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DIVISIBLE S-SYSTEMS AND R-MODULES

by VICTORIA GOULD

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1. Introduction

Throughout this paper S will denote a given monoid and R a given ring with unity. A set A is a *right S-system* if there is a map $\phi: A \times S \to A$ satisfying

$$\phi(a,1)=a$$

and

$$\phi(a,st)=\phi(\phi(a,s),t)$$

for any element a of A and any elements s,t of S. For $\phi(a,s)$ we write as and we refer to right S-systems simply as *S-systems*. One has the obvious definitions of an *S-subsystem*, an *S-homomorphism* and a *congruence* on an S-system. The reader is presumed to be familiar with the basic definitions concerning *right R-modules* over R. As with S-systems we will refer to right R-modules just as *R-modules*.

A number of papers have been published which classify monoids by properties of their S-systems, for example [**3**], [**4**], [**6**]. Many of the properties considered are inspired by the corresponding work in ring theory. In a previous paper [**5**] the author introduced a new concept of a coflat S-system, the definition used being a non-additive analogue of that of a coflat module, as in Proposition 1.3 of [**2**]. Proposition 3.3 and Corollary 3.4 of [**5**] together give a characterisation of a coflat S-system in terms of the existence of solutions of certain consistent equations. This suggests it might be of interest to study the connections between coflat and divisible S-systems.

It is easy to characterise monoids over which all S-systems are divisible. This we do in Section 2. We then give a detailed construction of a divisible S-system $\bar{A}$ containing any given S-system A. This construction enables us to classify those monoids for which all divisible S-systems are coflat. In an ensuing paper we generalise this method in order to characterise monoids over which all coflat S-systems are weakly f-injective and monoids over which all weakly f-injective S-systems are weakly injective.

The connections between injectivity and divisibility properties of R-modules have been well-researched (for example, [**8**]). In the last section we classify those rings R for which the notions of a divisible R-module and a weakly p-injective R-module coincide, using similar methods to those of Section 2.

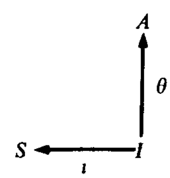
The relevant definitions for S-systems may be found in Section 2 and for R-modules in Section 3.

I would like to thank Dr J. B. Fountain for several particularly helpful suggestions with regard to this work.

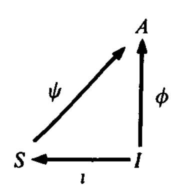
2. Divisible S-systems

As stated in the introduction, S will denote a fixed monoid. We remind the reader that an element s in S is *left* (*right*) *cancellable* if $sa=sb\,(as=bs)$, for any elements a, b of S, gives that $a=b$. Then an S-system A is said to be *torsion free* if, given any elements a, b of A and any right cancellable element s of S, $as=bs$ implies $a=b$. If $A=As$ for any left cancellable element s of S, then A is *divisible.*

An S-system A is *weakly* $(f-, p-)$ *injective* if, given any diagram of the form



where I is a (finitely generated, principal) ideal of S, $\iota: I \to S$ is the inclusion mapping and $\theta: I \to A$ is an S-homomorphism, then there exists an S-homomorphism $\psi: S \to A$ such that



commutes.

We now give the definition of a coflat S-system, proposed in [5].

An S-system A is *coflat* if, given any elements a of A and s of S with $a \notin As$, there exist elements h, k in S such that $sh=sk$ but $ah \neq ak$.

Proposition 2.1. *The following conditions are equivalent for an S-system A:*

(i) *A is coflat,*
(ii) *A is weakly p-injective,*
(iii) *if the equation $a=xs$, where $a \in A$ and $s \in S$ is soluble in some S-system B containing A, then it has a solution in A.*

This result follows from Proposition 3.3 and Corollary 3.5 of [5].

Let A be an S-system, $a \in A$ and $s \in S$, where s is left cancellable. It is immediate from Lemma 3.2 of [5] that the equation $a=xs$ has a solution in some S-system B containing A. Hence, if A is coflat, then $a=bs$ for some $b \in A$ and it follows that $A=As$. Thus we have proved

Proposition 2.2. *If A is a coflat S-system then A is divisible.*

The next result is equally straightforward. Before stating it we recall that an element s of S is *left* (*right*) *invertible* if there exists an element s' of S such that $s's=1$ ($ss'=1$).

