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DICHOTOMIES OF SOLUTIONS FOR A CLASS OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS*

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Abstract. This paper considers the long-time behavior of solutions for the Cauchy problem of a class of second-order nonlinear differential equations: $-x''+f(t,x,x')x'+g(x)=h(t)$. Under appropriate conditions it is shown that the solutions of the problem possess some dichotomy properties.

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1. Introduction. At issue is the Cauchy problem of a class of second-order nonlinear differential equations

$$-x''+f(t,x,x')x'+g(x)=h(t), \tag{1.1}$$

$$x(0)=x_0, \quad x'(0)=x_1, \tag{1.2}$$

where f, g and h are assumed throughout the paper to be continuous functions; moreover, f and h are ω-periodic in t. We are interested in the long-time behavior of the solutions of the problem. Before stating our results, let us pause for a moment to observe some simple facts concerning the linear equation

$$-x''+bx'+x=h(t) \tag{1.3}$$

where $h\in C(R^1)$ and is ω-periodic. It is well known that for any $b\in R^1$, (1.3) has an ω-periodic solution σ (see, for instance [**5**, Theorem 2.1]). Now any solution of (1.3) can be given by the formula

$$x(t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+\sigma(t) \tag{1.4}$$

by choosing appropriate constants C_1 and C_2, where

$$\lambda_1=(b+\sqrt{b^2+4})/2>0, \quad \lambda_2=(b-\sqrt{b^2+4})/2<0.$$

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From this we see that for any solution x of (1.3), either $x(t)$ and $x'(t) \to \pm\infty$ (this occurs when $C_1 \neq 0$), or $|x(t) - \sigma(t)| + |x'(t) - \sigma'(t)| \to 0$ (this occurs when $C_1 = 0$) as $t \to +\infty$. In this paper we try to extend this basic knowledge along with some other properties of (1.3) to the nonlinear equation (1.1). We will show that under appropriate conditions, the solutions of (1.1)-(1.2) exhibit some *dichotomy* properties.

Assume that f, g are locally Lipschitz. Then for any $x_0, x_1 \in R^1$, there exists a $0 < T(x_0, x_1) \leq +\infty$ which will be denoted simply by T hereafter, such that (1.1)-(1.2) has a unique solution x on $[0, T)$; moreover, if $T < +\infty$, $\sup_{[0,T)}(|x(t)| + |x'(t)|) = +\infty$. For convenience, we denote by $\phi(t; x_0, x_1)$ the solution of (1.1)-(1.2) with initial data (x_0, x_1). We prove the following interesting results:

THEOREM 1.1. *Assume f and g are locally Lipschitz and satisfy:*
(F_1) *there exists a constant $\kappa > 0$ such that*

$$|f(t, x, p)| \leq \kappa(1 + |p|);$$

(G_1) *$g(x) \to \pm\infty$ as $x \to \pm\infty$.*
Let x be the solution of (1.1)-(1.2). *Then*
(1) *if* $\sup_{[0,T)} |x(t)| < +\infty$, *then* $\sup_{[0,T)} |x'(t)| < +\infty$ (*hence* $T = +\infty$)*;*
(2) *if* $\sup_{[0,T)} x(t) = +\infty$, *then*

$$\lim_{t\to T} x(t) = +\infty, \quad \lim_{t\to T} x'(t) = +\infty; \tag{1.5}$$

if $\inf_{[0,T)} x(t) = -\infty$, *then*

$$\lim_{t\to T} x(t) = -\infty, \quad \lim_{t\to T} x'(t) = -\infty; \tag{1.6}$$

(3) *the sets*

$$I_{+\infty} = \{(x_0, x_1) \in R^2 : \ x(t) = \phi(t; x_0, x_1) \textit{ satisfies } (1.5)\},$$

$$I_{-\infty} = \{(x_0, x_1) \in R^2 : \ x(t) = \phi(t; x_0, x_1) \textit{ satisfies } (1.6)\}$$

are nonempty open subsets of R^2;
(4) *for any $x_0 \in R^1$, the sets*

$$D_{+\infty}(x_0) = \{x_1 \in R^1 : \ x(t) = \phi(t; x_0, x_1) \textit{ satisfies } (1.5)\},$$

$$D_{-\infty}(x_0) = \{x_1 \in R^1 : \ x(t) = \phi(t; x_0, x_1) \textit{ satisfies } (1.6)\}$$

are nonempty open subsets of R^1;
(5) *for any $x_0 \in R^1$, the set*

$$DB(x_0) = \{x_1 \in R^1 : \ x(t) = \phi(t; x_0, x_1) \textit{ is bounded on } R^+\}$$

is a nonempty closed subset of R^1.

