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ON UNIQUENESS SETS FOR EXPANSIONS IN SEQUENCES OF FUNCTIONS ARISING FROM SINGULAR GENERATING FUNCTIONS

JET WIMP

1. Introduction. Let $\{p_n(z)\}$ be a sequence of functions analytic in a region D. A problem in analysis which has received much attention is the following: describe the sets $Z \subset D$ for which

$$(1) \qquad \sum h_n p_n(z) = 0, \quad z \in Z,$$

implies h_n is 0 for all n. (To make the problem interesting, only those situations are studied where finite subsets of the $p_n(z)$ are linearly independent in D.) Another way of phrasing this is: Characterize the uniqueness sets of $p_n(z)$, a uniqueness set Z being a set in D such that the restriction of $\{p_n(z)\}$ to Z is linearly independent. If Z is not a uniqueness set then for some $\{h_n\}$ not all 0, we have

$$(2) \qquad \sum h_n p_n(z) = 0, \quad z \in Z.$$

This formula is called a non-trivial representation of 0 (on Z).

One reason for the interest in the problem is its relationship to the uniqueness problem for expansion of functions analytic in D in series of the $\{p_n(z)\}$, such an expansion being unique if and only if D is a uniqueness set for $\{p_n(z)\}$. Historically, this problem arose in the study of Fourier series.

A typical question asked is: how small can a uniqueness set be? (in terms of measure, cardinality, category, etc). Additionally, one may wish to find how the structure of uniqueness sets changes when constraints are put on the h_n, for instance, "gap"-type conditions. A finding along these lines is Colton's result, [**6**], on the Gegenbauer polynomials $C_n^{\nu}(z)$, $2\nu \neq -1, -2, -3, \ldots$. (All special functions in this paper are as defined in [**8**]. All sums are from 0 to ∞ unless indicated otherwise.)

THEOREM. *Let $\sum h_n n^{\nu}$ converge and*

$$(*) \qquad \begin{cases} h_n = 0 \text{ except (possibly) when } n \text{ belongs to a sequence } n_k \text{ with} \\ n_{k+1} > (1+\delta)n_k,\ \delta > 0. \end{cases}$$

Then there are no non-trivial representations of zero, $\sum h_n C_n^{\nu}(z)$, on $[-1, 1]$.

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If the condition (*) is removed, then the result is false, such representations being possible for certain values of ν.

A completely general approach to the problem of uniqueness sets is not very rewarding: examples have been devised to show anything can happen, and uniqueness sets have a recognizable structure only when the $\{p_n(z)\}$ fall into certain categories. Writers investigating the problem have found it most profitable to assume one of the following three situations prevails:

(1) the $\{p_n(z)\}$ are orthogonal on some curve in the complex plane;

(2) the $\{p_n(z)\}$ (or some closely associated function(s)) satisfy a differential equation;

(3) the $\{p_n(z)\}$ are generated by a generating function of Appell type

$$(3) \qquad A(w)\Psi[zg(w)] = \sum w^n p_n(z),$$

where $A(w)$, $g(w) \in \mathscr{H}(0)$ and $\Psi(t)$ is entire.

Colton's result (and related results for the Jacobi polynomials reported in [**13**]) are based on the fact that certain generating functions for the polynomials satisfy a singular partial differential equation (case 2)). Interesting things happen (also case (2)) when the $\{p_n(z)\}$ themselves satisfy a second order differential equation of Sturm-Liouville type. For instance, sets which are not uniqueness sets satisfy a reflection principle. Moreover (and very surprisingly) representations of 0 on an interval by $\{p_n(z)\}$ remain representations of 0 when the $\{p_n(z)\}$ are replaced by functions which satisfy other differential equations of Sturm-Liouville type, [**14**]. No work has yet been done to extend these results to Appell polynomials.

Situation (3) is probably the one for which most has been accomplished; much of the work was done to resolve questions raised in Boas and Buck's very influential treatise on polynomial expansions of analytic functions, [**1**]. In fact, we will treat situation (3) in this paper. The character of uniqueness sets for Appell polynomials reflects a fact which we formulate, rather loosely, as follows: Appell polynomial expansions behave like power series. We know that any bounded infinite set is a uniqueness set for $\{z^n\}$. Much the same seems to be true (it has not actually been proved) for Appell polynomials: either any bounded infinite set is a uniqueness set, or, no set is a uniqueness set. That both cases are possibilities is illustrated by the Bernoulli polynomials, defined by

$$(4) \qquad \frac{w^\nu e^{zw}}{(e^w - 1)^\nu} = \sum w^n \frac{B_n^{\nu}(z)}{n!}, \quad \nu \in \mathscr{C}, \ |w| < 2\pi.$$

