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ON ASYMPTOTIC BEHAVIOR OF INDUCED REPRESENTATIONS

LARRY BAGGETT AND KEITH F. TAYLOR

1. Introduction. This paper is devoted to the proof of the following theorem.

THEOREM 1.1. *Let H be a closed subgroup of a connected Lie group G, let N denote the largest (closed) subgroup of H which is normal in all of G, and suppose that π is a unitary representation of H whose restriction to N is a multiple of a character χ of N. Then every matrix coefficient of the induced representation U^π vanishes at infinity modulo the kernel of U^π providing that the following two conditions hold:*

i) *N is almost-connected (finite modulo its connected component).*

ii) *The subgroup H^k is "regularly related" to the diagonal subgroup D in G^k for at least one integer $k \geqq k_0$, where k_0 is determined by G and H.*

Remark 1. We recall that two subgroups G_1 and G_2 are "regularly related", in the sense of Mackey, if the double coset space is countably separated. Also, the assertion that the subgroups H^k and D are regularly related in G^k is equivalent to the statement that the orbit space for the action of G, via the diagonal map, on $(G/H)^k$ is countably separated. We shall need condition ii) precisely in order to employ Mackey's tensor product theorem.

Remark 2. We give below two examples which show the necessity of conditions i) and ii) or at least the necessity of some conditions like them.

Remark 3. Conditions i) and ii) both hold in a variety of cases, some quite general. For instance, if H is an algebraic sub-group of an algebraic group G, and if H is the stability subgroup for the character χ, then both conditions hold. For then N too must be algebraic (as a consequence of Proposition 2.1 below, for example) and therefore is itself almost connected. Also, the orbit space for the action of G on $(G/H)^k$ in this case is smooth because G is algebraic and the action is the restriction of a linear action. On the other hand, both of the examples below, which display the failure of conditions i) and ii), are nilpotent groups. Clearly these conditions are rather delicate. They are however easy to check in specific cases.

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Definition 1.2. Let U be a unitary representation of a locally compact group G. The *projective kernel* of U is the closed normal subgroup of G consisting of the elements g of G for which U_g is a scalar operator. It follows that if f and f' are elements of the space of U, then the absolute value of the matrix coefficient $(U_g(f'), f)$ is constant on the cosets of the projective kernel. We shall say that the representation U *vanishes at infinity*, *vanishes at infinity modulo its kernel*, or *vanishes at infinity modulo its projective kernel* if and only if the absolute value of each of its matrix coefficients vanishes at infinity, vanishes at infinity modulo its kernel, or vanishes at infinity modulo its projective kernel.

It is a conjecture of many experts that a locally compact group is of type I if and only if each of its irreducible representations vanishes at infinity modulo its projective kernel. Howe and Moore have made the initial step in verifying this. For, in [**3**], they show that each irreducible representation of an algebraic group does vanish at infinity modulo its projective kernel. Since algebraic groups are known to be of type I, their result supports the above-mentioned conjecture. The methods of [**3**] rely heavily on Mackey's theory and in particular on the tensor product theorem. Obviously we have borrowed this latter idea for our own purposes here.

In [**1**], the authors showed that a representation of a connected Lie group, induced from a character of a connected, closed, normal sub-group, actually vanishes at infinity modulo its kernel. The theorem of this paper is then a direct generalization of the older one, both conditions holding when $H = N$ and N is connected. The significance of the fact that these induced representations vanish at infinity modulo their kernels rather than just their projective kernels, i.e., that the projective kernel is compact modulo the kernel, is not yet clear to us. It does provide a slight improvement of the result in [**3**] for nilpotent groups. Indeed, one can argue by induction, à la Kirillov, to show that every irreducible representation of a connected nilpotent Lie group is induced from a character of an algebraic subgroup. Then our present theorem applies and we can conclude that each irreducible representation vanishes at infinity modulo its kernel. (The result of [**3**] would only have implied the vanishing at infinity modulo the projective kernel.)

Although our theorems are concerned at the outset with induced representations and not irreducible representations, it should be clear that they are formulated with the Mackey procedure in mind. A more obvious application of our ideas to Mackey's method can be found in Lemma D of [**2**].

1.3. *Some Integral Formulas*. Let G be a separable locally compact group with right Haar measure dg, and let m denote a probability measure on G

which is equivalent to Haar measure. Let ρ denote the Radon-Nikodym derivative of m with respect to Haar measure: $m(E) = \int_E \rho(g)dg$.