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Remark 1.1. An example satisfying the hypothesis in Theorem 1.1 is the well known Krall equation in atmosphere dynamics [**3**, **7**, **14**]:

$$-x'' + (a - |x'|)x' + \lambda x^3 - x = r\sin\omega t, \tag{1.7}$$

where a, λ, r and ω are constants with $\lambda > 0$.

THEOREM 1.2. *In addition to the hypothesis in Theorem* 1.1*, assume that* f, g *satisfy:*

(F_2) *f is bounded from below, i.e., there exists a constant $b > 0$ such that*

$$f(t, x, p) \geq -b, \quad \forall t, x, p \in R^1;$$

(G_2) $\liminf_{|x|\to+\infty} g(x)/x > 0$.

Suppose that the solution x of (1.1)-(1.2) *satisfies* (1.5) (*resp.* (1.6)). *Then there exist* $\lambda, C_0, C_1, C_2 > 0$ *such that*

$$x(t) \geq C_0 e^{\lambda t} - C_1 \quad (resp. \ \leq -C_0 e^{\lambda t} + C_1), \quad \forall t \geq 0,$$

$$x'(t) \geq \lambda C_0 e^{\lambda t} - C_2 \quad (resp. \ \leq -\lambda C_0 e^{\lambda t} + C_2), \quad \forall t \geq 0.$$

THEOREM 1.3. *In addition to the hypothesis in Theorem* 1.1*, if we further assume that $f(t, x, p)p + g(x)$ is strictly increasing in x, then when the solution x of* (*1.1*)-(*1.2*) *is bounded on* $[0, T)$ (*hence* $T = +\infty$)*, we have*

$$\lim_{t\to+\infty} (|x(t) - \sigma(t)| + |x'(t) - \sigma'(t)|) = 0, \tag{1.8}$$

where σ is the (*unique*) *ω-periodic solution of Eq.*(1.3). (*The existence of σ will be shown in Section 2.*)

An essential feature of the type of equations under our consideration is that the nonlinear terms $g(x)$ in these equations satisfy (G_1). For the study of long-time behavior of other types of second-order nonlinear differential equations, the interested reader is referred to, for instance [**2**, **10**, **12**, **15**, **16**, **17**] etc. and references therein for some recent developments.

This paper is organized as follows. In Section 2, we give some auxilary results. In Section 3, we prove Theorems 1.1-1.3 in detail.

2. Some auxiliary results. In this section we state some auxiliary results which will be used in the proofs of Theorems 1.1-1.3.

LEMMA 2.1. *Let $a, b \in R^1$ with $b - a \geq \tau > 0$, $x \in C^2((a, b))$. Suppose x satisfies the following differential inequality:*

$$|x''(t)| \leq c(1 + |x'(t)|^2), \quad t \in (a, b).$$

Assume that $\sup_{(a,b)} |x(t)| \leq M_0 < \infty$. *Then there exists a constant $C > 0$ depending only on c, M_0 and τ such that*

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$$\sup_{(a,b)} |x'(t)| \le C.$$

Lemma 2.1 is a particular case of Lemma 5.1 in [**6**, Ch. XII], which makes use of a Nagumo-type condition.

THEOREM 2.2. *Assume that f, g satisfies* (F_1) *and* (G_1)*, respectively. Then for any $a, b, \alpha, \beta \in R^1$ with $a < b$, the boundary value problem:*

$$\begin{cases} -x'' + f(t, x, x')x' + g(x) = h(t), & t \in (a, b); \\ x(a) = \alpha,\ x(b) = \beta \end{cases} \tag{BVP}$$

possesses at least one solution x.

THEOREM 2.3. *Assume that f, g satisfy the following conditions:*
$(F_1)^\circ$ *There exists a nonnegative and nondecreasing function $\mu \in C([0, +\infty))$ such that*

$$|f(t, x, p)| \le \mu(|x|)(1 + |p|).$$

$(G_1)^\circ$ $\exists\ A, B \in R^1$*, $A \le B$ such that*

$$g(A) \le h(t) \le g(B), \quad \forall t \in R^1.$$

Then Eq.(1.1) *has at least one ω-periodic solution x.*

The proofs of Theorems 2.2 and 2.3 are very standard using the well-known upper and lower solutions method, as given, e.g., in [**8**, **11**, **13**] etc., and thus are omitted. One can also derive these results by fully analogous argument as in the proofs of Lemma 3.1 in [**5**] and Lemma 3 in [**9**].

