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GROUP ALGEBRAS WITH ENGEL UNIT GROUPS

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Abstract

Let F be a field of characteristic $p \geq 0$ and G any group. In this article, the Engel property of the group of units of the group algebra FG is investigated. We show that if G is locally finite, then $\mathcal{U}(FG)$ is an Engel group if and only if G is locally nilpotent and G' is a p-group. Suppose that the set of nilpotent elements of FG is finite. It is also shown that if G is torsion, then $\mathcal{U}(FG)$ is an Engel group if and only if G' is a finite p-group and FG is Lie Engel, if and only if $\mathcal{U}(FG)$ is locally nilpotent. If G is nontorsion but FG is semiprime, we show that the Engel property of $\mathcal{U}(FG)$ implies that the set of torsion elements of G forms an abelian normal subgroup of G.

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1. Introduction

Let F be a field of characteristic $p \geq 0$ and G a group. Group identities on the group of units of the group algebra FG, $\mathcal{U}(FG)$, are of interest to many authors. The most famous result, known as Hartley's conjecture, asserts that if G is a torsion group and $\mathcal{U}(FG)$ satisfies a group identity, then FG satisfies a polynomial identity. The affirmative answer to his conjecture has been given in a series of papers [3–5, 7, 8]. We recommend the reader to refer to Lee's book [6], a good survey on group identities on units (and symmetric units) of group algebras.

Among other identities, the bounded Engel property is of much interest. If char $F = 0$ or char $F = p > 0$ and G has no p-elements, then the solution was found by Bovdi and Khripta in [2, Theorem 1.3] by showing that if $\mathcal{U}(FG)$ is (bounded) Engel, then the torsion elements of G form a (normal) abelian subgroup of G. They also presented solutions for other special cases. Subsequently, Riley solved the problem for torsion groups in [13]. He showed that if G is torsion and char $F = p > 0$, then the bounded Engel property of $\mathcal{U}(FG)$ implies that G is nilpotent and G has a p-abelian normal subgroup of finite p-power index (recall that for any prime $p \geq 0$, a group G is said to

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be *p-abelian* if its commutator subgroup G' is a finite p-group and that 0-abelian means abelian). The general result, showing that if G has a p-element and $\mathcal{U}(FG)$ is bounded Engel, then FG is bounded Lie Engel, was presented in Bovdi [1]. The converse had already been established in a much more general setting by Shalev in [16].

In this paper, instead of bounded Engel unit groups, we consider Engel unit groups and extend some earlier results. Our first result is for locally finite groups G such that $\mathcal{U}(FG)$ is Engel.

Theorem 1.1. *Let G be a locally finite group and F a field of characteristic $p \geq 0$. Then $\mathcal{U}(FG)$ is an Engel group if and only if G is locally nilpotent and G' is a p-group.*

Here, by 0-group we mean the identity group. Thus, if G is locally finite and F is of characteristic 0, then $\mathcal{U}(FG)$ is an Engel group if and only if G is abelian.

As indicated above, if F is a field and G is a torsion group, then $\mathcal{U}(FG)$ is bounded Engel if and only if FG is bounded Lie Engel (see [6, Corollary 5.2.13]). Our second main result is as follows.

Theorem 1.2. *Let G be a torsion group and F a field of characteristic $p \geq 0$. If the set of nilpotent elements of FG is finite, then the following conditions are equivalent:*

(1) *$\mathcal{U}(FG)$ is Engel;*
(2) *G is p-abelian and FG is Lie Engel;*
(3) *$\mathcal{U}(FG)$ is locally nilpotent.*

Finally, if G is nontorsion, we also have a partial result when FG is semiprime.

Theorem 1.3. *Let G be a group, T the set of torsion elements of G, and F a field such that FG is a semiprime ring. If the set of nilpotent elements of FG is finite and $\mathcal{U}(FG)$ is Engel, then T is an abelian normal subgroup of G.*

Note that, recently, in [10], the local nilpotency of the group of units of the group algebra FG was investigated by the author. He showed that if $\mathcal{U}(FG)$ is locally nilpotent, then the set of p-elements of G forms a subgroup P and the torsion elements of G/P form an abelian group. If, in addition, the set of nilpotent elements of FG is finite, every idempotent in $F(G/P)$ is central; a converse version was also indicated. As a result, it showed that if G is torsion, then $\mathcal{U}(FG)$ is locally nilpotent if and only if G is locally nilpotent and G' is a p-group, if and only if FG is Lie Engel and G is locally finite.