If H is a closed subgroup of G and θ denotes the natural map of G onto G/H, set μ equal to the measure $\theta_*(m)$ on G/H projected by θ from m:

$$\int_{G/H} f(s)d\mu(s) = \int_G f(\theta(g))dm(g) = \int_G f(\theta(g))\rho(g)dg.$$

The measure μ is quasi-invariant for the action of G on G/H, and we write p for the function on $(G/H) \times G$ which satisfies:

$$\int_{G/H} f(s \circ g)d\mu(s) = \int_{G/H} p(s, g)f(s)d\mu(s)$$

for all g in G and all measurable functions f on G/H.

Recall that any other quasi-invariant probability measure on G/H is equivalent to μ and that any such probability measure is $\theta_*(m')$ for some probability measure m' on G which is equivalent to Haar measure.

Now let k be a positive integer, and consider the measure space G^k equipped with the measure m^k. We again write θ, somewhat ambiguously, for the projection of G^k onto $(G/H)^k$, in which case the measure μ^k is $\theta_*(m^k)$. We let G act, via the diagonal map, on $(G/H)^k$:

$$(s_1, \ldots, s_k) \circ g = (s_1 \circ g, \ldots, s_k \circ g).$$

Denote by Ω the orbit space for this action and by σ the projection of $(G/H)^k$ onto Ω. Finally, let ν denote the measure $\sigma_*(\mu^k)$ on Ω. Then, for any measurable function f on Ω, we have

$$\int_\Omega f(\omega)d\nu(\omega) = \int_{[(G/H)^k]} f(\sigma(z))d\mu^k(z) = \int_{[G^k]} f(\sigma(\theta(g_1, \ldots, g_k)))$$
$$\times\, dm^k(g_1, \ldots, g_k) = \int_{[G^k]} f(\sigma(\theta(g_1, \ldots, g_k)))\left[\prod_{j=1}^k \rho(g_j)\right]dg_1 \ldots dg_k.$$

Of course Ω may well be a "bad" Borel space, in which case there will exist very few measurable functions on Ω. We shall say that this action of G on $(G/H)^k$ is *smooth* if Ω is countably separated in its quotient Borel structure.

1.4. *Formula for an Induced Representation.* We continue with the notation of the previous discussion. Fix a regular Borel cross-section γ of G/H into G. See [**4**]. If π is a unitary representation of H acting in the Hilbert space X, we define the induced representation U^π of G to act in the Hilbert space tensor product of $L^2(\mu)$ and X and to be given by the formula

$$(U_g^{\pi}(f \otimes \psi), (f \otimes \psi)) = \int_{G/H} [p(s, g)]^{-1/2} f(s \circ g)\bar{f}(s)$$
$$\times ([\pi_{(\gamma(s)g[\gamma(s \circ g)]^{-1})}](\psi), \psi) d\mu(s)$$

for f in $L^2(\mu)$ and ψ in X.

Of course the equivalence class of the induced representation U^{π} is independent of the choices of μ and γ.

We conclude this introduction by giving two examples, the first showing the importance of condition i) in Theorem 1.1 and the second showing the importance of condition ii).

Example 1.5. Let G be the four-dimensional group of real matrices of the form

$$\begin{matrix} 1 & t & t & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{matrix}$$

and let H be the closed, normal, abelian subgroup defined by the relations $t = 0$, y is an integer, and x is an integral multiple of 2π. We parameterize the elements of G by the natural quadruples (t, z, y, x), in which case H consists of the elements $(0, z, k, 2\pi n)$ for z arbitrary and k and n integers. The map θ, defined by

$$\theta(t, z, y, x) = (t, e^{2\pi i y}, e^{ix}),$$

can be taken as the natural map of G onto G/H, G/H having been identified with $\mathbf{R} \times \mathbf{T}^2$. A cross-section γ can then be defined by

$$\gamma(t, e^{2\pi i\alpha}, e^{i\beta}) = (t, 0, \alpha, \beta).$$

We choose μ to be the quasi-invariant probability measure

$$(2\pi)^{-3/2} e^{-(t^2/2)} dt d\alpha d\beta,$$

on G/H, and let χ be the representation (character) of H defined by

$$\chi(0, z, k, 2\pi n) = e^{iz}.$$

It follows from the formula in 1.4 that the projective kernel of the induced representation U^{χ} consists of the elements $(0, z, 0, 0)$.