3. Proof of Theorems 1.1-1.3. In this section we prove in detail our main results stated in the introduction.

Proof of Theorem 1.1.
(1) Assume that $\sup_{[0,T)} |x(t)| < +\infty$. Then by (F_1), one easily deduces that

$$|x''(t)| \le c(1 + |x'(t)|^2)$$

on $[0, T)$. If $T < \infty$, by Lemma 2.1 we obtain directly that $\sup_{[0,T)} |x'(t)| < +\infty$. Assume that $T = +\infty$. By applying Lemma 2.1 to x on any interval $(a, a+1)$ for $a \ge 0$, we conclude immediately that $\sup_{[0,T)} |x'(t)| < +\infty$.

(2) We only consider the case where $\sup_{[0,T)} x = +\infty$. The proof for the other one is analogous. By (G_1), there exists $A > 0$ such that for any $z \ge A$,

$$g(z) \ge h(t) + 1, \quad \forall t \in R^1. \tag{3.1}$$

Since $\sup_{[0,T)} x = +\infty$, we can find at least one $t_0 > 0$ such that $x(t_0) \ge A$ with $x'(t_0) > 0$. We claim that x is nondecreasing for $t \ge t_0$ and hence

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$$\lim_{t \to T} x(t) = +\infty. \tag{3.2}$$

Indeed, if there exist $t_1, t_2 \geq t_0$ with $t_1 < t_2$ such that $x(t_1) > x(t_2)$, then x attains its maximum on $[t_0, t_2]$ at some point $s \in (t_0, t_2)$, at which we have $x(s) > x(t_0) \geq A$ and $x'(s) = 0$, $x''(s) \leq 0$. By (3.1), one finds that at the point s,

$$-x'' + f(s, x, x')x' + g(x) \geq h(s) + 1 > h(s),$$

which leads to a contradiction (as x is a solution of (1.1)).

In the sequel we show that $\lim_{t \to T} x'(t) = +\infty$. We first prove that x' is unbounded on $[0, T)$. Suppose not. Then for some $C > 0$, $|x'(t)| < C$ for $\forall t \in [0, T)$. If $T < +\infty$, it follows that x is bounded on $[0, T)$. This contradicts to (3.2). Thus we assume $T = +\infty$. By (G_1), there exists $B > 0$ such that for any $x \geq B$,

$$g(x) > \kappa(1 + C)C + |h(t)| + 1, \quad \forall t \geq 0, \tag{3.3}$$

where κ is the constant in (F_1). By (3.2), we can take a $t^* > 0$ such that $x(t) \geq B$ for $t \geq t^*$. Because of (3.3) we have

$$x'' = g(x) + f(t, x, x')x' - h(t) \geq 1, \quad \forall t \geq t^*. \tag{3.4}$$

As a consequence, we see that $x'(t) \to +\infty$ as $t \to +\infty$. This a contradiction.

Since x' is unbounded on $[0, T)$, we can take a sequence $\{t_n\} \subset [0, T)$, $t_n \to T$ such that $x'(t_n) \to +\infty$ (as x is nondecreasing on $[t_0, T)$ and hence $x'(t) \geq 0$ for $t \in [t_0, T)$). Let $M > 0$ be given arbitrary. By the same argument as in (3.3)-(3.4), we can prove that there exists $t_M > 0$ such that if $t \geq t_M$ and $x'(t) \leq M$, then $x''(t) \geq 0$. It follows that if $t_n \geq t_M$ is such that $x'(t_n) \geq M$, then

$$x'(t) \geq M, \quad \forall t \geq t_n.$$

This completes the proof of the desired result.

(3) As above, we only show $I_{+\infty}$ is a nonempty open subset of R^2.

