ON A CONJECTURE OF LINDENSTRAUSS AND PERLES IN AT MOST 6 DIMENSIONS

by D. G. LARMAN

(Received 18 September, 1976)

1. Introduction. In [**1**] J. Lindenstrauss and M. A. Perles studied the extreme points of the set of all linear operators T of norm ≤ 1 from a finite dimensional Banach space X into itself. In particular they studied the question "When do these extreme points form a semigroup?".

Let X be a Banach space. Then $S(X)$ denotes the unit ball of X and $B(X)$ denotes the unit ball of all operators from X into itself (with the usual operator norm). Let ext A denote the set of extreme points of a set A. The two principal theorems of [**1**] are:

THEOREM 1. *The following three assertions, concerning a finite dimensional Banach space X, are equivalent:*

(1) $x \in \operatorname{ext} S(X)$, $T \in \operatorname{ext} B(X) \Rightarrow Tx \in \operatorname{ext} S(X)$;

(2) $T_1, T_2 \in \operatorname{ext} B(X) \Rightarrow T_1 T_2 \in \operatorname{ext} B(X)$;

(3) $\{T_i\}_{i=1}^m \in \operatorname{ext} B(X) \Rightarrow \|T_1 \ldots T_m\| = 1$, *for* $m = 1, 2, \ldots$.

THEOREM 2. *Let X be a Banach space of dimension* ≤ 4. *Then X has properties* (1) *to* (3) *of Theorem* 1 *if and only if one of the following conditions holds*:

(i) *X is an inner product space;*

(ii) *$S(X)$ is a polytope with the property that for every facet K of $S(X)$, $S(X)$ is the convex hull of* $K \cup -K$.

In 5 dimensions they give an example of a polytope $S(X)$ which satisfies (ii) of Theorem 2 but for which X does not have properties (1) to (3) of Theorem 1. However, they conjecture that any finite dimensional Banach space X which has properties (1) to (3) of Theorem 1 also satisfies (i) or (ii) of Theorem 2. The purpose of this note is to prove this conjecture for Banach spaces X of dimension at most 6. The methods probably work for higher dimensions but are limited by the large number of cases which need to be considered.

2. Pre-requisites. We state here the definitions and results of [**1**] which we shall use.

DEFINITION. Let pext $B(X)$ denote the subset of $B(X)$ consisting of all finite products of elements of ext $B(X)$ and let cl pext $B(X)$ denote its closure.

In [**1**] it was shown that if X satisfies Theorem 1 then

$$\operatorname{cl}\operatorname{pext} B(X) = \operatorname{ext} B(X).$$

DEFINITION. Let $k(X) = \min\{\dim TX : T \in \operatorname{cl}\operatorname{pext} B(X)\}$.

Let X be a Banach space of dimension n. Then X has properties (1) to (3) of Theorem 1 if and only if $k(X) > 0$. If $k(X) = n$ then X is an inner product space, and if

$k(X)=1$ then $S(X)$ is a polytope with the property that, for every facet K of $S(X)$, $S(X)$ is the convex hull of $K \cup -K$. The conjecture of Lindenstrauss and Perles therefore is that there does not exist X with $1<k(X)<n$. In [1] it was shown that $k(X) \neq n-1$ or $n-2$ which, of course, proves Theorem 2.

Furthermore it was shown that the following result holds.

LEMMA 1. *Let X have properties* (1) *to* (3) *of Theorem* 1 *and let* $1<k(X)<n$. *Say* $k(X)=k$. *Then* $\operatorname{ext} S(X)$ *is closed and is the union of an infinite number of k-dimensional ellipsoids, say* $\operatorname{ext} S(X)=\bigcup_{\alpha\in A} X_\alpha$ *where X_α is a k-dimensional ellipsoid. Also there is a projection P_α in* $\operatorname{ext} B(X)$ *from X to X_α and the restriction of every* $T\in \operatorname{ext} B(X)$ *to X_α is an isometry.*

Since $k(X)=k(X^*)$, Lemma 1 also holds for X^*, say $\operatorname{ext} S(X^*)$ is the union of an infinite number of k-dimensional ellipsoids $\{X^*_\beta\}_{\beta\in B}$. The various projections P_α induce circumscribing k-dimensional elliptic cylinders to $S(X)$, say $\{C_\beta\}_{\beta\in B}$, where C_β is the polar of X^*_β, $\beta\in B$. Consequently $\{C_\beta\}_{\beta\in B}$ is also infinite and closed in the obvious sense. Also each $X_\alpha(\alpha\in A)$ lies on the boundary of each $C_\beta(\beta\in B)$.

3. Additional lemmas. If, using the notation of the previous section, we consider a k-dimensional ellipsoid X_0 of the collection $\{X_\alpha\}_{\alpha\in A}$, we may consider X_0 as a base for C_β and let L_β denote the $(n-k)$-subspace of generators of C_β, i.e.

$$C_\beta = X_0 + L_\beta,\ \beta\in B.$$

Then, if $\mathbf{x}\in X_0$, $(\mathbf{x}+L_\beta)\cap S(X)$ is a face of $S(X)$ of dimension at most $n-k$. The collection $\{C_\beta\}_{\beta\in B}$ is closed and infinite, and consequently it contains a limit cylinder

$$C_{\beta_0} = X_0 + L_{\beta_0}.$$

Our first objective is to establish Lemma 3 which asserts that for each $\mathbf{x}\in X_0$, $(\mathbf{x}+L_{\beta_0})\cap S(X)$ has dimension less than $n-k$. To do this, we need to establish