Now if f is any nonzero element of $L^2(\mu)$, the space of U^{χ}, which is independent of the two circle variables $(f(t, e^{2\pi i\alpha}, e^{i\beta}) \equiv f(t, 1, 1))$, then

$$([U^{\chi}_{(0,0,k,2\pi n)}](f), f) = \int_{\mathbf{R}\times\mathbf{T}\times\mathbf{T}} |f(t, 1, 1)|^2 \chi$$
$$\times ((t, 0, \alpha, \beta)(0, 0, k, 2\pi n)[(t, 0, \alpha, \beta)]^{-1})(2\pi)^{-3/2} e^{-(t^2/2)} dt d\alpha d\beta$$
$$= \int_{\mathbf{R}} |f(t, 1, 1)|^2 e^{it(k+2\pi n)} (2\pi)^{-3/2} e^{-(t^2/2)} dt = \hat{F}(k + 2\pi n).$$

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where F is the nonzero L^1 function on $\mathbf{R}$ defined by

$$F(t) = (1/2\pi)|f(t, 1, 1)|^2 e^{-(t^2/2)}.$$

Clearly then $([U^\chi_{(0,0,k,2\pi n)}](f), f)$ does not vanish as $(0, 0, k, 2\pi n)$ tends to infinity, i.e., U^χ does not vanish at infinity modulo its projective kernel.

On the other hand, the action of G on $(G/K)^k$ is smooth for any integer k because H is normal. Of course the group N, which is the same as H here, is not almost-connected.

This example shows the significance of assumption i). The next example shows that, even when both H and N are connected, the assumption that G act smoothly on $(G/H)^k$ cannot entirely be dropped.

Example 1.6. Let G be the three-dimensional Heisenberg group modulo a discrete central subgroup. That is, as a manifold G is $\mathbf{T} \times \mathbf{R}^2$, where $\mathbf{T}$ is the unit circle in $\mathbf{C}$ and multiplication on G is given by, for (λ, q, p), $(\lambda', q', p') \in G$,

$$(\lambda, q, p)(\lambda', q', p') = (\lambda\lambda' e^{ipq'}, q + q', p + p').$$

Let $H = \{(1, 0, p) : p \in \mathbf{R}\}$, a closed subgroup of G. Then G/H can be identified with $\mathbf{T} \times \mathbf{R}$ and $\gamma: G/H \to G$ given by $\gamma(\lambda, q) = (\lambda, q, 0)$ is a regular cross-section. The action of G on $\mathbf{T} \times \mathbf{R}$ is given by

$$(\lambda, q) \cdot (\lambda', q', p') = (\lambda\lambda' e^{-ip'(q+q')}, q + q').$$

We want to investigate the action of G, via the diagonal map, on $(G/H)^k$ for various k to determine if it is smooth.

The case $k = 1$ is trivial and without significance. When $k = 2$, the action is given by

$$\begin{aligned}((\lambda_1, q_1), (\lambda_2, q_2)) &\cdot (\lambda', q', p') \\ &= ((\lambda_1\lambda' e^{-ip'(q_1+q')}, q_1 + q'), (\lambda_2\lambda' e^{-ip'(q_2+q')}, q_2 + q')).\end{aligned}$$

Then $(G/H)^2$ is the union of the two invariant Borel sets A and B, where

$$A = \{((\lambda_1, q), (\lambda_2, q)) : \lambda_1, \lambda_2 \in \mathbf{T}, q \in \mathbf{R}\} \quad \text{and}$$
$$B = (G/H)^2 - A = \{((\lambda_1, q_1), (\lambda_2, q_2)) : q_1 \neq q_2\}.$$

The Borel set $\{((\lambda, 0), (1, 0)) : \lambda \in \mathbf{T}\}$ is a cross-section for the orbits in A and $\{(1, 0), (1, q) : q \in \mathbf{R} - 0\}$ is a cross-section for B. So the orbit space is countably separated and the action is smooth.

