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Effect of the location of a protection zone in a reaction–diffusion model

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In this paper, we consider the dynamical behaviour of a reaction–diffusion model for a population residing in a one-dimensional habit, with emphasis on the effects of boundary conditions and protection zone. We assume that the population is subjected to a strong Allee effect in its natural domain but obeys a monostable nonlinear growth in the protection zone $[L_1, L_2]$ with two constants satisfying $0 \leqslant L_1 < L_2$, and the general Robin condition is imposed on $x = 0$ (i.e. $u(t, 0) = bu_x(t, 0)$ with $b \geqslant 0$). We show the existence of two critical values $0 < L_* \leqslant L^*$, and prove that a vanishing–transition–spreading trichotomy result holds when the length of protection zone is smaller than L_*; a transition–spreading dichotomy result holds when the length of protection zone is between L_* and L^*; only spreading happens when the length of protection zone is larger than L^*. Based on the properties of L_*, we obtain the precise strategies for an optimal protection zone: if b is large (i.e. $b \geqslant 1/\sqrt{-g'(0)}$), the protection zone should start from somewhere near 0; while if b is small (i.e. $b < 1/\sqrt{-g'(0)}$), then the protection zone should start from somewhere away from 0, and as far away from 0 as possible.

Keywords: Reaction–diffusion equation; strong Allee effect; protection zone; species spreading; long-time behaviour

2020 *Mathematics Subject Classification:* Primary: 35K15; 35K55; 35B40; 92D25

1. Introduction

It follows from [29] that more than 99% of all species amounting to over 5 billion species that ever lived on Earth are estimated to be extinct. Some scientists estimate that up to half of presently existing plant and animal species may become extinct by 2100 [41]. Humans can cause extinction of a species through overharvesting, pollution, habitat destruction and other influences. A variety of conservation programmes have been designed in order to prevent further extinctions. One of the most effective method to protect endangered species from extinction is establishing protection zones. The role of protection zones in preventing population from extinction has been investigated in [7–11, 13, 15–18, 24–26, 30, 40, 42] and the references therein for reaction–diffusion models.

In the present work, we are interested in the effects of the location of a protection zone and the boundary condition on the long-time behaviour of an endangered species whose spatiotemporal evolution was described by a reaction–diffusion model

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with a protection zone. That is our model is given by the following form:

$$\begin{cases} u_t = u_{xx} + f(u), & t > 0,\ x \in (L_1, L_2), \\ u_t = u_{xx} + g(u), & t > 0,\ x \in (0, L_1) \cup (L_2, \infty), \\ u(t, 0) = bu_x(t, 0), & t > 0, \\ u(0, x) = u_0(x) \geqslant 0, & x \geqslant 0, \end{cases} \tag{1.1}$$

where the general Robin condition is imposed on $x = 0$ with $b \geqslant 0$; $L_2 > L_1 \geqslant 0$ are two constants and the protection zone is $[L_1,\ L_2]$. Throughout this paper, we set

$$L = L_2 - L_1,$$

then L and L_1 are the length and the starting point of the protection zone, respectively. In the protection zone, the growth of the species is governed by a monostable nonlinearity $f(u)$ which is a C^1 function satisfying

$$f(0) = f(1) = 0 < f'(0), \quad f'(1) < 0,\ (1 - u)f(u) > 0,\ \forall u > 0,\ u \neq 1. \tag{1.2}$$

The nonlinearity $g(u)$ is used to describe the evolution species which obeys the strong Allee effect [1] out of the protection zone. In order to describe the strong Allee effect, a typical reaction function is the so-called 'bistable' nonlinear terms; see, for example, [6, 20, 23, 28, 39] and the references therein. Here, we assume that the bistable nonlinear term $g(u)$ is a C^1 function and satisfies

$$g(0) = g(\theta) = g(1) = 0, \quad g(u) \begin{cases} < 0 & \text{in } (0, \theta), \\ > 0 & \text{in } (\theta, 1), \\ < 0 & \text{in } (1, \infty), \end{cases} \tag{1.3}$$

for some $\theta \in (0,\ 1)$, $g'(0) < 0$, $g'(1) < 0$ and

$$\int_0^1 g(s)\mathrm{d}s > 0. \tag{1.4}$$

Next, we need to give the interface conditions at $x = L_i$ for $i = 1,\ 2$. Here, we assume that the population density is continuous and the population flux is conserved at the interface points $x = L_i$. Then the interface conditions at $x = L_i$ are given by

$$\begin{cases} u(t, L_i - 0) = u(t, L_i + 0), & t > 0, \quad i = 1, 2, \\ u_x(t, L_i - 0) = u_x(t, L_i + 0), & t > 0, \quad i = 1, 2, \end{cases} \tag{1.5}$$

where $u(t,\ L_i - 0)$ and $u_x(t,\ L_i - 0)$ represent, respectively, the left limit value and the left derivative of u with respect to x at $x = L_i$, and $u(t,\ L_i + 0)$ and $u_x(t,\ L_i + 0)$ are respectively the right limit value and the right derivative of u with respect to x at $x = L_i$. Especially, if $L_1 = 0$, then the conditions for $x = L_1$ in (1.5) should be removed automatically.

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Combining (1.1) and (1.5), we are led to the following system:

$$\begin{cases} u_t = u_{xx} + f(u), & t > 0, \quad x \in (L_1, L_2), \\ u_t = u_{xx} + g(u), & t > 0, \quad x \in (0, L_1) \cup (L_2, \infty), \\ u(t,0) = bu_x(t,0), & t > 0, \\ u(t, L_i - 0) = u(t, L_i + 0), & t > 0, \quad i = 1, 2, \\ u_x(t, L_i - 0) = u_x(t, L_i + 0), & t > 0, \quad i = 1, 2, \\ u(0,x) = u_0(x) \geqslant 0, & x \geqslant 0. \end{cases} \tag{P}$$

The initial function u_0 belongs to $\mathscr{X}(h)$ for some $h > 0$, where

$$\mathscr{X}(h) := \{\phi \in L^\infty([0,\infty)) : \ \phi(0) = b\phi'(0), \ \phi > 0 \text{ in } (0,h), \ \phi = 0 \text{ in } [h, \infty)\}. \tag{1.6}$$

Problem (***P***) with $b = \infty$ (i.e. $u_x(t,\,0) = 0$) has been studied recently in [**10**]. The authors in [**10**] obtained that there are two critical values $0 < L_* \leqslant L^*$ which affect the long-time behaviour of the solutions significantly. More precisely, in the small protection zone case ($L < L_*$), there is a vanishing–transition–spreading trichotomy result; in the medium-sized protection zone case ($L_* < L < L^*$), there is a transition–spreading dichotomy result; only spreading happens in the large protection zone case ($L > L^*$). They also found that L_* is increasing in L_1, which yields that the protection zone should start from somewhere near 0, see [**10**] for more details. And any other relevant works with free-boundary conditions can be found, for instance, in [**34**–**36**].

When $L = 0$, that is to say, there is no protection zone in the environment, some special cases of (***P***) were studied by many authors. Among them, Du and Matano [**14**] considered the case where $b = \infty$ and obtained a rather comprehensive analysis of the dynamical behaviour of solutions by introducing a parameter in the initial value. And any other relevant works can be found, for instance, in [**2**, **5**, **43**] and the references therein. In the case where $L_1 = 0$ and $L_2 = \infty$, the authors in [**2**] also studied problem (***P***), and obtained the asymptotic behaviour of solutions and the existence of the travelling wave solutions, see [**2**] for more details.

Our primary goal in this paper is to examine the role of the protection zone by studying the dynamics of the reaction–diffusion model (***P***).

Throughout the paper, unless otherwise specified, in addition to the previously imposed conditions (1.2)–(1.4) on f, g, we further assume that

(H) The functions f, g are *globally Lipschitz* and $g(u) < f(u)$ for all $0 < u < 1$.

For any given $h > 0$ and $u_0 \in \mathscr{X}(h)$, it is known from [**12**, **19**, **38**] that (***P***) admits a unique nonnegative solution $u \in C^{1,2}((0,\,\infty) \times ([0,\,\infty) \setminus (\{L_1\} \cup \{L_2\}))) \cap C^{\gamma/2, 1+\gamma}((0,\,\infty) \times [0,\,\infty))$ for any $\gamma \in (0,\,1)$, and u exists for all time $t > 0$. By the comparison principle and classical theory for parabolic equations, we see that u is uniformly bounded with respect to both space and time. Therefore, one may expect that the long-time behaviour of solutions will be determined by nonnegative and bounded stationary solutions of (***P***), that is, the solutions of the

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following elliptic equation:

$$\begin{cases} v'' + f(v) = 0, & x \in (L_1, L_2), \\ v'' + g(v) = 0, & x \in (0, L_1) \cup (L_2, \infty), \\ v(0) = bv'(0), & \\ v(L_i - 0) = v(L_i + 0), & i = 1, 2, \\ v'(L_i - 0) = v'(L_i + 0), & i = 1, 2. \end{cases} \tag{1.7}$$

A phase-plane analysis shows that all nonnegative and bounded stationary solutions of (1.7) can be classified as follows (cf. § 2):

(1) Trivial solution: $v \equiv 0$;

(2) Active states: $v(x) = u^*(x)$ is a positive and increasing solution of (1.7), subject to $u^*(\infty) = 1$;

(3) Ground states: $v(x) = U(x)$ is a positive solution of (1.7), and when $x > L_2$, $U(\cdot) = V(\cdot - z)$ where $z \in \mathbb{R}$ and V is the unique positive symmetrically decreasing solution of

$$V'' + g(V) = 0, \quad V(0) = \theta^*, \ V(\pm\infty) = 0,$$

where $\theta^* \in (\theta,\, 1)$ is uniquely determined by the condition

$$\int_0^{\theta^*} g(s)\mathrm{d}s = 0;$$

(4) Positive periodic solutions: $v(x)$ is a positive solution of (1.7) and when $x > L_2$, $v(x) = P(x - z_1)$, where $z_1 \in \mathbb{R}$ and P is a periodic solution of $v'' + g(v) = 0$ satisfying $0 < \min P < \theta < \max P < \theta^*$.