LEMMA 2. *Let* $Y_m=\{\mathbf{x}:(\mathbf{x}-\mathbf{y}_m)^t A_m(\mathbf{x}-\mathbf{y}_m)\leq \alpha_m\}$ $(\alpha_m>0)$ *be a closed convex elliptic cylinder in E^n, where $\mathbf{x}^t A_m \mathbf{x}$ is a positive semi-definite quadratic form,* $m=1,2,\ldots$. *Suppose that there exist $n+1$ affinely independent points $\mathbf{x}_1,\ldots,\mathbf{x}_{n+1}$ which lie on the boundary of each Y_m, $m=1,2,\ldots$. Then there exist a subsequence M and a closed convex n-dimensional set Y such that*

(i) *$Y_m\cap B\to Y\cap B$ as $m\to\infty$ through M for any closed ball B,*

(ii) *$\mathbf{x}_1,\ldots,\mathbf{x}_{n+1}$ lie on the boundary of Y and at least one of the line segments $[\mathbf{x}_i,\mathbf{x}_j]$ does not lie in the boundary of Y.*

Proof. By using the Blaschke selection theorem and a standard diagonalisation argument we choose a subsequence M and a closed convex n-dimensional set Y such that $Y_m\cap B\to Y\cap B$ as $m\to\infty$ through M for any closed ball B. We suppose that Lemma 2 is false, i.e., that $[\mathbf{x}_i,\mathbf{x}_j]$ lies on the boundary of Y for $1\leq i<j\leq n+1$. As $\mathbf{x}_1,\ldots,\mathbf{x}_{n+1}$ are affinely independent, $\operatorname{conv}(\mathbf{x}_1,\ldots,\mathbf{x}_{n+1})$ meets the interior of Y. Consequently we may

pick a $(d+1)$-membered subset $(2 \leq d \leq n)$, say $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$, so that $\text{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_{d+1})$ meets the interior of Y but $\text{conv}(\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_d})$ is contained in the boundary of Y for $1 \leq i_1 < \ldots < i_d \leq d+1$.

Let D be the affine space spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_{d+1}$. Then $D \cap Y_m \to \text{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_{d+1})$ as $m \to \infty$ in M. Consequently $D \cap Y_m$ is bounded for sufficiently large m in M, and so $D \cap Y_m$ is a d-dimensional ellipsoid for sufficiently large m in M. But then $D \cap Y_m$ is centrally symmetric and so $\text{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_{d+1})$ is centrally symmetric, which is not so. This contradiction establishes Lemma 2.

LEMMA 3. *The subset* $X_0^{n-k} = \{\mathbf{x} : (\mathbf{x} + L_{\beta_0}) \cap S(X)$ *has dimension* $n-k\}$ *of* X_0 *is empty.*

Proof. We suppose that the lemma is false. Let $\mathbf{y}_0 \in X_0^{n-k}$. Then $(\mathbf{y}_0 + L_{\beta_0}) \cap S(X)$ contains $n-k+1$ affinely independent extreme points $\mathbf{y}_0, \ldots, \mathbf{y}_{n-k}$. Each of these extreme points $\mathbf{y}_0, \ldots, \mathbf{y}_{n-k}$ is contained in (at least) one k-dimensional ellipsoid, $X_0, \ldots, X_{n-k}$ respectively say, from amongst the collection $\{X_\alpha\}_{\alpha \in A}$.

Now $C_{\beta_0} = X_0 + L_{\beta_0}$ is a limit cylinder of the collection $\{C_\beta\}_{\beta \in B}$, and so we can choose distinct cylinders

$$C_{\beta_m} = X_0 + L_{\beta_m}, \qquad m = 0, 1, 2, \ldots$$

so that $C_{\beta_m} \to C_{\beta_0}$ as $m \to \infty$. The set $Y_m = (\mathbf{y}_0 + L_{\beta_0}) \cap (X_0 + L_{\beta_m})$ is the intersection of the flat $\mathbf{y}_0 + L_{\beta_0}$ with the elliptic cylinder $X_0 + L_{\beta_m}$ and consequently Y_m is also an elliptic cylinder (possibly an ellipsoid) in $\mathbf{y}_0 + L_{\beta_0}$.

By Lemma 2, there exist a subsequence M and a closed convex $(n-k)$-dimensional set Y in $\mathbf{y}_0 + L_{\beta_0}$ so that

(i) $Y_m \cap B \to Y \cap B$ as $m \to \infty$ through M for any closed ball B in $\mathbf{y}_0 + L_{\beta_0}$,

(ii) $\mathbf{y}_0, \ldots, \mathbf{y}_{n-k}$ lie on the relative boundary of Y and at least one of the line segments $[\mathbf{y}_i, \mathbf{y}_j]$ does not lie in the boundary of Y.

We may suppose, without loss of generality, that $[\mathbf{y}_0, \mathbf{y}_1]$ does not lie in the relative boundary of Y. Now, by continuity there exists a neighbourhood U of $\mathbf{y}_0$ in X_0 such that U is contained in X_0^{n-k}. Let $\mathbf{x}_m$ be that point of X_0 such that $\mathbf{y}_1$ and $\mathbf{x}_m$ lie on the same face of C_m, i.e., $\mathbf{x}_m = (\mathbf{y}_1 + L_{\beta_m}) \cap X_0$. Then, since $C_{\beta_m} \to C_{\beta_0}$ as $m \to \infty$, $\mathbf{x}_m \to \mathbf{y}_0$ as $m \to \infty$. Also, if $\mathbf{x}_m = \mathbf{y}_0$ for all but finitely many $m \in M$ then the line segment $[\mathbf{y}_1, \mathbf{y}_0]$ lies on the same face of $C_{\beta m} \cap (\mathbf{y}_0 + L_{\beta_0}) = Y_m$ for all but finitely many $m \in M$. So $[\mathbf{y}_1, \mathbf{y}_0]$ is on the boundary of Y, which yields a contradiction. Consequently, we may suppose that $\mathbf{x}_m \neq \mathbf{y}_0$ for all $m \in M$.

