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AN INTEGRAL REPRESENTATION OF A ${}_{10}\phi_9$ AND CONTINUOUS BI-ORTHOGONAL ${}_{10}\phi_9$ RATIONAL FUNCTIONS

MIZAN RAHMAN

1. Introduction. One of the most remarkable q-extensions of the classical beta integral was recently introduced by Askey and Wilson [**1**]

$$(1.1)\quad \frac{1}{2\pi i}\int_C \frac{(z^2;\, q)_\infty(z^{-2};\, q)_\infty}{(az;\, q)_\infty(a/z;\, q)_\infty(bz;\, q)_\infty(b/z;\, q)_\infty(cz;\, q)_\infty} \times \frac{1}{(c/z;\, q)_\infty(dz;\, q)_\infty(d/z;\, q)_\infty}\frac{dz}{z}$$

$$= \frac{2(abcd;\, q)_\infty}{(q;\, q)_\infty(ab;\, q)_\infty(ac;\, q)_\infty(ad;\, q)_\infty(bc;\, q)_\infty} \times \frac{1}{(bd;\, q)_\infty(cd;\, q)_\infty} \equiv 2h_0, \text{ say,}$$

where $|q| < 1$ and the pairwise products of $\{a, b, c, d\}$ as a multiset do not belong to the set $\{q^j, j = 0, -1, -2, \dots\}$. The contour C is the unit circle described in the positive direction, but with suitable deformations to separate the sequences of poles converging to zero from the sequences of poles diverging to infinity. The symbol $(A;\, q)_\infty$ is an infinite product defined by

$$(1.2)\quad (A;\, q)_\infty = \prod_{k=0}^{\infty} (1 - Aq^k)$$

whenever it converges.

Under the stronger condition

$$\max(\, |a|, |b|, |c|, |d|\,) < 1$$

(1.1) reduces to a real integral

$$(1.3)\quad \frac{1}{2\pi}\int_{-1}^{1} \frac{h(x;\, 1)h(x;\, -1)h(x;\, \sqrt{q})h(x;\, -\sqrt{q})}{h(x;\, a)h(x;\, b)h(x;\, c)h(x;\, d)}\frac{dx}{\sqrt{1-x^2}} = h_0$$

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where

$$(1.4)\qquad h(x; a) = \prod_{k=0}^{\infty} (1 - 2axq^k + a^2q^{2k})$$

$$= (ae^{i\theta}; q)_\infty (ae^{-i\theta}; q)_\infty, \quad x = \cos\theta.$$

If we set $a = -d = \sqrt{q}$, $b = q^{\alpha+1/2}$, $c = -q^{\beta+1/2}$ in (1.3) and take the limit $q \to 1-$ then it is easy to see it reduces to the beta integral

$$(1.5)\qquad \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta dx = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}$$

once we have made use of Jackson's q-gamma function

$$(1.6)\qquad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty}(1 - q)^{1-x}$$

and the property that

$$\Gamma(x) = \lim_{q\to 1-} \Gamma_q(x).$$

Askey and Wilson [**1**] also showed that the polynomials

$$(1.7)\qquad p_n\left(\frac{z + z^{-1}}{2}; a, b, c, d\right) = {}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, az, a/z \\ ab, ac, ad \end{matrix}; q, q\right]$$

are orthogonal with respect to the complex weight function defined in (1.1), i.e.,

$$(1.8)\qquad \frac{1}{2\pi i}\int_C p_n\left(\frac{z + z^{-1}}{2}; a, b, c, d\right)$$

$$\times\, p_m\left(\frac{z + z^{-1}}{2}; a, b, c, d\right) f(z)dz = 2\delta_{m,n}h_n,$$

where

$$(1.9)\qquad h_n = h_0\left[\frac{\left(\frac{abcd}{q}; q\right)_n (1 - abcdq^{2n-1})}{(q; q)_n(1 - abcd/q)(cd; q)_n}\right.$$

$$\left.\times\, \frac{(ab; q)_n(ac; q)_n(ad; q)_n}{(bd; q)_n(bc; q)_n}a^{-2n}\right]^{-1},$$

and

(1.10) $$f(z) = \frac{z^{-1}(z^2; q)_\infty}{(az; q)_\infty(a/z; q)_\infty(bz; q)_\infty} \times \frac{(z^{-2}; q)_\infty}{(b/z; q)_\infty(cz; q)_\infty(c/z; q)_\infty(dz; q)_\infty(d/z; q)_\infty},$$

with

(1.11) $$(A; q)_n = (A; q)_\infty/(Aq^n; q)_\infty.$$

The basic hypergeometric function on the right hand side of (1.7) is defined by the general formula

(1.12) $${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, q_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \dots (a_{r+1}; q)_n}{(q; q)_n(b_1; q)_n \dots (b_r; q)_n} z^n$$

whenever the series converges.