2. The proofs

In this section we prove the above results. Occasionally, we borrow our methods from [6].

Let G be a group. For x, y in G, define

$$(x, {}_1y) = (x, y) = x^{-1}y^{-1}xy, \quad (x, {}_{n+1}y) = ((x, {}_ny), y).$$

The group G is an Engel group if for each $x, y \in G$, there exists an integer $n = n(x, y)$, depending on x and y, such that $(x, {}_ny) = 1$.

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Lemma 2.1. *Let D be a division ring with $\dim_{Z(D)} D < \infty$ and n a natural number. If $\mathrm{GL}_n(D)$ is an Engel group, then $n = 1$ and D is a field.*

Proof. See [11, Theorem 1.3]. □

Lemma 2.2. *Let F be a field of characteristic $p \geq 0$ and G a torsion group. Then $\mathcal{U}(FG)$ is nilpotent if and only if FG is Lie nilpotent, if and only if G is nilpotent and p-abelian.*

Proof. See [6, Corollary 4.2.7] and [15, Theorem V.4.4]. □

For any ring R, by $J(R)$ we mean the Jacobson radical of R. We now prove our first result.

Proof of Theorem 1.1. First, assume that $\mathcal{U}(FG)$ is an Engel group. Then, clearly, G is an Engel group. But, by a famous result of Zorn, every finite Engel group is nilpotent, so G is locally nilpotent. Now let $g \in G'$ and assume that $g = (x_1, y_1)^{n_1} \cdots (x_s, y_s)^{n_s}$, where x_i and y_i are in G and $n_i \in \mathbb{Z}$. Let H denote the subgroup of G generated by all the x_i and y_i, $1 \leq i \leq s$. Then H is a finite group and $\mathcal{U}(FH)$; hence, $\mathcal{U}(FH)/(1 + J(FH))$ is Engel. Now we can apply the Wedderburn–Artin theorem to deduce that

$$\mathcal{U}(FH)/(1 + J(FH)) \simeq \mathcal{U}(FH/J(FH)) \simeq \bigoplus_{i=1}^{r} \mathrm{GL}_{n_i}(D_i),$$

where each $n_i \geq 1$ and each D_i is a division ring. Since each D_i is a finite-dimensional division algebra over F, Lemma 2.1 yields that each $n_i = 1$ and D_i is a field. Consequently, $\mathcal{U}(FH/J(FH))$ is abelian. If $p = 0$, then $J(FH) = 0$, so H is abelian and hence $g = 1$; that is, G is abelian and the result follows. Let $p > 0$. Then we have $g \in 1 + J(FH)$, so $g - 1$ is nilpotent. Hence, g is a p-element, so G' is a p-group, as desired.

Conversely, let G' be a p-group and G be locally nilpotent. Let $\alpha, \beta \in \mathcal{U}(FG)$, and let H be the subgroup of G generated by the supports of all of the $\alpha, \beta, \alpha^{-1}$ and β^{-1}. Then H is a finite nilpotent group. Thus, $\mathcal{U}(FH)$ is nilpotent by Lemma 2.2. Therefore, $(\alpha, {}_n\beta) = 1$ for some positive integer n; that is, $\mathcal{U}(FG)$ is an Engel group, and the proof is completed. □

To prove Theorem 1.2, we need several lemmas. If G is a group and $p \geq 0$ a prime number, we let P be the set of all p-elements of G (here, of course, if $p = 0$, we let $P = 1$).