Now, let us list some possible situations on the asymptotic behaviour of the solutions to problem (***P***):

- **vanishing**: $\lim_{t\to\infty} u(t,\, \cdot) = 0$ uniformly in $[0,\, \infty)$;
- **spreading**: $\lim_{t\to\infty} |u(t,\, \cdot) - u^*(\cdot)| = 0$ locally uniformly in $[0,\, \infty)$;
- **transition**: $\lim_{t\to\infty} |u(t,\, \cdot) - U(\cdot)| = 0$ locally uniformly in$[0, \infty)$,

where $u^*(x)$ and $U(x)$ are the active state and ground state, respectively.

Denote

$$L_* := \frac{1}{\sqrt{f'(0)}} \arctan\left[\sqrt{-\frac{g'(0)}{f'(0)}} \cdot \frac{(b\sqrt{-g'(0)}+1)\mathrm{e}^{2\sqrt{-g'(0)}L_1} - (b\sqrt{-g'(0)}-1)}{(b\sqrt{-g'(0)}+1)\mathrm{e}^{2\sqrt{-g'(0)}L_1} + (b\sqrt{-g'(0)}-1)}\right] + \frac{1}{\sqrt{f'(0)}} \arctan\sqrt{-\frac{g'(0)}{f'(0)}}. \tag{1.8}$$

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By the proof of assertion (I) in theorem 1.1, we know that if $0 < L < L_*$, then problem (1.7) has a ground state. This allows us to define

$$L^* := \sup\{L_0 > 0 : \ \text{problem (1.7) with } L := L_2 - L_1 = L_0 \text{ has a ground state}\}. \tag{1.9}$$

In § 3, one obtains that L^* is bounded.

We are now in a position to give a satisfactory description of the long-time dynamical behaviour of problem (***P***).

THEOREM 1.1. *Assume that **(H)** holds. For any given $b \geqslant 0$ and $L_2 > L_1 \geqslant 0$, let $L := L_2 - L_1$, u be the solution of (**P**) with $u_0 = \sigma\phi$ for some $\phi \in \mathscr{X}(h)$, $h > 0$ and $\sigma > 0$. Moreover, let L_* and L^* be given in (1.8) and (1.9), respectively. The following assertions hold.*

(I) *(Small protection zone case) If $0 < L < L_*$, then there exist $\sigma_*,\ \sigma^* \in (0, \infty)$ with $\sigma_* \leqslant \sigma^*$ such that the following trichotomy holds:*
 (i) *Vanishing happens when $0 < \sigma < \sigma_*$;*
 (ii) *Transition happens when $\sigma \in [\sigma_*, \sigma^*]$;*
 (iii) *Spreading happens when $\sigma > \sigma^*$.*

(II) *(Medium-sized protection zone case) If $L_* < L^*$ and $L_* < L < L^*$, then there exists $\sigma^* \in (0, \infty)$ such that the following dichotomy holds:*
 (i) *Transition happens when $\sigma \in (0, \sigma^*]$;*
 (ii) *Spreading happens when $\sigma > \sigma^*$.*

(III) *(Large protection zone case) If $L > L^*$, then spreading happens for all $\sigma > 0$.*

It is known from theorem 1.1 that if the length of protection zone L satisfies $L > L_*$, then the species will survive in $(0, \infty)$ all the time regardless of its initial data. Thus, L_* is called *the effective length of protection zone*.

Next, based on the property of the effective length L_* of protection zone, we give the following result.

THEOREM 1.2. *Assume that **(H)** holds. For any given $b \geqslant 0$ and $L_2 > L_1 \geqslant 0$, let $L := L_2 - L_1$. The following assertions hold.*

(i) *When $0 \leqslant b < 1/\sqrt{-g'(0)}$, then the bigger L_1 is, the shorter effective length of protection zone is, and the shortest effective length is $(2/\sqrt{f'(0)}) \arctan \sqrt{-(g'(0)/f'(0))}$, which is independent of b.*

(ii) *When $b > 1/\sqrt{-g'(0)}$, then the smaller L_1 is, the shorter effective length of protection zone is, and the shortest effective length is $(1/\sqrt{f'(0)})[\arctan \sqrt{-(g'(0)/f'(0))} + \arctan(1/b\sqrt{f'(0)})]$.*

(iii) *When $b = 1/\sqrt{-g'(0)}$, then the effective length of protection zone is identically equal to $(2/\sqrt{f'(0)}) \arctan \sqrt{-(g'(0)/f'(0))}$, regardless of the choice of L_1.*

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From theorem 1.2, we can deduce that if b is large (i.e. $b \geqslant 1/\sqrt{-g'(0)}$), then the protection zone should start from somewhere near 0; while if b is small (i.e. $b < 1/\sqrt{-g'(0)}$), then the protection zone should start from somewhere away from 0, and as far away from 0 as possible.

The rest of our paper is organized as follows. In § 2, we prepare some preliminary results, including the analysis of the associated stationary solution problems, the comparison principle and a general convergence result. Section 3 covers the dynamical behaviour of solutions of ($\boldsymbol{P}$) and the proofs of theorems 1.1 and 1.2. In § 4, we end the paper with some discussion on our results.

2. Some preliminary results

In this section, we present some preliminary results which will be frequently used later.

2.1. Stationary solutions

Clearly, a so-called stationary solution of ($\boldsymbol{P}$) is a solution of (1.7). We will use the phase-plane analysis to describe the solutions of (1.7). In the qp-plane, each solution of $q'' + f(q) = 0$ corresponds to a trajectory $p^2 = F(q) - C_1$, where $p := q'$, C_1 is a constant and $F(q) := -2\int_0^q f(v)\mathrm{d}v$. At the same time, each solution of $q'' + g(q) = 0$ gives a trajectory $p^2 = G(q) - C_2$ where C_2 is a constant and $G(q) := -2\int_0^q g(v)\mathrm{d}v$. Moreover, for any solution of (1.7), the connection condition at $x = L_i$ ($i = 1,\ 2$) is fulfilled whenever there are some C_1 and C_2 such that the trajectory $q'^2 = F(q) - C_1$ intersects the trajectory $q'^2 = G(q) - C_2$. Noting that such an intersection point may not be unique, thus several stationary solutions of ($\boldsymbol{P}$) can be derived from different trajectories.

It is easy to check that the trajectory $p^2 = -2\int_0^q g(v)\mathrm{d}v$ passes through the point $(\theta^*, 0)$ in the phase plane, which is denoted by Γ^*; for any $\beta \in (0, 1)$, the trajectory $p^2 = 2\int_q^\beta f(v)\mathrm{d}v$ passes through the point $(\beta, 0)$, which is denoted by Γ_β. Using the phase-plane analysis, together with condition **(H)**, we are able to obtain the following lemma.

LEMMA 2.1. *For any $\beta \in (0, \theta^*)$, there are exactly two points of intersection of Γ^* and Γ_β. If $\beta = \theta^*$, there is a unique point $(\theta^*, 0)$ of intersection of Γ^* and Γ_{θ^*}. If $\beta \in (\theta^*, 1)$, then Γ^* does not intersect Γ_β.*

Based on lemma 2.1, we shall list all possible bounded and nonnegative stationary solutions of ($\boldsymbol{P}$) in the following lemma; one can also refer to Figure 1 for the structure of active states and ground states.