In [4] Nassrallah and Rahman found the integral representation

(1.13) $$\int_{-1}^{1} w(x; \lambda, \mu, \nu, \rho)\frac{h(x; \tau)}{h(x; \sigma)}dx$$

$$= \frac{(\lambda\mu\nu\rho; q)_\infty(\lambda\mu\nu\sigma; q)_\infty}{(q; q)_\infty(\lambda\mu; q)_\infty(\lambda\nu; q)_\infty(\lambda\rho; q)_\infty(\lambda\sigma; q)_\infty}$$

$$\times \frac{(\lambda\tau; q)_\infty(\mu\tau; q)_\infty(\nu\tau; q)_\infty}{(\mu\nu; q)_\infty(\mu\rho; q)_\infty(\mu\sigma; q)_\infty(\nu\rho; q)_\infty(\nu\sigma; q)_\infty(\lambda\mu\nu\tau; q)_\infty}$$

$$\times {}_8\phi_7\left[\begin{matrix} \lambda\mu\nu\tau q^{-1}, q\sqrt{\ }, -q\sqrt{\ }, \lambda\mu, \lambda\nu, \mu\nu, \tau\sigma^{-1}, \tau\rho^{-1} \\ \sqrt{\ }, -\sqrt{\ }, \nu\tau, \mu\tau, \lambda\tau, \lambda\mu\nu\sigma, \lambda\mu\nu\rho \end{matrix}; \rho\sigma\right],$$

where

(1.14) $$w(x; \lambda, \mu, \nu, \rho) = \frac{(1 - x^2)^{-1/2}}{2\pi} \frac{h(x; 1)h(x; -1)h(x; \sqrt{q})h(x; -\sqrt{q})}{h(x; \lambda)h(x; \mu)h(x; \nu)h(x; \rho)},$$

(1.15) $$\max(|\lambda|, |\mu|, |\nu|, |\rho|, |\rho\sigma|) < 1,$$

and the open square root is taken to be the positive root over the top left hand parameter which, in this case, is $\lambda\mu\nu\tau q^{-1}$. If we take $\lambda = -\rho = \sqrt{q}$, $\mu = q^{\alpha+1/2}$, $\nu = -q^{\beta+1/2}$, $\tau = \sigma q^\gamma$ and take the limit $q \to 1-$ then (1.13) reduces to Euler's integral formula for a ${}_2F_1$, namely,

(1.16) $${}_2F_1\left[\begin{matrix} \gamma, \alpha + 1 \\ \alpha + \beta + 2 \end{matrix}; \sigma\right]$$

$$= \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \int_0^1 x^\alpha(1 - x)^\beta(1 - x\sigma)^{-\gamma} dx.$$

However, there is another important limiting case of (1.13) that the authors failed to point out in [**4**]. Let us replace λ, μ, ν, ρ, σ, τ by q^λ, q^μ, q^ν, q^ρ, q^σ and q^τ, respectively, and, more importantly, replace $e^{i\theta}$ in the integral by q^{ix}. If we now take the limit $q \to 1-$, use (1.6) and simplify, we obtain

$$(1.17)\quad \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \frac{\Gamma(\lambda + ix)\Gamma(\lambda - ix)\Gamma(\mu + ix)\Gamma(\mu - ix)}{\Gamma(2ix)\Gamma(-2ix)} \times \frac{\Gamma(\nu + ix)\Gamma(\nu - ix)\Gamma(\rho + ix)\Gamma(\rho - ix)\Gamma(\sigma + ix)\Gamma(\sigma - ix)}{\Gamma(\tau + ix)\Gamma(\tau - ix)}$$