Lemma 2.3. *Let G be a group and F a field such that the set of nilpotent elements of FG is finite. Suppose that FG is semiprime and, for all $\alpha, \beta, \gamma \in FG$ with $\alpha^2 = \beta\gamma = 0$, we have $\beta\alpha\gamma = 0$. If $\mathcal{U}(FG)$ is an Engel group, then the set of torsion elements of G forms a normal abelian subgroup T of G.*

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Proof. First we claim that $P = 1$. To prove this, let $p > 0$ and $g, h \in P$. Then $(g-1)^{p^t} = 0$ for some $t \geq 0$. Also, $(1-h)\hat{h} = 0$, where $\hat{h} = 1 + h + \cdots + h^{o(h)-1}$. Thus, by [6, Lemma 1.2.11], $(1-h)(g-1)\hat{h} = 0$. As $(1-h)\hat{h} = 0$, we have $(1-h)g\hat{h} = 0$, so $g\hat{h} = hg\hat{h}$. Since $g \in \operatorname{supp}(g\hat{h})$, we have $g = hgh^i$ for some i. That is, $g^{-1}hg \in \langle h\rangle$ and hence $\langle h\rangle$ is normal in $\langle P\rangle$ and, similarly, so is $\langle g\rangle$. Thus, $gh \in \langle g\rangle\langle h\rangle$, which is a subgroup of order dividing $o(g)o(h)$ and therefore a p-group. That is, P is a (normal) subgroup of G. But, for any $g \in P$, $g-1$ is nilpotent; thus, P is finite since the set of nilpotent elements of FG is finite. Consequently, by [9, Theorem 4.2.13], semiprimeness of FG implies $P = 1$ and the claim is established.

Take any $g \in T$. Since $p \nmid o(g)$, we have an idempotent $(1/o(g))\hat{g}$ and, by [6, Lemma 1.2.10], this idempotent is central. That is, $\langle g\rangle$ is normal in G, so T is a normal subgroup of G. By the Dedekind–Baer theorem (see [14, Theorem 5.3.7]), either T is abelian or $T \simeq Q_8 \times E \times O$, where Q_8 is the quaternion group of order eight, E is an elementary abelian 2-group, and O is an abelian group in which every element has odd order. Since T is also torsion, in either case, this implies that T is locally finite, and then the result follows from Theorem 1.1. □

It is easy to see that if $n \geq 2$, then

$$(x, {}_n y) = y^{1-n}x^{-1}y\cdots y^{-1}xy^n,$$

and the number of terms appearing in the right-hand side is $2^{n+1} + 1$.

Lemma 2.4. *Let F be a field and R an F-algebra whose unit group is Engel. If the set of nilpotent elements of R is finite, then there exists a nonzero polynomial $f(t) \in F[t]$ such that for every a and b in R satisfying $a^2 = b^2 = 0$, we have $f(ab) = 0$.*

Proof. First notice that for any integer z, $(1+a)^z = 1 + za$, and similarly for b, so, in particular, $1+a$ and $1+b$ are units. Thus, there exists a natural number $n = n(a,b) \geq 2$ so that

$$\begin{aligned} 1 &= (1+a, {}_n 1+b) \\ &= (1+b)^{1-n}(1+a)^{-1}(1+b)\cdots(1+a)(1+b)^n \\ &= (1+(1-n)b)(1-a)(1+b)\cdots(1+a)(1+nb). \end{aligned}$$

As R has only finite nilpotent elements, we can choose n so large such that the above equations hold for all such a and b. Define a polynomial

$$g(x,y) = (1+(1-n)y)(1-x)(1+y)\cdots(1+x)(1+ny) - 1.$$

Then we can write $g(x,y) = g_1(x,y) + g_2(x,y) + g_3(x,y)$, where g_1 is the sum of all of the monomials in g in which either x^2 or y^2 appears, g_2 is the sum of the other monomials starting with y and ending with x, and g_3 is the sum of the remaining terms. Now g_2 is a linear combination of terms of the form $(yx)^i$ for various i (otherwise, an x^2 or a y^2 must appear) and, indeed, the unique term of highest degree in g_2 is $\pm(1-n)(yx)^{2^n}$.

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Assume that the characteristic of F is such that $\pm(1-n)(yx)^{2^n} \neq 0$, that is, g_2 is not a zero polynomial. Now $xg_2(x,y)y$ is a polynomial in xy, so let us take $f(t)$ such that $f(xy) = xg_2(x,y)y$. Notice that $g_1(a,b) = 0$, since a^2 or b^2 will appear, and both are zero. Also, $ag_3(a,b)b = 0$, since each monomial in g_3 starts with x or ends with y; hence, we will have a^2 at the beginning or b^2 at the end. Thus, $f(ab) = ag_2(a,b)b = ag(a,b)b = 0$, as required.