Proposition 2.3. *The following conditions are equivalent for the monoid S.*

(i) *all right S-systems are divisible,*
(ii) *all right ideals of S are divisible*
(iii) *S is divisible (as an S-system),*
(iv) *left cancellable elements of S are left invertible.*

Proof. (*i*)⇒(*ii*)⇒(*iii*). Clear.
(*iii*)⇒(*iv*). Let $s\in S$ be left cancellable. Then as S is a divisible S-system there exists an element s' of S with $1=s's$. Thus s is left invertible.
(*iv*)⇒(*i*). Let a be an element of an S-system A and let s be a left cancellable element of S. From (iv) there is an element s' of S with $1=s's$. Then

$$a=a1=a(s's)=(as')s.$$

Hence $A=As$ and A is divisible.

In Theorem 2.2 of [6] Knauer and Petrich show that all right S-systems are torsion free if and only if all right cancellable elements are right invertible. Hence

Corollary 2.4. *All right S-systems are divisible if and only if all left S-systems are torsion free.*

For an S-system A and a subset H of $A\times A$ we denote by $\rho(H)$ the congruence generated by H, that is, the smallest congruence ν over A such that $H\subseteq\nu$.

Lemma 2.5. [**10**]. *The ordered pair (a,b) is in $\rho(H)$ if and only if $a=b$ or there exists a natural number n and a sequence*

$$a=c_1t_1,\ d_1t_1=c_2t_2,\dots,d_{n-1}t_{n-1}=c_nt_n,\ d_nt_n=b,$$

where $t_1,\dots,t_n$ are elements of S and for each $i\in\{1,\dots,n\}$ either (c_i,d_i) or (d_i,c_i) is in H.

A sequence as in Lemma 2.5 will be referred to as a *$\rho(H)$-sequence of length n*. For any congruence ρ on A, the set of congruence classes of ρ can be made into an S-system, with the obvious action of S. We write A/ρ to denote this S-system and $[a]_\rho$, or simply $[a]$ where ρ is understood, for the ρ-class of an element a of A.

We say that an element s of the monoid S is *almost regular* if there exist elements $r,r_1,\dots,r_m,s_1,\dots,s_m$ of S and left cancellable elements $c_1,\dots,c_m$ of S such that

$$(AR)\ s=srs_1,\ c_is_i=r_is_{i+1},\ (i=1,\dots,m-1),\ c_ms_m=r_ms.$$

If $s\in S$ in regular, then taking $m=1$, $s_1=s$, $c_1=r_1=1$ and $r=s'$ for some inverse s' of s it is clear that s is an almost regular element. However, we note that non-regular elements may be almost regular. For example, a left cancellable element s of a monoid need not be regular but putting $m=1$, $r=s_1=r_1=1$ and $c_1=s$ one sees that s is almost regular.

If all elements of S are almost regular, then we say that S is an *almost regular* monoid.

We make immediate use of the above ideas in the next proposition, which classifies those monoids for which the notions of a divisible S-system and a coflat S-system coincide.

We point out that in view of the remarks above, all regular monoids and all left cancellative monoids have this property.

Proposition 2.6. *All divisible S-systems over the monoid S are coflat if and only if S is almost regular.*

Proof. Assume that S is an almost regular monoid. Let A be a divisible S-system and $\theta: sS \to A$ be an S-homomorphism from a principal right ideal sS of S to A. By hypothesis s is an almost regular element and so there exist elements $r, r_1, \ldots, r_m, s_1, \ldots, s_m$ of S and left cancellable elements $c_1, \ldots, c_m$ of S satisfying (AR). Then

$$\theta(s) = \theta(srs_1) = \theta(sr)s_1$$

and as A is divisible, $\theta(sr) = a_1c_1$ for some element a_1 of A. Hence

$$\theta(s) = (a_1c_1)s_1 = a_1(c_1s_1) = a_1(r_1s_2) = (a_1r_1)s_2.$$

Again by the divisibility of A there is an element a_2 in A such that $a_1r_1 = a_2c_2$. This gives

$$\theta(s) = (a_2c_2)s_2 = a_2(c_2s_2) = a_2(r_2s_3) = (a_2r_2)s_3.$$

Continuing in this manner we obtain

$$\theta(s) = a_m(c_ms_m) = a_m(r_ms) = (a_mr_m)s.$$

Hence θ is given by left multiplication with an element of A; it is easy to see from this that A must be weakly p-injective. Thus A is coflat by Proposition 2.1.

To prove the converse we begin by detailing a construction of a divisible S-system $\bar{A}$ containing an arbitrary given S-system A.