Let $A > 0$ be such that (3.1) holds for any $z \geq A$. Assume that $(x_0, x_1) \in I_{+\infty}$. Take a $t_0 > 0$ such that

$$x(t_0) > A, \quad x'(t_0) > 0,$$

where $x = \phi(t; x_0, x_1)$. By continuity of ϕ with respect to initial data (x_0, x_1), there exists a $\delta > 0$ such that for any (y_0, y_1) with $|y_0 - x_0|, |y_1 - x_1| < \delta$, $y = \phi(t; y_0, y_1)$ satisfies

$$y(t_0) > A, \quad y'(t_0) > 0. \tag{3.5}$$

By the same argument as in showing (3.2), we can show that y is nondecreasing for $t \geq t_0$. Now we claim that $\lim_{t \to T} y(t) = +\infty$ and hence $(y_0, y_1) \in I_{+\infty}$. Indeed, if this is not the case, then

$$A < \lim_{t \to T} y(t) = c^* < +\infty. \tag{3.6}$$

By the first conclusion (1), we know that $y'(t)$ is bounded on $[0, T)$ and hence $T = +\infty$. Define

$$y_n(t) = y(t + n\omega), \quad t \in [0, \omega], \; n \in N.$$

By periodicity, we have

$$-y_n'' + f(t, y_n, y_n')y_n' + g(y_n) = h(t), \quad t \in (0, \omega). \tag{3.7}$$

Since $y(t)$ and $y'(t)$ are bounded on $[0, +\infty)$, by (1.1), one sees that $y''(t)$ is also bounded on $[0, +\infty)$. By the classical Arzela-Ascoli's Theorem, y_n has a subsequence y_{n_i} that converges to a function y^* in $C^1([0, \omega])$. Invoking (3.7), one also deduces that y_{n_i} converges to y^* in $C^2([0, \omega])$. In view of (3.6), we have $y^* \equiv c^*$ on $[0, \omega]$. Now we pass to the limit in (3.7) for y_{n_i} to obtain that

$$g(c^*) = h(t), \qquad t \in (0, \omega),$$

which contradicts (3.1).

The nonemptiness will be shown in the following argument.

(4) Let $x_0 \in R^1$. Let $A > 0$ be the constant such that (3.1) holds with any $z \geq A$. We take a $y_0 > \max(A, x_0)$ and consider the boundary value problem

$$\begin{cases} -x'' + f(t, x, x')x' + g(x) = h(t), & t \in (0, 1); \\ x(0) = x_0, \quad x(1) = y_0. \end{cases} \tag{3.8}$$

According to Theorem 2.2 (3.8) has at least a solution $x^* \in C^2([0, 1])$. Since $y_0 > \max(A, x_0)$, we can find a $t_0 \in (0, 1]$ such that $x^*(t_0) > A$ and $x^{*\prime}(t_0) > 0$. Let $x_1 = x^{*\prime}(0)$, $x(t) = \phi(t; x_0, x_1)$. By uniqueness of (1.1)-(1.2), we have $x(t) = x^*(t)$ on $[0, 1]$; therefore $x(t_0) > A$, $x'(t_0) > 0$. Repeating the argument below (3.5) in the proof of the third conclusion (3), one can show that $\sup_{[0,T)} x(t) = +\infty$ and hence $(x_0, x_1) \in I_{+\infty}$. Therefore $I_{+\infty}$ and $D_{+\infty}(x_0)$ are nonempty. The openness of $D_{+\infty}(x_0)$ is a consequence of that of $I_{+\infty}$.

Similarly we can show that $D_{-\infty}(x_0)$ is a nonempty open subset of R^1.

(5) Let $x_0 \in R^1$. By (1) and (4), one sees that $D_{+\infty}(x_0) \bigcup D_{-\infty}(x_0) \bigcup DB(x_0) = R^1$. As a topological consequence, we deduce immediately from (4) that $DB(x_0)$ is a nonempty closed subset of R^1.

The proof of the theorem is complete.

Now we turn to the proof of Theorem 1.2. We only consider the case when the solution of (1.1)-(1.2) satisfies (1.5). For the other one the argument is parallel and thus is omitted. We start with the following basic lemma.

LEMMA 3.1. *Let $b, k \in R^1$ with $k > 0$,*

$$\lambda_1 = (\sqrt{b^2 + 4k} - b)/2, \quad \lambda_2 = -(\sqrt{b^2 + 4k} + b)/2.$$

Assume that $x \in C^2([0, T))$ satisfies

$$x'' + bx' - kx \geq 0, \quad \forall t \in (0, T); \tag{3.9}$$

$$x(0) \geq 0, \quad x'(0) - \lambda_1 x(0) \geq 0. \tag{3.10}$$

Then $x(t)$, $x'(t) \geq 0$ *for* $t \in [0, T)$.