When $\operatorname{Re} \nu < 0$, no set is a uniqueness set because

$$(5) \qquad \sum \frac{(2\pi i)^n}{n!} B_n^{\nu}(z) = 0, \quad z \in \mathscr{C}.$$

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When $\operatorname{Re} \nu \geqq 0$, we conjecture that every bounded infinite set (but no finite set) is a uniqueness set. In this paper we will establish the fact for $\operatorname{Re} \nu > 3/2$. Buckholtz's work shows it is true for $\nu = 1$, and $\nu = 0$ is the power series case. For other values of ν, $\operatorname{Re} \nu \geqq 0$, the question is still open.

The first general theorem dealing with uniqueness sets for Appell polynomials is Theorem 8.8 in Boas and Buck: let $g(w) \in \mathscr{H}(\Omega)$ and be univalent in Ω, where Ω contains the disk $\bar{U}_R = \{w|\,|w| < R\}$. Let $A(w) \in \mathscr{H}(\bar{U}_R)$, $A(w) \neq 0$ in $\bar{U}_R$, and

$$(6) \qquad \sum h_n p_n(z) = 0, \quad z \in \mathscr{C},$$

for some sequence $\{h_n\}$ with $h_n = O(R^n)$. Then h_n is 0 for all n. In other words, $\mathscr{C}$ is a uniqueness set for $\{p_n(z)\}$.

This theorem has several shortcomings. But its shortcomings have generated a great amount of productive research.

First, a priori estimates of the h_n cannot, generally, be known. One would like to omit any such order estimate. Although it is unlikely that this can always be done, Buckholtz [**3**], following the earlier work of Read, [**10**] has, by the use of differential operators, shown a satisfying result for an important case: when $\Psi(t) = e^t$ and $A(w)$ is analytic in U (the unit disk) but has poles and is non-zero on T any bounded infinite set is a uniqueness set for $\{p_n(z)\}$. See also [**2**]. This result can be easily extended to cover the case $\Psi =$ arbitrary entire, $g(w) = w$ (Brenke polynomials). T is the unit circle.

Second, much smaller sets than $\mathscr{C}$ are usually uniqueness sets. Buckholtz and Shaw [**4, 5**] have shown that when $p_n(z)$ has a certain asymptotic property, call it P, then convergence on a finite set of $\sum h_n p_n(z)$ entails uniform convergence on every compact set. By the Vitali convergence theorem, then, "$z \in \mathscr{C}$" in (6) can be replaced by the weaker "$z \in Z$, Z bounded and infinite". In my experience, P always seems to hold for Appell polynomials. This has neither been proved nor disproved, however.

Third, the theorem requires that $A(w)$ be analytic on the boundary of U_R. In most cases, $A(w)$ has only a finite circle of convergence, say U_S, and those sequences $\{h_n\}$ which can produce nontrivial representations of zero on infinite bounded sets are usually the ones which are $O(S^n)$ or $O(n^a S^n)$ for some a. The Bernoulli polynomials are a case in point, and there $S = 2\pi$. Remedying this defect is no easy matter, however. The machinery of contour integration, which produces such an elegant proof of Boas and Buck's Theorem 8.8 can no longer be employed, since integration is along ∂U_S of a kernel which is singular there. What has to be used instead is mean convergence in $L^p(\partial U_S)$. The proofs required are longer though no less elegant.

In this paper, $p_n(z)$ need not be a polynomial. In fact the basic assumptions are rather meager. Let

(7) $\quad A(w)\Psi(z, w) = \sum w^n p_n(z),$

$A(w) \in \mathscr{H}(0)$, $\Psi(z, w)$ analytic in $\Lambda \times \Omega$ where

(8) $\quad \Lambda = \{z| \quad |z| < R\} = U_R,$

and Ω is a region containing 0.

The notation $\Psi_n(w)$ will denote the Taylor coefficients of $\Psi(z, w)$.

(9) $\quad \Psi(z, w) = \sum z^n \Psi_n(w), \quad z \in \Lambda.$

We give several examples. The most ambitious is when $A(w)$ has algebraic singularities on its circle of convergence. In this case both the second and third problems above can be surmounted. The uniqueness sets (bounded infinite sets) are really optimal since it is known such expansions can represent functions with an infinite number of unbounded zeros. (Without loss of generality, it is always assumed $R = 1$.)

The development relies heavily on the theory of H^p spaces. The book by Duren, [**7**], contains all the necessary background information.