LEMMA 2.2. *Assume that **(H)** holds. For any given $L_2 > L_1 \geqslant 0$ and $b \geqslant 0$, all solutions of the stationary problem* (1.7) *are one of the following types:*

(1) *Trivial solution: $v \equiv 0$;*

(2) *Active states: $v(x) = u^*(x)$ (see Figure 1(b)) is a positive and increasing solution of (1.7), subject to $u^*(\infty) = 1$;*

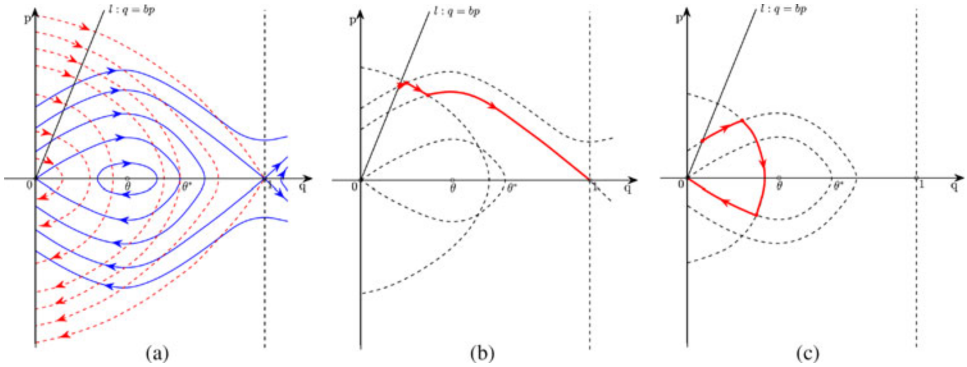

Figure 1. qp-plane. (a) The red dotted curves are the trajectories for $q'' + f(q) = 0$, the blue solid curves are the trajectories for $q'' + g(q) = 0$ and the black solid line l: $q = bp$ with $b > 0$. (b) The red solid curve is the trajectory for an active state of (1.7). (c) The red solid curve is the trajectory for a ground state of (1.7).

(3) *Ground states:* $v(x) = U(x)$ *(see Figure 1(c)) is a positive solution of (1.7), and when* $x > L_2$, $U(\cdot) = V(\cdot - z)$ *where* $z \in \mathbb{R}$ *and* V *is the unique positive symmetrically decreasing solution of*

$$V'' + g(V) = 0, \quad V(0) = \theta^*, \ V(\pm\infty) = 0;$$

(4) *Positive periodic solutions:* $v(x)$ *is a positive solution of (1.7) and when* $x > L_2$, $v(x) = P(x - z_1)$, *where* $z_1 \in \mathbb{R}$ *and* P *is a periodic solution of* $v'' + g(v) = 0$ *with* $0 < \min P < \theta < \max P < \theta^*$.

It follows from the phase-plane analysis that any ground state U of (1.7) satisfies that

$$\|U\|_{L^\infty([0,\infty))} \leqslant \theta^*. \tag{2.1}$$

We also observe that, for any $\alpha \in (\theta^*, 1)$, the trajectory for $q'' + g(q) = 0$ passing through the point $(\alpha, 0)$ in the phase plane gives a function v_α satisfying

$$v_\alpha'' + g(v_\alpha) = 0 < v_\alpha \leqslant \alpha \quad \text{in } (0, 2l_\alpha), \quad v_\alpha(0) = v_\alpha(2l_\alpha) = 0, \quad v_\alpha(l_\alpha) = \alpha, \tag{2.2}$$

with

$$l_\alpha = \int_0^\alpha \frac{\mathrm{d}s}{\sqrt{2\int_s^\alpha g(v)\mathrm{d}v}} \in (0, \infty). \tag{2.3}$$

2.2. Comparison principle

In this section, we give the following useful comparison principle, which is stated as follows.

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LEMMA 2.3. *Assume that* $\bar{u}(t,\, x) \in C^{1,2}((0,\, \infty) \times ([0,\, \infty) \setminus \{L_i\}))$ $(i = 1,\ 2)$ *satisfies*

$$\begin{cases} \bar{u}_t \geqslant \bar{u}_{xx} + f(\bar{u}), & t > 0,\ x \in (L_1, L_2), \\ \bar{u}_t \geqslant \bar{u}_{xx} + g(\bar{u}), & t > 0,\ x \in (0, L_1) \cup (L_2, \infty), \\ \bar{u}(t,0) \geqslant b\bar{u}_x(t,0), & t > 0, \\ \bar{u}(t, L_i + 0) = \bar{u}(t, L_i - 0), & t > 0,\ i = 1,\ 2, \\ \bar{u}_x(t, L_i + 0) \geqslant \bar{u}_x(t, L_i - 0), & t > 0,\ i = 1,\ 2. \end{cases}$$

When $\bar{u}_0(x) \geqslant u_0(x)$ *for* $x \in [0,\, \infty)$ *and* u *is a solution of* (***P***), *then*

$$\bar{u}(t,x) \geqslant u(t,x) \quad \text{for } t > 0 \text{ and } x \in [0, \infty).$$

The proof of lemma 2.3 is similar as that of [**12**, lemma 2.2], so the details are omitted here.

REMARK 2.4. The function $\bar{u}$ in lemma 2.3 is often called a supersolution to (***P***). A subsolution can be defined analogously by reversing all the inequalities. The corresponding comparison principle for subsolutions holds in the above case.

2.3. A general convergence theorem

By similar analysis to [**10**, **14**], we can present a general convergence result, which reads as follows.

THEOREM 2.5 (Convergence theorem for system (***P***)). *Let* $u(t,\, x)$ *be a solution of* (***P***) *with* $u_0 \in \mathscr{X}(h)$ *for some* $h > 0$. *Then* u *converges to a stationary solution* v *of (1.7) as* $t \to \infty$ *locally uniformly in* $[0,\, \infty)$. *Moreover, the limit* v *is either* 0, *or an active state, or a ground state of (1.7).*

Proof. Denote by $\omega(u)$ the ω-limit set of $u(t,\, \cdot)$ in the topology of $L^\infty_{loc}([0,\, \infty))$. By local parabolic estimates, the definition of $\omega(u)$ remains unchanged if the topology of $L^\infty_{loc}([0,\, \infty))$ is replaced by that of $C^2_{loc}([0,\, L_1) \cup (L_1,\, L_2) \cup (L_2,\, \infty)) \cap C^1_{loc}([0,\, \infty))$. It is well-known that $\omega(u)$ is a compact, connected and invariant set.

Since the conditions in (***P***) at the interface points $x = L_i$ for $i = 1,\ 2$ do not change the number of sign changes of the functions defined similarly as in [**14**, Lemmas 2.7, 2.9], then by the argument of [**14**, theorem 1.1] with slight modifications, we can show that $\omega(u)$ consists of only one element, which is either a constant solution or a nonnegative solution of (1.7) which is decreasing with respect to $x > h$. In view of lemma 2.2, $\omega(u)$ contains either 0, or an active state, or a ground state of (1.7). Consequently, we obtain that as $t \to \infty$, u converges to either 0, or an active state, or a ground state of (1.7) locally uniformly in $[0,\, \infty)$. □

3. Classification of dynamical behaviour

In this section, we obtain a complete description of the long-time dynamical behaviour of (***P***). Firstly, we study the properties of the principal eigenvalue of a linear eigenvalue problem and prove theorem 1.2. Next, we give some sufficient conditions for vanishing and for spreading, and obtain the boundedness of L^*. Finally, we give the proof of theorem 1.1.

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3.1. A linear eigenvalue problem

We first study the following eigenvalue problem:

$$\begin{cases} -\varphi'' - f'(0)\varphi = \lambda\varphi, & x \in (L_1, L_2), \\ -\varphi'' - g'(0)\varphi = \lambda\varphi, & x \in (0, L_1) \cup (L_2, \infty), \\ \varphi(0) = b\varphi'(0), \quad \varphi(\infty) = 0, & \\ \varphi(L_i - 0) = \varphi(L_i + 0), & i = 1, 2, \\ \varphi'(L_i - 0) = \varphi'(L_i + 0), & i = 1, 2, \end{cases} \tag{3.1}$$

and analyse the properties of its principal eigenvalue. It will turn out that these preliminary results are crucial in determining the dynamics of (***P***). Set

$$h(x) = \begin{cases} -f'(0), & x \in [L_1, L_2], \\ -g'(0), & x \in (0, L_1) \cup (L_2, \infty). \end{cases}$$

As $h \in L^\infty([0,\, \infty))$, it is well-known that the principal eigenvalue (or the so-called first eigenvalue) of (3.1) exists. Thus, we use $\lambda_1(L,\, b)$ to denote the principal eigenvalue of (3.1). The corresponding eigenfunction φ_1^L of (3.1) satisfies $\varphi_1^L \in C^1([0,\, \infty)) \cap C^2([0,\, \infty) \setminus \{L_i\})$ $(i = 1,\ \ 2)$, $\varphi_1^L > 0$ on $(0,\, \infty)$ and $\varphi_1^L(0) = b(\varphi_1^L)'(0)$.