First we let C be the set of left cancellable elements of S and define Σ_0, F_0, K_0 and A_1 as follows:

$$\Sigma_0 = C \times A,$$

F_0 is the free S-system on the set $\{x_\sigma : \sigma \in \Sigma_0\}$, that is is $F_0 = \bigcup_{\sigma \in \Sigma_0} x_\sigma S$,

$$K_0 = \{(x_\sigma c, a) : \sigma = (c, a) \in \Sigma_0\},$$

$$A_1 = (A \cup F_0)/\rho(K_0).$$

Suppose now that $a_1, a_2 \in A$ and $[a_1]=[a_2]$ in A_1. Thus $a_1=a_2$ or a_1 and a_2 are connected via a $\rho(K_0)$-sequence, which it is easy to see must be of even length. If

$$a_1=b_1t_1, \quad d_1t_1=b_2t_2 \quad d_2t_2=a_2$$

is a $\rho(K_0)$-sequence, then $b_1 \in A$ and $d_1=x_\sigma c$ for some $\sigma=(c,b_1)\in\Sigma_0$. Thus $b_2=x_\sigma c$ and $d_2=b_1$. From $d_1t_1=b_2t_2$ it follows that $ct_1=ct_2$ and so $t_1=t_2$ as c is left cancellable. Hence

$$a_1=b_1t_1=b_1t_2=d_2t_2=a_2.$$

We now choose $n\in\mathbb{N}$, $n>0$ and make the induction assumption that if m_1, m_2 are elements of A connected by a $\rho(K_0)$-sequence of (necessarily even) length less than $2n$, then $m_1=m_2$.

Suppose that

$$a_1=b_1t_1, \quad d_1t_1=b_2t_2,\dots,d_{2n}t_{2n}=a_2$$

is a $\rho(K_0)$-sequence connecting a_1 and a_2. As above, $a_1=d_2t_2$ and so

$$a_1=b_3t_3, \quad d_3t_3=b_4t_4,\dots,d_{2n}t_{2n}=a_2$$

is a $\rho(K_0)$-sequence of length $2(n=1)$ connecting a_1 and a_2, thus $a_1=a_2$ by the induction assumption. Hence A is embedded in A_1 and we may identify the element a of A with the element $[a]$ of A_1.

In a similar manner one constructs a sequence $A_1\subseteq A_2\subseteq A_3\subseteq\dots$ using $\Sigma_1,\Sigma_2,\dots$, $F_1,F_2,\dots$ and $K_1,K_2,\dots$ where Σ_i, F_i and K_i are defined using A_i in the same way that Σ_0, F_0 and K_0 are defined in terms of A. Although $\Sigma_0\subseteq\Sigma_1\subseteq\dots$ at each stage we choose a basis for F_i which is disjoint from the bases used for $F_0,F_1,\dots,F_{i-1}$. For ease of notation we make the convention that for $n\in\mathbb{N}$ the $\rho(K_n)$-class of an element a of $A_n\cup F_n$ will be denoted by $[a]_n$.

Now put $\bar{A}=\bigcup_{i\in\mathbb{N}}A_i$, where A_0 is identified with A. We claim that $\bar{A}$ is divisible.

Let $\bar{a}\in\bar{A}$ and $c\in C$. Then $\bar{a}\in A_n$ for some $n\in\mathbb{N}$ and so $\sigma=(c,\bar{a})\in\Sigma_n$ and $(y_\sigma c,\bar{a})\in K_n$, where $\{y_\sigma:\sigma\in\Sigma_n\}$ is the basis for F_n. In A_{n+1},

$$\bar{a}=[\bar{a}]_n=[y_\sigma c]_n=[y_\sigma]_n c.$$

Now $[y_\sigma]_n$ is an element of A_{n+1} and hence of $\bar{A}$. Thus $\bar{A}$ is a divisible S-system containing A.

We now assume that all divisible S-systems are coflat. Let s be an element of S. We wish to show that s is almost regular.

The S-system $\overline{sS}$ is divisible and hence is coflat by assumption. Thus the inclusion mapping $\iota:sS\to\overline{sS}$ can be extended to an S-homomorphism $\psi:S\to\overline{sS}$. This gives that

$$s=\iota(s)=\psi(s)=\psi(1)s.$$

Now $\psi(1)\in(sS)_n$ for some $n\in\mathbb{N}$. If $n=0$ then s is a regular element, hence s is almost regular. Thus we may assume that $n\geqq 1$.

From the construction of $(sS)_n$, $\psi(1)$ is either of the form $\psi(1)=[z_v r_n]_{n-1}$ where $v=(c_n, a_{n-1})$, $v\in\Sigma_{n-1}$, $r_n\in S$ and $\{z_v : v\in\Sigma_{n-1}\}$ is the basis of F_{n-1}, or the form $\psi(1)=[m_{n-1}]_{n-1}$ where $m_{n-1}\in(sS)_{n-1}$. In this latter case we note that $\tau=(1,m_{n-1})\in\Sigma_{n-1}$ and so $\psi(1)=[x_\tau]_{n-1}$, hence we may assume that $\psi(1)$ is of the first form.