There will be a hyperplane of support, say H_m, to $X_0 + L_{\beta_m}$, and hence to $S(X)$, which contains both $\mathbf{x}_m$ and $\mathbf{y}_1$. Then $(\mathbf{y}_0 + L_{\beta_0}) \cap H_m$ is a hyperplane in $\mathbf{y}_0 + L_{\beta_0}$ which supports $(\mathbf{y}_0 + L_{\beta_0}) \cap S(X)$ at $\mathbf{y}_1$. Since $[\mathbf{y}_0, \mathbf{y}_1]$ does not lie in the relative boundary of Y, $(\mathbf{y}_0 + L_{\beta_0}) \cap H_m$ may be supposed to converge to a hyperplane $(\mathbf{y}_0 + L_{\beta_0}) \cap H$, and H_m converges to H, which supports Y at $\mathbf{y}_1$ and $\mathbf{y}_0 \notin (\mathbf{y}_0 + L_{\beta_0}) \cap H$.

Now consider a line segment $[\mathbf{y}_0, \mathbf{z}_1]$ passing through the relative interior of $(\mathbf{y}_0 + L_{\beta_0}) \cap S(X)$, where $\mathbf{z}_1 \in \text{relbdy}\{(\mathbf{y}_0 + L_{\beta_0}) \cap S(X)\}$ and $\mathbf{z}_1$ is chosen so close to $\mathbf{y}_1$ as to ensure that the hyperplane H cuts the line through $[\mathbf{z}_1, \mathbf{y}_0]$ in a point $\mathbf{b}_1$, where $\mathbf{b}_1, \mathbf{z}_1, \mathbf{y}_0$ occur in that order. Consequently, we may suppose that for $m \in M$, H_m cuts the line through $[\mathbf{z}_1, \mathbf{y}_0]$ in a point $\mathbf{b}_m$, where $\mathbf{b}_m$, $\mathbf{z}_1$, $\mathbf{y}_0$ occur in that order.

https://doi.org/10.1017/S0017089500003426 Published online by Cambridge University Press

Consider next the 3-dimensional subspace G_m generated by $\mathbf{z}_1$, $\mathbf{y}_0$ and $\mathbf{x}_m$. In G_m the 2-plane $H_m \cap G_m$ contains $\mathbf{b}_m$ and $\mathbf{x}_m$ and is tangent to $X_0 \cap G_m$. Consequently $H_m \cap G_m$ strictly separates the point $\mathbf{y}_0$ from the ray

$$\ell_m = \{\mathbf{x}_m + t(\mathbf{z}_1 - \mathbf{y}_0),\ t > 0\}$$

So ℓ_m does not meet the face $(\mathbf{x}_m + L_{\beta_0}) \cap S(X)$. But

$$\ell_0 = \lim_{m\to\infty} \ell_m = \{\mathbf{y}_0 + t(\mathbf{z}_1 - \mathbf{y}_0),\ t > 0\}$$

meets $(\mathbf{y}_0 + L_{\beta_0}) \cap S(X)$ in a relatively interior point $\mathbf{z} = (\mathbf{z}_1 + \mathbf{y}_0)/2$. So there exists a finite set $\mathbf{q}_1, \dots, \mathbf{q}_p$ of extreme points of $S(X)$ in $(\mathbf{y}_0 + L_{\beta_0}) \cap S(X)$ whose convex hull is $(n-k)$-dimensional and contains $\mathbf{z}$ as a relatively interior point. Let $X_{\alpha_1}, \dots, X_{\alpha_p}$ be ellipsoids amongst $\{X_\alpha\}_{\alpha \in A}$ such that $\mathbf{q}_i \in X_{\alpha_i}$ $(i = 1, \dots, p)$, and let $q_i^m = X_{\alpha_i} \cap (\mathbf{x}_m + L_{\beta_0})$ $(i = 1, \dots, p)$. Then $\mathbf{q}_i^m \to \mathbf{q}_i$ as $m \to \infty$ $(i = 1, \dots, p)$ and so, for sufficiently large m in M, $\mathbf{z}^m = \mathbf{x}_m + (\mathbf{z}_1 - \mathbf{y}_0)/2$ lies in the relative interior of the $(n-k)$-dimensional set $\operatorname{conv}(\mathbf{q}_1^m, \dots, \mathbf{q}_p^m)$ which is contained in $(\mathbf{x}_m + L_{\beta_0}) \cap S(X)$. This contradicts the previous result that ℓ_m does not meet $(\mathbf{x}_m + L_{\beta_0}) \cap S(X)$ and completes the proof of Lemma 3.

The next lemma uses an extension of the methods used to prove Proposition 4.3 of [**1**].

