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ON THE SET OF LIMITS OF RIEMANN SUMS

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1. Introduction

Let F map $[0, 1]$ into a Banach space $\boldsymbol{B}$ and let $R(F)$ denote the set of all limits of Riemann sums of F. The set $R(F)$ need not be convex in general (Nakamura and Amemiya (1966)) but is always convex when B is finite dimensional as first shown by Hartman (1947). A proof of Hartman's result, based on a description of $R(F)$ when the range of F is finite, was given in Ellis (1959). In this note this description is refined, the extreme points of $R(F)$ are determined and the following complete characterization of $R(F)$ is obtained (where $N_n = \{1, 2, \cdots, n\}$).

THEOREM 1.1. *Let* F: $[0, 1] \to \{\alpha_i, i \in N_n\}$ *and let* $E_i = \{t: F(t) = \alpha_i\}$, $i \in N_n$. *Then* $R(F)$ *coincides with the points* $\sum_1^n a_i \alpha_i$ *for which the coefficients* a_i *satisfy, for each* $N' \subset N_n$,

$$(1.1) \qquad m\left(\bigcup_{i \in N'} E_i\right)^0 \leqq \sum_{i \in N'} a_i \leqq m\left(\overline{\bigcup_{i \in N'} E_i}\right)$$

In the theorem A^0 and $\bar{A}$ denote the interior and closure of a set A respectively and m denotes Lebesgue measure. Note that (1.1) implies that

$$0 \leqq a_i \leqq 1,\ i \in N_n;\ \sum_1^n a_i = 1.$$

2. The closure of R′(F)

In Ellis (1959) it was shown that $R(F)$ is the closure of a set denoted by $R'(F)$. In this section we show that $R'(F)$ is closed.

We first describe the notation. By $\mathscr{D}$ we denote a partition $0 = t_0 < t_1 < \cdots < t_n = 1$; $\mathscr{E}$ a set of intermediary values $\xi_i, t_i \leqq \xi_i < t_{i+1}$, $i = 0, 1, \cdots, n-1$; $t_{n-1} \leqq \xi_{n-1} \leqq t_n$; $|\mathscr{D}| = \max(t_{i+1} - t_i)$, the norm of $\mathscr{D}$ and

$$\Sigma(\mathscr{D},\mathscr{E}) = \sum_{i=0}^{n-1} F(\xi_i)(t_{i+1} - t_i)$$

will be called a Riemann sum for F on $\mathscr{D}$. Note that F may be non-measurable. If Range F is contained in the Banach space $\boldsymbol{B}$, $P \in \boldsymbol{B}$ will be called a limit of Riemann sums if for some sequence $\{\mathscr{D}_n, \mathscr{E}_n\}$, $|\mathscr{D}_n| \to 0$ and $\| P - \Sigma(\mathscr{D}_n, \mathscr{E}_n) \| \to 0$ as $n \to \infty$.

Our results are based on the following elementary result.

LEMMA 2.1. *Let U be an arbitrary open subset of* $[0, 1]$ *and, for $\mathscr{D}$ any partition of* $[0, 1]$, *let $A(\mathscr{D})$ denote the union of those intervals of $\mathscr{D}$ that fall inside U. Then as $|\mathscr{D}| \to 0$, $m[A(\mathscr{D})] \to m(U)$.*

In Ellis (1959) Lemma 2.1 was used in showing that if $N' \subset N_n$ and $A_{N'}(\mathscr{D})$ denotes the union of those intervals of $\mathscr{D}$ on each of which Range $F = \{\alpha_i, i \in N'\}$ then $\lim_{|\mathscr{D}| \to 0} m[A_{N'}(\mathscr{D})]$ exists. We now denote this limit by $K_{N'}$. It is easy to verify that

$$K_{N'} = \lim_{|\mathscr{D}| \to 0} m[A_n(\mathscr{D})] = m\left[\left(\bigcup_{i \in N'} E_i\right)^0 - \bigcup_{j \in N'} \left(\bigcup_{i \in N' \setminus j} E_i\right)^0\right]. \tag{2.1}$$