2. General theorems.

THEOREM 1. *Let* $\Omega \supset \bar{U}$ *and let* $\{\Psi_n(w)\}$ *be complete in* $\mathscr{H}(\bar{U})$. *Let*

(10) $\quad \sum |h_n|^q n^{q-2} < \infty$

for some $q \geqq 2$ *and* $A(w) \in H^p$, $1/p + 1/q = 1$. *Let*

(11) $\quad \sum h_n p_n(z) = 0$

on a set Z *having a limit point in* Λ (*see* (9)).

Then if $A(w) \neq 0$ *in* U, h_n *is* 0 *for all* n, *while if* $A(w)$ *has a finite number of zeros in* U, *the representation* (11) *must result from taking linear combinations of the (finite number of) representations of* 0 *formed by evaluating* $A(w)\Psi(z, w)$ *and its derivatives at the zeros of* $A(w)$ *in* U.

Proof. Define

(12) $\quad h(w) := \sum \dfrac{h_n}{w^{n+1}}, \quad |w| > 1,$

then $h(1/w) \in H^q$. Denote the outer radial limit of $h(w)$ by $h^*(w)$, the radial limit of $A(w)$ by $A^*(w)$. Then $A^*(w)h^*(w) \in L^1(T)$.

A straightforward argument based on Cauchy's inequality shows we can find a circle $N \subset \Lambda$ containing an infinite number of points of Z on which (9) converges uniformly in z and uniformly for $t \in T$.

Let

(13) $$F(z) := \int_T A^*(w)h^*(w)\Psi(z, w)dw$$
$$= \sum F_n z^n, \quad z \in N.$$

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Choose $r > 1$ and let $z \in Z \cap N$. Then

$$(14)\quad |F(z)| \leqq \left| \int_T A^*(w)(h^*w) - h(rw))\Psi(z, w)dw \right| + \left| \int_T A^*(w)h(rw)\Psi(z, w)dw \right|$$
$$\leqq \|A^*(w)\Psi(z, w)\|_p \|h^*(w) - h(rw)\|_q + \left| \int_{|w|=\delta} A(w)h(rw)\Psi(z, w)dw \right|, \; 1/r < \delta < 1,$$
$$\leqq C\|h^*(w) - h(rw)\|_q + |\sum h_n p_n(z)/r^{n+1}|,$$

and by mean convergence and Abel's theorem, the right hand side $\rightarrow 0$ as $r \rightarrow 1$. Thus $F(z)$ is 0 on a bounded infinite set, so $F(z) \equiv 0$ in N, or F_n is 0 for all n, or

$$(15)\quad \int_T A^*(w)h^*(w)\Psi_n(w)dw = 0, \quad n = 0, 1, 2, \dots.$$

The $\{\Psi_n(w)\}$ are complete in $\mathscr{H}(\bar{U})$, so we may uniformly approximate w^k, $k = 0, 1, 2, \dots$, by a finite linear combination of them. (Obviously, having $\{\Psi_n\}$ complete in $C(T)$ is sufficient here, but generally more difficult in practice to verify. See the example following.) This means

$$(16)\quad \int_T A^*(w)h^*(w)w^k dw = 0, \quad k = 0, 1, 2, \dots,$$

so $A^*(w)h^*(w)$ must be the boundary function of some H^1 function, $u(w)$. Define

$$(17)\quad \hat{h}(w) := u(w)/A(w), \quad w \in U.$$

Consider first the case where $A(w) \not\equiv 0$ in U. Then $\hat{h}(w)$ is analytic in U. Employing the canonical factorization of $u(w)$ and $A(w)$, we can represent $\hat{h}(w)$ as

$$(18)\quad \hat{h}(w) = B(w)S(w)K(w),$$

where $B(w)$ is a Blaschke product, $S(w)$ a singular inner function and

$$(19)\quad K(w) = e^{i\tau} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w} \ln \left| \frac{u^*(e^{it})}{A^*(e^{it})} \right| dt \right\}.$$

Now $A^*(e^{it})$ can be 0 at most on a set of measure 0. Thus $u^*(e^{it}) = A^*(e^{it})h^*(e^{it})$ a.e., so

$$(20)\quad K(w) = e^{i\tau} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w} \ln |h^*(e^{it})| dt \right\}.$$

Since $h^*(w) \in L^q(T)$, the above, used in (18) shows, by the canonical factorization theorem, [**7**, Theorem 2.8], that $\hat{h}(w) \in H^q$ and, in fact, $h^*(w)$ is its boundary function.