LEMMA 4. *Let X be a finite dimensional Banach space with $0 < k(X) < n$. If $[\mathbf{a}, \mathbf{b}]$ is an edge of $S(X)$ then there must be at least 2 members of $\{X_\alpha\}_{\alpha \in A}$ which contain $\mathbf{b}$.*

Proof. If the lemma is false, then there is an edge $[\mathbf{a}, \mathbf{b}]$ of $S(X)$ such that $\mathbf{b}$ is contained in exactly one member X_1 of $\{X_\alpha\}_{\alpha \in A}$. Let Z be the 2-dimensional subspace of X spanned by $[\mathbf{a}, \mathbf{b}]$ and let $k = k(X)$. Then, if $B(E^k, X)$ denotes the set of linear operators of norm at most 1 from E^k to X, there exists $T \in B(E^k, X)$ such that

$$T\mathbf{e}_1 = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}),\ T(\alpha \mathbf{e}_1 + \beta \mathbf{e}_2) = \mathbf{y} \in Z$$

with $\alpha^2 + \beta^2 = \|\mathbf{y}\| = 1$, $\beta \neq 0$ and $T\mathbf{e}_i = 0$ for $i > 2$ (here $\{\mathbf{e}_i\}_{i=1}^k$ denotes the usual coordinate basis of E^k). Let $T = \sum_{i=1}^{q} \lambda_i T_i$ with $\lambda_i > 0$ $(i = 1, \dots, q)$, $\sum_{i=1}^{q} \lambda_i = 1$ and $T_i \in \operatorname{ext} B(E^k, X)$ $(i = 1, \dots, q)$. Then, since T_i takes extreme points to extreme points (see Lemmas 3.11–13 of [**1**]), $T_i\mathbf{e}_1 = \mathbf{a}$ or $\mathbf{b}$ for $i = 1, \dots, q$. We assume that $T_i\mathbf{e}_1 = \mathbf{a}$ for $i = 1, \dots, p$ and $T_i\mathbf{e}_1 = \mathbf{b}$ for $i = p+1, \dots, q$. Then we have $1/2 = \sum_{i=1}^{p} \lambda_i = \sum_{i=p+1}^{q} \lambda_i$, and T_i is an isometry from E^k to X_1 for $i = p+1, \dots, q$. Let

$$\mathbf{y}_0 = 2 \sum_{i=1}^{p} \lambda_i T_i(\alpha \mathbf{e}_1 + \beta \mathbf{e}_2)$$

and

$$\mathbf{y}_1 = 2 \sum_{i=p+1}^{q} \lambda_i T_i(\alpha \mathbf{e}_1 + \beta \mathbf{e}_2).$$

Then $y=(\mathbf{y}_0+\mathbf{y}_1)/2$, and consequently $\mathbf{y}_0$ and $\mathbf{y}_1$ lie on the boundary of $S(X)$. Also $\mathbf{y}_1$ lies on the relative boundary of X_1 and so

$$T_i(\alpha\mathbf{e}_1+\beta\mathbf{e}_2)=\mathbf{y}_1, \qquad p+1\leq i\leq q.$$

Since $\beta\neq 0$, $\mathbf{y}_1\neq\pm\mathbf{b}$.

Since X_1 meets the subspace spanned by $\mathbf{a},\mathbf{b}$ only at $\pm\mathbf{b}$, it follows that $\mathbf{a},\mathbf{b},\mathbf{y}_0,\mathbf{y}_1$ span a 3-dimensional subspace F. Using the methods of Lemma 3.8 of [**1**], we see that there exists $V\in\operatorname{ext}B(X)$ such that $V(\mathbf{a})=V(\mathbf{b})=\mathbf{b}$. Consequently, if M denotes the $(k+1)$-dimensional subspace generated by X_1 and $\mathbf{b}-\mathbf{a}$, $V(S(X)\cap M)=X_1$. So the cylinder

$$C=\{(F\cap X_1)+t(\mathbf{b}-\mathbf{a}),\ t\text{ real}\}$$

supports $F\cap S(X)$ and contains $F\cap X_1$ on its boundary; further, $[\mathbf{a},\mathbf{b}]$ is contained in a generator of C.

Similarly, considering $[\mathbf{y}_0,\mathbf{y}_1]$ and $W\in\operatorname{ext}B(X)$ with $W(\mathbf{y}_0)=W(\mathbf{y}_1)=\mathbf{b}$, we see that there exists a cylinder

$$C'=\{(F\cap X_1)+t(\mathbf{y}_1-\mathbf{y}_0),\ t\text{ real}\}$$

which supports $F\cap S(X)$ and contains $F\cap X_1$ on its boundary; also $[\mathbf{y}_0,\mathbf{y}_1]$ is contained in one of the generators of C'. Since $\mathbf{0}\in\operatorname{lin}(\mathbf{a},\mathbf{b},(\mathbf{y}_0+\mathbf{y}_1)/2)$, $\mathbf{y}_1-\mathbf{y}_0$ is parallel to $\mathbf{b}-\mathbf{a}$ only if $\mathbf{y}_1=\pm\mathbf{b}$, which is impossible. So C' is not C and again $\mathbf{0}\in\operatorname{lin}(\mathbf{a},\mathbf{b},(\mathbf{y}_0+\mathbf{y}_1)/2)$ only if $\mathbf{y}_1$ is $\pm\mathbf{b}$, which is impossible. This establishes Lemma 4.

LEMMA 5. *Let C be a convex body in E^n such that* ext C *is contained in $L_1\cup L_2$, where L_1 and L_2 are hyperplanes. Then, if $\mathbf{y}$ belongs to* $(\operatorname{ext}C)\cap(L_1\backslash L_2)$, *there is an edge of C which contains $\mathbf{y}$.*

Proof. The result is trivial when $n=2$ and, proceeding by induction, it is enough to find a proper face F of C which contains $\mathbf{y}$ but which is not contained in L_1.

