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SOME GOOD SEQUENCES OF INTERPOLATORY POLYNOMIALS: ADDENDUM

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In 1974 [**2**] we used the $n + 2$ zeros of $(1 - x^2)P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$, where $P_n^{(\alpha,\beta)}(x)$ denotes Jacobi polynomials, to construct a sequence of linear operators $\{A_n^{(\alpha,\beta)}(f, x)\}$ which has the following properties:

(i) $A_n^{(\alpha,\beta)}(f, x)$ is a linear polynomial operator mapping $C[-1, 1]$ into polynomials of degree $\leqq n(1 + c)$, ($c > 0$ fixed but arbitrary)

(ii) $A_n^{(\alpha,\beta)}(f, x_{kn}) = f(x_{kn})$, $k = 1, \ldots, n$ where x_{kn} is the kth zero of $P_n^{(\alpha,\beta)}(x)$,

(iii) $|f(x) - A_n^{(\alpha,\beta)}(f, x)| \leqq C \cdot [\omega((1 - x^2)^{1/2}/n) + \omega(1/n^2)]$ where $\omega(\delta)$ is the modulus of continuity of f. Also

(*) $A_n^{(\alpha,\beta)}(f, x)$ is expressed in terms of $f(-1)$, $f(1)$ and $f(x_{kn})$, $(k = 1, 2, \ldots, n)$.

The polynomials $A_n^{(\alpha,\beta)}(f, x)$ use the function values $f(\pm 1)$, but do not interpolate $f(x)$ at these end-points. They do so only at the zeros of $P_n^{(\alpha,\beta)}(x)$. In view of this, it appears that our claim that "$A_n^{(\alpha,\beta)}(f, x)$ have the interpolatory property" is a little misleading.

However it is possible, at least for $-\frac{1}{2} \leqq \alpha, \beta \leqq \frac{3}{2}$ to modify the operator slightly so that the properties (1), (iii) and (*) remain valid while (ii) can be replaced by

(ii)$^+$ $\quad A_n(f, x_{kn}) = f(x_{kn}), \quad k = 1, \ldots, n, \quad A_n(f, \pm 1) = f(\pm 1).$

Even more, we prove that our modified interpolation polynomials satisfy the more precise Teliakovski-Gopengauz type estimate (iii)$^+$ (see below).

In order to do so we set

$$(1) \quad \tilde{A}_n^{(\alpha,\beta)}(f, x) = A_n^{(\alpha,\beta)}(f, x) + P_n^{(\alpha,\beta)}(x) \cdot \Lambda(f, x)$$

where

$$(2) \quad \Lambda(f, x) = \frac{1 + x}{2} \cdot \frac{f(1) - A_n^{(\alpha,\beta)}(f, 1)}{P_n^{(\alpha,\beta)}(1)} + \frac{1 - x}{2} \cdot \frac{f(-1) - A_n^{(\alpha,\beta)}(f, -1)}{P_n^{(\alpha,\beta)}(-1)}$$

It is easy to verify that $\tilde{A}_n^{(\alpha,\beta)}(f, x)$ interpolates $f(x)$ at all the zeros of

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$(1 - x^2)' P_n^{(\alpha,\beta)}(x)$ and the degree of $\tilde{A}_n^{(\alpha,\beta)}$ is $< n(1 + c)$. Our aim is to show that

$$\text{(iii)}^+ \quad |f(x) - \tilde{A}_n^{(\alpha,\beta)}(f, x)| \leqq c\omega\left(\frac{(1 - x^2)^{1/2}}{n}\right).$$

By virtue of (1) and (2) we have

$$\begin{aligned} |f(x) - \tilde{A}_n^{(\alpha,\beta)}(f, x)| &\leqq |f(x) - A_n^{(\alpha,\beta)}(f, x)| \\ &+ |f(1) - A_n^{(\alpha,\beta)}(f, 1)| \cdot (1 + x) \left| \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} \right| \\ &+ |f(-1) - A_n^{(\alpha,\beta)}(f, -1)| \cdot (1 - x) \left| \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)} \right|. \end{aligned}$$

It follows from Theorem 2 of [**2**] that

$$\begin{aligned} |f(x) - \tilde{A}_n^{(\alpha,\beta)}(f, x)| &\leqq c\left[\omega\left(\frac{(1 - x^2)^{1/2}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right] \\ &+ c\omega\left(\frac{1}{n^2}\right)\left[(1 + x)\left|\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}\right| + (1 - x)\left|\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)}\right|\right] \end{aligned}$$

Since

$$P_n^{(\alpha,\beta)}(1) = \binom{n + \alpha}{n}, \quad P_n^{(\alpha,\beta)}(-1) = \binom{n + \beta}{n}$$

and

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-1/2}\, O(n^{-1/2}), & cn^{-1} \leqq \theta \leqq \pi/2 \\ O(n^\alpha), & 0 \leqq \theta \leqq cn^{-1} \end{cases}$$

it follows that if $-\frac{1}{2} \leqq \alpha, \beta \leqq \frac{3}{2}$ we have

$$(3) \qquad |f(x) - \tilde{A}_n^{(\alpha,\beta)}(f, x)| \leqq c\left[\omega\left(\frac{(1 - x^2)^{1/2}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right].$$

This is the Timan-type estimate for our modified polynomials. In order to prove the Teliakovski-Gopengauz type estimate, we proceed as follows.