Thus $\hat{h}(w)$ has the Cauchy integral representation

$$(21)\quad \hat{h}(w) = \frac{1}{2\pi i}\int_T \frac{h^*(\zeta)d\zeta}{\zeta - w} = \frac{1}{2\pi i}\sum \mu_{-n-1}w^n,\ |w| < 1,$$

$$\mu_k: = \int_T h^*(\zeta)\zeta^k d\zeta,\quad k = 0, \pm 1, \pm 2, \dots .$$

But

$$(22)\quad \mu_{-n-1} = \lim_{\substack{r\to 1\\ r>1}} \int_T \frac{h(r\zeta)d\zeta}{\zeta^{n+1}} = \lim_{\substack{r\to 1\\ r>1}} \lim_{N\to\infty} \int_T \sum_{k=0}^{N} \frac{h_k d\zeta}{r^{k+1}\zeta^{n+k+2}} .$$
$$= 0.$$

So $\hat{h}(w) \equiv 0$. This means $\hat{h}^*(w) = 0$, or $h^*(w) = 0$ a.e. But $h(1/w)$ is itself the Cauchy integral of its boundary function, so $h(w)$ must be zero identically for $|w| > 1$, or h_n is 0 for all n.

It is surprising that when $A(w)$ has zeros in U, the convergence to 0 of $\sum h_n p_n(z)$ places such a strong requirement on the sequence $\{h_n\}$ that $h(w)$ can actually be analytically continued across T. When this is established it will follow that $h(w)$, which is continued into U by $\hat{h}(w)$, is a rational function with poles at the zeros of $A(w)$, and the rest of the argument proceeds as in [**2**, p. 26]. In fact, all we need to do is to show

$$(23)\quad \lim_{\substack{r\to 1\\ r>1}} \|h(rw) - h^*(w)\|_q = \lim_{\substack{r\to 1\\ r>1}} \|\hat{h}(rw) - h^*(w)\|_q = 0.$$

We can then invoke a generalized Schwarz reflection principle (the one-variable edge-of-the-wedge theorem, [**11**]). For details, see the Appendix.

The first limit in (23) follows at once from mean convergence. To show the second we represent $A(w)$ as

$$(24)\quad A(w) = B(w)\tilde{A}(w),\quad w \in U,$$

where $B(w)$ is the (finite) Blaschke product corresponding to $A(w)$, and choose $r > |\alpha|$ for any zero, α, of $A(w)$ in U.

A canonical factorization argument shows

$$(25)\quad \tilde{A}(w)^{-1} \in H^p; B(w)\hat{h}(w) = u(w)\tilde{A}(w)^{-1} \in H^q; B^2(w)\hat{h}(w) \in H^q.$$

So

$$(26)\quad \begin{aligned} \|\hat{h}(rw) - h^*(w)\|_q = \|\{&[B(w) - B(rw)]\\ &\times [B(rw)\hat{h}(rw) - B(w)h^*(w)]\\ &-[B^2(rw)\hat{h}(rw) - B^2(w)h^*(w)]\}[B(w)B(rw)]^{-1}\|_q\\ &< M\{\|B(rw)\hat{h}(rw) - B(w)h^*(w)\|_q\\ &+ \|B^2(rw)\hat{h}(rw) - B^2(w)h^*(w)\|_q\} \end{aligned}$$

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and by mean convergence the right hand side $\to 0$ as $r \to 1$. Thus $h(w)$ may be continued analytically across T and the theorem is proved.

The theorem has an interesting consequence. The equation

$$(27) \quad \int_T A^*(w)h^*(w)\Psi_n(w)dw = 0, \quad n = 0, 1, 2, \ldots,$$

results from the fact that $\Psi(z, w)w^k$, being analytic in $\bar{U}$, may be uniformly approximated by linear combinations of the $\Psi_n(w)$. Thus

$$(28) \quad \int_T A^*(w)h^*(w)\Psi(z, w)w^k dw = 0, \quad k = 0, 1, 2, \ldots$$

or

$$(29) \quad \sum h_{n+k}p_n(z) = 0, \quad z \in Z, \quad k = 0, 1, 2, \ldots$$

Thus, any representation of 0 induces a class of such representations for, corresponding to the difference operator,

$$(30) \quad \mathscr{L} = \sum_{s=0}^{\sigma} b_s E^s, \quad Ep_n = p_{n+1},$$

we have

$$(31) \quad \sum \mathscr{L}(h_n)p_n(z) = 0, \quad z \in Z.$$

For the sake of completeness, we give an extension of Boas and Buck's theorem. The proof follows closely their proof, and is omitted.