Let H be a hyperplane of support to C at $\mathbf{y}$. If $L_1\cap L_2\neq\emptyset$, we may suppose, by taking a projective transformation if necessary, that $H\cap L_1$ contains a translate of $L_2\cap L_1$. Then, if Π denotes the orthogonal projection of E^n along $L_1\cap L_2$, $\mathbf{y}$ is an extreme point of the 2-dimensional convex body ΠC. The point $\Pi\mathbf{y}$ is not in ΠL_2 and $\operatorname{ext}\Pi C$ is contained in $\Pi L_1\cup\Pi L_2$. So there exists an edge F^* of ΠC which contains $\Pi\mathbf{y}$ but which is not contained in ΠL_1. Then $F=C\cap\Pi^{-1}F^*$ is the required face of C.

If $L_1\cap L_2=\emptyset$ i.e., L_1 is parallel to L_2, then it is possible to choose H so that $H\neq L_1$. Then we project along $H\cap L_1$ and argue as before.

LEMMA 6. *Let X be a 6-dimensional Banach space with $k(X)=2$. Then there are no points on $S(X)$ which lie on two distinct members of $\{X_\alpha\}_{\alpha\in A}$. Consequently $S(X)$ does not contain any edges.*

Proof. We suppose that the lemma is false. Let X_1, X_2 be two ellipses of $\{X_\alpha\}_{\alpha\in A}$ which intersect. Without loss of generality we may suppose that

$$X_1: x_1^2+x_2^2=1, \qquad x_3=x_4=x_5=x_6=0,$$

https://doi.org/10.1017/S0017089500003426 Published online by Cambridge University Press

and

$$X_2 : x_2^2 + x_3^2 = 1, \qquad x_1 = x_4 = x_5 = x_6 = 0,$$

which intersect in the point (0, 1, 0, 0, 0, 0)'. Then the cylinders $\{C_\beta\}_{\beta \in B}$ which arise from the dual ellipses $\{X_\beta^*\}_{\beta \in B}$ of $S(X^*)$ meet the 3-dimensional space $x_4 = x_5 = x_6 = 0$ in cylinders of the form

$$x_2^2 + (x_1 \pm x_3)^2 = 1,$$

and hence their generators contain one of (1, 0, ±1, 0, 0, 0)'. This means that each of the ellipses X_β^* is orthogonal to one of $(1, 0, \pm 1, 0, 0, 0)'$ and hence the extreme points of $S(X^*)$ are contained in two 5-dimensional subspaces L_1^*, L_2^*. Consequently, if X_1^* is one of the collection $\{X_\beta^*\}_{\beta \in B}$ such that $X_1^* \cap (L_1^* \setminus L_2^*) \neq \emptyset$, then, using Lemma 5, if $\mathbf{y}^* \in X_1^* \cap (L_1^* \setminus L_2^*)$ there exists an edge of $S(X^*)$ which contains $\mathbf{y}^*$. So, using Lemma 4, there exists an ellipse of $\{X_\beta^*\}_{\beta \in B}$, different from X_1^*, which contains $\mathbf{y}^*$.

Let $\mathbf{y}_2^*$, $\mathbf{y}_3^*$ be distinct points of $X_1^* \cap (L_1 \setminus L_2)$ and let X_2^*, X_3^* be distinct from X_1^* and contain $\mathbf{y}_2^*$, $\mathbf{y}_3^*$ respectively. We now disregard the special forms, assumed previously, for X_1 and X_2 and we may instead assume that

$$X_1^* : x_1^2 + x_2^2 = 1, \qquad x_3 = x_4 = x_5 = x_6 = 0,$$
$$X_2^* : x_2^2 + x_3^2 = 1, \qquad x_1 = x_4 = x_5 = x_6 = 0,$$
$$X_3^* : x_1^2 + x_4^2 = 1, \qquad x_2 = x_3 = x_5 = x_6 = 0,$$

and hence that

$$\mathbf{y}_2^* = (0, 1, 0, 0, 0, 0)',$$
$$\mathbf{y}_3^* = (1, 0, 0, 0, 0, 0)'.$$

Then each cylinder arising from the ellipses in $\{X_\alpha\}_{\alpha \in A}$ meets the 4-dimensional subspace $x_5 = x_6 = 0$ in a cylinder of the form

$$(x_1 \pm x_3)^2 + (x_2 \pm x_4)^2 = 1.$$

So each cylinder arising from $\{X_\alpha\}_{\alpha \in A}$ contains amongst its generators one of the four 2-dimensional subspaces

$$x_1 = \pm x_3, \qquad x_2 = \pm x_4, \qquad x_5 = x_6 = 0,$$

and not all of these cylinders can share a common generator. This means that the extreme points of $S(X)$ are contained in at least two and at most four 4-dimensional subspaces $L_{i_1}, \ldots, L_{i_j}$ and $L_{i_1} \cap \ldots \cap L_{i_j}$ is the 2-dimensional subspace $L : x_1 = x_2 = x_3 = x_4 = 0$. We may suppose that

$$(\text{ext}\, S(X)) \setminus \bigcup_{\substack{t=1 \\ t \neq k}}^{j} L_{i_t} \neq \emptyset \ (k = 1, \ldots, j),$$

for otherwise L_k is redundant. For each L_{i_k} we may pick X_1, X_2, X_3 as X_1^*, X_2^*, X_3^* were chosen above, and we deduce that the cylinders arising from $\{X_\beta^*\}_{\beta \in B}$ contain, amongst

https://doi.org/10.1017/S0017089500003426 Published online by Cambridge University Press

their generators, one of four 2-dimensional subspaces $L_{i_k,1}, \ldots, L_{i_k,4}$, at most one of which can be L and all of which lie in L_{i_k}.