$$= \frac{2\Gamma(\lambda + \mu)\Gamma(\lambda + \nu)\Gamma(\lambda + \rho)\Gamma(\lambda + \sigma)\Gamma(\mu + \nu)\Gamma(\mu + \rho)}{\Gamma(\lambda + \tau)\Gamma(\mu + \tau)\Gamma(\nu + \tau)\Gamma(\lambda + \mu + \nu + \rho)} \times \frac{\Gamma(\mu + \sigma)\Gamma(\nu + \rho)\Gamma(\nu + \sigma)\Gamma(\lambda + \mu + \nu + \tau)}{\Gamma(\lambda + \mu + \nu + \sigma)}$$

$$\times {}_7F_6\left[\begin{matrix}\lambda + \mu + \nu + \tau - 1, \frac{\lambda + \mu + \nu + \tau + 1}{2}, \lambda + \mu, \lambda + \nu, \mu + \nu, \tau - \sigma, \tau - \rho \\ \frac{\lambda + \mu + \nu + \tau - 1}{2}, \nu + \tau, \mu + \tau, \lambda + \tau, \lambda + \mu + \nu + \sigma, \lambda + \mu + \nu + \rho\end{matrix}; 1\right],$$

provided $\mathrm{Re}(\lambda, \mu, \nu, \rho, \sigma) > 0$. Integral representations of ${}_7F_6$ in terms of Mellin-Barnes type integrals are well-known (see, for example [**2**, p. 44]), but representation as a real integral seems to be a new result. One can see, of course, that in the special case $\tau = \rho$ or σ (1.17) reduces to Wilson's [**6**] result.

The first objective of this paper is to consider the special case of (1.13) when the ${}_8\phi_7$ becomes a ${}_6\phi_5$ and exploit the result to obtain an integral representation of a very well-poised and balanced ${}_{10}\phi_9$. The second objective is to generalize (1.7) and (1.8) and find a continuous analogue to Wilson's [7] bi-orthogonal ${}_{10}\phi_9$ rational functions in a discrete variable.

2. An integral representation of a ${}_{10}\phi_9$. In (1.13) let us set $\tau = \lambda\mu\nu\rho\sigma$. Then the ${}_8\phi_7$ series becomes a ${}_6\phi_5$ which is summable by [**5**, (IV.7), p. 247]. Using this sum and simplifying we obtain

$$(2.1)\quad \int_{-1}^{1} v(x; \lambda, \mu, \nu, \rho, \sigma)dx$$

$$= \frac{(\lambda\mu\nu\rho)_\infty(\lambda\mu\nu\sigma)_\infty(\lambda\mu\rho\sigma)_\infty(\lambda\nu\rho\sigma)_\infty(\mu\nu\rho\sigma)_\infty}{(q)_\infty(\lambda\mu)_\infty(\lambda\nu)_\infty(\lambda\rho)_\infty(\lambda\sigma)_\infty(\mu\nu)_\infty(\mu\rho)_\infty(\mu\sigma)_\infty(\nu\rho)_\infty(\nu\sigma)_\infty(\rho\sigma)_\infty}$$

$$= g_0, \text{ say,}$$

where

$$(2.2)\quad v(x; \lambda, \mu, \nu, \rho, \sigma)$$

$$= w(x; \lambda, \mu, \nu, \rho)\frac{h(x; \lambda\mu\nu\rho\sigma)}{h(x; \sigma)}$$

$$= \frac{(1 - x^2)^{-1/2}}{2\pi}\,\frac{h(x; 1)h(x; -1)h(x; \sqrt{q})h(x; -\sqrt{q})h(x; \lambda\mu\nu\rho\sigma)}{h(x; \lambda)h(x; \mu)h(x; \nu)h(x; \rho)h(x; \sigma)}.$$

Note that in (2.1) we have used $(a)_\infty$ to mean $(a; q)_\infty$ for the sake of printing economy. Since we shall be using the same basic q all through we shall continue to use this shorthand notation throughout the rest of the paper. A sufficient condition for the existence of the integral in (2.1) is

$$(2.3)\quad \max(\,|\lambda|, |\mu|, |\nu|, |\rho|, |\sigma|\,) < 1.$$

However, if one or more of the parameters exceed 1 in numerical value but the pairwise products of $\{\lambda, \mu, \nu, \rho, \tau\}$ as a multiset do not belong to the set $\{q^j, j = 0, 1, 2, \dots\}$, then (2.1) must be replaced by the complex integral