THEOREM 2 (Boas and Buck). *Let*

$$(32) \quad \sum h_n p_n(z) = 0$$

on some set Z having a limit point in Λ. Let $\Omega \supset \bar{U}_R$ and let $\{\Psi_n(w)\}$ be complete in $\mathscr{H}(\bar{U}_R)$ and

$$(33) \quad |h_n| < MR^n, \; n > n_0.$$

Then if $A(w) \neq 0$ in $\bar{U}_R$, h_n is 0 for all n, while if $A(w)$ has zeros in $\bar{U}_R$, the representation of zero arises from these zeros as in Theorem 1.

Example. Here $A \equiv 1$,

$$(34) \quad \Psi(z, w) = (1 - 4zw)^{-1/2}2^{c-1}[1 + (1 - 4zw)^{1/2}]^{1-c} \times \exp[1 - (1 - 4zw)^{1/2}/2z]$$

and

$$(35) \quad p_n(z) = \frac{1}{n!}\, {}_2F_0(-n, c + n | -z),$$

(the Bessel polynomials). For $|z|$ sufficiently small, $\Psi(z, w)$ is analytic for $|w| \leqq R$. Interchanging the order of summation shows

$$(36)\quad \Psi(z,w) = \sum \frac{w^n z^n}{n!}(c+n)_{n\,1}F_1\left(\begin{matrix} c+2n \\ c+n \end{matrix}\middle| w\right)$$

so

$$(37)\quad \Psi_n(w) = \frac{(c+n)_n}{n!} w^n {}_1F_1\left(\begin{matrix} c+2n \\ c+n \end{matrix}\middle| w\right).$$

($(a)_n$ denotes Pochhammer's notation; see [**8**].) We will show that the sequence $\{\Psi_n(w)\}$ is complete in $\mathscr{H}(B)$ for any compact subset B of $\mathscr{C}$ provided $c \neq 0, -1, -2, \ldots$.

Let

$$(38)\quad \begin{aligned} A_{m,n} &= 0, \quad n < m \\ &= \frac{n!(-1)^{m+n}}{(n-m)!(c+m)_m(c+2m+1)_{n-m}}, \; n \geqq m. \end{aligned}$$

The series $\sum A_{m,n}\Psi_n(w)$ converges absolutely and uniformly in any bounded region, since

$$(39)\quad \Psi_n(w) = \frac{2^{c-1}(4w)n^{-1/2}}{\sqrt{\pi}} e^{2w}[1 + O(n^{-1})],$$

uniformly. Interchanging the order of summation and evaluating a ${}_2F_1$ of unit argument shows

$$(40)\quad w^m = \sum A_{m,n}\Psi_n, \quad m = 0, 1, 2, \ldots$$

which established the completeness of $\{\Psi_n(w)\}$.

Thus, if $c \neq 0, -1, -2, \ldots$, and

$$(41)\quad \sum \frac{h_n}{n!} {}_2F_0(-n, n+c|-z) = 0,$$

on a set of points having 0 as a limit point, and $h_n = O(R^n)$ for some R, then h_n is 0 for all n. (The restriction on c may be removed by appeal to a theorem of Carleman, [**9**]).

3. Applications to functions algebraic on T. These functions, called $\mathscr{A}$-functions, are analytic in $\bar{U}$ except at a finite number of points on T, where the function has "algebraic" singularities.

To make this precise, label these points

$$(42)\quad \sigma_r = e^{i\theta_r}, \quad 1 \leqq r \leqq m; \theta_r \in [0, 2\pi), \theta_i \neq \theta_j, i \neq j,$$

and assume that, for w sufficiently close to σ_r in U,

$$(43)\quad A(w) = \sum_{k=0}^{\infty} c_{r,k}\left(1 - \frac{w}{\sigma_r}\right)^{\alpha_r+k}, \quad 1 \leqq r \leqq m,$$

$$\alpha_r \neq 0, 1, 2, \ldots, c_{r,0} \neq 0.$$

We further require that the exponents of the singularities, $\{\alpha_r\}$, may be divided into at most three groups as follows:

$$(44)\quad \begin{aligned}&(1)\ \alpha_1 = \alpha_r, \quad 2 \leqq r \leqq m_1;\\ &(2)\ \left.\begin{aligned}\operatorname{Im}\alpha_1 &= \operatorname{Im}\alpha_r\\ \operatorname{Re}\alpha_1 &< \operatorname{Re}\alpha_r\end{aligned}\right\} \quad m_1 + 1 \leqq r \leqq m_2;\\ &(3)\ \operatorname{Re}(\alpha_1 + 1) < \operatorname{Re}\alpha_r, \quad m_2 + 1 \leqq r \leqq m.\end{aligned}$$

For $\mathscr{A}$ functions, $p_n(z)$ has simple asymptotic properties, in fact, properties similar to property P mentioned in the introduction.