We may classify the cylinders arising from the $\{X^*_\beta\}_{\beta\in B}$ into a finite number of classes according to which of the 2-dimensional spaces $L_{i_k,1}, \ldots, L_{i_k,4}$ are contained amongst its generators $(k=1,\ldots,j)$. It is only in the class (if it exists) in which L occurs as the 2-dimensional subspace for each k that a 3-dimensional subspace of generators is not determined.

In $S(X^*)$, this means that the extreme points of $S(X^*)$ are contained in finitely many 3-dimensional subspaces $M_1, \ldots, M_p$ and at most one 4-dimensional subspace N. Now $M_1, \ldots, M_p$ can contain at most two members each of $\{X^*_\beta\}_{\beta\in B}$, and so there are only finitely many X^*_β that are not wholly contained in N.

There are two 5-dimensional subspaces N_1, N_2 which contain ext $S(X^*)$ and we may suppose that $(\text{ext}\, S(X))\bigcap(N_1\backslash N)\neq\emptyset$ and hence is infinite. By Lemmas 4, 5 it follows that for each point $\mathbf{y}\in(\text{ext}\, S(X))\bigcap(N_1\backslash N)$ there are at least two members of $\{X^*_\beta\}_{\beta\in B}$ which contain $\mathbf{y}$. Consequently, there are infinitely many $\{X^*_\beta\}_{\beta\in B}$ which are not contained in N. This contradiction establishes Lemma 6.

LEMMA 7. *Let X be a 5- or 6-dimensional Banach space. Then $k(X)\neq 2$.*

Proof. We only prove the lemma in the harder 6-dimensional case. We choose $C_{\beta_0}=X_0+L_{\beta_0}$ as in Lemma 3 with $k(X)=2$, and deduce that the subset

$$X_0^4=\{\mathbf{x}:(\mathbf{x}+L_{\beta_0})\cap S(X)\quad\text{has dimension } 4\}$$

of X_0 is empty.

If $(\mathbf{x}+L_{\beta_0})\cap S(X)$ has dimension 3, let H be the affine hull of $(\mathbf{x}+L_{\beta_0})\cap S(X)$. Any cylinder $C_\beta=X_0+L_\beta$, with $L_\beta\neq L_{\beta_0}$, meets H in a cylinder $H\cap C_\beta$ which is either the product of an ellipse and a line or the product of a line segment and a plane. The extreme points of $(\mathbf{x}+L_{\beta_0})\cap S(X)$ must lie on the relative boundary of $H\cap C_\beta$ and so $(\mathbf{x}+L_{\beta_0})\cap S(X)$ must contain edges of $S(X)$, which contradicts Lemma 6.

So, $(\mathbf{x}+L_{\beta_0})\cap S(X)$ is either the single point $\mathbf{x}$ or a 2-dimensional ellipse, for each $\mathbf{x}\in X_0$. Since $\{X_\alpha\}_{\alpha\in A}$ is infinite, $(\mathbf{x}+L_{\beta_0})\cap S(X)$ is an ellipse, except for possibly two opposite points of X_0.

Consider next a sequence of distinct cylinders $C_{\beta_m}=X_0+L_{\beta_m}$ $(m=0,1,2,\ldots)$, which converge to C_{β_0} as $m\to\infty$, and an ellipse $E=(\mathbf{x}+L_{\beta_0})\cap S(X)$. Unless $\mathbf{x}+L_{\beta_m}$ contains E, the projection of E along L_{β_m}, into X_0, must be an ellipse on X_0 and so must coincide with X_0. But, as $m\to\infty$, this projection must converge to $\mathbf{x}$, which would be impossible. So we conclude that there exists $M(\mathbf{x})$, such that if $m\geq M(\mathbf{x})$, $\mathbf{x}+L_{\beta_m}$ contains E. So L_{β_m} contains the 2-dimensional subspace $D(\mathbf{x})=\text{lin}\{E-x\}$. As $S(X)$ is 6-dimensional and X_0 is only 2-dimensional, we must be able to choose $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ in X_0 such that $D(\mathbf{x}_1)$, $D(\mathbf{x}_2)$, $D(\mathbf{x}_3)$ arise from ellipses $(\mathbf{x}_i+L_{\beta_0})\cap S(X)$ $(i=1,2,3)$ and span the 4-dimensional subspace L_{β_0}. Then, if $m\geq\max_{1\leq i\leq 3} M(x_i)$, $L_{\beta_m}=L_{\beta_0}$ and so $C_{\beta_m}=C_{\beta_0}$, which contradicts the fact that the cylinders $\{C_{\beta_m}\}_{m=0}^\infty$ are distinct.

https://doi.org/10.1017/S0017089500003426 Published online by Cambridge University Press

LEMMA 8. *Let X be a 6-dimensional Banach space. Then $k(X) \neq 3$.*

Proof. We suppose that $k(X)=3$. Then, using Lemma 3, $(\mathbf{x}+L_{\beta_0}) \cap S(X)$ is at most 2-dimensional for all $\mathbf{x} \in X_0$. If two ellipses do not coincide then they meet in at most four points. So, if $(\mathbf{x}+L_{\beta_0}) \cap S(X)$ is not an ellipse then it is either a single point, an edge or a 2-dimensional convex set whose boundary consists of at most four edges. Hence, as $\{X_\alpha\}_{\alpha \in A}$ is infinite, for almost all $\mathbf{x}$ in X_0, $(\mathbf{x}+L_{\beta_0}) \cap S(X)$ is a 2-dimensional ellipse.