Now let the characteristic of F be such that $\pm(1-n)(yx)^{2^n} = 0$; then we surely have $\pm n(yx)^{2^n} \neq 0$. But we also have

$$\begin{aligned} 1 &= (1+b)^{-1}(1+a, {}_n 1+b)(1+b) \\ &= (1-nb)(1-a)(1+b)\cdots(1+a)(1+(n+1)b), \end{aligned}$$

and a similar argument works. □

LEMMA 2.5. *Let R be an F-algebra, and suppose that R contains a right ideal I such that I satisfies a polynomial identity of degree n, but $I^n \neq 0$. Then R satisfies a nondegenerate multilinear generalized polynomial identity.*

PROOF. See [6, Lemma 1.2.16]. □

For any group G, we write $\Delta(G)$ for the FC center; that is, the subgroup of G consisting of elements with only finitely many conjugates.

LEMMA 2.6. *Let G be a torsion group and F a field of characteristic $p \geq 0$ such that FG is semiprime. If the set of nilpotent elements of FG is finite and $\mathcal{U}(FG)$ is an Engel group, then G is abelian.*

PROOF. Take $\alpha, \beta, \gamma \in FG$ such that $\alpha^2 = \beta\gamma = 0$. Now $(\gamma\rho\beta)^2 = 0$ for all $\rho \in FG$; thus, by Lemma 2.4, there exists a nonzero polynomial $f(t) \in F[t]$ such that $f(\alpha\gamma\rho\beta) = 0$. Therefore, $\beta f(\alpha\gamma\rho\beta)\alpha\gamma\rho = 0$, and this is a nonzero polynomial in $\beta\alpha\gamma\rho$. That is, the right ideal $I = \beta\alpha\gamma FG$ satisfies a nonzero polynomial of degree n. If $I^n = 0$, then semiprimeness of FG implies that $I = 0$; thus, $\beta\alpha\gamma = 0$, and Lemma 2.3 does the jobs. So, assume that $I^n \neq 0$; then, by Lemma 2.5, FG satisfies a nondegenerate multilinear generalized polynomial identity.

Now, by [9, Theorem 5.3.15], $(G : \Delta(G)) < \infty$ and $|\Delta(G)'| < \infty$. Now $\Delta(G)/\Delta(G)'$, as a torsion abelian group, is locally finite. Hence, $\Delta(G)$ and so G is locally finite and, by Theorem 1.1, we obtain that G' is a p-group and, since FG has a finite number of nilpotent elements, G' is finite. But, then, by [9, Theorem 4.2.13], $G' = 1$ and thus G is abelian. □

LEMMA 2.7. *Let F be a field of characteristic $p > 0$, G a group, and H a finite normal p-subgroup of G. If the set of nilpotent elements of FG is finite and $\mathcal{U}(FG)$ is an Engel group, then the set of nilpotent elements of $F(G/H)$ is finite and $\mathcal{U}(F(G/H))$ is an Engel group, too.*

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Proof. Let I be the kernel of the natural ring homomorphism $FG \longrightarrow F(G/H)$. By [6, Lemma 1.1.1], I is a nilpotent ideal. Therefore, the number of nilpotent elements of $F(G/H)$ is also finite.

On the other hand, let $\bar{\alpha}_i \in \mathcal{U}(FG/I)$ for $i = 1, 2$. Then, if $\bar{\beta}_i = (\bar{\alpha}_i)^{-1}$, we can lift $\bar{\alpha}_i$ and $\bar{\beta}_i$ up to α_i and β_i in FG. If we let $u_i = \alpha_i\beta_i - 1$, then $\bar{u}_i = 0$, so $u_i \in I$ and therefore $1 + u_i = \alpha_i\beta_i \in \mathcal{U}(FG)$. Thus, α_i has a right (and similarly left) inverse; so $\alpha_i \in \mathcal{U}(FG)$. Now there exists a natural number $n = n(\alpha_1, \alpha_2)$ so that $(\alpha_1, {}_n\alpha_2) = 1$ and, therefore, $(\bar{\alpha}_1, {}_n\bar{\alpha}_2) = 1$; that is, $\mathcal{U}(F(G/H))$ is Engel. □

For any prime p, we write $\Delta^p(G)$ for the subgroup generated by the p-elements in $\Delta(G)$.