The interested reader can now fill in the details of the case $k = 3$. There are several different kinds of orbits in $(G/H)^3$ which can be easily parametrized as above and then there is the set,

$$C = \{((\lambda_1, q), (\lambda_2, q + \alpha), (\lambda_3, q + \beta)) : \lambda_1, \lambda_2, \lambda_3 \in \mathbf{T}, q, \alpha, \beta \in \mathbf{R},$$
$$\alpha \neq 0, \beta \neq 0, \alpha/\beta \text{ irrational}\}.$$

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This is an invariant co-null set in $(G/H)^3$. The orbit of any point $((\lambda_1, q), (\lambda_2, q + \alpha), (\lambda_3, q + \beta))$ in C is

$$\{((\lambda, s), (\lambda_2\lambda_1^{-1}\lambda e^{it\alpha}, s + \alpha), (\lambda_3\lambda_1^{-1}\lambda e^{it\beta}, s + \beta))\},$$

where (λ, t, s) runs through $\mathbf{T} \times \mathbf{R} \times \mathbf{R}$. The orbit space in question is, up to a null set, the same as the one for the action of G restricted to C. This orbit space is not countably separated. The details of showing this are messy and will be left to the reader. We remark only on the obvious similarity with the well known non-smooth action of $\mathbf{R}$ on $\mathbf{T}^2$.

Thus, for $k = 3$, H^k is not regularly related to the diagonal in G^k for this group G, likewise for any larger k. In this connection the following remark is very significant:

Remark 1.7. With G and H as in example 1.6 and χ, the trivial character on H, U^χ does not vanish at infinity. (The projective kernel is trivial.) This is easy to check.

Thus, unwieldy as it is, the hypothesis that H^k is regularly related to the diagonal in G^k for suitable k, which appears in Theorem 1.1 is necessary or, at least, it cannot be totally eliminated.

2. Proof of theorem 1.1. Let H be a closed subgroup of a connected Lie group G, and let N denote the largest subgroup of H which is normal in all of G. We have then $N = \bigcap_{g \in G} (gHg^{-1})$. The following proposition may very well be a part of the Lie theory folklore. For completeness, however, we present a short proof in Section 3.

PROPOSITION 2.1. *There exists an integer k_0, with $k_0 \leqq 2^{(\dim H - \dim N + 1)}$, such that for almost all k_0-tuples $(g_1, \ldots, g_{k_0})$ in G^{k_0} we have that*

$$N = \bigcap_{i=1}^{k_0} (g_i H g_i^{-1}).$$

The integer k_0 of the last proposition is the one for which we must have the subgroups H^k and D regularly related in G^k for some $k \geqq k_0$, in Theorem 1.1. Only one such k is required.

Proof of Theorem 1.1. Let G, H, N, and k be as in the above discussion. We suppose that π is a unitary representation of H whose restriction to N is a multiple of a character χ of N. We assume that N is almost-connected and that the action of G, via the diagonal map, on $(G/H)^k$ is smooth. By Mackey's tensor product theorem [**4**], we may express the k-fold tensor power $[U^\pi]^{\otimes k}$ of the induced representation U^π as a direct integral over the orbit space Ω of $(G/H)^k$ under the action of G:

$$[U^\pi]^{\otimes k} \equiv \int_\Omega \oplus\ \pi^\omega d\nu(\omega).$$

Since ν is the measure on Ω projected by $\sigma \cdot \theta$ from the measure m^k on G^k, see 1.3, it follows, by Schur's Lemma for instance, that $[U^\pi]^{\otimes k}$ is equivalent to a subrepresentation of the representation V given by

$$V = \int_{[G^k]} \oplus\ \pi^{\sigma(\theta(g_k,\ldots,g_k))}\, dm^k(g_1, \ldots, g_k).$$

It is clear too that the kernel of U^π is contained in the kernel of $[U^\pi]^{\otimes k}$, and also that the kernel of $[U^\pi]^{\otimes k}$ equals the kernel of V. We will be done then if we can show that the matrix coefficients for V vanish at infinity modulo the kernel of U^π.

Now it follows from Mackey's development of the tensor product theorem that the representation $\pi^{\sigma(\theta(g_1,\ldots,g_k))}$ is equivalent to an induced representation:

$$\pi^{\sigma(\theta(g_1,\ldots,g_k))} \equiv \operatorname{ind}^G{}_{[\cap_{i=1}^k (g_iHg_i^{-1})]}(\rho^{(g_1,\ldots,g_k)}),$$

where $\rho^{(g_1,\ldots,g_k)}$ is defined on $\cap_{i=1}^k (g_iHg_i^{-1})$ by

$$\rho_h^{(g_1,\ldots,g_k)} = \prod_{i=1}^k \otimes\ \pi_{(g_i^{-1}hg_i)}.$$

Hence V is equivalent to

$$\int_{[G^k]} \oplus\ [\operatorname{ind}^G{}_{[\cap_{i=1}^k (g_iHg_i^{-1})]}(\rho^{(g_1,\ldots,g_k)})]dm^k(g_1, \ldots, g_k).$$