We may suppose that X_0 is the 3-sphere

$$x_1^2+x_2^2+x_3^2=1, \qquad x_4=x_5=x_6=0$$

and that one of these ellipses $(\mathbf{x}+L_{\beta_0}) \cap S(X)$ is

$$(x_4-1)^2+x_5^2=1, \qquad x_1=1, \qquad x_2=x_6=0,$$

where $\mathbf{x}=(1, 0, 0, 0, 0, 0)'$.

Consider any 3-cylinder arising from $\{X_\beta^*\}_{\beta \in B}$ intersected with the 5-dimensional subspace $x_6=0$. This has equation

$$(x_1+\alpha_1x_4+\beta_1x_5)^2+(x_2+\alpha_2x_4+\beta_2x_5)^2+(x_3+\alpha_3x_4+\beta_3x_5)^2=1.$$

If we consider the subset lying in the 2-dimensional affine subspace

$$x_1=1. \qquad x_2=x_3=x_6=0,$$

we obtain

$$(1+\alpha_1x_4+\beta_1x_5)^2+(\alpha_2x_4+\beta_2x_5)^2+(\alpha_3x_4+\beta_3x_5)^2=1,$$

which must be equivalent to

$$(x_4-1)^2+x_5^2=1.$$

So $\alpha_1=-1$, $\beta_1=0$, $\alpha_2=\alpha_3=0$, $\beta_2^2+\beta_3^2=1$. Hence if we write $\beta_2=\cos\lambda$, $\beta_3=\sin\lambda$ the 3-cylinder, intersected with $x_6=0$, then has the form

$$(x_1-x_4)^2+(x_2+x_5\cos\lambda)^2+(x_3+x_5\sin\lambda)^2=1,$$

or

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2-1-2x_1x_4=-2x_5(x_2\cos\lambda+x_3\sin\lambda).$$

If there is an extreme point of $S(X)$ in the 5-dimensional subspace $x_6=0$ which does not lie in either $x_5=0$ or $x_2=x_3=0$, then λ can take one of two values λ_1, λ_2 in $[0, 2\pi]$. Say

$$\mathbf{y}=(y_1, y_2, y_3, y_4, y_5, y_6)'$$

with

$$y_1^2+y_2^2+y_3^2+y_4^2+y_5^2-1-2y_1y_4=-2y_5(y_2\cos\lambda+y_3\sin\lambda).$$

Then the two sets of 2-dimensional generators for the cylinders are given by

$$x_1=x_4, \qquad x_2=-x_5\cos\lambda_1, \qquad x_3=-x_5\sin\lambda_1$$

and

$$x_1=x_4, \qquad x_2=-x_5\cos\lambda_2, \qquad x_3=-x_5\sin\lambda_2.$$

So both sets of generators lie in the 3-space

$$x_1 = x_4, \qquad x_2 y_2 + x_3 y_3 = c x_5,$$

where c is a constant determined by $\mathbf{y}$. Hence the two sets of generators intersect, i.e., all the cylinders $\{C_\beta\}_{\beta \in B}$ have a common generator, which is impossible.

So any point of ext $S(X)$ in $x_6 = 0$ must lie in either the set $x_5 = 0$, or in $x_2 = x_3 = 0$, or in both. Each of the 3-spheres meets $x_6 = 0$ in at least a 2-sphere. If one of these 3-spheres X_γ, other than X_0, meets $x_5 = 0$, $x_6 = 0$ in a 2-sphere, then X_0 and X_γ intersect. Otherwise, any two 3-spheres of $\{X_\alpha\}_{\alpha \in A}$ meet the 3-dimensional subspace $x_2 = x_3 = x_6 = 0$ in at least a 2-sphere and so intersect. So we may suppose, in any event, that there are two 3-spheres X_1, X_2 of the collection $\{X_\alpha\}_{\alpha \in A}$ which intersect. If X_1 is

$$x_1^2 + x_2^2 + x_3^2 = 1, \qquad x_4 = x_5 = x_6 = 0,$$

then we may suppose that the other 3-sphere X_2 is one of

$$\text{(i)} \quad x_2^2 + x_3^2 + x_4^2 = 1, \qquad x_1 = x_5 = x_6 = 0,$$

$$\text{(ii)} \quad x_3^2 + x_4^2 + x_5^2 = 1, \qquad x_1 = x_2 = x_6 = 0.$$

Consider first case (i). Any cylinder arising from $\{X_\beta^*\}_{\beta \in B}$ meets the 4-dimensional subspace $x_5 = x_6 = 0$ in a cylinder of the form

$$(x_1 + \alpha_1 x_4)^2 + (x_2 + \alpha_2 x_4)^2 + (x_3 + \alpha_3 x_4)^2 = 1.$$

In the 3-dimensional subspace $x_1 = x_5 = x_6 = 0$, this reduces to

$$\alpha_1^2 x_4^2 + (x_2 + \alpha_2 x_4)^2 + (x_3 + \alpha_3 x_4)^2 = 1,$$

which must be equivalent to

$$x_2^2 + x_3^2 + x_4^2 = 1.$$

So $\alpha_1 = \pm 1$, $\alpha_2 = \alpha_3 = 0$, i.e., all the cylinders have one of $(\pm 1, 0, 0, 0, 0, 0)'$ amongst their generators. Dually, this means that the extreme points of $S(X^*)$ are contained in two 5-dimensional subspaces L_1 and L_2. So the cylinders arising from $\{X_\alpha\}_{\alpha \in A}$ give rise to faces of $S(X^*)$ whose extreme points are (almost always) disconnected. So these faces cannot (almost always) be ellipses, which gives the required contradiction in case (i).