From (2.1), $K_i = K_{\{i\}} = m(E_i^0)$. By induction, for any N':

$$\sum_{N^* \subset N'} K_{N^*} = m\left(\bigcup_{i \in N'} E_i\right)^0; \tag{2.2}$$

$$\sum_{N^* \subset N_n} K_{N^*} = 1. \tag{2.2'}$$

For each $N' \subset N_n$ let $c_{N'i}$, $i \in N'$, be any set of non-negative real numbers satisfying $\sum_{i \in N'} c_{N'i} = 1$. As in Ellis (1959) let $R'(F)$ be the set of points $P \in \boldsymbol{B}$ of the form

$$P = \sum_{i=1}^{n} \left(\sum_{\substack{N' \subset N_n \\ i \in N'}} c_{N'i} K_{N'} \right) \alpha_i \tag{2.3}$$

$$= \sum_{N' \subset N_n} \sum_{i \in N'} (c_{N'i} K_{N'} \alpha_i).$$

In Ellis (1959) it was shown that $R(F) = \bar{R}'(F)$.

PROPOSITION 2.1. *$R'(F)$ is closed and thus every $P \in R(F)$ is of the form* (2.3).

PROOF. Let $R_0(F)$ denote the set of points P satisfying (2.3) for which each $c_{N'i}$ is 0 or 1, a finite set of points. By Day (1962) (Lemma 2, p. 79) the convex hull of $R_0(F)$ is compact and therefore closed. Since $R_0(F) \leqq R'(F)$ and $R'(F)$ is convex, the convex hull of $R_0(F)$ is contained in $R'(F)$. On the other hand it is easy to verify that each $P \in R'(F)$ can be expressed as a convex combination of points in $R_0(F)$ and thus is contained in and so coincides with the convex hull of $R_0(F)$.

3. The extreme points of $R(F)$ and Theorem 1.1

We denote by $R^*(F)$ the set of points in $\boldsymbol{B}$ for which (1.1) holds for every $N' \subset N_n$. It is easy to verify that $R^*(F)$ is convex.

PROPOSITION 3.1. $R^*(F) \supset R(F)$.

PROOF. From (2.3), if $P \in R(F)$,

$$P = \sum_1^n a_i\alpha_i: a_i = \sum_{N' \subset N_n; i \in N'} c_{N'i}K_{N'}, \; i \in N_n.$$

Thus, for any $N' \subset N_n$,

$$\begin{aligned}\sum_{i \in N'} a_i &= \sum_{i \in N'} \Big(\sum_{N'' \subset N_n; i \in N''} c_{N''i}K_{N''} \Big) \\ &\geqq \sum_{N'' \subset N'} \sum_{i \in N''} c_{N''i}K_{N''} = \sum_{N'' \subset N'} K_{N''} = m\Big(\bigcup_{i \in N'} E_i\Big)^0,\end{aligned}$$

using (2.2).

On the other hand

$$\sum_{i \in N'} a_i = 1 - \sum_{i \notin N'} a_i \leqq 1 - m\Big(\bigcup_{i \notin N'} E_i\Big)^0 = m\Big(\overline{\bigcup_{i \in N'} E_i}\Big).$$

For $\{p_i, i \in N_n\}$ any permutation of N_n let $P = \sum_{i=1}^n a_{p_i}\alpha_{p_i}$, where for each $k \leqq n$,

$$\sum_{i=1}^k a_{p_i} = m\Big(\bigcup_{i=1}^k E_{p_i}\Big)^0.$$

Then

$$\sum_{i=k+1}^n a_{p_i} = m\Big(\overline{\bigcup_{i=k+1} E_{p_i}}\Big)$$

and P is a limit of Riemann sums for which α_{p_i} is used as intermediary value in $\mathscr{E}$ only when necessary (i.e. on intervals falling inside $E^0_{p_i}$), α_{p_2} only where necessary after the intermediary values α_{P_1} have been assigned, etc. Likewise P is a limit of Riemann sums for waich α_{p_n} is used as intermediary value whenever possible, $\alpha_{p_{n-1}}$ whenever possible after the values α_{p_n} have been assigned, etc. Let $E(F)$. be the set of all such P for all permutations of N_n.