Thus $[s]_{n-1}=[z_\sigma r_n s]_{n-1}$ for some $\sigma=(c_n, a_{n-1})\in\Sigma_{n-1}$ and $r_n\in S$. As $s\neq z_\sigma r_n s$ there is a $\rho(K_{n-1})$-sequence

$$z_\sigma r_n s=b_1t_1,\quad d_1t_1=b_2t_2,\dots,d_pt_p=s$$

connecting $z_\sigma r_n s$ and s in $(sS)_{n-1}\cup F_{n-1}$. Hence $b_1=z_\sigma c_n$ and so $r_n s=c_n t_1$. Further, $d_1=a_{n-1}$ and as $a_{n-1}t_1$, s are both in $(sS)_{n-1}$ and any two $\rho(K_{n-1})$-related elements in $(sS)_{n-1}$ are equal in $(sS)_{n-1}$, it follows that $a_{n-1}t_1=s$.

Either $n=1$ and so $a_{n-1}=sr$ for some $r\in S$, or $n>1$. In the latter case we obtain as above $a_{n-2}\in(sS)_{n-2}$, t_2, $r_{n-1}\in S$ and $c_{n-1}\in C$ such that $r_{n-1}t_1=c_{n-1}t_2$, $a_{n-2}t_2=s$. Clearly we may continue in this manner to obtain $s=a_0t_n$ where $a_0\in sS$ and $t_n\in S$. Thus $s=srt_n$ for some $r\in S$. Then by putting $t_1=s_n$, $t_2=s_{n-1},\dots,t_n=s_1$ we see that s is almost regular.

Corollary 2.7[7]. *All S-systems of the monoid S are coflat if and only if S is regular.*

Proof. If S is regular then as noted above, S is almost regular and so all divisible S-systems are coflat. Let s be a left cancellable element of S. Then $s=ss's$ for some $s'\in S$, hence $1=s's$ and s is left invertible. Proposition 2.3 gives that all S-systems are divisible, hence all S-systems are coflat.

Conversely, assume that all S-systems are coflat. By Proposition 2.2, all S-systems are divisible and so by Proposition 2.3, left cancellable elements are left invertible.

Let $s\in S$. Since all divisible S-systems are coflat, s is almost regular. Let $r, r_1,\dots,r_m, s_1,\dots,s_m$ be elements of S and let $c_1,\dots,c_m$ be left cancellable elements of S satisfying (AR). For $i\in\{1,\dots,m\}$ choose $c_i'\in S$ with $c_i'c_i=1$. Then $s_m=c_m'c_ms_m=c_m'r_ms$ and for $i\in\{1,\dots,m-1\}$ $s_i=c_i'r_is_{i+1}$. Now

$$s=srs_1=src_1'r_1s_2=\cdots=src_1'r_1c_2'r_2\dots c_{m-1}'r_{m-1}c_m'r_mS$$

and so s is regular.

3. Divisible *R*-modules

The definition of a *weakly* $(f\text{-},p\text{-})$ *injective* R-module corresponds directly to that of a weakly $(f\text{-},p\text{-})$ injective S-system. However, the notion of coflatness in R-modules coincides with that of weak f-injectivity [2] and not with weak p-injectivity as in the semigroup case. Further, every weakly-injective R-module is *injective* [1], whereas this is not true for S-systems. Finally, an R-module M is *divisible* if $M=Mr$ for every non zero-divisor r of R.

The relations between the above properties of R-modules have been extensively investigated. In [8], Ming considers rings for which the properties of divisibility, weak p-injectivity and injectivity coincide. The proof of Proposition 2.6, in particular the construction of a divisible S-system $\bar{A}$ containing any given S-system A, suggests that a similar method might be used to obtain an elementary characterisation of rings over which all divisible R-modules are weakly p-injective. Such a characterisation is obtained in Proposition 3.3.

First we have the straightforward analogues of Propositions 2.2 and 2.3.

Proposition 3.1[8]. *If M is a weakly p-injective R-module then M is divisible.*

Proposition 3.2. *The following conditions are equivalent for a ring R.*

(i) *all right R-modules are divisible,*
(ii) *all right ideals of R are divisible,*
(iii) *R is divisible (as a right R-module),*
(iv) *non-zero-divisors in R are left invertible.*

A ring R is *Von Neumann regular* if the multiplicative semigroup of R is regular. We shall refer to Von Neumann regular rings simply as *regular* rings.

We now state the analogue of Proposition 2.6.