Lemma 2.8. *Let G be a torsion group and F a field of characteristic $p > 0$ such that $\Delta^p(G)$ is finite. If the set of nilpotent elements of FG is finite and $\mathcal{U}(FG)$ is an Engel group, then G is p-abelian.*

Proof. Letting $H = C_G(\Delta^p(G))$, we first show that H is p-abelian. Since $(G : C_G(a)) < \infty$ for each $a \in \Delta^p(G)$, and $\Delta^p(G)$ is finite, we have $(G : H) < \infty$. Now, if $h \in \Delta^p(H)$ is a p-element, then h has finitely many conjugates in H. As H has finite index in G, there can be only finitely many conjugates in G as well. Thus, $\Delta^p(H) \leq \Delta^p(G)$. Also, H centralizes $\Delta^p(H)$, so $\Delta^p(H)$ is abelian and hence a finite p-group.

Now, by Lemma 2.7, the set of nilpotent elements of $F(H/\Delta^p(H))$ is finite and $\mathcal{U}(F(H/\Delta^p(H)))$ is an Engel group. But, $F(H/\Delta^p(H))$ is semiprime. So, $H/\Delta^p(H)$ is abelian by Lemma 2.6; thus, $H' \subseteq \Delta^p(H)$ is a finite p-group. This implies that H is p-abelian, as desired.

Now H/H', being an abelian torsion group, is locally finite. Thus, H is a locally finite group and, since $(G : H) < \infty$, G is also locally finite. Now Theorem 1.1 completes the proof. □

For any ring R, let $N(R)$ denote the nilpotent radical of R; that is, the sum of all nilpotent ideals in R.

Lemma 2.9. *Let G be a torsion group and F a field of characteristic $p \geq 0$. If the set of nilpotent elements of FG is finite and $\mathcal{U}(FG)$ is an Engel group, then G is p-abelian.*

Proof. If $p = 0$, the result follows from Lemma 2.6; so let $p > 0$. If $N(FG)$ is a nilpotent ideal, then $\Delta^p(G)$ is finite by a result of Passman [9, Theorem 8.1.12]. Thus, by Lemma 2.8, we may assume that $N(FG)$ is not nilpotent. Since for each nilpotent element $a \in FG$, $1 + a \in \mathcal{U}(FG)$ and, since there exists a finite number of such nilpotent elements, we can fix a natural number n such that for each pair of nilpotent elements $a, b \in FG$, we have $(1 + a, {}_n1 + b) = 1$.

We use similar methods to those used in the proof of [6, Lemma 1.2.26]. Let $F\{x_1, x_2\}$ be the free algebra on noncommuting indeterminates x_1 and x_2, and let $R = F\{x_1, x_2\}[[z]]$ be its power series ring. Then it is known that $1 + x_1z$ and $1 + x_2z$

generate a free subgroup of $\mathcal{U}(R)$. Thus, $0 \neq (1 + x_1 z, {}_n 1 + x_2 z) - 1$. Expanding this expression,

$$0 \neq (1 + x_1 z, {}_n 1 + x_2 z) - 1 = \sum_{i \geq 1} f_i(x_1, x_2) z^i,$$

where each f_i is a homogeneous polynomial of degree i. Let r be the smallest integer such that f_r is not the zero polynomial. Since $N(FG)$ is not nilpotent, we can choose a nilpotent ideal I of FG and $s \geq r$ such that $I^s \neq 0$, but $I^{s+1} = 0$ (see [6, Lemma 1.2.24]). Choose any $\alpha_1, \alpha_2, \alpha_3 \in I$. Then

$$0 = (1 + \alpha_1, {}_n 1 + \alpha_2) - 1 = \sum_{i=r}^{s} f_i(\alpha_1, \alpha_2)$$