It follows from the formula in 1.4 that the kernel of U^π is contained in N and that the commutator subgroup of N belongs to this kernel. We may as well factor out this kernel, in which case we may assume that N is an almost-connected, closed, normal, abelian subgroup of G. Thus write N as a direct product $N = JA$ for J a compact central subgroup of G and A a closed, not necessarily normal, vector, subgroup. We represent the character χ as a pair (α, λ) for α in $\hat{J}$ and λ in $\hat{A}$. Then the action of G on the character χ is given by $(\alpha, \lambda) \cdot g = (\alpha, \lambda^g)$, where the map $g \to \lambda^g$ is an analytic map of G into the Euclidean space $\hat{A}$.

Since the integer k is the one guaranteed by Proposition 2.1, we have that

$$\bigcap_{i=1}^k (g_iHg_i^{-1}) = N$$

for almost all k-tuples $(g_1, \ldots, g_k)$. The representation $\rho^{(g_1,\ldots,g_k)}$ is, for such a k-tuple, a multiple of a character of N:

$$\rho_n^{(g_1,\ldots,g_k)} = \prod_{i=1}^k \chi(g_i^{-1}ng_i)I,$$

where I denotes the identity operator on the space of $\pi^{\otimes k}$. Therefore V is equivalent to a multiple of the representation W defined by

$$W = \int_{[G^k]} \oplus \, [\mathrm{ind}_N{}^G \, (\chi^{(g_1,\ldots,g_k)})] dm^k(g_1, \ldots, g_k),$$

where $\chi^{(g_1,\ldots,g_k)}$ is the character of N given by

$$\chi^{(g_1,\ldots,g_k)}(n) = \prod_{i=1}^{k} \chi(g_i^{-1} n g_i).$$

It will suffice then to show that W vanishes at infinity on G.

Let us show first that the matrix coefficients $(W_n(f'), f)$ vanish at infinity on N for any two elements f, f' of the space of W. We let γ denote a fixed regular Borel cross-section of G/N into G, and we write P for a probability measure on G/N which is equivalent to Haar measure. Then the Hilbert space for each of the induced representations in the integrand for W can be taken to be the space $L^2(P)$. Letting f, f' be two elements of the space of W, i.e., two square-integrable functions from the measure space (G^k, m^k) into the Hilbert space $L^2(P)$, we have

$$\begin{aligned}
&(W_n(f'), f) \\
&\quad = \int_{[G^k]} ([\mathrm{ind}_H{}^G \, (\chi^{(g_1,\ldots,g_k)})]_n [f'(g_1, \ldots, g_k)], f(g_1, \ldots, g_k))_{L^2(P)} \\
&\quad \times dm^k(g_1, \ldots, k) = \int_{[G^k]} \int_{G/N} [f'(g_1, \ldots, g_k)](s) [\bar{f}(g_1, \ldots, g_k)](s) \\
&\quad \times \chi^{(g_1,\ldots,g_k)}(\gamma(s) n [\gamma(s)]^{-1}) dP(s) dm^k(g_1, \ldots, g_k) \\
&\quad = \int_{[G^k]} \int_{G/N} [f'(g_1, \ldots, g_k)](s) [\bar{f}(g_1, \ldots, g_k)](s) \\
&\quad \times \prod_{i=1}^{k} \chi(g_i^{-1} \gamma(s) n [g_i^{-1} \gamma(s)]^{-1}) dP(s) dm^k(g_1, \ldots, g_k) \\
&\quad = \int_{G/N} \int_{[G^k]} [f'(g_1, \ldots, g_k)](s) [\bar{f}(g_1, \ldots, g_k)](s) \\
&\quad \times \prod_{i=1}^{k} \chi(g_i^{-1} \gamma(s) n [g_i^{-1} \gamma(s)]^{-1}) \prod_{j=1}^{k} \rho(g_j) dg_1 \ldots dg_k dP(s) \\
&\quad = \int_{G/N} \int_{[G^k]} [f'(\gamma(s) g_1, \ldots, \gamma(s) g_k)](s) [\bar{f}(\gamma(s) g_1, \ldots, \gamma(s) g_k)](s) \\
&\quad \times \prod_{i=1}^{k} \chi(g_i^{-1} n g_i) \prod_{j=1}^{k} \rho(\gamma(s) g_j) [\delta(\gamma(s))]^k dg_1 \ldots dg_k dP(s) \\
&\qquad\qquad\qquad = \int_{[G^k]} \chi^{(g_1,\ldots,g_k)}(n) F_{(f,f')}(g_1, \ldots, g_k) dg_1 \ldots dg_k,
\end{aligned}$$