LEMMA 1. *Let $A(w)$ be an $\mathscr{A}$-function. Suppose the determinant*

$$(45)\quad |\Psi_{k-1}(\sigma_j)| \neq 0, \quad j, k = 1, 2, \ldots, m_1,$$

and let

$$(46)\quad \sum h_n p_n(z)$$

converge on a set of points Z having a limit point in Λ. Then all the series

$$(47)\quad \sum h_n \sigma_r^{-n}(n+1)^{-\alpha_r - j - 1}, \quad r = 1, 2, \ldots, m; j = 0, 1, 2, \ldots,$$

converge, and the series (46) *converges uniformly on compact z-sets and represents an entire function of z.*

Proof. Let $z \in \mathscr{B}$, $\mathscr{B}$ compact. Darboux's method (see [**12**]) when applied to $A(w)\Psi(z, w)$ gives a result that may be written in the form

$$(48)\quad p_n(z) = \sum_{j=0}^{s} \sum_{r=1}^{m} b_{r,j}\sigma_r^{-n}(n+1)^{-\alpha_r - j - 1}\Psi(z, \sigma_r) + \xi_n(z)/(n+1)^2$$

where $\xi_n(z)$ is bounded in n uniformly in $\mathscr{B}$ and

$$b_{r,0} = c_{r,0}\Psi(z, \sigma_r).$$

By (44), each of the series

$$(49)\quad \sum h_n \sum_{r=1}^{m_1} c_{r,0}\sigma_r^{-n}(n+1)^{-\alpha_r - 1}\Psi(z_k, \sigma_r), \quad z_k \in Z,$$

must converge. If the determinant

$$(50)\quad |\Psi(z_{l_k}, \sigma_j)| \neq 0, \quad j, k = 1, 2, \ldots, m_1,$$

for all choices of $z_{l_k} \in Z$ then we can form linear combinations of the above series to obtain the convergent series

$$(51)\quad \sum h_n \sigma_r^{-n}(n+1)^{-\alpha_r - 1}$$

and, by Abel's test and the conditions (44), all the series (47) will converge. Assume (50) is true. Denote the (p, q) segment of the series (47),

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$|\sum_p^q|$, by $R_{p,q}^{n,j}$. Then there are constants $B_{r,j}$ such that

$$(52)\quad \left|\sum_p^q h_n p_n(z)\right| \leqq \sum_{j=0}^{s} \sum_{r=1}^{m_1} B_{r,j} R_{p,q}^{r,j} + M \sum_p^q \frac{1}{(n+1)^2}.$$

Thus the left hand side is a uniform Cauchy sequence and the theorem is proved once we establish (50).

It is sufficient to take the 2×2 case, the others being treated similarly. Assume the contrary of (50), and that

$$(53)\quad \begin{vmatrix} \Psi(z_k, \sigma_1) & \Psi(z_k, \sigma_2) \\ \Psi(z_j, \sigma_1) & \Psi(z_j, \sigma_2) \end{vmatrix} = 0, \quad \text{for all } z_k, z_j \in Z.$$

Subtract the first row from the second, divide by $z_k - z_j$ and let z_k, z_j be members of a subsequence of Z approaching some point $a \in \Lambda$. Let first $j \to \infty$ and then $k \to \infty$. This gives

$$(54)\quad \begin{vmatrix} \Psi_0(\sigma_1) & \Psi_0(\sigma_2) \\ \Psi_1(\sigma_1) & \Psi_1(\sigma_2) \end{vmatrix} = 0$$

which cannot be.

Note Lemma 1 is a straightforward generalization of the case when $A(w)$ is meromorphic in $\mathscr{H}(\bar{U})$, see [**2**].

LEMMA 2. *Let $A(w) \in \mathscr{H}(0)$, $g(w) \in \mathscr{H}(\Omega)$, $\Omega \supset \{0\}$, and*

$$(55)\quad A(w)e^{zg(w)} = \sum w^n p_n(z)$$

and let $w_0 \in \Omega$, $g(w_0) \neq 0$. Then the system of polynomials

$$(56)\quad \hat{p}_n(z) = \int_z^{z+\tau} p_n(u)du, \quad \tau = 2\pi i/g(w_0),$$

is an Appell set with

$$(57)\quad A(w)B(w)e^{zg(w)} = \sum w^n \hat{p}_n(z),$$

where $B(w) \in \mathscr{H}(\Omega)$, $B(w) \not\equiv 0$, and

$$(58)\quad B(w) = O(w - w_0), \quad w \to w_0.$$

Proof. The proof is obvious.