Consider next X_2 as in (ii). Any cylinder arising from $\{X_\beta^*\}_{\beta \in B}$ meets $x_6 = 0$ in a cylinder of the form

$$(x_1 + \alpha_1 x_4 + \beta_1 x_5)^2 + (x_2 + \alpha_2 x_4 + \beta_2 x_5)^2 + (x_3 + \alpha_3 x_4 + \beta_3 x_5)^2 = 1,$$

which, when also $x_1 = x_2 = 0$, has the form

$$(\alpha_1 x_4 + \beta_1 x_5)^2 + (\alpha_2 x_4 + \beta_2 x_5)^2 + (x_3 + \alpha_3 x_4 + \beta_3 x_5)^2 = 1,$$

which must be

$$x_3^2 + x_4^2 + x_5^2 = 1.$$

https://doi.org/10.1017/S0017089500003426 Published online by Cambridge University Press

Consequently,

$$\alpha_3=\beta_3=0, \qquad \alpha_1^2+\alpha_2^2=1, \qquad \beta_1^2+\beta_2^2=1, \qquad \alpha_1\beta_1+\alpha_2\beta_2=0.$$

Let $\alpha_1=\cos\lambda$, $\alpha_2=\sin\lambda$, $\beta_1=\cos\rho$, $\beta_2=\sin\rho$. Then

$$\cos\lambda\cos\rho+\sin\lambda\sin\rho=0,$$

that is,

$$\cos(\lambda-\rho)=0.$$

So $\rho=\lambda+3\pi/2$ or $\rho=\lambda+\pi/2$. Hence the cylinder has the form

$$(x_1+x_4\cos\lambda+x_5\sin\lambda)^2+(x_2+x_4\sin\lambda-x_5\cos\lambda)^2+x_3^2=1,$$

or

$$(x_1+x_4\cos\lambda-x_5\sin\lambda)^2+(x_2+x_4\sin\lambda+x_5\cos\lambda)^2+x_3^2=1.$$

i.e., either

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2-1=-2\sin\lambda(x_1x_5+x_2x_4)-2\cos\lambda(x_1x_4-x_2x_5), \tag{1}$$

or

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2-1=2\sin\lambda(x_1x_5-x_2x_4)-2\cos\lambda(x_1x_4+x_2x_5). \tag{2}$$

If (1) occurs and there exists $\mathbf{y}_1=(y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, 0)'$ in ext $S(X)$ such that at least one of $y_{11}y_{15}+y_{12}y_{14}$ or $y_{11}y_{14}-y_{12}y_{15}$ is non-zero, then λ can take at most two values in $[0, 2\pi]$. Consequently, the generators of the cylinders $\{C_\beta\}_{\beta\in B}$ arising from $\{X_\beta^*\}_{\beta\in B}$ contain at least one of four 2-dimensional subspaces. Hence the extreme points ext $S(X^*)$ of $S(X^*)$ lie in the union of at most four 4-dimensional subspaces. So the cylinders arising from $\{X_\alpha\}_{\alpha\in A}$ give rise to faces of $S(X^*)$ whose extreme points are (almost always) disconnected. So these faces cannot (almost always) be ellipses, which gives a contradiction.

So, if (1) occurs, then, for all extreme points in ext $S(X)$,

$$x_1x_5+x_2x_4=0, \qquad x_1x_4-x_2x_5=0 \tag{3}$$

and, if (2) occurs,

$$x_1x_5-x_2x_4=0, \qquad x_1x_4+x_2x_5=0. \tag{4}$$

We deal only with the case when (1), and hence (3), occurs; the argument when (2), and hence (4), occurs is similar.

From (3) we obtain

$$(x_1^2+x_2^2)x_5=0.$$

Hence either $x_1=x_2=0$ or $x_5=0$. If $x_1\neq 0$ and $x_5=0$, then $x_4=0$. If $x_2\neq 0$ and $x_5=0$, then $x_4=0$. Consequently, either $x_1=x_2=0$, or $x_4=x_5=0$. So, if X_α is a 3-sphere amongst $\{X_\alpha\}_{\alpha\in A}$, but different from X_1 and X_2, then X_α meets one of X_1, X_2 in a 2-sphere and we are again in case (i), which completes the proof of Lemma 8.

Combining Lemmas 7 and 8 and Proposition 4.4 of **[1]** (which says that if dim $X=n$,

https://doi.org/10.1017/S0017089500003426 Published online by Cambridge University Press

$k(X) \neq n-1$ or $n-2$) we obtain

THEOREM 3. *Let X be a Banach space of dimension at most six. Then X has properties* (1) *to* (3) *of Theorem* 1 *only if one of the following conditions holds:*

(i) *X is an inner product space;*

(ii) *$S(X)$ is a polytope with the property that for every facet K of $S(X)$, $S(X)$ is the convex hull of $K \cup -K$.*

REFERENCE

1. J. Lindenstrauss and M. A. Perles, On extreme operators in finite dimensional spaces, *Duke Math. J.* **36** (1969), 301–314.

DEPT. OF MATHEMATICS
UNIVERSITY COLLEGE LONDON
GOWER STREET, LONDON WC1E 6BT

https://doi.org/10.1017/S0017089500003426 Published online by Cambridge University Press