where $F_{(f,f')}$ is defined on G^k by

$$F_{(f,f')}(g_1, \ldots, g_k) = \int_{G/N} [f'(\gamma(s)g_1, \ldots, \gamma(s)g_k)](s)[\bar{f}(\gamma(s)g_1, \ldots, \gamma(s)g_k)](s) \times \prod_{j=1}^{k} \rho(\gamma(s)g_j)[\delta(\gamma(s))]^k dP(s),$$

and where δ denotes the modular function on G.

Writing the elements of N as products $n = ja$ for j in J and a in A, we have

$$\begin{aligned}
|(W_{ja}(f'), f)| &= \left| \int_{[G^k]} F_{(f,f')}(g_1, \ldots, g_k) \prod_{i=1}^{k} (\alpha, \lambda)(g_i^{-1} n g_i) dg_1 \ldots dg_k \right| \\
&= \left| \int_{[G^k]} F_{(f,f')}(g_1, \ldots, g_k) \prod_{i=1}^{k} ((\alpha, \lambda) \cdot (g_i))(n) dg_1, \ldots, dg_k \right| \\
&= \left| \int_{[G^k]} F_{(f,f')}(g_1, \ldots, g_k) \prod_{i=1}^{k} \lambda^{(g_i)}(a) dg_1, \ldots, dg_k \right| \\
&= \left| \int_{[G^k]} F_{(f,f')}(g_1, \ldots, g_k) e^{i((a,\ldots,a),\varphi(g_1,\ldots,g_k))} dg_1, \ldots, dg_k \right|,
\end{aligned}$$

where φ is the analytic map of G^k into the Euclidean space $\hat{A}^k$ defined by

$$\varphi(g_1, \ldots, g_k) = \lambda^{(g_1)}, \ldots, \lambda^{(g_k)}.$$

The following lemma, which follows either from the results of [**5**] or [**1**], is now useful.

LEMMA 2.2. *Let φ be an analytic map of an analytic manifold M into a Euclidean space $\mathbf{R}^n$. Suppose that η is a finite measure on M which is locally absolutely continuous with respect to Lebesgue measure on M, and write $\varphi_*(\eta)$ for the finite measure on $\mathbf{R}^n$ which is projected by φ from η. Then the Fourier-Stieltjes transform of $\varphi_*(\eta)$ vanishes at infinity if and only if the range of φ belongs to no proper hyperplane. Explicitly, the map*

$$y \rightarrow \int_{[\mathbf{R}^n]} e^{i(y,x)} d\varphi_*(\eta)(x) = \int_M e^{i(y,\varphi(m))} d\eta(m)$$

vanishes at infinity if and only if the range of φ belongs to no proper hyperplane.

In the case at hand, we let M be the manifold G^k and set η equal to the locally absolutely continuous measure $F_{(f,f')} dg_1 \ldots dg_k$. It follows immediately that η is finite. ($|\eta|(G^k) \leq \|f\| \, \|f'\|$.) Take φ to be the map which sends $(g_1, \ldots, g_k)$ to the vector in $\hat{A}^k$ whose i'th component is $\lambda^{(g_i)}$.

Let us check that the range of φ is not contained in any proper hyperplane. By way of contradiction, suppose that $(a_1, \ldots, a_k)$ is a nonzero element of A^k such that $((a_1, \ldots, a_k), \varphi(g_1, \ldots, g_k))$ equals a constant c for all k-tuples $(g_1, \ldots, g_k)$. Without loss of generality, suppose that $a_1 \neq 0$. Fixing the variables $g_2, \ldots, g_k$ all to be equal to the identity, we find that (a_1, λ^g) equals a constant c' for all g in G. But then there would exist a nonzero element b_1 of A, in fact a multiple of a_1, such that (b_1, λ^g) is a multiple of 2π for all g in G. But then, directly from the formula in 1.4, we see that the kernel of U^π is nontrivial. This is a contradiction to our assumption, and hence the range of φ belongs to no proper hyperplane.

Now fix an $\epsilon > 0$. There exists, by the lemma, a compact subset C of N (depending of course on f and f') such that

$$|(W_n(f'), f)| < \epsilon$$

whenever n is outside C. This shows at least that $W|_N$ vanishes at infinity. We deduce from this the following important observation.