Proposition 3.3 *The following conditions are equivalent for a ring R with set of non-zero-divisors C:*

(i) *all divisible R-modules are weakly p-injective,*
(ii) *for any element r of R there exist a positive integer n and n finite sets*

$$\{s_{i1},\dots,s_{i,p(i)}\} \qquad (1 \leqq i \leqq n)$$

of elements of R and n finite sets

$$\{c_{i1},\dots,c_{i,p(i)}\} \qquad (1 \leqq i \leqq n)$$

of elements of C such that if $I_j = Rs_{j1} + \cdots + Rs_{j,p(j)}$ $(j=1,\dots,n)$ *and* $I_{n+1} = Rr$, *then*

(a) $r \in rI_1$,

(b) $c_{jk}s_{jk} \in I_{j+1}$ $(j=1,\dots,n;\ k=1,\dots,p(j))$.

Before giving the proof we make some comments on this result. If r is a regular element of R, then putting $n=1$, $p(1)=1$, $s_{11}=r$, $c_{11}=1$, one sees that r satisfies conditions (ii) above. As in the semigroup case, a non-regular element may satisfy (ii). For if $c \in C$, then taking $n=1$, $p(1)=1$, $s_{11}=1$, $c_{11}=c$ we have that c satisfies (ii). Thus all non zero-divisors satisfy (ii).

We now prove the propostion.

(ii)⇒(i). Let M be a divisible R-module and let $\theta: rR \to M$ be an R-homomorphism from a principal right ideal rR of R to M. By assumption there exist $n \in \mathbb{N}$ and finite sets of elements

$$\{s_{i1},\dots,s_{i,p(i)}\}(1 \leqq i \leqq n),\ \{c_{i1},\dots,c_{i,p(i)}\}\ (1 \leqq i \leqq n),$$

of R, C respectively, satisfying the conditions of (ii).

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We have $r\in rI_1=Rs_{11}+\cdots+Rs_{1,p(1)}$ and so there are elements $r_1,\ldots,r_{p(1)}$ of R such that $r=rr_1s_{11}+\cdots+rr_{p(1)}s_{1,p(1)}$. Since M is divisible, for any $k\in\{1,\ldots,p(1)\}$ there is an element $m_{1,k}$ in M such that $\theta(rr_k)=m_{1,k}c_{1,k}$. Thus

$$\theta(r)=\theta(rr_1)s_{11}+\cdots+\theta(rr_{p(1)})s_{1,p(1)}$$

$$=\sum_{k=1}^{p(1)} m_{1,k}c_{1,k}s_{1,k}.$$

Now $I_2=Rs_{21}+\cdots+Rs_{2,p(2)}$ so using (b) there are elements $u_{k,l}$ of R, $k\in\{1,\ldots,p(1)\}$, $l\in\{1,\ldots,p(2)\}$ such that for $k\in\{1,\ldots,p(1)\}$,

$$c_{1,k}s_{1,k}=u_{k,1}s_{21}+\cdots+u_{k,p(2)}s_{2,p(2)}.$$

Then

$$\theta(r)=\sum_{k=1}^{p(1)} m_{1,k}\sum_{l=1}^{p(2)} u_{k,l}s_{2,l}$$

$$=\sum_{k=1}^{p(1)}\sum_{l=1}^{p(2)} m_{1,k}u_{k,l}s_{2,l}$$

$$=\sum_{l=1}^{p(2)} v_{2,l}s_{2,l}$$

for some $v_{21},\ldots,v_{2,p(2)}\in M$.

Again using the divisibility of M, there are elements $m_{21},\ldots,m_{2,p(2)}$ of M such that $v_{2,l}=m_{2,l}c_{2,l}$ for $l\in\{1,\ldots,p(2)\}$. Then

$$\theta(r)=\sum_{l=1}^{p(2)} m_{2,l}c_{2,l}s_{2,l}=\sum_{l=1}^{p(2)} m_{2,l}\sum_{k=1}^{p(3)} w_{l,k}s_{3,k}$$

for some elements $w_{l,k}$ of $R, l\in\{1,\ldots,p(2)\}$, $k\in\{1,\ldots,p(3)\}$. It follows that there are elements $z_{31},\ldots,z_{3,p(3)}$ of M with

$$\theta(r)=\sum_{k=1}^{p(3)} z_{3,k}s_{3,k}.$$

Clearly we may continue in this way to obtain

$$\theta(r)=\sum_{k=1}^{p(n)} x_{n,k}s_{n,k}$$

for some $x_{n,1},\ldots,x_{n,p(n)}\in M$. Then there are elements $m_{n,1},\ldots,m_{n,p(n)}$ of M with $x_{n,k}=m_{n,k}c_{n,k}, k\in\{1,\ldots,p(n)\}$. This gives that

$$\theta(r)=\sum_{k=1}^{p(n)} m_{n,k}c_{n,k}s_{n,k}.$$

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But for $k\in\{1,\dots,p(n)\}$, $c_{n,k}s_{n,k}=t_kr$ for some $t_k\in R$. Hence

$$\theta(r)=\sum_{k=1}^{p(n)} m_{n,k}t_kr=\left(\sum_{k=1}^{p(n)} m_{n,k}t_k\right)r.$$

Thus θ is given by left multiplication with an element of M. It is then easy to see that θ can be extended to an R-homomorphism $\psi:R\to M$. Since rR and θ were chosen arbitrarily it follows that M is weakly p-injective.