Theorem 1 cannot be applied directly to expansions in polynomials generated by $A(w)\Psi(z, w)$ when $A(w)$ is an $\mathscr{A}$-function, for the following reason. Suppose

$$(59)\quad \sum h_n p_n(z)$$

converges on a set of points having a limit point in Λ. Then by Lemma 1,

$$(60)\quad \sum h_n \sigma_r^{-n}(n+1)^{-\alpha_r - 1}, \quad r = 1, 2, \ldots, m_1,$$

converges. All one can say is

(61) $h_n = O[n^{\alpha_1+1}]$,

or

(62) $|h_n|^q n^{q-2} = O[n^{q(\alpha_1+2)-2}]$.

To guarantee the convergence of $\sum |h_n|^q n^{q-2}$ for some $q > 2$ it is necessary to take $\operatorname{Re} \alpha_1 < -3/2$. But then $A(w) \notin H^p$ for any p. In fact, we will not be able to treat the problem of algebraic $A(w)$ for all possible $\Psi(z, w)$. When Ψ is of the form $\Psi(zg(w))$, where $\Psi(t)$ is an entire function of order 1 satisfying certain conditions, it is possible to apply an operator to the series $\sum h_n p_n(z)$ to reduce the problem to one involving polynomials generated by $A(w)e^{zg(w)}$, and then to use Lemma 2 to integrate the series repeatedly until the exponent α_1 with largest real part is such that $A(w) \in H^p$. This operator, the Φ-operator is defined as follows: let $\{\phi_n\}$ be a complex sequence, $\phi_n \neq 0$, $\phi_n = O(R^n)$ for some R. Let

(63) $\Phi[f(z)] = \sum f_n \phi_n z^n$,

when

(64) $f(z) = \sum f_n z^n$.

Then Φ is a one to one linear mapping of $\mathscr{H}(\mathscr{C})$ onto $\mathscr{H}(\mathscr{C})$. Further, Φ takes polynomials into polynomials. If a sequence of functions $E_n(z) \in \mathscr{H}(\mathscr{C})$ converges to 0 uniformly on compact sets, then so does $\Phi[E_n(z)]$, as a straightforward argument using Cauchy's integral formula for $E_n(z)$ reveals. Thus if $\sum h_n p_n(z)$ converges to 0 uniformly on compact sets, so does $\sum h_n \Phi[p_n(z)]$.

We can now show

THEOREM 3. *Let*

(65) $A(w)\Psi[zg(w)] = \sum w^n p_n(z)$

where

i) $A(w)$ *is an* $\mathscr{A}$*-function,* $A(w) \neq 0$ *in* U, $\operatorname{Re} \alpha_1 < -3/2$;

ii) $\Psi(t)$ *is an entire function,*

(66) $\Psi(t) = \sum \Psi_n t^n, \quad \Psi_n \neq 0,$

with $(n!\,\Psi_n)^{-1} = O(R^n)$ *for some* R.

iii) $g(w)$ *is analytic and univalent in* $\Omega \supset \bar{U}$ *and* $g(\sigma_r) \neq 0$, $r = 1, 2, \ldots, m$ (*see* (42)).

Then if

(67) $\sum h_n p_n(z) = 0$

on a set of points having a limit point in $\mathscr{C}$, h_n *is* 0 *for all* n.

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Proof. First, we wish to establish that $\sum h_n p_n(z)$ converges uniformly on compact sets. Here, $\Psi_n(w) = [g(w)]^n \Psi_n$. The determinant (45) is a Vandermonde whose factors are $g(\sigma_j) - g(\sigma_k)$ and so cannot vanish, by the univalence of $g(w)$. In this case, $\Lambda = \mathscr{C}$.

We also have the fact that $h_n = O(n^{\alpha_1+1})$ and a $q \geqq 2$ can be found which assures the convergence of (10) provided $\operatorname{Re} \alpha_1 < -3/2$. To $\sum h_n p_n(z)$ we apply the ϕ operator with $\phi_n = (n!\,\Psi_n)^{-1}$. This gives

(68) $\sum h_n \hat{p}_n(z) = 0$, uniformly on compact sets where

(69) $A(w)e^{zg(w)} = \sum w^n \hat{p}_n(z)$.

We now integrate the expansion (68) repeatedly between z and and $z + \tau$, $\tau = 2\pi i/g(\sigma_r)$, $r = 1, 2, \ldots, m$, to get

(70) $\sum h_n p_n^*(z) = 0$, uniformly on compact sets

where $p_n^*(z)$ is generated by $A^*(w)e^{zg(w)}$ with

(71) $$A^*(w) = O\left[\left(1 - \frac{w}{\sigma_r}\right)^{\alpha_r+K}\right], w \to \sigma_r \text{ in } U, r = 1, 2, \ldots, m.$$

For K sufficiently large, $A^*(w) \in H^p$ for all $p > 1$.