Proof. Equation (3.9) can be rewritten as

$$\frac{d}{dt}(x' - \lambda_1 x) - \lambda_2(x' - \lambda_1 x) \geq 0, \quad \forall t \in (0, T),$$

from which we infer that

$$x' - \lambda_1 x \geq (x'(0) - \lambda_1 x(0))e^{\lambda_2 t} \geq 0, \quad \forall t \in [0, T). \tag{3.11}$$

It follows that

$$x(t) \geq x(0)e^{\lambda_1 t} \geq 0, \quad \forall t \in [0, T). \tag{3.12}$$

Noting that $\lambda_1 > 0$, by (3.11) and (3.12),

$$x'(t) \geq \lambda_1 x(t) \geq 0, \quad \forall t \in [0, T).$$

The proof is complete.

Proof of Theorem 1.2. Assume that $x(t) = \phi(t; x_0, x_1)$ satisfies (1.5). By Theorem 1.1, we know that $\lim_{t\to T} x(t) = +\infty$, $\lim_{t\to T} x'(t) = +\infty$. Let $t_0 \geq 0$ be such that

$$x(t) > 0, \quad x'(t) > 0, \quad \forall t \geq t_0. \tag{3.13}$$

By (F_2) and (G_2), there exist k, $c_0 > 0$ such that

$$x'' + bx' - kx - c_0 \geq 0, \quad \forall t \geq t_0. \tag{3.14}$$

Let $\lambda_1 = (\sqrt{b^2 + 4k} - b)/2$, $\lambda_2 = -(\sqrt{b^2 + 4k} + b)/2$. We denote by y the solution of the initial-value problem:

$$y'' + by' - ky - c_0 = 0, \quad y(0) = x(t_0),\ y'(0) = x'(t_0).$$

Then

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} - \frac{c_0}{k}.$$

We claim that $c_1 > 0$. Indeed, if $c_1 \leq 0$, noting that $\lambda_1 > 0$ and $\lambda_2 < 0$, we will have either $y(0) = c_1 + c_2 - c_0/k < 0$ (in case $c_2 \leq 0$) or $y'(0) = \lambda_1 c_1 + \lambda_2 c_2 < 0$ (in case $c_2 > 0$), which yields a contradiction (as $x(t_0)$, $x'(t_0) > 0$). Write $u(t) = x(t) - y(t - t_0)$. Then $u(t_0) = u'(t_0) = 0$. u satisfies

$$u'' + bu' - ku \geq 0, \quad \forall t \geq t_0.$$

By virtue of Lemma 3.1, we have $u(t)$, $u'(t) \geq 0$ for $t \in [t_0, T)$, i.e.,

$$x(t) \geq c_1 e^{-\lambda_1 t_0} e^{\lambda_1 t} + c_2 e^{-\lambda_2 t_0} e^{\lambda_2 t} - \frac{c_0}{k}, \quad \forall t \geq t_0,$$

$$x'(t) \geq \lambda_1 c_1 e^{-\lambda_1 t_0} e^{\lambda_1 t} + \lambda_2 c_2 e^{-\lambda_2 t_0} e^{\lambda_2 t}, \quad \forall t \geq t_0,$$

which completes the proof of the desired results.

Finally we give a detailed proof of Theorem 1.3.

Proof of Theorem 1.3. Let x be the solution of (1.1)-(1.2). Assume that $\sup_{R^+} |x(t)| < +\infty$. By Theorem 1.1 (1), $\sup_{R^+} |x'(t)| < +\infty$. Further by (1.1), one sees that $\sup_{R^+} |x''(t)| < +\infty$. From Theorem 2.3, we know that under the assumptions of Theorem 1.3, (1.1) has at least a ω-periodic solution σ. In the sequel we show that (1.8) holds. If $x \equiv \sigma$ on R^+, then the proof is complete. Thus we may assume, without loss of generality that there exists $t_1 \in R^+$ such that $x(t_1) - \sigma(t_1) > 0$. In the following we first prove that $x - \sigma$ is monotone on $[t^*, +\infty)$ for some $t^* > 0$. We divide the argument into two cases.

Case 1. There exists $t_2 > t_1$ such that $x(t_2) - \sigma(t_2) < 0$. In this case, one can easily see that there exists $t^* > t_1$ such that

$$x(t^*) - \sigma(t^*) \leq 0, \quad x'(t^*) - \sigma'(t^*) < 0.$$

We show that $x - \sigma$ nonincreasing on $[t^*, +\infty)$. Suppose not, then (noting that $x - \sigma$ is stictly decreasing on $[t^*, t^* + \delta)$ for some small $\delta > 0$) $x - \sigma$ will have a local minimum point s, at which we have $x(s) < \sigma(s)$. Noting that $x'(s) = \sigma'(s)$, $x''(s) \geq \sigma''(s)$, by the strict monotonicity assumption on $f(t, z, p)p + g(z)$ in z, one finds at the point s that

$$-x'' + f(s, x, x')x' + g(x) < -\sigma'' + f(s, \sigma, \sigma')\sigma' + g(\sigma) = h(s),$$

which is a contradiction.