Let $\lambda_1^R(L,\, b)$ be the principal eigenvalue of

$$\begin{cases} -\varphi'' + h(x)\varphi = \lambda\varphi, & 0 < x < R, \\ \varphi(0) = b\varphi'(0), & \varphi(R) = 0. \end{cases} \tag{3.2}$$

It follows from [**3**, proposition 6.11] (or [**4**, theorem 4.1]) that

$$\lambda_1^R(L,\, b) \text{ is decreasing in } R > 0 \text{ and } \lim_{R\to\infty} \lambda_1^R(L, b) \leqslant \lambda_1(L, b). \tag{3.3}$$

Let L_* be given as in § 1. Then we are able to conclude that

Lemma 3.1. *For any given $b \geqslant 0$ and $0 \leqslant L_1 < L_2$, let $L := L_2 - L_1$ and $\lambda_1(L,\, b)$ be the principal eigenvalue of (3.1). Then we have for any given $0 \leqslant L_1 < L_2$ and any $b \geqslant 0$,*

$$\lambda_1(L, b) \in (-f'(0), -g'(0)),$$

and

$$L = \frac{1}{\theta_2}\left\{ \arctan\left[\frac{\theta_1}{\theta_2} \cdot \frac{(b\theta_1 + 1)e^{\theta_1 L_1} - (b\theta_1 - 1)e^{-\theta_1 L_1}}{(b\theta_1 + 1)e^{\theta_1 L_1} + (b\theta_1 - 1)e^{-\theta_1 L_1}}\right] + \arctan\frac{\theta_1}{\theta_2} \right\},$$

where

$$\theta_1 = \sqrt{-g'(0) - \lambda_1(L, b)} \text{ and } \theta_2 = \sqrt{f'(0) + \lambda_1(L, b)}.$$

Moreover, for any given $b \geqslant 0$, $\lambda_1(L,\, b)$ is decreasing in $L > 0$, and there exists a unique $L_* := L_*(L_1,\, b)$ such that $\lambda_1(L,\, b)$ is negative (resp. 0, or positive) when $L > L_*$ (resp. $L = L*$, or $L < L_*$).

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Proof. Let us write $\lambda_1 = \lambda_1(L, b)$ for simplicity. It follows from lemma A.1 in Appendix A that $\varphi'(x) > 0$ for $0 < x \ll 1$ and $\lambda_1 \in (-f'(0), -g'(0))$ for any given $0 \leqslant L_1 < L_2$ and any given $b \geqslant 0$.

We only sketch the proof for the case where $L_1 > 0$; the analysis for the case where $L_1 = 0$ is similar.

Since $\lambda_1 \in (-f'(0), -g'(0))$, then

$$0 < \theta_i < \sqrt{f'(0) - g'(0)}, \quad \textit{for } i = 1, 2. \tag{3.4}$$

For $x \in [0, L_1)$, it follows from the second equation of (3.1) that

$$-\varphi'' = (\lambda_1 + g'(0))\varphi \text{ with } g'(0) + \lambda_1 < 0,$$

which implies that there are two constants C_1 and C_2 such that

$$\varphi(x) = C_1 \mathrm{e}^{\theta_1 x} + C_2 \mathrm{e}^{-\theta_1 x}, \quad \forall x \in [0, L_1).$$

This, together with $\varphi(0) = b\varphi'(0)$ and $\varphi(x) > 0$ for $x > 0$, yields that

$$C_1 > 0, \quad C_2 = \frac{b\theta_1 - 1}{b\theta_1 + 1} C_1,$$

and for $x \in (0, L_1)$,

$$\varphi(x) = \frac{C_1}{b\theta_1 + 1}[(b\theta_1 + 1)\mathrm{e}^{\theta_1 x} + (b\theta_1 - 1)\mathrm{e}^{-\theta_1 x}],$$

and

$$\varphi'(x) = \frac{C_1\theta_1}{b\theta_1 + 1}[(b\theta_1 + 1)\mathrm{e}^{\theta_1 x} - (b\theta_1 - 1)\mathrm{e}^{-\theta_1 x}] > 0.$$

Then we have

$$\frac{\varphi'(L_1 - 0)}{\varphi(L_1 - 0)} = \theta_1 \cdot \frac{(b\theta_1 + 1)\mathrm{e}^{\theta_1 L_1} - (b\theta_1 - 1)\mathrm{e}^{-\theta_1 L_1}}{(b\theta_1 + 1)\mathrm{e}^{\theta_1 L_1} + (b\theta_1 - 1)\mathrm{e}^{-\theta_1 L_1}} > 0. \tag{3.5}$$

Similarly, for $x \in (L_2, \infty)$, there are two constants C_3 and C_4 such that

$$\varphi(x) = C_3 \mathrm{e}^{\theta_1 x} + C_4 \mathrm{e}^{-\theta_1 x}, \quad \forall x \in (L_2, \infty).$$

Using $\varphi(\infty) = 0$, we infer that $C_4 > 0 = C_3$, and so for $x \in (L_2, \infty)$,

$$\varphi(x) = C_4 \mathrm{e}^{-\theta_1 x},$$

and

$$\varphi'(x) = -C_4\theta_1 \mathrm{e}^{-\theta_1 x} < 0.$$

Then we obtain

$$\frac{\varphi'(L_2 + 0)}{\varphi(L_2 + 0)} = -\theta_1 < 0. \tag{3.6}$$

Since $\varphi'(L_i + 0) = \varphi'(L_i - 0)$ $(i = 1, 2)$, it follows from (3.5) and (3.6) that

$$\varphi'(L_1 + 0) > 0 > \varphi'(L_2 - 0). \tag{3.7}$$

By the first equation of (3.1), we have $\varphi''(x) < 0$ for $x \in (L_1, L_2)$. This, together with (3.7), yields that there is a unique $a \in (L_1, L_2)$ satisfying $\varphi'(a) = 0$.

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When $x \in (L_1, L_2)$, we get from the first equation of (3.1) that

$$-\varphi'' = (f'(0) + \lambda_1)\varphi \text{ with } f'(0) + \lambda_1 > 0,$$

thus there are two constants C_5 and C_6 such that

$$\varphi(x) = C_5 \cos[\theta_2(x-a)] + C_6 \sin[\theta_2(x-a)], \quad \forall x \in (L_1, L_2).$$

Since $\varphi'(a) = 0$ and $\varphi(x) > 0$ for $x > 0$, then $C_5 > 0 = C_6$. In turn, it holds

$$\varphi(x) = C_5 \cos[\theta_2(x-a)], \quad \forall x \in (L_1, L_2). \tag{3.8}$$

Moreover, basic computation gives that

$$\frac{\varphi'(L_1+0)}{\varphi(L_1+0)} = -\theta_2 \tan[\theta_2(L_1-a)], \quad \frac{\varphi'(L_2-0)}{\varphi(L_2-0)} = -\theta_2 \tan[\theta_2(L_2-a)].$$

By virtue of (3.5) and (3.6), it then follows that

$$\frac{\theta_1}{\theta_2} \cdot \frac{(b\theta_1+1)\mathrm{e}^{\theta_1 L_1} - (b\theta_1-1)\mathrm{e}^{-\theta_1 L_1}}{(b\theta_1+1)\mathrm{e}^{\theta_1 L_1} + (b\theta_1-1)\mathrm{e}^{-\theta_1 L_1}} = \tan[\theta_2(a-L_1)] > 0 \tag{3.9}$$

and

$$\frac{\theta_1}{\theta_2} = \tan[\theta_2(L_2-a)] > 0. \tag{3.10}$$

Thanks to (3.8)–(3.10), we may have that

$$0 < \theta_2(L_2-a) < \frac{\pi}{2}, \quad 0 < \theta_2(a-L_1) < \frac{\pi}{2}. \tag{3.11}$$

By a similar argument as the proof of [**10**, lemma 4.1], we have

$$a \to L_1, \quad \lambda_1 \to -g'(0) > 0, \text{ as } L_2 \to L_1, \tag{3.12}$$

and as $L_2 \to \infty$,

$$a \to \infty, \quad \lambda_1 \to -f'(0) < 0, \ \theta_2 \to 0, \quad \theta_2(a-L_1) \to \frac{\pi}{2} \quad \text{and} \quad \theta_2(L_2-a) \to \frac{\pi}{2}. \tag{3.13}$$

Furthermore, making use of (3.9)–(3.11) again, we deduce that

$$a - L_1 = \frac{1}{\theta_2} \arctan\left[\frac{\theta_1}{\theta_2} \cdot \frac{(b\theta_1+1)\mathrm{e}^{\theta_1 L_1} - (b\theta_1-1)\mathrm{e}^{-\theta_1 L_1}}{(b\theta_1+1)\mathrm{e}^{\theta_1 L_1} + (b\theta_1-1)\mathrm{e}^{-\theta_1 L_1}}\right]$$

and

$$L_2 - a = \frac{1}{\theta_2} \arctan\frac{\theta_1}{\theta_2}.$$

Adding these two identities infers

$$L = L_2 - L_1 = \frac{1}{\theta_2}\left\{\arctan\left[\frac{\theta_1}{\theta_2} \cdot \frac{(b\theta_1+1)\mathrm{e}^{\theta_1 L_1} - (b\theta_1-1)\mathrm{e}^{-\theta_1 L_1}}{(b\theta_1+1)\mathrm{e}^{\theta_1 L_1} + (b\theta_1-1)\mathrm{e}^{-\theta_1 L_1}}\right] + \arctan\frac{\theta_1}{\theta_2}\right\}. \tag{3.14}$$

It is noted that θ_1 is decreasing while θ_2 is increasing with respect to λ_1. By virtue of (3.14), some basic analysis shows that λ_1 is decreasing with respect to

$L > 0$. In addition, by (3.12) and (3.13), we conclude that there is a unique value

$$L_*(L_1, b) := \frac{1}{\sqrt{f'(0)}} \arctan \left[\sqrt{-\frac{g'(0)}{f'(0)}} \cdot \frac{\begin{array}{c}(b\sqrt{-g'(0)}+1)\mathrm{e}^{\sqrt{-g'(0)}L_1} \\ -(b\sqrt{-g'(0)}-1)\mathrm{e}^{-\sqrt{-g'(0)}L_1}\end{array}}{\begin{array}{c}(b\sqrt{-g'(0)}+1)\mathrm{e}^{\sqrt{-g'(0)}L_1} \\ +(b\sqrt{-g'(0)}-1)\mathrm{e}^{-\sqrt{-g'(0)}L_1}\end{array}} \right] + \frac{1}{\sqrt{f'(0)}} \arctan \sqrt{-\frac{g'(0)}{f'(0)}}$$

such that $\lambda_1 < 0$ if $L > L_* := L_*(L_1, b)$, $\lambda_1 = 0$ if $L = L_*$ and $\lambda_1 > 0$ if $0 < L < L_*$. The proof is complete now. □

We obtain the following property of $L_*(L_1, b)$.