$$(2.4)\quad \frac{1}{2\pi i}\int_C \frac{(z^2)_\infty(z^{-2})_\infty(\lambda\mu\nu\rho\sigma z)_\infty}{(\lambda z)_\infty(\lambda/z)_\infty(\mu z)_\infty(\mu/z)_\infty(\nu z)_\infty}$$

$$\times \frac{(\lambda\mu\nu\rho\sigma/z)_\infty}{(\nu/z)_\infty(\rho z)_\infty(\rho/z)_\infty(\sigma z)_\infty(\sigma/z)_\infty}\,\frac{dz}{z}$$

$$= 2g_0,$$

where C is essentially the same contour used in (1.1).

One can easily see that if any one of the parameters is equated to zero, (2.1) reduces to (1.3) while (2.4) goes to (1.1).

Since (2.4) is more general than (2.1) we shall henceforth be working with (2.4) and will indicate, whenever necessary, special results corresponding to (2.1).

Let $\sigma \to \sigma q^r$, $r = 0, 1, 2, \dots$. Since

$$(A\sigma q^r)_\infty = (A\sigma)_\infty/(A\sigma)_r, \quad (A\sigma)_r = (A\sigma; q)_r,$$

(2.4) gives

$$(2.5)\quad \frac{1}{2\pi i}\int_C g(z)\frac{(\sigma z)_r(\sigma/z)_r}{(\lambda\mu\nu\rho\sigma z)_r(\lambda\mu\nu\rho\sigma/z)_r}dz$$
$$= 2g_0\frac{(\lambda\sigma)_r(\mu\sigma)_r(\nu\sigma)_r(\rho\sigma)_r}{(\lambda\mu\nu\sigma)_r(\lambda\mu\rho\sigma)_r(\lambda\nu\rho\sigma)_r(\mu\nu\rho\sigma)_r}$$

where

$$(2.6)\quad g(z) = \frac{z^{-1}(z^2)_\infty(z^{-2})_\infty(\lambda\mu\nu\rho\sigma z)_\infty}{(\lambda z)_\infty(\lambda/z)_\infty(\mu z)_\infty(\mu/z)_\infty(\nu z)_\infty}$$
$$\times\frac{(\lambda\mu\nu\rho\sigma/z)_\infty}{(\nu/z)_\infty(\rho z)_\infty(\rho/z)_\infty(\sigma z)_\infty(\sigma/z)_\infty}.$$

Let us multiply (2.5) by

$$\frac{(\lambda\mu\nu\rho\sigma^2q^{-1})_r(1-\lambda\mu\nu\rho\sigma^2q^{2r-1})(A)_r(B)_r(C)_r}{(q)_r(1-\lambda\mu\nu\rho\sigma^2q^{-1})(\lambda\mu\nu\rho\sigma^2/A)_r(\lambda\mu\nu\rho\sigma^2/B)_r(\lambda\mu\nu\rho\sigma^2/C)_r}x^r,$$
$$|x| < 1,$$

and sum over r to get

$$(2.7)\quad \frac{1}{2\pi i}\int_C dz g(z)\,{}_8\phi_7\left[\begin{matrix}\lambda\mu\nu\rho\sigma^2q^{-1}, q\sqrt{\ }, -q\sqrt{\ }, & \sigma z,\\ \sqrt{\ }, -\sqrt{\ }, \lambda\mu\nu\rho\sigma/z,\end{matrix}\right.$$
$$\left.\begin{matrix}\sigma/z, & A, & B, & C\\ \lambda\mu\nu\rho\sigma z, & \dfrac{\lambda\mu\nu\rho\sigma^2}{A}, & \dfrac{\lambda\mu\nu\rho\sigma^2}{B}, & \dfrac{\lambda\mu\nu\rho\sigma^2}{C}\end{matrix}; q, x\right]$$
$$= 2g_0\,{}_{10}\phi_9\left[\begin{matrix}\lambda\mu\nu\rho\sigma^2q^{-1}, q\sqrt{\ }, -q\sqrt{\ }, & \lambda\sigma, & \mu\sigma, & \nu\sigma,\\ \sqrt{\ }, -\sqrt{\ }, & \mu\nu\rho\sigma, & \lambda\nu\rho\sigma, & \lambda\mu\rho\sigma,\end{matrix}\right.$$
$$\left.\begin{matrix}\rho\sigma, & A, & B, & C\\ \lambda\mu\nu\sigma, & \dfrac{\lambda\mu\nu\rho\sigma^2}{A}, & \dfrac{\lambda\mu\nu\rho\sigma^2}{B}, & \dfrac{\lambda\mu\nu\rho\sigma^2}{C}\end{matrix}; q, x\right],$$