LEMMA 1. *Let*

$$(4) \qquad |f(x_1) - f(x_2)| \leqq A|x_1 - x_2| \quad x_1, x_2 \in [-1, 1]$$

and $P_n \in \pi_n$ *be such that*

$$(5) \qquad |f(x) - P_n(x)| \leqq B\left(\frac{\sqrt{1 - x^2}}{n} + \frac{1}{n^2}\right).$$

Then

$$(6) \qquad |P_n'(x)| \leqq c_1 A + c_2 B.$$

Proof. Let $|P_n'(\cos\theta_0)| = ||P_n'|| = M$. In switching form f to $-f$ if necessary, we can assume that $P_n'(\cos\theta_0) = ||P_n'||$. It follows from Bernstein's inequality that

$$(7)\qquad P_n'(\cos\theta) \geqq \frac{1}{2}M \quad \theta \in \left(\theta_0 - \frac{1}{2n}, \theta_0 + \frac{1}{2n}\right)$$

We infer that there exists an interval $I_n = [\theta_1, \theta_2]$ of length $1/2n$ inside $[0, \pi]$ containing θ_0, so that (7) holds for every $\theta \in I_n$. Consequently

$$P_n(\cos\theta_1) - P_n(\cos\theta_2) = \int_{\theta_1}^{\theta_2} P_n'(\cos\theta)\sin\theta\, d\theta$$
$$\geqq \frac{1}{2}M\int_{\theta_1}^{\theta_2}\sin\theta\, d\theta = \frac{1}{2}M(\cos\theta_1 - \cos\theta_2),$$

i.e.

$$(8)\qquad \begin{aligned}\frac{1}{2}M \leqq \frac{P_n(\cos\theta_1) - P_n(\cos\theta_2)}{\cos\theta_1 - \cos\theta_2} &= \frac{f(\cos\theta_1) - f(\cos\theta_2)}{\cos\theta_1 - \cos\theta_2}\\ &- \frac{f(\cos\theta_1) - P_n(\cos\theta_1)}{\cos\theta_1 - \cos\theta_2} + \frac{f(\cos\theta_2) - P_n(\cos\theta_2)}{\cos\theta_1 - \cos\theta_2}\end{aligned}$$

Now the second two terms together are smaller in modulus than

$$(9)\qquad \frac{B\left(\frac{\sin\theta_1}{n} + \frac{\sin\theta_2}{n}\right) + 2Bn^{-2}}{\cos\theta_1 - \cos\theta_2} \leqq \frac{B}{n\sin\frac{\theta_2 - \theta_1}{2}} + \frac{2Bn^{-2}}{1 - \cos\frac{1}{2n}} \leqq c_3 B$$

By virtue of (8), (4), and (9) we have

$$M \leqq 2A + 2c_3 B.$$

LEMMA 2. *Let $f(x)$ satisfy* (4) *and let $P_n \in \pi_n$ interpolate $f(x)$ at ± 1 and satisfy* (5). *Then*

$$(10)\qquad |f(x) - P_n(x)| \leqq c\frac{\sqrt{1 - x^2}}{n}.$$

Proof. If $1/n^2 < (\sqrt{1 - x^2})/n$, the result is trivial and follows at once from (5). If $(\sqrt{1 - x^2})/n < 1/n^2$, then x is close to $+1$ or -1, say $+1$. From Lemma 1, we have

$$(11)\qquad \begin{aligned}|f(x) - P_n(x)| &\leqq |f(x) - f(1)| + |P_n(x) - P_n(1)|\\ &\leqq (1 - x)A + (1 - x)(c_1 A + c_2 B).\end{aligned}$$

Now $1 - x < 1 - x^2 < (\sqrt{1 - x^2})/n$ and (10) follows from (11). By Lemma 2, (4) implies that $(\text{iii})^+$ holds whenever $f \in \text{Lip } 1$.

By virtue of a theorem of deVore [**1**], it follows that $(\text{iii})^+$ must be valid for arbitrary $f \in C[-1, 1]$.

In our original version of this paper we were concerned only with showing that the Timan-type estimate (3) is valid. We thank the referee for suggesting the generalization to the estimate $(\mathrm{iii})^+$. The transition from (3) to $(\mathrm{iii})^+$ is carried out as it was indicated by the referee in his report. It was also noted by the referee that the result possibly could be extended to higher continuity moduli.

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References

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