PROPOSITION 3.2. *Let $L_*(L_1, b)$ be given in lemma 3.1. There exists $B^* := 1/\sqrt{-g'(0)}$ such that $L_*(L_1, b)$ is decreasing (resp. increasing) with respect to $L_1 \geqslant 0$ if $0 \leqslant b < B^*$ (resp. $b > B^*$). Moreover, for any given $L_1 \geqslant 0$, $L_*(L_1, b)$ is decreasing with respect to $b \geqslant 0$.*

Proof. The basic computation gives that

$$\frac{\partial L_*(L_1, b)}{\partial L_1} = \frac{4(b^2\beta^2 - 1)}{\left\{ \frac{\gamma^2}{\beta^2} + \left[\frac{(b\beta+1)\mathrm{e}^{\beta L_1} - (b\beta-1)\mathrm{e}^{-\beta L_1}}{(b\beta+1)\mathrm{e}^{\beta L_1} + (b\beta-1)\mathrm{e}^{-\beta L_1}} \right]^2 \right\} \left[(b\beta+1)\mathrm{e}^{\beta L_1} + (b\beta-1)\mathrm{e}^{-\beta L_1} \right]^2},$$

with $\gamma := \sqrt{f'(0)}$ and $\beta := \sqrt{-g'(0)}$, which implies that

$$\frac{\partial L_*(L_1, b)}{\partial L_1} \begin{cases} < 0, & \text{if } 0 \leqslant b < B^*, \\ = 0, & \text{if } b = B^*, \\ > 0, & \text{if } b > B^*. \end{cases}$$

Moreover, we obtain the following results: for any fixed $b \geqslant 0$,

(i) $\min\limits_{L_1 \geqslant 0} L_*(L_1, b) = \lim\limits_{L_1 \to \infty} L_*(L_1, b) = \frac{2}{\sqrt{f'(0)}} \arctan \sqrt{-\frac{g'(0)}{f'(0)}}$ if $0 \leqslant b < B^*$;

(ii) $\max\limits_{L_1 \geqslant 0} L_*(L_1, b) = L_*(0, b) = \frac{1}{\sqrt{f'(0)}} [\arctan \sqrt{-\frac{g'(0)}{f'(0)}} + \arctan \frac{1}{b\sqrt{f'(0)}}]$ if $0 \leqslant b < B^*$;

(iii) $L_*(L_1, b) \equiv \frac{2}{\sqrt{f'(0)}} \arctan \sqrt{-\frac{g'(0)}{f'(0)}}$ if $b = B^*$;

(iv) $\min\limits_{L_1 \geqslant 0} L_*(L_1, b) = L_*(0, b) = \frac{1}{\sqrt{f'(0)}} [\arctan \sqrt{-\frac{g'(0)}{f'(0)}} + \arctan \frac{1}{b\sqrt{f'(0)}}]$ if $b > B^*$;

(v) $\max\limits_{L_1 \geqslant 0} L_*(L_1, b) = \lim\limits_{L_1 \to \infty} L_*(L_1, b) = \frac{2}{\sqrt{f'(0)}} \arctan \sqrt{-\frac{g'(0)}{f'(0)}}$ if $b > B^*$.

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Similarly, we can compute that

$$\frac{\partial L_*(L_1,b)}{\partial b}=\frac{-4}{\left\{\frac{\gamma^2}{\beta^2}+\left[\frac{(b\beta+1)\mathrm{e}^{\beta L_1}-(b\beta-1)\mathrm{e}^{-\beta L_1}}{(b\beta+1)\mathrm{e}^{\beta L_1}+(b\beta-1)\mathrm{e}^{-\beta L_1}}\right]^2\right\}\times\left[(b\beta+1)\mathrm{e}^{\beta L_1}+(b\beta-1)\mathrm{e}^{-\beta L_1}\right]^2}<0,$$

which yields that for any fixed $L_1\geqslant 0$, $L_*(L_1,\, b)$ is decreasing with respect to $b\geqslant 0$, and

(i) $\max\limits_{b\geqslant 0} \quad L_*(L_1,\, b)=L_*(L_1,\, 0)=\frac{1}{\sqrt{f'(0)}}\left\{\arctan\left[\sqrt{-\frac{g'(0)}{f'(0)}}\cdot\frac{\mathrm{e}^{2\sqrt{-g'(0)}L_1}+1}{\mathrm{e}^{2\sqrt{-g'(0)}L_1}-1}\right]\right.$
$\left.+\arctan\sqrt{-\frac{g'(0)}{f'(0)}}\right\}$,

(ii) $\min\limits_{b\geqslant 0} \; L_*(L_1,\, b)=\lim\limits_{b\to\infty}L_*(L_1,\, b)=\frac{1}{\sqrt{f'(0)}}\left\{\arctan\left[\sqrt{-\frac{g'(0)}{f'(0)}}\cdot\frac{\mathrm{e}^{2\sqrt{-g'(0)}L_1}-1}{\mathrm{e}^{2\sqrt{-g'(0)}L_1}+1}\right]+\right.$
$\left.\arctan\sqrt{-\frac{g'(0)}{f'(0)}}\right\}$.

The proof is now complete. □

REMARK 3.3. From above proposition and its proof, we obtain that

$$\min_{0\leqslant b\leqslant B^*,L_1\geqslant 0}L_*(L_1,b)=\frac{2}{\sqrt{f'(0)}}\arctan\sqrt{-\frac{g'(0)}{f'(0)}}=2\min_{b>B^*,L_1\geqslant 0}L_*(L_1,b),$$
$$\max_{b\geqslant 0,L_1\geqslant 0}L_*(L_1,b)=\frac{1}{\sqrt{f'(0)}}\left[\frac{\pi}{2}+\arctan\sqrt{-\frac{g'(0)}{f'(0)}}\right].$$

Proof of theorem 1.2. Theorem 1.2 follows from proposition 3.2 immediately. □

Finally, we derive the following estimate for L^*. That is, we have

PROPOSITION 3.4. *For any given $b\geqslant 0$ and $0\leqslant L_1<L_2$, let L^* be given in (1.9), then L^* is bounded.*

Proof. Firstly, we consider the case where f is a Fisher–KPP type of nonlinearity (i.e. $f(u)/u$ is decreasing with respect to $u\geqslant 0$) and prove that $L^*\leqslant 2L^0$ with

$$L^0:=\int_0^{\theta^*}\frac{1}{\sqrt{2\int_r^{\theta^*}f(s)\mathrm{d}s}}\,\mathrm{d}r<\infty.$$

It follows from [**10**, proposition 3.10] that the following auxiliary problem:

$$\begin{cases}q''+f(q)=0,\ q(x)>0, & x\in(-l,l),\\ q'(0)=q(\pm l)=0,\end{cases}\tag{3.15}$$

admits a unique positive symmetrically decreasing solution $q_l(x)$ when $l>l_0:=\pi/2\sqrt{f'(0)}$, and $q_l(x)$ is increasing in $l>l_0$, that is, when $l_2>l_1>l_0$, then $q_{l_2}(x)>$

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$q_{l_1}(x)$ for $x \in [-l_1, l_1]$. It is easy to check that $L^0 > l_0$. Moreover, when $l = L^0$, the unique positive solution $q_l(x)$ of problem (3.15) satisfies $q_l(0) = \theta^*$.

