow subgroup of G_i and K^cP_i is a twisted group ring of P_i over K with respect to the factor set c for each $i=1,\ldots,n$. Since K is an algebraically closed field with characteristic $p>0$ and P_i is a p-group, $K^cP_i\cong KP_i$ as K-algebras for each i (cf. [**7**, Lemma 2.1]). Hence

$$B_i\cong KHe'_{i1}\otimes_K KP_i\otimes_K K_{t_i}, \text{ as } K\text{-algebras}.$$

Put $h_i^2=[KHe'_{i1}:K]$ and $h_i>0$. By [**6**, Theorem 3], $f_i=h_it_i$. This shows that $B_i\cong(KP_i)_{f_i}$ as K-algebras and that

$$J(B_i)\cong(J(KP_i))_{f_i} \tag{*}$$

for each i. Now, let us divide $B_1,\ldots,B_n$ into the following three types:

(*a*) $J(B_i)=0$.
(*b*) $J(B_i)\neq 0$, $J(B_i)^2=0$.
(*c*) $J(B_i)^2\neq 0$.

When B_i is of type (*a*) or (*b*), f_i is divisible by p. Indeed, if B_i is of type (*a*), $u_i=f_i$ and so p^a divides f_i from [**1**, (18)]. If B_i is of type (*b*), by [**11**, Lemma 7], $p=2$ and $u_i=2f_i$, and so f_i is divisible by 2 since $a\geqq 2$ and 2^a divides u_i from [**1**, (18)]. Hence the principal block B_1 is of type (*c*). By rearranging the numbers $2,\ldots,n$, we may assume that $B_1,\ldots,B_m$ are of type (*c*) and that $B_{m+1},\ldots,B_n$ are of type (*a*) or (*b*) for some $m\leqq n$. If B_i is of type (c), since $J(KG)$ is commutative, it follows from (*) that $f_i=1$ and so $h_i=t_i=1$. This implies that $B_1,\ldots,B_m$ are all blocks of KG with defect a.

Next, since P is an abelian p-Sylow subgroup of G and G is p-nilpotent, by [**5**, §3 (p. 184)], $N_G(P)=C_G(P)$. Set $N=N_G(P)$ and $\tilde{H}=H\cap N=O_{p'}(N)$. Since N is p-nilpotent, it follows from [**6**, Theorem 1] and [**5**, Lemma 2] that m is equal to the number of blocks of KN. Let $1=\sum_{i=1}^{m}\sum_{j=1}^{\tilde{f}_i}\tilde{e}_{ij}$ be a decomposition of the unit element of KN into a sum of mutually orthogonal primitive idempotents of KN such that $KN\tilde{e}_{ij}\cong KN\tilde{e}_{i'j'}$ if and only if $i=i'$. Put $\tilde{e}_i=\tilde{e}_{i1}$, $\tilde{U}_i=KN\tilde{e}_i$, $\tilde{F}_i=\tilde{U}_i/J(KN)\tilde{U}_i$ and $\tilde{u}_i=[\tilde{U}_i:K]$, and hence

$\tilde{f}_i=[\tilde{F}_i:K]$ for each i. Set $\tilde{B}_i=\sum_{j=1}^{\tilde{f}_i}\oplus KN\tilde{e}_{ij}$ for each i, and so $\tilde{B}_1,\dots,\tilde{B}_m$ are all blocks of KN. Since N is p-nilpotent, as for B_i, we can write

$$\tilde{B}_i\cong K\tilde{H}\tilde{e}'_{i1}\otimes_K K\tilde{P}_i\otimes_K K_{\tilde{t}_i},\text{ as } K\text{-algebras,}$$

where $\tilde{e}'_{i1}$ is a centrally primitive idempotent of $K\tilde{H}$, $\tilde{G}_i=\{y\in N\mid y^{-1}\tilde{e}'_{i1}y=\tilde{e}'_{i1}\}$, $\tilde{t}_i=|N:\tilde{G}_i|$ and $\tilde{P}_i$ is a p-Sylow subgroup of $\tilde{G}_i$ for each $i=1,\dots,m$. Since P is normal in N, all blocks of KN have defect a. Put $\tilde{h}_i^2=[K\tilde{H}\tilde{e}'_{i1}:K]$ and $\tilde{h}_i>0$. By [**6**, Theorem 3], $\tilde{f}_i=\tilde{h}_i\tilde{t}_i$. Hence $\tilde{t}_i$ is not divisible by p and this shows that $\tilde{t}_i=1$ for all i. This implies that $\tilde{e}'_{i1}$ is a centrally primitive idempotent of KN and that $\tilde{P}_i=P$ for all i.