Case 2. $x(t) - \sigma(t) \geq 0$ for all $t \geq t_1$. If $x'(t) - \sigma'(t) \leq 0$ for $t \geq t_1$, then $x - \sigma$ is nonincreasing on $[t_1, +\infty)$ and thus $t^* = t_1$. Now assume that there exists a $t^* \geq t_1$ such that $x'(t^*) - \sigma'(t^*) > 0$. Since $x(t^*) - \sigma(t^*) \geq 0$, by fully analogous argument as in Case 1, we can show that $x - \sigma$ is nondecreasing on $[t^*, +\infty)$.

Now since x is bounded, we conclude that $\lim_{t \to +\infty}(x(t) - \sigma(t))$ exists. Let

$$\lim_{t \to +\infty} (x(t) - \sigma(t)) = c^*. \tag{3.15}$$

We show that $c^* = 0$ and thus

$$\lim_{t \to +\infty} (x(t) - \sigma(t)) = 0. \tag{3.16}$$

Define $x_n(t) = x(t + n\omega)$ for $t \in [0, \omega]$. By periodicity, we have

$$-x_n'' + f(t, x_n, x_n')x_n' + g(x_n) = h(t), \quad t \in (0, \omega). \tag{3.17}$$

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By similar argument as in showing the convergence of y_n in (3.7), we know that x_n has a subsequence x_{n_k} that converges in $C^2([0, \omega])$ to a function x^*. In view of (3.15) and the definition of x_n, we see that $x^*(t) = \sigma(t) + c^*$. We pass to the limit in (3.17) and obtain

$$-\sigma'' + f(t, \sigma + c^*, \sigma')\sigma' + g(\sigma + c^*) = h(t), \quad \forall t \in (0, \omega), \tag{3.18}$$

which implies by the strict monotonicity assumption on $f(t, z, p)p + g(z)$ in z that $c^* = 0$.

Finally we show that

$$\lim_{t\to+\infty} (x'(t) - \sigma'(t)) = 0. \tag{3.19}$$

Assume that $x - \sigma$ is nonincreasing on $[t^*, +\infty)$. In this case we infer from (3.16) that

$$x(t) - \sigma(t) \geq 0, \quad t \geq t^*. \tag{3.20}$$

Note that $x'(t) - \sigma'(t) \leq 0$ for any $t \geq t^*$. If (3.19) is not true, then there exists $\varepsilon_0 > 0$ and a sequence $t_n \in R^+$, $t_n \to +\infty$ such that $x'(t_n) - \sigma'(t_n) < -2\varepsilon_0$ for any $n \in N$. Since x'' and σ'' are bounded, we deduce that $x' - \sigma'$ is uniformly continuous on R^+, therefore there exists $\delta > 0$ such that for $\forall n$,

$$x'(t) - \sigma'(t) < -\varepsilon_0, \quad \forall t \in [t_n, t_n + \delta]. \tag{3.21}$$

Equations (3.16) and (3.21) imply that $x(t_n + \delta) - \sigma(t_n + \delta) < 0$ for n sufficiently large. This contradicts (3.20) and proves (3.19).

In a quite similar manner we can prove (3.19) in the case when $x - \sigma$ is nondecreasing on $[t^*, +\infty)$. The proof is complete.

Remark 3.1. When the function f in equation (1.1) is independent of t, x' and does not change sign, some results similar to Theorem 1.3 can be found in a recent paper [**1**, Theorem 3.2 and Proposition 3.6]). The method in [**1**] makes use of some topological tools. In contrast, ours seems to be more simple and direct.

A significant feature of the results in [**1**] is that the functions f and g are allowed to be defined only on an interval $(a, b) \subset R^1$, thus one can consider even some singular equations. We also point out that in the case in which f is a negative constant, the conclusion of Theorem 1.3 here is included in Theorem 3.2 of [**1**].

Remark 3.2. When a solution x of (1.1) satisfying (1.5) or (1.6) blows up in finite time (i.e., $T < +\infty$) is an interesting problem. It has been considered recently in [**4**]. For instance, we have proved that for the Krall equation (1.7), any solution satisfying (1.5) or (1.6) blows up in finite time.

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