where A, B, C are arbitrary parameters, real or complex. This, in itself, is of not much interest unless the series on both sides are balanced. Accordingly we set

$$x = q \quad\text{and}\quad C = (\lambda\mu\nu\rho\sigma)^2/ABq.$$

We may now write down a second such formula by rearranging the parameters in such a way that the two ${}_8\phi_7$ series can be connected by the non-terminating form of Jackson's theorem [**5**, (IV.15), p. 248]:

$$ {}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & f \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix}; q, q\right] $$

$$ + \frac{(bq/c)_\infty(bq/d)_\infty(bq/e)_\infty(bq/f)_\infty(aq)_\infty(c)_\infty}{(b^2q/a)_\infty(bc/a)_\infty(bd/a)_\infty(be/a)_\infty(bf/a)_\infty} $$

$$ \times \frac{(d)_\infty(e)_\infty(f)_\infty(b/a)_\infty}{(aq/c)_\infty(aq/d)_\infty(aq/e)_\infty(aq/f)_\infty(a/b)_\infty} $$

$$ \times {}_8\phi_7\left[\begin{matrix} b^2/a, qb/\sqrt{a}, -qb/\sqrt{a}, & b, bc/a, bd/a, & be/a, bf/a \\ b/\sqrt{a}, & -b/\sqrt{a}, bq/a, bq/c, bq/d, & bq/e, bq/f \end{matrix}; q, q\right] $$

$$ = \frac{(aq)_\infty(b/a)_\infty(aq/de)_\infty(aq/ce)_\infty(aq/cd)_\infty}{(aq/c)_\infty(aq/d)_\infty(aq/e)_\infty(aq/f)_\infty(bc/a)_\infty} $$

$$ \times \frac{(aq/cf)_\infty(aq/df)_\infty(aq/ef)_\infty}{(bd/a)_\infty(be/a)_\infty(bf/a)_\infty}, $$

where $a^2q = bcdef$. A tedious but straightforward calculation then yields

(2.9)
$$ \frac{1}{2\pi i}\int_C \frac{dz}{z}\frac{(z^2)_\infty(z^{-2})_\infty(\lambda\mu\nu\rho\sigma z/B)_\infty(\lambda\mu\nu\rho\sigma/Bz)_\infty}{(\lambda z)_\infty(\lambda/z)_\infty(\mu z)_\infty(\mu/z)_\infty(\nu z)_\infty(\nu/z)_\infty} $$

$$ \times \frac{(ABqz/\lambda\mu\nu\rho\sigma)_\infty(ABq/\lambda\mu\nu\rho\sigma z)_\infty}{(\rho z)_\infty(\rho/z)_\infty(\sigma z)_\infty(\sigma/z)_\infty\left(\frac{Aqz}{\lambda\mu\nu\rho\sigma}\right)_\infty\left(\frac{Aq}{\lambda\mu\nu\rho\sigma z}\right)_\infty} $$

$$ = 2g_0\frac{\left(\frac{\lambda\mu\nu\rho}{B}\right)_\infty\left(\frac{\lambda\mu\nu\rho\sigma^2}{B}\right)_\infty\left(\frac{ABq}{\lambda\mu\nu\rho}\right)_\infty\left(\frac{ABq}{\lambda\mu\nu\rho\sigma^2}\right)_\infty}{(\lambda\mu\nu\rho)_\infty(\lambda\mu\nu\rho\sigma^2)_\infty\left(\frac{Aq}{\lambda\mu\nu\rho}\right)_\infty\left(\frac{Aq}{\lambda\mu\nu\rho\sigma^2}\right)_\infty} $$