By rearranging the numbers $1,\dots,m$, we can assume that B_i corresponds to $\tilde{B}_i$ through the Brauer homomorphism for each $i=1,\dots,m$ (cf. [**2**, Lemma 56.1, Theorem 58.3 (Brauer's first main theorem)]). Fix any i such that $1\leqq i\leqq m$. Since $t_i=1$, e'_{i1} is a centrally primitive idempotent of KG, and so we may write $e'_{i1}=e_i$ since $f_i=1$. Put $B=B_i$, $e=e'_{i1}=e_i$, $\tilde{B}=\tilde{B}_i$ and $\tilde{e}=\tilde{e}'_{i1}$. Let $\{K_r\}$ be the set of all conjugate classes of G. The Brauer homomorphism $\sigma:Z(KG)\to Z(KN)$ is defined as $\sigma(\widehat{K_r})=\widehat{K_r\cap N}$ for each r. We know that $\sigma(e)=\tilde{e}$. e is a centrally primitive idempotent of KH and $\tilde{e}$ is a centrally primitive idempotent of $K\tilde{H}$. Thus, if we let $\{L_t\}$ be the set of all conjugate classes of H and if we define $\sigma':Z(KH)\to Z(K\tilde{H})$ as $\sigma'(\widehat{L_t})=\widehat{L_t\cap N}$ for each t, it follows that $\sigma'(e)=\tilde{e}$. On the other hand, $[KHe:K]=1$, and so $KHe=Ke$. Take any $h\in\tilde{H}$. We shall claim that $h\tilde{e}\in K\tilde{e}$. Since $KHe=Ke$, we can write $he=\delta e$ for some $\delta\in K$. Let $e=\sum_t\alpha_t\,.\,\widehat{L_t}$, where $\alpha_t\in K$. Thus, $\tilde{e}=\sigma'(e)=\sum_t\alpha_t\,.\,(\widehat{L_t\cap N})$. Since $\sum_t\alpha_t\,.\,h\widehat{L_t}=\sum_t\delta\alpha_t\,.\,\widehat{L_t}$, it is seen that

$$\sum_t\alpha_t\,.\,h(\widehat{L_t\cap N})+\sum_t\alpha_t\,.\,h(\widehat{L_t\backslash N})=\sum_t\delta\alpha_t\,.\,(\widehat{L_t\cap N})+\sum_t\delta\alpha_t\,.\,(\widehat{L_t\backslash N}).$$

For each $x\in H$, $hx\in\tilde{H}$ if and only if $x\in N$. Hence

$$\sum_t\alpha_t\,.\,h(\widehat{L_t\cap N})=\sum_t\delta\alpha_t\,.\,(\widehat{L_t\cap N}).$$

This implies that $h\tilde{e}\in K\tilde{e}$. Hence $K\tilde{H}\tilde{e}\subseteqq K\tilde{e}$ and so $K\tilde{H}\tilde{e}=K\tilde{e}$. Therefore $[K\tilde{H}\tilde{e}:K]=1$.

Consequently, $\tilde{h}_i=1$ for all i. This shows that every irreducible $K\tilde{H}$-module is of K-dimension one, and so $\tilde{H}$ is abelian. Hence N is abelian since $N=\tilde{H}\times P$. This completes the proof.

REMARK 1. Assume $|P|=2$ or 1 if $p=2$, and assume $|P|=1$ if p is odd. In this case, from [**10**, Theorem] and the proof of Theorem 1, $J(KG)^2=0$, and so $J(KG)$ is commutative. Since G is a p-nilpotent group with an abelian p-Sylow subgroup P, by the proof of Theorem1, $N_G(P)=C_G(P)$. But $N_G(P)$ is nonabelian in general. Indeed, if we set that H is a nonabelian finite group of order prime to p and that $G=H\times P$, then $G=N_G(P)=C_G(P)$ and $N_G(P)$ is nonabelian.