PROPOSITION 3.2. *If* Range $F = \{\alpha_i, i \in N_n\}$ *and the points* $\{\alpha_i\}$ *are linearly independent then every point of* $E(F)$ *is an extreme point of* $R^*(F)$.

PROOF. We assume for convenience that $P = \sum_1^n a_i\alpha_i$, with

$$\sum_1^k a_i = m\Big(\bigcup_1^k E_i\Big)^0, \; k = 1,2,\cdots,n. \quad \text{Let } P_j = \sum_1^n a_i^j\alpha_i, \quad j = 1,2,$$

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be in $R^*(F)$ and suppose that $P = (P_1 + P_2)/2$. The linear independence implies that $a_i = (a_i^1 + a_i^2)/2$, $i = 1, 2, \cdots, n$. Since $P_j \in R^*(F)$, $j = 1, 2$; $a_i^j \geqq m(E_i^0) = a_i$, $j = 1, 2$ so that $a_1 = a_1^1 = a_1^2$. Similarly $a_i = a_i^1, a_i^2$; $i = 2, 3, \cdots, n$. Thus $P_1 = P_2 = P$ and P is an extreme point of $R^*(F)$.

Note that when the set $\{\alpha_i\}$ is not linearly independent, $E(F)$ may contain points that are not extreme points. For example if $\boldsymbol{B} = \boldsymbol{R}$, $R(F)$ is a point or line segment and contains one or two distinct extreme points. However, $E(F)$ may contain $n!$ distinct points.

PROPOSITION 3.3. *For $P \in R^*(F)$ assume that there exist $i, j \in N_n$ with strict inequality holding in* (1.1) *for $N' = \{i, j\}$ and for every $N' \subset N_n$ that contains one but not both of i, j. Then P is not an extreme point of $R^*(F)$.*

PROOF. Let $P = \sum_1^n a_r \alpha_r$, assume the hypotheses satisfied for i, j and let $d > 0$ be less than the minimum difference in the inequalities in (1.1) for all N' in the hypotheses. Define

$$P_k = \sum_1^n a_r^k \alpha_r, \qquad k = 1, 2;$$

with

$$\begin{aligned} a_r^k &= a_r,\ r \neq i, j; \qquad k = 1, 2; \\ a_i^1 &= a_i + d,\ a_j^1 = a_j - d; \\ a_i^2 &= a_i - d,\ a_j{}^2 = a_j + d. \end{aligned}$$

Then, if $i, j \in N' \subset N_n$ or $(i, j) \cap N' = \varnothing$, $\sum_{N'} a_r^k = \sum_{N'} a_r$. $k = 1, 2$ and (1.1) is satisfied for N'. For the remaining $N' \subset N_n$ (1.1) is a consequence of the choice of d. Thus $P_1, P_2 \in R^*(F)$. From the definition, $P = (P_1 + P_2)/2$. Assume that $P_1 = P$. Then $P_1 - P = d(\alpha_i - \alpha_j) = 0$, implying that $\alpha_i = \alpha_j$, a contradiction.

PROPOSITION 3.4. *$E(F)$ contains the extreme points of $R^*(F)$.*

PROOF. The proof is trivial for $n = 2$. Assume that $n > 2$, that $P = \sum_1^n a_r \alpha_r \in R^*(F)$ and that, for every $N' \underset{\neq}{\subset} N_n$,

$$m\left(\bigcup_{r \in N'} E_r\right)^0 < \sum_{r \in N'} a_r.$$

By complementation

$$\sum_{r \in N'} a_r < m\left(\overline{\bigcup_{r \in N'} E_r}\right).$$

The hypotheses of Proposition 3.3 are satisfied for any pair i, j and thus P is not an extreme point.