We claim that $L^* \leqslant 2L^0$. Let us use an indirect argument and suppose that $L^* > 2L^0$. For any fixed $L_0 \in (2L^0, L^*)$ and any $0 \leqslant L_1 < L_2$ satisfying $L_0 = L_2 - L_1$, consider the following problem:

$$\begin{cases} v_t = v_{xx} + f(v), & t > 0, \quad x \in (L_1, L_2), \\ v(t, L_1) = v(t, L_2) = 0, & t > 0, \\ v(0, x) = v_0(x), & x \in [L_1, L_2]. \end{cases} \tag{3.16}$$

It follows from some standard analysis that for any $v_0(x) \geqslant, \not\equiv 0$, the unique positive solution $v(t, x)$ of (3.16) satisfies

$$\left\| v(t, \cdot) - q_{(L_2-L_1)/2}\left(\cdot - \frac{L_1 + L_2}{2}\right) \right\|_{L^\infty([L_1, L_2])} \to 0, \quad \text{as } t \to \infty,$$

where $q_{(L_2-L_1)/2}(x)$ is the unique positive solution of (3.15) with $l = (L_2 - L_1)/2 = L_0/2 > L^0$. Since the solution $u(t, x)$ of (***P***) satisfies that $u(1, x) > 0$ for all $x \geqslant 0$, we can take $v_0(x)$ small enough such that $u(1, x) > v_0(x)$ in $[L_1, L_2]$. Hence, the comparison principle can be used to obtain that

$$u(t + 1, x) \geqslant v(t, x), \quad \text{for all } t > 0, \quad x \in [L_1, L_2].$$

Combining this, the fact that $q_l(0) > \theta^*$ with $l > L^0$, and lemma 2.2, we see that $u(t, x) - u^*(x) \to 0$ locally uniformly in $[0, \infty)$ as $t \to \infty$, which means that only spreading can happen for u. That is, problem (***P***) does not have a ground state for any $L_0 > 2L^0$. This contradicts the definition of L^*, and so $L^* \leqslant 2L^0$.

Later, we consider the case where f is a generally monostable nonlinearity. In this case we need to construct a Fisher–KPP type of nonlinearity $\underline{f}(u)$ satisfying $\underline{f}(0) = \underline{f}(1) = 0$ and $\underline{f}(u) \leqslant f(u)$ for $u \in [0, 1]$. Once this is done, we then use the same argument as above to obtain that $L^* \leqslant 2\underline{L}^0$ with

$$\underline{L}^0 := \int_0^{\theta^*} \frac{1}{\sqrt{2\int_r^{\theta^*} \underline{f}(s)\mathrm{d}s}} \mathrm{d}r < \infty.$$

Now, let us construct such $\underline{f}$. As f is a C^1 function, there exist two small positive constants δ_0 and δ_1 such that

$$f(u) \geqslant \frac{1}{2} f'(0)u \text{ for } u \in [0, \delta_0] \text{ and } f(u) \geqslant \frac{1}{2} f'(1)(u - 1) \text{ for } u \in [1 - 2\delta_1, 1].$$

Choose $k := \min\{\frac{1}{2} f'(0), \min\limits_{s \in [\delta_0, 1-\delta_1]} f(s)/s\}$ and construct the following function $\underline{f} \in C^1$ such that:

$$\underline{f}(u) \begin{cases} = ku & \text{for } \ u \in [0, 1 - 2\delta_1], \\ > 0 & \text{for } \ u \in [1 - 2\delta_1, 1 - \delta_1], \\ = \dfrac{1}{2} f'(1)(u - 1) & \text{for } \ u \in [1 - \delta_1, \infty), \end{cases}$$

and that $\underline{f}(u)$ is a Fisher–KPP type of nonlinearity with

$$\underline{f}(u) \leqslant f(u) \text{ for } u \in [0, 1],$$

which completes the proof. □

REMARK 3.5. It follows from the above proof that the Fisher–KPP condition for $f(u)$ is the key point, which guarantees that problem (3.15) admits a unique positive symmetrically decreasing solution $q_l(x)$ when $l > l_0$. The proof of this proposition is similar to those of [**10**, propositions 3.10 and 4.2]. However, it appeared that the proofs of [**10**, propositions 3.10 and 4.2] were not completely correct, because the authors in [**10**] only considered the case where $f(u)$ is a Fisher–KPP type of nonlinearity. By a similar argument as above, one can fill this gap and give the correct proof.

3.2. Vanishing and spreading phenomena

Now, we give some sufficient conditions for vanishing and for spreading. We begin this section with the following result.

LEMMA 3.6. *For any $L_2 > L_1 \geqslant 0$ and any $b \geqslant 0$, let $\lambda_1(L, b)$ be the principal of the eigenvalue problem (3.1) and u be a solution of (**P**) with $u_0 \in \mathscr{X}(h)$ for some $h > 0$. The following assertions hold:*

(i) *When $\lambda_1(L, b) > 0$ and $\|u_0\|_{L^\infty}$ is small, then vanishing happens;*

(ii) *When $\lambda_1(L, b) < 0$, then vanishing does not happen for any $u_0 \not\equiv 0$.*

Proof. (i) Let φ_1^L be the corresponding positive eigenfunction, which can be normalized satisfying $\|\varphi_1^L\|_{L^\infty} = 1$. Define

$$\bar{u}(t, x) := \delta \mathrm{e}^{-((\lambda_1(L,b)/2)t)} \varphi_1^L(x) \quad \text{for } t \geqslant 0,\ x \geqslant 0,$$

with some $\delta > 0$ so small that

$$f(s) \leqslant \left(f'(0) + \frac{\lambda_1(L, b)}{2}\right)s \quad \text{and} \quad g(s) \leqslant \left(g'(0) + \frac{\lambda_1(L, b)}{2}\right)s \text{ for } s \in [0, \delta]. \tag{3.17}$$

A simple calculation yields that

$$\begin{cases} \bar{u}_t - \bar{u}_{xx} - f(\bar{u}) \geqslant \left(f'(0) + \dfrac{\lambda_1(L, b)}{2}\right)\bar{u} - f(\bar{u}) \geqslant 0, & t > 0, x \in (L_1, L_2), \\ \bar{u}_t - \bar{u}_{xx} - g(\bar{u}) \geqslant \left(g'(0) + \dfrac{\lambda_1(L, b)}{2}\right)\bar{u} - g(\bar{u}) \geqslant 0, & t > 0, x \in (0, L_1) \cup (L_2, \infty), \\ \bar{u}(t, 0) = b\bar{u}_x(t, 0), & t > 0, \\ \bar{u}(t, L_i - 0) = \bar{u}(t, L_i + 0), & t > 0, i = 1, 2, \\ \bar{u}_x(t, L_i - 0) = \bar{u}_x(t, L_i + 0), & t > 0, i = 1, 2. \end{cases}$$

Choose $\|u_0\|_{L^\infty}$ small such that $u_0(x) \leqslant \bar{u}(0, x)$ in $[0, h]$, then $\bar{u}$ is a supersolution of (**P**). Lemma 2.3 yields that $u(t, x) \leqslant \bar{u}(t, x)$ for $t \geqslant 0$ and $x \geqslant 0$. Combining this

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with the fact that $\|\bar{u}\|_{L^\infty([0,\infty))} \to 0$ as $t \to \infty$, we obtain that vanishing happens for u.

(ii) As $\lambda_1(L, b) < 0$, in light of (3.3), one obtains that $\lambda_1^R(L, b) < 0$ for all large R with $R > L_2 + h$. The positive eigenfunction corresponding to $\lambda_1^R(L, b)$, denoted by $\varphi_1^{L,R}$, solves (3.2) and can be normalized so that $\|\varphi_1^{L,R}\|_{L^\infty} = 1$. Set

$$w(x) = \begin{cases} \varrho\varphi_1^{L,R}(x), & x \in [0, R], \\ 0, & x \in (R, \infty), \end{cases}$$

where the constant $\varrho > 0$ can be chosen to be sufficiently small such that

$$f(s) \geqslant \left(f'(0) + \frac{\lambda_1^R(L,b)}{2}\right)s \quad \text{and} \quad g(s) \geqslant \left(g'(0) + \frac{\lambda_1^R(L,b)}{2}\right)s \quad \text{for } s \in [0, \varrho].$$

Consequently, we deduce that

$$\begin{cases} -w_{xx} - f(w) \leqslant \dfrac{\lambda_1^R(L,b)}{2} w \leqslant 0, & x \in (L_1, L_2), \\ -w_{xx} - g(w) \leqslant \dfrac{\lambda_1^R(L,b)}{2} w \leqslant 0, & x \in (0, L_1) \cup (L_2, \infty), \\ w(0) = bw_x(0), & \\ w(L_i - 0) = w(L_i + 0), & i = 1, 2, \\ w_x(L_i - 0) = w_x(L_i + 0), & i = 1, 2. \end{cases}$$

Furthermore, since $u(1, x) > 0$ for all $x \geqslant 0$, we can take ϱ to be smaller if necessary such that $u(1, x) > w(x)$ for all $x \geqslant 0$. Hence, w is a generalized subsolution of (***P***) for $t \geqslant 1$, $x \geqslant 0$. By lemma 2.3, we obtain $u(t, x) \geqslant w(x)$ for $t > 1$ and $x \geqslant 0$. This apparently implies that vanishing cannot happen for u, which completes the proof of this lemma. □

Based on the phase-plane analysis we can give the following sufficient condition for spreading.