Next, we shall have the following main theorem of this paper. This gives a group-theoretical condition of G for $J(KG)$ to be commutative.

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THEOREM 2. *For an arbitrary prime number p, $J(KG)$ is commutative if and only if G is a group of the following two types*:

(*i*) $|G|$ *is not divisible by* 2^2 *when* $p=2$, *and* $|G|$ *is not divisible by p when p is odd.*

(*ii*) *G is a p-nilpotent group with an abelian p-Sylow subgroup P.* $b_0=|O_{p'}(G):G'|$, $b_1=\ldots=b_{a-2}=0$, *and if p is odd,* $b_{a-1}=0$, *where* $|P|=p^a$ *and* b_k *is the number of p-regular conjugate classes* K_j *of G such that* $|K_j|$ *is divisible by* p^k *and not by* p^{k+1} *for* $k=0,\ldots,a$.

Proof. From the proof of Theorem 1 we can assume that K is algebraically closed. So we use notations n, U_i, F_i, u_i and f_i as in the proof of Theorem 1. Put $H=O_{p'}(G)$.

Suppose that $J(KG)$ is commutative and that $|G|$ is divisible by 2^2 when $p=2$, and is divisible by p when p is odd. By [**11**, Theorem 2], G is a p-nilpotent group with an abelian p-Sylow subgroup P. From the proof of Theorem 1, the number of blocks of KG with defect a is equal to the number of nonisomorphic irreducible KG-modules F_i such that $f_i=1$. Thus, by [**1**, Theorem 2] and [**1**, p. 588], $b_0=|G:G'P|=|O_{p'}(G):G'|$. Since G is p-nilpotent, we use notations B_i, P_i, h_i and t_i as in the proof of Theorem 1. Let $C=(c_{ii'})_{1\leq i,i'\leq n}$ be the Cartan matrix for KG. It follows from [**6**, Theorem 3] that $f_i=h_it_i$ and $u_i=p^ah_i=f_i\,|P_i|$, hence $c_{ii}=|P_i|$ and $c_{ii'}=0$ if $i\neq i'$. Since $p^a=t_i\,|P_i|$ and $(h_i,p)=1$, a block B_i has defect d if and only if $|P_i|=p^d$. Put $|P_i|=p^{d_i}$ for each i. We say that B_i is of type (a), (b) or (c) as in the proof of Theorem 1. If B_i is of type (a), B_i has defect 0 since p^a divides $u_i=f_i$. If B_i is of type (b), by [**11**, Lemma 7], $p=2$ and $u_i=2f_i$, and so B_i has defect 1 since 2^a divides u_i. If B_i is of type (c), B_i has defect a from the proof of Theorem 1. Let $\{K_1,\ldots,K_n\}$ be the set of all p-regular conjugate classes of G and let K_i have p-defect k_i, that is to say, $|K_i|$ is divisible by p^{a-k_i} and not by p^{a-k_i+1}, for each i.

Case 1. $p=2$. Since every d_i is 0, 1 or a and

$$C=\begin{bmatrix}2^{d_1} & & 0\\ & \ddots & \\ 0 & & 2^{d_n}\end{bmatrix},$$

it follows from [**1**, §16] that every k_i is also 0, 1 or a. This implies that $b_1=\ldots=b_{a-2}=0$.

Case 2. p is odd. Since every d_i is 0 or a, as in Case 1, every k_i is also 0 or a. Hence $b_1=\ldots=b_{a-2}=b_{a-1}=0$.

Conversely, suppose that (i) or (ii) holds. If (i) holds, by [**10**, Theorem], $J(KG)^2=0$, and so $J(KG)$ is commutative. So we can assume that (ii) holds. Since G is p-nilpotent, we use notations B_i and P_i as in the proof of Theorem 1. From (ii), [**1**, Theorem 2] and [**1**, p. 588] we have that the number of blocks of KG with defect a is equal to the number of nonisomorphic irreducible KG-modules F_i such that $f_i=1$. This shows that for a block B_i, B_i has defect a if and only if $f_i=1$. From the proof of Theorem 1, $B_i\cong(KP_i)_{f_i}$ for each i. If B_i has defect 0, $J(B_i)=0$. If $p=2$ and B_i has defect 1, $|P_i|=2$ and so $J(B_i)^2\cong(J(KP_i)^2)_{f_i}=0$ from [**10**, Theorem]. If B_i has defect a, $f_i=1$ and hence $J(B_i)\cong J(KP)$.