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Excluding this case there is a maximal $N' \subsetneqq N_n$ with

$$(3.1)\qquad \sum_{r\in N'} a_r = m\Big(\bigcup_{r\in N'} E_r\Big)^0; \quad \sum_{r\notin N'} a_r = m\Big(\overline{\bigcup_{r\notin N'} E_i}\Big).$$

For convenience of notation we assume that $N' = N_k$, $1 \leqq k < n$. We first show that if $n-k \geqq 3$ then P is not an extreme point of $R^*(F)$.

We note that with each point in $R^*(F)$ and each $N' \subset N_n$ we can associate numbers $K_{N'}$ by (2.1) (in terms of the sets E_r). These numbers will satisfy (2.2) and (2.2)′ and, by complementation,

$$(3.2)\qquad m\Big(\overline{\bigcup_{r\in N'} E_r}\Big) = \Sigma\{K_{N^*} : N^* \cap N' \neq \varnothing\}.$$

Let $N^* \subset N_n \backslash N_k$. Then

$$\sum_{r\in N^*} a_r + \sum_{r\in N_k} a_r > m\Big(\bigcup_{r\in N_k\cup N^*} E_r\Big)^0 = \sum_{N'\subset N_k\cup N^*} K_{N'}$$
$$\sum_{N'\subset N_k} K_{N'} + \Sigma\{K_{N'} : N' \subset (N_k \cup N^*);\ N' \cap N^* \neq \varnothing\}$$

since $\geqq$ holds by (1.1) and equality would contradict the maximality of N_k. Thus

$$(3.3)\qquad \begin{aligned}\sum_{r\in N^*} a_r &> \Sigma\{K_{N'} : N' \subset (N_k \cup N^*);\ N' \cap N^* \neq \varnothing\}\\ &\geqq \sum_{N'\subset N^*} K_{N'} = m\Big(\bigcup_{r\in N^*} E_r\Big)^0.\end{aligned}$$

Now let $N'' \subset N_k$, $\varnothing \neq N^* \subset N_n \backslash N_k$, $N^\# = N'' \cup N^{*\prime}$ Then

$$(3.4)\qquad \begin{aligned} m\Big(\bigcup_{r\in N^\#} E_r\Big)^0 &= \sum_{N'\subset N^\#} = \sum_{N'\subset N''} K_{N'} + \Sigma\{K_{N'} : N' \subset N^\#;\ N' \cup N^* \neq \varnothing\}\\ &< \sum_{i\in N''} a_i + \sum_{i\in N^*} a_i,\end{aligned}$$

using (3.3). It follows that if $i, j \in N_n \backslash N_k$ and $N^\# = \{i, j\}$ or contains one but not both of i, j, then

$$m\Big(\bigcup_{r\in N^\#} E_r\Big)^0 < \sum_{r\in N^\#} a_r.$$

With $N^\# = N'' \cup N^*$, $N^* \neq \varnothing$ as before;

$$N_n \backslash N^\# = (N_k \backslash N'') \cup [(N_n \backslash N_k) \backslash N^*],$$

(3.4) holds for $N_n \backslash N^\#$ and

$$\sum_{r\in N^\#} a_r = 1 - \sum_{r\in N_n\backslash N^\#} a_r < 1 - m\Big(\bigcup_{r\in N_n\backslash N^\#}\Big)E_r)^0 = m\Big(\overline{\bigcup_{r\in N^\#} E_r}\Big).$$

Proposition 3.3 then implies that P is not an extreme point.