2.3. For any f in the space of W and any g in G there exists a compact subset D of N and a neighborhood U of g (both depending on f and g) such that

$$|(W_{ng'}(f), f)| < 2\epsilon$$

whenever g' is in U and n is outside D.

Finally, let us write the elements of G as products $g = n\gamma(t)$ for n in N and t in G/N, and let us fix an element f in the space of W. Then

$$\begin{aligned}
|(W_g(f), f)| &= |(W_{n\gamma(t)}(f), f)| \\
&= \Big| \int_{[G^k]} \int_{G/N} [p(s, g)]^{-1/2} [f(g_1, \ldots, g_k)](st) \\
&\quad \times [\bar{f}(g_1, \ldots, g_k)](s) \chi^{(g_1, \ldots, g_k)} (\gamma(s) g [\gamma(st)]^{-1}) dP(s) dm^k(g_1, \ldots, g_k) \Big| \\
&\leqq \int_{[G^k]} \int_{G/N} [p(s, g)]^{-1/2} |[f(g_1, \ldots, g_k)](st)| \\
&\qquad \times |[f(g_1, \ldots, g_k)](s)| dP(s) dm^k(g_1, \ldots, g_k).
\end{aligned}$$

It follows from the dominated convergence theorem, applied to the finite measure m^k, that this integral tends to zero as t tends to infinity in G/N. Hence there exists a compact subset K of G/N such that

$$|(W_{n\gamma(t)}(f), f| < \epsilon$$

whenever t is outside K. Let L be the compact subset $\overline{\gamma(K)}$ of G. Then, using a compactness argument and 2.3, we conclude that there exists a

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compact subset E of N such that

$$|(W_{n\gamma(t)}(f), f)| < 2\epsilon$$

whenever $n\gamma(t)$ is outside EL.

This of course completes the proof.

3. Proof of 2.1. Proposition 2.1 will be established by means of a series of lemmas.

LEMMA 3.1. *Suppose L and M are Lie subgroups of a connected Lie group G. If the connected components of these groups are not normal in G, then for almost all g in G,*

$$\dim (gLg^{-1} \cap M) < \max (\dim L, \dim M).$$

Proof. It can clearly be assumed that $\dim L = \dim M$. Let L_0 and M_0 be the respective connected components of L and M. Suppose B were a subset of G of positive measure such that

$$\dim (gLg^{-1} \cap M) = \dim M \quad \text{for all } g \in B.$$

Then $gL_0g^{-1} \cap M_0 = M_0$, for all $g \in B$ and, consequently, $hL_0h^{-1} = L_0$, for all $h \in B^{-1}B$. Thus the normalizer of L_0 in G is a closed subgroup of positive measure in a connected Lie group and so must be all of G. But L_0 is not normal in G, hence such a set B cannot exist.

In the situation we are concerned with, H is a closed subgroup of a connected Lie group G and N is the maximal normal subgroup of G contained in H.

LEMMA 3.2. *There exists an integer $k \leqq 2^{(\dim H - \dim N)}$ such that*

$$\dim \left(\bigcap_{i=1}^{k} g_iHg_i^{-1}\right) = \dim N,$$

for almost all k-tuples $(g_1, \ldots, g_k) \in G^k$.

Proof. Recall that m is a probability measure on G which is mutually absolutely continuous with respect to right Haar measure and hence with respect to left Haar measure. Then m^k is quasi-invariant under the action of G on G^k via left multiplication of diagonal elements. That is, if $g \in G$ and $y = (g_1, \ldots, g_k) \in G^k$, then

$$g \cdot y = (gg_1, \ldots, gg_k).$$

Let $p(\cdot, g)$ denote the Radon-Nikodym derivative of the measure $E \rightarrow m^k(g \cdot E)$ with respect to m^k. Recall that $p(y, g)$ is jointly measurable in y and g.

Observe that for any k, $N \subseteq \bigcap_{i=1}^{k} g_iHg_i^{-1}$, for all $(g_1, \ldots, g_k) \in G^k$.

Hence, it suffices to prove that if $j \leqq (\dim H - \dim N)$ and $k = 2^j$, then for almost all k-tuples $(g_1, \ldots, g_k)$ in G^k we have

$$\dim \left(\bigcap_{i=1}^{k} g_i H g_i^{-1} \right) \leqq \dim H - j.$$

(Where no confusion can arise, we use $x = (g_1, \ldots, g_k)$ and $H^x = \bigcap_{i=1}^k g_i H g_i^{-1}$.)