Case 1. $p=2$. From (ii) every block B_i of KG has defect 0, 1 or a. Hence $J(KG)$ is commutative.

Case 2. p is odd. By (*ii*), every block B_i of KG has defect 0 or a, and so $J(KG)$ is commutative. This finishes the proof of Theorem 2.

REMARK 2. When p is odd, by [**11**, Theorem 1] (cf. [**3**, Corollary 5.2]), $J(KG)$ is commutative if and only if $J(KG)\subseteqq Z(KG)$. But when $p=2$, this does not hold in general. We can assume that K is algebraically closed from the proof of Theorem 1. Though $J(KG)$ is commutative, $J(KG)\nsubseteq Z(KG)$ if $a\geqq 2$ and there exists a block B_i of KG of type (b) (cf. the proof of Theorem 1). Indeed, suppose that $a\geqq 2$, $J(KG)$ is commutative and there is a block B_i of KG of type (b). By [**11**, Lemma 7], $p=2$. Since $J(KG)$ is commutative and $a\geqq 2$, we use notations n, m, B_i and P_i as in the proof of Theorem 1. We can write $J(KG)=\sum_{i=1}^{n}\oplus J(B_i)$. Since there is a block B_i of type (b), $\sum_{i=m+1}^{n}\oplus J(B_i)\neq 0$. It follows from the proof of Theorem 2 that $m=|G:G'P|$. If B_i is of type (c), by the proofs of Theorems 1 and 2, it is seen that $J(B_i)\cong J(KP)$, and so $[J(B_i):K]=2^a-1$. Thus $[J(KG):K]=\sum_{i=1}^{m}[J(B_i):K]+\sum_{i=m+1}^{n}[J(B_i):K]>m(2^a-1)=|G:G'P|\,(2^a-1)$. Hence $[J(K(G'P)):K]>2^a-1$ by the proof of [**11**, Theorem 1]. Therefore $J(KG)\nsubseteq Z(KG)$ by [**8**, Theorem 2] and [**9**, Theorem].

An example of the above case is as follows.

EXAMPLE. Assume that K is algebraically closed and $p=2$. Put $G=\langle x, y \mid x^4=y^3=1, x^{-1}yx=y^2\rangle$. G is a 2-nilpotent group with a cyclic 2-Sylow subgroup $P=\langle x\rangle$. The decomposition matrix D for G and the Cartan matrix C for KG are given as

$$D=\begin{bmatrix}1&0\\1&0\\1&0\\1&0\\0&1\\0&1\end{bmatrix},\qquad C=\begin{bmatrix}4&0\\0&2\end{bmatrix}.$$

We use notations U_i, f_i, B_i and P_i as in the proof of Theorem 1. From the proof of Theorem 2, $|P_1|=4$ and $|P_2|=2$. Hence $B_1\cong KP$ and $B_2\cong(KP_2)_2$. This shows that $J(KG)$ is commutative by [**10**, Theorem]. On the other hand, we have that $f_1=1$, $f_2=2$, $KG\cong U_1\oplus U_2\oplus U_2$, B_1 is of type (c) and B_2 is of type (b). Since $y^{-1}x^{-1}yx=y$ and $G/O_{2'}(G)\cong P$, it follows that $O_{2'}(G)=G'$, and so $G'P=G$. Hence $[J(K(G'P)):K]=[J(KG):K]=|G|-(f_1^2+f_2^2)=7>3=2^2-1$. Thus $J(KG)\nsubseteq Z(KG)$ by [**8**, Theorem 2] and [**9**, Theorem]. Indeed, $\{(1+x)e, (1+x^2)e, (1+x^3)e, 1+x^2, x(1+x^2), x(1+x^2)y, (1+x^2)y^2\}$ is a K-basis of $J(KG)$, where $e=1+y+y^2$. Using this we have that $J(KG)$ is commutative but $J(KG)\nsubseteq Z(KG)$ since $\{x(1+x^2)\}y\neq y\{x(1+x^2)\}$.

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