$(i)\Rightarrow(ii)$. We parallel the proof of Proposition 2.6 by constructing a divisible R-module $\bar{M}$ containing an arbitrary given R-module M.

Let $\Sigma_0=C\times M$ and let $X_0=\{x_\sigma:\sigma\in\Sigma_0\}$ be a set in one-one correspondence with Σ_0. Let F_0 be the free R-module on X_0 and put $G_0=M\oplus F_0$. Now let H_0 be the R-submodule of G_0 generated by K_0 where

$$K_0=\{x_\sigma c-m:\sigma=(c,m)\in\Sigma_0\}.$$

Finally, put $M_1=G_0/H_0$.

We claim that M is embedded in M_1. Suppose that $m_1,m_2\in M$ and $m_1+H_0=m_2+H_0$. Thus $m_1-m_2\in H_0$ and so either $m_1=m_2$ or m_1-m_2 can be expressed as

$$m_1-m_2=\sum_{i=1}^{n}(x_{\sigma_i}c_i-a_i)r_i$$

where $\sigma_i=(c_i,a_i)\in\Sigma_0, r_i\in R\setminus\{0\}$, $1\leqq i\leqq n$. Hence

$$m_1-m_2=\sum_{i=1}^{n}x_{\sigma_i}c_ir_i-\sum_{i=1}^{n}a_ir_i$$

and as $c_1,\dots,c_n$ are cancellable, $c_ir_i\neq 0$ for $i\in\{1,\dots,n\}$. Clearly this is impossible. Thus $m_1=m_2$ and $\phi:M\to M_1$ defined by $\phi(m)=m+H_0$ is an embedding of M into M_1. We will identify the element m of M with its image $\phi(m)$ in M_1 and consider M as an R-submodule of M_1.

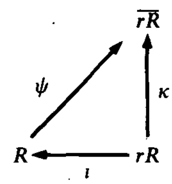
In a similar manner one constructs a sequence $M_1\subseteq M_2\subseteq\dots$ using $\Sigma_1,\Sigma_2,\dots$, $F_1,F_2,\dots,G_1,G_2,\dots,K_1,K_2,\dots$ and $H_1,H_2,\dots$ where Σ_i,F_i,G_i,K_i and H_i are defined using M_i in the same way that Σ_0, F_0, G_0, K_0 and H_0 are defined in terms of M. Although $\Sigma_0\subseteq\Sigma_1\dots$, at each stage we choose for the basis of F_i a set of symbols $\{y_\sigma:\sigma\in\Sigma_i\}$ not occurring in $G_0,\dots,G_{i-1}$.

We put $\bar{M}=\bigcup_{i=0}^{\infty}M_i$ where $M_0=M$. Then $\bar{M}$ is an R-module containing M, further we claim that $\bar{M}$ is divisible. For let $c\in C$ and $\bar{m}\in\bar{M}$. Then $\bar{m}\in M_n$ for some $n\in\mathbb{N}$ and so $\sigma=(c,\bar{m})\in\Sigma_n$. Thus $y_\sigma c-\bar{m}\in K_n$ where $\{y_\sigma:\sigma\in\Sigma_n\}$ is used in the construction of G_n. Now in M_{n+1} we are identifying $\bar{m}$ with its image $\bar{m}+H_n$ and so

$$\begin{aligned}\bar{m}+H_n&=\bar{m}+y_\sigma c-\bar{m}+H_n\\&=y_\sigma c+H_n=(y_\sigma+H_n)c.\end{aligned}$$

As $y_\sigma+H_n\in M_{n+1}$ and $M_{n+1}\subseteq\bar{M}$, we have shown that $\bar{M}$ is divisible.

Now let R be a ring with all divisible R-modules weakly p-injective. Let $r \in R$ and form the divisible R-module $\overline{rR}$ containing rR as above. By assumption $\overline{rR}$ is weakly p-injective and so there exists an R-homomorphism $\psi: R \to \overline{rR}$ such that



commutes, where $\iota: rR \to R$ and $\kappa: rR \to \overline{rR}$ are the inclusion mappings. Thus

$$r = \kappa(r) = \psi\iota(r) = \psi(r) = \psi(1)r.$$

By the construction of $\overline{rR}$, either $\psi(1) \in rR$ or $\psi(1) \in (rR)_n$ for some $n \in \mathbb{N}\setminus\{0\}$. In the former case it is clear that r is a regular element and so (ii) holds for r.