and therefore

$$\sum_{i=r}^{s} f_i(\alpha_1, \alpha_2) \alpha_3^{s-r} = 0.$$

Also, as $I^{s+1} = 0$, the above equation shows that

$$f_r(\alpha_1, \alpha_2) \alpha_3^{s-r} = 0.$$

That is, $f_r(x_1, x_2) x_3^{s-r}$ is a polynomial identity of degree s for I. But $I^s \neq 0$, so, by Lemma 2.5, FG satisfies a nondegenerate multilinear generalized polynomial identity. Now the same argument as in the second paragraph of the proof of Lemma 2.6 completes the proof. □

Lemma 2.10. *Every left (right) Artinian Lie Engel ring is Lie nilpotent.*

Proof. See [12, Theorem 6]. □

Let R be a ring and let $x, y \in R$. Define the generalized Lie commutators as follows: $[x, {}_0 y] = x$ and $[x, {}_n y] = [x, {}_{n-1} y]y - y[x, {}_{n-1} y]$, $n = 1, 2, \ldots$. We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\mathcal{U}(FG)$ be an Engel group. By Lemma 2.9, G is p-abelian and, being torsion, G is locally finite. Given α and β in FG, let H be the subgroup of G generated by the supports of these elements. Since every finite Engel group is nilpotent, H is nilpotent and p-abelian. Therefore, FH is Lie nilpotent by Lemma 2.2 and, since $\alpha, \beta \in FH$, we have $[\alpha, {}_n \beta] = 0$ for some positive integer n. Consequently, FG is Lie Engel; thus, (1) implies (2).

Suppose that G is p-abelian and FG is Lie Engel. Then, again, G is locally finite. Let $\alpha_1, \ldots, \alpha_n$ be a finite number of elements of $\mathcal{U}(FG)$. We have to show that the subgroup $U = \langle \alpha_1, \ldots, \alpha_n \rangle$ of $\mathcal{U}(FG)$ is nilpotent. Let H be the subgroup of G generated by the supports of all of the α_i and α_i^{-1}. Since G is locally finite, H is a finite group and thus FH is an Artinian ring. Thus, by Lemma 2.10, FH is Lie nilpotent. Thereby, $\mathcal{U}(FH)$ is nilpotent by Lemma 2.2 and thus $U \subseteq \mathcal{U}(FH)$ is also nilpotent. Thus, (2) yields (3) and, clearly, (3) implies (1). We are done. □

The proof of Theorem 1.3 relies on the following lemma.

LEMMA 2.11. *Let G be a group containing an element of infinite order and F a field. Suppose that FG satisfies a nondegenerate multilinear generalized polynomial identity. Then there exists $\alpha \in FG$ so that $F[\alpha]$ is an infinite central subring of FG containing no zero divisors in FG.*

PROOF. See [6, Lemma 1.4.6]. □

Now our last result can be proved.

PROOF OF THEOREM 1.3. Take $\alpha, \beta, \gamma \in FG$ such that $\alpha^2 = \beta\gamma = 0$. We claim that $\beta\alpha\gamma = 0$. Otherwise, by a similar argument as in the first paragraph of the proof of Lemma 2.6, FG satisfies a nondegenerate multilinear generalized polynomial identity and then Lemma 2.11 implies that FG is a D-algebra, where D is an infinite commutative F-algebra having no zero divisors in FG. In particular, as in the proof of Lemma 2.6, if $\lambda \in D$, then $0 = f(\alpha\gamma\lambda\rho\beta) = f(\lambda\alpha\gamma\rho\beta)$. Since there are infinitely many such λ, we may apply the Vandermonde argument to conclude that $(\alpha\gamma\rho\beta)^n = 0$. Thus, $(\beta\alpha\gamma\rho)^{n+1} = 0$, that is, $\beta\alpha\gamma FG$ is a nil right ideal of bounded exponent. Thus, by a known result of Herstein and Levitzki, FG contains a nonzero nilpotent ideal, contracting semiprimeness, and thus $\beta\alpha\gamma FG = 0$, so $\beta\alpha\gamma = 0$, as claimed. Now the result follows from Lemma 2.3. □

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