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Thus if P is an extreme point of $R^*(F)$, k is either $n-1$ or $n-2$. Assume that $k = n-2$. Then

$$\begin{aligned} a_{n-1} + a_n = m(\overline{E_{n-1} \cup E_n}) &= \Sigma\{K_{N'}: N' \subset N_n, N' \cap (n-1, n) \neq \emptyset\} \\ &= \Sigma\{K_{N'}: (n-1, n) \subset N'\} + \Sigma\{K_{N'}: n \in N', n-1 \notin N'\} \\ &\qquad + \Sigma\{K_{N'}: n-1 \in N'; n \notin N'\}, \\ &= A + A_n + A_{n-1}, \end{aligned}$$

defining A, A_n and A_{n-1}.

From (3.2) $A + A_i = m(\bar{E}_i)$, $i = n-1, n$. Thus if $A = 0$, $a_{n-1} + a_n = m(\bar{E}_{n-1}) + m(\bar{E}_n)$ and (1.1) implies that $a_i = m(\bar{E}_i)$, $i = n-1, n$. This contradicts the assumption that $k = n-2$. Thus we may assume that $A \neq 0$.

Assuming that $A \neq 0$ let $P = \Sigma_{r=1}^n a_r^i \alpha_r$, $i = 1,2$; where $a_r^i = a_r$, $i = 1,2$; $r < n-1$; $a_n^1 = A + A_n a_{n-1}^1 = A_{n-1}$; $a_n^2 = A_n$, $a_{n-1}^2 = A + A_{n-1}$. Then $P_i \in R^*(F)$, $i = 1,2$ and there exists λ, $0 < \lambda < 1$ with $a_n = A_n + \lambda A$; $a_{n-1} = A_{n-1} + (1-\lambda)A$. It follows that $P = \lambda P_1 + (1-\lambda)P_2$, showing that P is not an extreme point.

We have shown that the assumption that P is an extreme point of $R^*(F)$ implies that for some $n_1 \leqq n$, $a_{n_1} = m(\bar{E}_{n_1})$, $\Sigma_{r \neq n_1} a_r = m(\bigcup_{r \neq n_1} E_r)^0$. Similar considerations applied to $\Sigma_{r \neq n_1} a_r$ show that if P is an extreme point of $R^*(F)$ there exists $n_2 \neq n_1$ with $a_{n_1} + a_{n_2} = m(\overline{E_{n_1} \cup E_{n_2}})$ and, continuing this process, that $P \in E(F)$.

Corollary. *The set of extreme points of $R^*(F)$ is contained in $E(F)$ and coincides with $E(F)$ when the set $\{\alpha_i, i \in N_n\}$ is linearly independent.*

Proposition 3.5. *$R^*(F)$ is compact.*

Proof. The part B of R^n defined by the points $(a_1, a_2, \cdots, a_n)$, $a_i \geqq 0$; $\Sigma_1^n a_i = 1$ is compact. The subset B^* of B for which the additional inequalities in (1.1) are satisfied is compact as a closed subset of B.

If the function ϕ mapping $B \times \Pi_1^n(\alpha_i\} \subset R^n \times B^n$ into B is defined by the formula

$$(a_1, a_2, \cdots a_n, \alpha_1, \alpha_2, \cdots \alpha_n) \to \sum_{i=1}^n a_i \alpha_i,$$

(Bourbaki (1953), Proposition 1, p. 80) then ϕ is continuous and $R^*(F)$ is compact as the image of the compact subset $B^* \times \Pi_1^n\{\alpha_i\}$.

Proof of Theorem 1.1. Since $R^*(F)$ is a compact, convex subset of the locally convex space B it is the closed convex hull of its extreme points by the Krein-Mil'man Theorem (Day (1962), Theorem 1, p. 78) and thus of $E(F)$ since

$E(F)$ contains the set of extreme points of $R^*(F)$. Since $R_0(F) \supset E(F)$ and $R(F)$ is the closed convex hull of $R_0(F)$, $R^*(F) \subset R(F)$ and thus $R(F)$ and $R^*(F)$ coincide.

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