LEMMA 3.7. *Assume that* ***(H)*** *holds. For any $L_2 > L_1 \geqslant 0$ and any $b \geqslant 0$, let u be the solution of* (***P***) *with $u_0 \in \mathscr{X}(h)$ for some $h > 0$. If for any $\alpha \in (\theta^*, 1]$, $u_0 \geqslant \alpha$ on $[r, r + 2l_\alpha]$ for some $r \geqslant L_2$, where l_α is given in (2.3), spreading happens for u.*

Proof. It follows from [**5**, lemmas 3.1 and 3.2] that the solution w of the following problem:

$$\begin{cases} w_t = w_{xx} + g(w), & t > 0,\ x \in (r, \infty), \\ w(t, r) = 0, & t > 0, \\ w(0, x) = u_0(x), & x \in (r, \infty), \end{cases}$$

satisfies

$$\lim_{t\to\infty} w(t, x) = W^*(x) \text{ locally uniformly in } x \in [r, \infty), \tag{3.18}$$

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where W^* is the unique solution of

$$w'' + g(w) = 0 < w \text{ in } (r, \infty), \quad w(r) = 0,\ w(\infty) = 1.$$

The comparison principle gives that $u(t,\, x) \geqslant w(t,\, x)$ for $t > 0$ and $x \geqslant r$. Since active states are only solutions of (1.7) bigger than $W^*(x)$ for $x \geqslant r$, then the conclusion follows from theorem 2.5 immediately. □

3.3. Proof of theorem 1.1

Based on the preparation of the previous subsections, we are now ready to give

Proof of theorem 1.1. Let u_σ be a solution of ($\boldsymbol{P}$) with $u_0 = \sigma\phi$ for some $\phi \in \mathscr{X}(h)$, $h > 0$ and $\sigma > 0$, and define

$$\Sigma_1 = \{\sigma > 0 : \text{ spreading happens for } u_\sigma\}.$$

We claim that for any $L_2 > L_1 \geqslant 0$ and any $b \geqslant 0$, Σ_1 is a nonempty open interval.

Firstly, we show that Σ_1 is nonempty. As f and g are globally Lipschitz on $[0,\, \infty)$, there is $K > 0$ such that

$$f(u),\ g(u) \geqslant -Ku, \text{ for all } u \geqslant 0.$$

Consider the following problem:

$$\begin{cases} \underline{u}_t = \underline{u}_{xx} - K\underline{u}, & t > 0,\ x \in [0, \infty), \\ \underline{u}(t, 0) = 0, & t > 0, \\ \underline{u}(0, x) = \sigma\phi(x), & x \in [0, h], \\ \underline{u}(0, x) = 0, & x \in [h, \infty). \end{cases}$$

Clearly, this problem admits a unique positive solution $\underline{u}$ and the comparison principle yields that, for $t \geqslant 0$, $x \geqslant 0$,

$$u_\sigma(t, x) \geqslant \underline{u}(t, x) = \sigma \int_0^h \frac{\mathrm{e}^{-((x-y)^2/4t) - Kt}}{\sqrt{4\pi t}} (1 - \mathrm{e}^{-(xy/t)}) \phi(y) \mathrm{d}y.$$

Then for any $\alpha \in (\theta^*,\, 1)$, we have $u_\sigma(1,\, x) > \alpha$ in $[L_2,\, L_2 + 2l_\alpha]$ provided that σ is sufficiently large. This and lemma 3.7 yield that $\sigma \in \Sigma_1$, which implies that Σ_1 is nonempty.

Later we show that Σ_1 is open. Choose any $\sigma_1 \in \Sigma_1$, then for any $\alpha \in (\theta^*,\, 1)$ and l_α given in (2.3), we can find $T_1 > 0$ such that

$$u_{\sigma_1}(T_1, x; \phi_{\sigma_1}) > \alpha \text{ in } [L_2, L_2 + 2l_\alpha]. \tag{3.19}$$

By the continuous dependence of the solution of ($\boldsymbol{P}$) on its initial values, if $\epsilon > 0$ is sufficiently small, then the solution u_ϵ of ($\boldsymbol{P}$) with $u_0 = \phi_{\sigma_1 - \epsilon}$ satisfies (3.20). It then follows from lemma 3.7 that spreading happens for u_ϵ, which infers that $\sigma_1 - \epsilon \in \Sigma_1$. On the contrary, the comparison principle implies that $\sigma \in \Sigma_1$ for any $\sigma > \sigma_1$. Thus, Σ_1 is open. Define $\sigma^* := \inf \Sigma_1$, then $\Sigma_1 = (\sigma^*,\, \infty)$.

(I) When $0 < L < L_*$, it follows from lemma 3.1 that the eigenvalue problem (3.1) with $L \in (0,\, L_*)$ admits a positive principal eigenvalue $\lambda_1(L,\, b)$. Combining

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this with lemma 3.6(i), we have that vanishing happens for all small $\sigma > 0$, thus

$$\Sigma_0 = \{\sigma > 0 : \text{ vanishing happens for the solution } u_\sigma \text{ of } (\mathbf{??})\} \neq \emptyset.$$

Moreover, by the same argument of [**10**, lemma 3.6(i)], we see that Σ_0 is an open interval. Define $\sigma_* := \sup \Sigma_0$, then $\Sigma_0 = (0, \sigma_*)$. Recalling that $\sigma^* := \inf \Sigma_1$ and $\Sigma_1 = (\sigma^*, \infty)$, then we have $\sigma_* \leqslant \sigma^*$ and neither spreading nor vanishing happen for $u_\sigma(t, x)$ with $\sigma \in [\sigma_*, \sigma^*]$. Thus, each solution $u_\sigma(t, x)$ with $\sigma \in [\sigma_*, \sigma^*]$ is a transition one.

(II) When $L_* < L < L^*$, it follows from lemma 3.1 that principal eigenvalue $\lambda_1(L, b)$ of eigenvalue problem (3.1) with $L \in (L_*, L^*)$ is negative. This, together with lemma 3.6(ii), implies that vanishing does not happen for any $\sigma > 0$. On the contrary, we have proved that $\Sigma_1 = (\sigma^*, \infty)$ with $\sigma^* := \inf \Sigma_1 < \infty$. Moreover, it follows from the proof of [**10**, lemma 3.8] that problem (***P***) admits a ground state for any $0 < L < L^*$. Thus, we obtain that $\sigma^* > 0$ and each solution $u_\sigma(t, x)$ with $\sigma \in (0, \sigma^*]$ is a transition one.

(III) When $L > L^*$, it follows from lemma 3.1 that principal eigenvalue $\lambda_1(L, b)$ of eigenvalue problem (3.1) with $L > L^*$ is negative. This, together with lemma 3.6(ii), implies that vanishing does not happen for any $\sigma > 0$. Combining with the definition of L^* and the proved fact that $\Sigma_1 = (\sigma^*, \infty)$ with $\sigma^* := \inf \Sigma_1 < \infty$, we obtain that $\sigma^* = 0$ and spreading happens for all $\sigma > 0$.

The whole proof of theorem 1.1 is thus complete. □

4. Discussion

In the present work, we have been concerned with a reaction–diffusion model with a bounded protection zone for an endangered single species, living in a one-dimensional habit, where the species is subjected to a strong Allee effect in its natural habitat, but within the protection zone the species growth is governed by the monostable nonlinear reaction.

Assume that the protection zone is $[L_1, L_2]$, and the general Robin condition is imposed on $x = 0$ (i.e. $u(t, 0) = bu_x(t, 0)$ with $b \geqslant 0$). Our results (theorem 1.1) have shown that there are two critical values $0 < L_* \leqslant L^*$, and proved that a vanishing–transition–spreading trichotomy result holds when the length $L := L_2 - L_1$ of protection zone is smaller than L_*; a transition–spreading dichotomy result holds when $L_* < L < L^*$; only spreading happens when $L > L^*$. As a consequence, our results suggest that the protection zone works only when its length L is larger than the critical value L_*. Furthermore, in light of theorem 1.2, we obtained that L_* is an increasing function of L_1 when $b < 1/\sqrt{-g'(0)}$; while L_* is decreasing with respect to L_1 when it holds $b \geqslant 1/\sqrt{-g'(0)}$. This suggests that the precise strategies for an optimal protection zone is that if b is large (i.e. $b \geqslant 1/\sqrt{-g'(0)}$), in order to make L_* small, then the protection zone should start from somewhere near 0; while if b is small (i.e. $b < 1/\sqrt{-g'(0)}$), then the protection zone should start from somewhere away from 0, and as far away from 0 as possible.

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In this paper, we have assumed that the species live in a one-dimensional space. In fact, the habitat of a biological population, in general, can be rather complicated. For example, natural river systems are often in a spatial network structure. The topological structure of a river network can greatly influence the species spreading and vanishing. Therefore, as in [**12**, **19**, **21**, **22**, **27**, **31**–**33**, **37**], it would be interesting to consider a more general river habitat (bounded or unbounded) consisting of one branch or more than one branch. Then a reaction–diffusion model with strong Allee effect and a protection zone in a river network should be an interesting problem. We leave it for future work.

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Appendix A.

LEMMA A.1. *For any given $b \geqslant 0$ and $0 \leqslant L_1 < L_2$, let $L = L_2 - L_1$ and $\lambda_1(L, b)$ be the principal eigenvalue of (3.1). Then we have*

$$\lambda_1(L, b) \in (-f'(0), -g'(0)).$$

Proof. For simplicity, we write $\lambda_1 = \lambda_1(L, b)$, and $\varphi(x)$ is denoted to be a corresponding positive eigenfunction. First of all, we claim that

$$\varphi'(x) > 0 \text{ for } \ 0 < x \ll 1. \tag{A.1}$$

In fact, when $b = 0$, then $\varphi(0) = 0$. This, together with $\varphi(x) > 0$ for $x > 0$, yields that $\varphi'(x) > 0$ for $0 < x \ll 1$. Let us consider the case where $b > 0$. If $\varphi(0) = 0$, then $\varphi'(0) = 0$, as $\varphi(x) > 0$, so $\varphi'(x) > 0$ for $0 < x \ll 1$. If $\varphi(0) > 0$, then $\varphi'(0) > 0$, so $\varphi'(x) > 0$ for $0 < x \ll 1$. If $\varphi(0) < 0$, then $\varphi'(0) < 0$, thus $\varphi(x) < 0$ for $0 < x \ll 1$, which is a contradiction. Thus, our claim is proved.