We prove the above statement by induction on j. It is clearly true when $j = 0$. Suppose it holds for some j, $0 \leqq j < (\dim H - \dim N)$. Thus, there exists a measurable $A \subseteq G^k$ such that

$$m^k(G^k - A) = 0 \quad \text{and}$$
$$(\dim H^y) \leqq \dim H - j \quad \text{for all } y \in A.$$

We claim that for almost all $(x, y) \in A \times A$,

$$\dim (H^x \cap H^y) \leqq \dim H - j - 1.$$

To see this, define f on $G^k \times G^k$ by

$$f(x, y) = \max \{0, [\dim (H^x \cap H^y) - \dim H + j + 1]\}.$$

Then, if $f(x, y) = 0$ almost everywhere in $G^k \times G^k$ we will have established the above claim. Consider

$$\begin{aligned} &\int_{G^k} \int_{G^k} f(x, y) dm^k(x) dm^k(y) \\ &\quad = \int_G \int_{G^k} \int_{G^k} f(x, g \cdot y) p(y, g) dm^k(x) dm^k(y) dm(g) \\ &\quad = \int_{G^k} \int_{G^k} \int_G f(x, g \cdot y) p(y, g) dm(g) dm^k(x) dm^k(y) \\ &\quad = \int_A \int_A \int_G f(x, g \cdot y) p(y, g) dm(g) dm^k(x) dm^k(y). \end{aligned}$$

For each $(x, y) \in A \times A$, either $\dim H^x = \dim H^y = \dim N$ or for almost all $g \in G$

$$\dim (H^x \cap gH^y g^{-1}) < \max (\dim H^x, \dim H^y) \leqq \dim H - j - 1,$$

by Lemma 3.1. But, $gH^y g^{-1} = H^{(g \cdot y)}$ so, in either case, $f(x, g \cdot y) = 0$ for almost all $g \in G$. Thus,

$$\int_{G^k} \int_{G^k} f(x, y) dm^k(x) dm^k(y) = 0.$$

This establishes the inductive step and therefore the lemma.

So we have shown that, for some integer $k \leqq 2^{(\dim H - \dim N)}$ and for almost all $x \in G^k$, $\dim H^x = \dim N$. In particular, the connected com-

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ponent of H^x is contained in N for almost all $x \in G^k$. Hence, H^x/N is a countable subgroup of G/N containing no nontrivial normal subgroup of G/N, for almost all $x \in G^k$.

LEMMA 3.3. *Let L and M be countable subgroups of a connected Lie group G such that neither L nor M contains a nontrivial normal subgroup of G. Then, for almost all g in G,*

$$gLg^{-1} \cap M = \{e\}.$$

Proof. Suppose that B is a subset of G with positive measure and $gLg^{-1} \cap M \neq \{e\}$, for all $g \in B$. Since M is countable, there exists $m \in M$, $m \neq e$ and a subset B_1 of B such that measure of B_1 is positive and $m \in gLg^{-1}$ for all $g \in B_1$. Since L is countable, there exists an $l \in L$, $l \neq e$, and a subset B_2 of B_1 such that measure of B_2 is positive and $glg^{-1} = m$ for all $g \in B_2$. Hence, for all $h \in B_2^{-1}B_2$, we have $h^{-1}lh = l$. This implies that the centralizer of l has positive measure and so is all of G. But, then l is in the center of G which is impossible since L contains no normal subgroup of G other than $\{e\}$. This proves the lemma.

Returning to the situation described before the statement of Lemma 3.3, we have a co-null set C in G/N and a co-null set A in G^k such that for all $Ng \in C$ and $x, y \in A$, $gH^xg^{-1} \cap H^y = N$. That is, if C' is the inverse image of C under the canonical projection, then C' is a co-null set in G and for all $g \in C', x, y \in A$,

$$H^{(g.x,y)} = H^{g.x} \cap H^y = N. \tag{3.4}$$

A Fubini argument, similar to that used in the proof of Lemma 3.2, shows that $(C' \cdot A) \times A = \{(g \cdot x, y) \colon g \in C', x, y \in A\}$ is a co-null set in G^{2k}. Therefore, the equation 3.4 completes the proof of Proposition 2.1 with $k_0 = k$.

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University of Colorado,
Boulder, Colorado;
University of Saskatchewan,
Saskatoon, Saskatchewan