Suppose then that $\psi(1) \in (rR)_n$ where $n > 0$. We note that we may assume that $r \neq 0$, since 0 is a regular element of R. From the construction of $(rR)_n, \psi(1) = g_{n-1} + H_{n-1}$ for some $g_{n-1} \in G_{n-1}$. Now in $(rR)_n$ we identify r with its image $r + H_{n-1}$ and so

$$r + H_{n-1} = (g_{n-1} + H_{n-1})r = g_{n-1}r + H_{n-1}$$

giving that $g_{n-1}r - r \in H_{n-1}$.

Suppose that $\{z_\sigma : \sigma \in \Sigma_{n-1}\}$ is the basis of F_{n-1} used in the construction of G_{n-1}. Then

$$g_{n-1} = m_{n-1} + \sum_{i=1}^{f(n)} z_{\sigma_i} r_i$$

for some $f(n) \in \mathbb{N}$, $m_{n-1} \in (rR)_{n-1}$, $r_1, \ldots, r_{f(n)} \in R$ and distinct $\sigma_1, \ldots, \sigma_{f(n)} \in \Sigma_{n-1}$. However, if $\sigma = (1, m_{n-1})$ then

$$\begin{aligned} g_{n-1} + H_{n-1} &= g_{n-1} + z_\sigma - m_{n-1} + H_{n-1} \\ &= z_\sigma + \sum_{i=1}^{f(n)} z_{\sigma_i} r_i + H_{n-1}. \end{aligned}$$

Thus we may assume that g_{n-1} has the form

$$g_{n-1} = \sum_{i=1}^{f(n)} z_{\sigma_i} r_i$$

for some $f(n) \in \mathbb{N}, r_1, \ldots, r_n \in R$ and distinct $\sigma_1, \ldots, \sigma_{f(n)} \in \Sigma_{n-1}$.

We have $g_{n-1}r-r\in H_{n-1}$ and H_{n-1} is generated by K_{n-1}, hence

$$g_{n-1}r-r=\sum_{k=1}^{p(n)}(z_{v_k}c_{n,k}-\bar{m}_{n-1,k})s_{n,k} \tag{1}$$

for some $p(n)\in\mathbb{N}$, $s_{n,k}\in R$ and distinct $v_k=(c_{n,k},\bar{m}_{n-1,k})\in\Sigma_{n-1}$, $k\in\{1,\ldots,p(n)\}$. Thus

$$\sum_{i=1}^{f(n)}z_{\sigma_i}r_ir-r=\sum_{k=1}^{p(n)}z_{v_k}c_{n,k}s_{n,k}-\sum_{k=1}^{p(n)}\bar{m}_{n-1,k}s_{n,k}.$$

Now $G_{n-1}=(rR)_{n-1}\oplus F_{n-1}$ so that

$$r=\sum_{k=1}^{p(n)}\bar{m}_{n-1,k}s_{n,k}$$

and

$$\sum_{i=1}^{f(n)}z_{\sigma_i}r_ir=\sum_{k=1}^{p(n)}z_{v_k}c_{n,k}s_{n,k}.$$

As $r\neq 0$, $s_{n,k}\neq 0$ for some $k\in\{1,\ldots,p(n)\}$ and so from considering the form of (1) we may assume that $s_{n,k}\neq 0$ for all $k\in\{1,\ldots,p(n)\}$. Hence $c_{n,k}s_{n,k}\neq 0$ for all $k\in\{1,\ldots,p(n)\}$. This gives that $f(n)=p(n)$ and for $k\in\{1,\ldots,p(n)\}$ we have that $c_{n,k}s_{n,k}\in I_{n+1}$ where $I_{n+1}=Rr$.

If $n=1$ then there exist $a_1,\ldots,a_{p(1)}\in R$ with $\bar{m}_{n-1,k}=ra_k$ for $k\in\{1,\ldots,p(1)\}$. Then

$$r=r\sum_{k=1}^{p(1)}a_ks_{1,k}$$

so that $r\in rI_1$ where $I_1=Rs_{11}+\cdots+Rs_{1,p(1)}$ and r satisfies (ii).