Next, we divide the proof into four steps as follows.

Step 1: $\lambda_1 \neq -g'(0)$. If there are $0 \leqslant L_1^0 < L_2^0$ such that $\lambda_1 = -g'(0)$, then for $x > L_2^0$, it follows from the second equation of (3.1) that

$$-\varphi'' = (g'(0) + \lambda_1)\varphi = 0,$$

which implies that there is a constant C such that $\varphi'(x) \equiv C$ for $x > L_2^0$. This, combining with $\varphi(\infty) = 0$, shows that $\varphi(x) \equiv 0$, a contradiction.

Step 2: $\lambda_1 < -g'(0)$. Suppose that $\lambda_1 > -g'(0)$ for some $0 \leqslant L_1^0 < L_2^0$. Then, for $x > L_2^0$, it holds

$$-\varphi'' = (g'(0) + \lambda_1)\varphi \text{ and } g'(0) + \lambda_1 > 0.$$

As $\varphi(x) > 0$ for $x > 0$, then $\varphi'' < 0$ in (L_2^0, ∞), which yields that $\varphi'(x)$ is decreasing in $x > L_2^0$. If $\varphi'(\infty) \geqslant 0$, φ is increasing in $x > L_2^0$. Since $\varphi > 0$ on $[L_2^0, \infty)$, we arrive at a contradiction with $\varphi(\infty) = 0$. If $\varphi'(\infty) < 0$, it is easily shown that there

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exists a large $x_0 > 0$ such that $\varphi(x) < 0$ for $x \in [x_0, \infty)$, which again leads to a contradiction.

Step 3: $\lambda_1 \neq -f'(0)$. If there are $0 \leqslant L_1^0 < L_2^0$ such that $\lambda_1 = -f'(0)$. It then follows from the first equation of (3.1) that

$$-\varphi'' = (f'(0) + \lambda_1)\varphi \equiv 0 \quad \text{for all} \;\; x \in (L_1^0, L_2^0),$$

which yields that there is a constant C_0 such that

$$\varphi'(x) \equiv C_0 \quad \text{for all} \;\; x \in (L_1^0, L_2^0).$$

We claim that $C_0 > 0$. If this is proved, for $x > L_2^0$, it follows from the second equation of (3.1) that

$$-\varphi'' = (g'(0) + \lambda_1)\varphi = (g'(0) - f'(0))\varphi,$$

with $g'(0) - f'(0) < 0$, then there exist two constants $\tilde{C}_1$ and $\tilde{C}_2$ such that

$$\varphi(x) = \tilde{C}_1 \mathrm{e}^{\sqrt{f'(0)-g'(0)}\,x} + \tilde{C}_2 \mathrm{e}^{-\sqrt{f'(0)-g'(0)}\,x} \quad \text{for } x > L_2^0.$$

Noting that $\varphi(\infty) = 0 < \varphi(x)$ for $x > 0$, we have $\tilde{C}_2 > 0 = \tilde{C}_1$. In turn, $\varphi(x) = \tilde{C}_2 \mathrm{e}^{-\sqrt{f'(0)-g'(0)}\,x}$ for $x > L_2^0$. Hence, we obtain

$$\varphi'(L_2^0 + 0) = -\tilde{C}_2\sqrt{f'(0) - g'(0)}\mathrm{e}^{-\sqrt{f'(0)-g'(0)}\,L_2^0} < 0 < C_0 = \varphi'(L_2^0 - 0).$$

This leads to a contradiction with the condition $\varphi'(L_2 - 0) = \varphi'(L_2 + 0)$.

Now, let us prove $C_0 > 0$. In fact, when $L_1^0 = 0$, it then follows from (A.1) that $C_0 > 0$. When $L_1^0 > 0$, then for $x \in (0, L_1^0)$, it follows from the second equation of (3.1) that

$$-\varphi'' = (g'(0) + \lambda_1)\varphi = (g'(0) - f'(0))\varphi.$$

As $g'(0) - f'(0) < 0$, one can find two constants $\tilde{C}_3$ and $\tilde{C}_4$ such that

$$\varphi(x) = \tilde{C}_3 \mathrm{e}^{\sqrt{f'(0)-g'(0)}\,x} + \tilde{C}_4 \mathrm{e}^{-\sqrt{f'(0)-g'(0)}\,x} \quad \text{for } x \in (0, L_1^0).$$

This, together with $\varphi(0) = b\varphi'(0)$ and $\varphi(x) > 0$ for $x > 0$, yields that

$$\tilde{C}_4 = \frac{b\sqrt{f'(0) - g'(0)} - 1}{b\sqrt{f'(0) - g'(0)} + 1}\tilde{C}_3 \quad \text{and} \quad \tilde{C}_3 > 0,$$

which implies that

$$\varphi(x) = \tilde{C}_3\left[\mathrm{e}^{\sqrt{f'(0)-g'(0)}x} + \frac{b\sqrt{f'(0) - g'(0)} - 1}{b\sqrt{f'(0) - g'(0)} + 1}\mathrm{e}^{-\sqrt{f'(0)-g'(0)}x}\right] > 0 \quad \text{for} \quad x \in (0, L_1),$$

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and

$$\begin{aligned}
&\varphi'(L_1^0-0)\\
&\quad=\tilde{C}_3\sqrt{f'(0)-g'(0)}\left[\mathrm{e}^{\sqrt{f'(0)-g'(0)}L_1^0}-\frac{b\sqrt{f'(0)-g'(0)}-1}{b\sqrt{f'(0)-g'(0)}+1}\mathrm{e}^{-\sqrt{f'(0)-g'(0)}L_1^0}\right]\\
&\qquad>0.
\end{aligned}$$

Thanks to the condition $\varphi'(L_1^0-0)=\varphi'(L_1^0+0)$, we see that $C_0=\varphi'(L_1^0-0)>0$. Thus, we prove that $C_0>0$ in this case.

Step 4: $\lambda_1>-f'(0)$. If there are $0\leqslant L_1^0<L_2^0$ such that $\lambda_1<-f'(0)$. From the first equation of (3.1), we see that

$$\varphi''=-(f'(0)+\lambda_1)\varphi>0 \text{ for } \quad x\in(L_1^0,L_2^0). \tag{A.2}$$

We claim that

$$\varphi'(L_2^0-0)>0. \tag{A.3}$$

If this is done, then when $x>L_2^0$, it follows from the second equation of (3.1) and the results proved in steps 1 and 2 that

$$-\varphi''=(g'(0)+\lambda_1)\varphi \text{ with } g'(0)+\lambda_1<0.$$

Then $\varphi''(x)>0$ for $x>L_2^0$ and we can find two constants $\tilde{C}_5$ and $\tilde{C}_6$ such that

$$\varphi(x)=\tilde{C}_5\mathrm{e}^{\sqrt{-(g'(0)+\lambda_1)}\,x}+\tilde{C}_6\mathrm{e}^{-\sqrt{-(g'(0)+\lambda_1)}\,x} \quad \text{for } x>L_2^0.$$

As $\varphi(\infty)=0$, it is necessary that $\tilde{C}_6>0=\tilde{C}_5$. Hence, $\varphi(x)=\tilde{C}_6\mathrm{e}^{-\sqrt{-(g'(0)+\lambda_1)}x}$ for $x>L_2^0$ and

$$\varphi'(L_2^0+0)=-\tilde{C}_6\sqrt{-(g'(0)+\lambda_1)}\mathrm{e}^{-\sqrt{-(g'(0)+\lambda_1)}L_2^0}<0.$$

Using this, (A.3) and the condition $\varphi'(L_2^0-0)=\varphi'(L_2^0+0)$, we arrive at a contradiction.

It is remaining to prove (A.3). If $L_1^0=0$, it follows from (A.1) and (A.2) that

$$\varphi'(L_2^0-0)>0.$$

Now, we consider the case where $L_1^0>0$. For $x\in(0,\,L_1^0)$, it follows from the second equation of (3.1) and the results proved in steps 1 and 2 that

$$\varphi''=-(g'(0)+\lambda_1)\varphi>0.$$

This, together with (A.1), yields that

$$\varphi'(L_1^0-0)>0.$$

Since $\varphi'(L_1^0-0)=\varphi'(L_1^0+0)$, one obtains that

$$\varphi'(L_1^0+0)>0. \tag{A.4}$$

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Combining this with (A.2), we have

$$\varphi'(L_2^0 - 0) > \varphi'(L_1^0 + 0) > 0,$$

which ends the proof of (A.3).

The proof of this lemma is complete. □

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