To the expansion $\sum h_n p_n^*(z)$ we now apply Theorem 1. Since the powers $[g(w)]^n$ are complete in $\mathscr{H}(\bar{U})$, the proof of Theorem 3 is completed.

COROLLARY. *Let* $\operatorname{Re} \nu > 3/2$ *or* $\nu = 0$ *or* 1. *If either of the expansions*

(72) $$\begin{cases} \sum h_n B_n^{\nu}(z) = 0 \\ \sum h_n E_n^{\nu}(z) = 0 \ (\textit{the Euler polynomials}) \end{cases}$$

holds on a bounded infinite set of points, then h_n *is* 0 *for all* n.

4. Non-algebraic functions. The character of uniqueness sets for polynomials generated by $A(w)\Psi[zg(w)]$ where $A(w) \in \mathscr{H}(U)$ and has singularities of a non-algebraic kind on T can sometimes be determined by an application of the Φ-operator, Theorem 1, and asymptotic information about $p_n(z)$ (which is often available; see the numerous references in [**12**]). Since no simple characterization of such functions is possible, we treat an example.

First, note that when α is real, $\alpha < 0$, and $\operatorname{Re} \mu < 1/p$,

(73) $$F(w) = \frac{e^{\alpha/(1-w)}}{(1 - w)^{\mu}}$$

is in H^p.

Consider the polynomials generated by

(74) $$\frac{e^{\alpha/(1-w)}}{(1 - w)^{\beta}}\Psi[zg(w)] = \sum w^n p_n(z), \quad \alpha < 0, \operatorname{Re} \beta > 5/2,$$

where $\Psi(t)$ is entire of order 1, $(\Psi_n\, n!)^{-1} = O(R^n)$ for some R and $g(w)$ is analytic and univalent in $\bar{U}$, $g(1) \neq 0$. This kind of generating function was discussed by Wright [**15**] who showed

$$(75)\quad p_n(z) = n^{\beta/2-3/4}\{e^{2i|\alpha|^{1/2}n^{1/2}}[f_1(z) + f_2(z)n^{-1/2} + \ldots] + e^{-2i|\alpha|^{1/2}n^{1/2}}[g_1(z) + g_2(z)n^{-1/2} + \ldots]\},\quad n \to \infty,$$

uniformly on compact z-sets. Let

$$(76)\quad \sum h_n p_n(z) = 0$$

on a bounded infinite set of points. The same kind of arguments used previously show the series converges to 0 uniformly on compact sets. Since $\operatorname{Re}\beta > 5/2$ we can find a $q \geqq 2$ so that $\sum |h_n|^q n^{q-2}$ is convergent. Proceeding as before we apply the Φ operator and then integrate repeatedly to obtain the expansion

$$(77)\quad \sum h_n p_n^*(z) = 0,\quad \text{uniformly on compact sets,}$$

where $p_n^*(z)$ is generated by

$$(78)\quad e^{\alpha/(1-w)}(1 - w)^{m-\beta}K(w)e^{zg(w)},$$

$K(1) \neq 0$, $K \in \mathscr{H}(\bar{U})$. To the expansion (77) we now apply Theorem 1, picking m suitably large. We conclude that h_n is 0 for all n.

Appendix. We use Theorem 8 in reference [**11**] where (without loss of generality) $E = [0, 2\pi]$ and

$$W^+ = \{z | 0 \leqq \operatorname{Re} z \leqq 2\pi,\ \operatorname{Im} z > 0\}$$
$$W^- = \bar{W}^+.$$

The theorem says if $f(z)$ is analytic in $W^+ \cup W^-$ and

$$\lim_{y\to 0} \int_0^{2\pi} f(x + iy)\phi(x)dx$$

exists for every infinitely differentiable function $\phi(x)$, then $f(z)$ has an analytic extension to $W^+ \cup E \cup W^-$. If $f^*(x) = \lim_{y\to 0} f(x + iy)$ exists almost everywhere, this is equivalent to requiring

$$\lim_{\substack{y\to 0\\ y>0}} \int_0^{2\pi} |f(x + iy) - f^*(x)|dx = \lim_{\substack{y\to 0\\ y<0}} \int_0^{2\pi} |(f(x + iy) - f^*(x)|dx = 0.$$

For our problem, we map $[0, 2\pi]$ into T by $w = e^{ix}$ and then the above condition translates into the condition (23).

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Drexel University,
Philadelphia, Pennsylvania