Otherwise, $n>1$ and

$$r+H_{n-2}=\sum_{k=1}^{p(n)}m_{n-1,k}s_{n,k}+H_{n-2},$$

where $m_{n-1,k}+H_{n-2}=\bar{m}_{n-1,k}$, $k\in\{1,\ldots,p(n)\}$. Thus

$$\sum_{k=1}^{p(n)}m_{n-1,k}s_{n,k}-r\in H_{n-2}.$$

For $k\in\{1,\ldots,p(n)\}$, $m_{n-1,k}\in G_{n-2}$ and as above we may assume that

$$m_{n-1,k}=\sum_{i=1}^{h(k)}y_{\rho_{k,i}}r_{k,i}$$

where $h(k)\in\mathbb{N}$, $\rho_{k,i}\in\Sigma_{n-2}$, $r_{k,i}\in R, i\in\{1,\ldots,h(k)\}$ and $\{y_\rho:\rho\in\Sigma_{n-2}\}$ is the basis of F_{n-2}

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used in the construction of G_{n-2}. Further, we may express $\sum_{k=1}^{p(n)} m_{n-1,k}s_{n,k}-r$ as

$$\sum_{k=1}^{p(n)} m_{n-1,k}s_{n,k}-r=\sum_{j=1}^{p(n-1)}(y_{\mu_j}c_{n-1,j}-\bar{m}_{n-2,j})s_{n-1,j}$$

where $p(n-1)\in\mathbb{N}$, $s_{n-1,1},\ldots,s_{n-1,p(n-1)}\in R$ and $\mu_1,\ldots,\mu_{p(n-1)}$ are distinct elements of Σ_{n-2}, where $\mu_j=(c_{n-1,j},\bar{m}_{n-1})$, $j\in\{1,\ldots,p(n-1)\}$ and as above we may assume that $s_{n-1,j}\neq 0$ for all $j\in\{1,\ldots,p(n-1)\}$. Thus

$$\sum_{k=1}^{p(n)}\sum_{i=1}^{h(k)} y_{\rho_{k,i}}r_{k,i}s_{n,k}-r=\sum_{j=1}^{p(n-1)} y_{\mu_j}c_{n-1,j}s_{n-1,j}-\sum_{j=1}^{p(n-1)}\bar{m}_{n-2,j}s_{n-1,j}.$$

Then

$$r=\sum_{j=1}^{p(n-1)}\bar{m}_{n-2,j}s_{n-1,j}$$

Also, for any $j\in\{1,\ldots,p(n-1)\}$

$$c_{n-1,j}s_{n-1,j}\in I_n$$

where

$$I_n=Rs_{n,1}+\cdots+Rs_{n,p(n)}.$$

Clearly we may continue in this way to obtain

$$r=\sum_{k=1}^{p(1)} b_k s_{1,k}$$

where $b_1,\ldots,b_{p(1)}\in rR$. Then there exist $d_1,\ldots,d_{p(1)}\in R$ with $b_k=rd_k$, $k\in\{1,\ldots,p(1)\}$ so that

$$r=\sum_{k=1}^{p(1)} rd_k s_{1,k}$$

hence $r\in rI_1$ where

$$I_1=Rs_{11}+\cdots+Rs_{1,p(1)}$$

and so (ii) holds.

Corollary 3.4[8]. *If R is an integral domain then all divisible R-modules are weakly p-injective.*

Corollary 3.5[9]. *The ring R is regular if and only if all R-modules are weakly p-injective.*

Proof. If R is a regular ring then it follows as in the case for monoids that all R-modules are weakly p-injective.

Conversely, assume that all R-modules are weakly p-injective. By Propositions 3.2 and 3.3, the non zero-divisors of R are left invertible are R satisfies condition (ii) of Proposition 3.3.

Let $r \in R$. Then there is a positive integer n and n finite sets

$$\{s_{i,1}, \ldots, s_{i,p(i)}\} \qquad (1 \leqq i \leqq n)$$

of elements of R and n finite sets

$$\{c_{i,1}, \ldots, c_{i,p(i)}\} \qquad (1 \leqq i \leqq n)$$

of non-zero-divisors of R, satisfying condition (ii). For $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, p(j)\}$,

$$c_{j,k} s_{j,k} \in I_{j+1}$$

and as $c_{j,k}$ is left invertible, $1 = c'_{j,k} c_{j,k}$ for some $c'_{j,k} \in R$, giving

$$s_{j,k} \in c'_{j,k} I_{j+1} \subseteq I_{j+1}.$$

Hence for $j \in \{1, \ldots, n\}$,

$$\begin{aligned} I_j &= Rs_{j,1} + \cdots + Rs_{j,p(j)} \\ &\subseteq RI_{j+1} \\ &\subseteq I_{j+1}. \end{aligned}$$

Thus

$$r \in rI_1 \subseteq rI_2 \subseteq \cdots \subseteq rI_{n+1} = rRr,$$

giving that r is regular.

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School of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW
England