$$ \times {}_{10}\phi_9\left[\begin{matrix} \lambda\mu\nu\rho\sigma^2q^{-1}, q\sqrt{\ }, -q\sqrt{\ }, & \lambda\sigma, & \mu\sigma, & \nu\sigma, & \rho\sigma, & A, & B, & \lambda^2\mu^2\nu^2\rho^2\sigma^2/ABq \\ \sqrt{\ }, -\sqrt{\ }, & \mu\nu\rho\sigma, & \lambda\nu\rho\sigma, & \lambda\mu\rho\sigma, & \lambda\mu\nu\sigma, & \lambda\mu\nu\rho\sigma^2/A, & \lambda\mu\nu\rho\sigma^2/B, & ABq/\lambda\mu\nu\rho \end{matrix}; q, q\right] $$

$$ + 2g_0\frac{(\lambda\sigma)_\infty(\mu\sigma)_\infty(\nu\sigma)_\infty(\rho\sigma)_\infty(Aq/\lambda\sigma)_\infty(Aq/\mu\sigma)_\infty}{(\lambda\mu\rho\sigma)_\infty(\lambda\nu\rho\sigma)_\infty(\mu\nu\rho\sigma)_\infty(\lambda\mu\nu\sigma)_\infty\left(\frac{Aq}{\lambda\mu\nu\sigma}\right)_\infty} $$

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$$\times \frac{(Aq/\nu\sigma)_\infty(Aq/\rho\sigma)_\infty(B)_\infty(Aq/B)_\infty}{\left(\frac{Aq}{\lambda\mu\rho\sigma}\right)_\infty\left(\frac{Aq}{\lambda\nu\rho\sigma}\right)_\infty\left(\frac{Aq}{\mu\nu\rho\sigma}\right)_\infty(\lambda\mu\nu\rho)_\infty\left(\frac{Aq}{\lambda\mu\nu\rho}\right)_\infty}$$

$$\times \frac{(\lambda^2\mu^2\nu^2\rho^2\sigma^2/ABq)_\infty(BA^2q^2/\lambda^2\mu^2\nu^2\rho^2\sigma^2)_\infty}{(\lambda\mu\nu\rho\sigma^2/Aq)_\infty(A^2q^2/\lambda\mu\nu\rho\sigma^2)_\infty}$$

$$\,_{10}\phi_9\left[\begin{matrix} qA^2/\lambda\mu\nu\rho\sigma^2,\ q\sqrt{\ },\ -q\sqrt{\ },\ \frac{Aq}{\mu\nu\rho\sigma},\ \frac{Aq}{\lambda\mu\rho\sigma},\ \frac{Aq}{\lambda\nu\rho\sigma},\ \frac{Aq}{\lambda\mu\nu\sigma},\ A,\ \frac{\lambda\mu\nu\rho}{B},\ \frac{ABq}{\lambda\mu\nu\rho\sigma^2} \\ \sqrt{\ },\ -\sqrt{\ },\ \frac{Aq}{\lambda\sigma},\ \frac{Aq}{\nu\sigma},\ \frac{Aq}{\mu\sigma},\ \frac{Aq}{\rho\sigma},\ \frac{Aq^2}{\lambda\mu\nu\rho\sigma^2},\ \frac{BA^2q^2}{\lambda^2\mu^2\nu^2\rho^2\sigma^2},\ \frac{Aq}{B} \end{matrix};q,q\right]$$

The left hand side is clearly symmetric in λ, μ, ν, ρ and σ, so the right hand side must also be. But this is precisely the content of Bailey's [**3**] four-term transformation formula for non-terminating $_{10}\phi_9$'s.

If we set $B = q^{-n}$, $n = 0, 1, 2, \ldots$ (2.9) takes a particularly simple form

(2.10)
$$\frac{1}{2\pi i}\int_C \frac{dz}{z}\frac{(z^2)_\infty(z^{-2})_\infty(\lambda\mu\nu\rho\sigma z)_\infty}{(\lambda z)_\infty(\lambda/z)_\infty(\mu z)_\infty(\mu/z)_\infty(\nu z)_\infty(\nu/z)_\infty}$$

$$\times \frac{(\lambda\mu\nu\rho\sigma/z)_\infty}{(\rho z)_\infty(\rho/z)_\infty(\sigma z)_\infty(\sigma/z)_\infty}$$

$$\times \frac{(\lambda\mu\nu\rho\sigma z/A)_n(\lambda\mu\nu\rho\sigma/Az)_n}{(\lambda\mu\nu\rho\sigma/z)_n(\lambda\mu\nu\rho\sigma z)_n}$$

$$= 2g_0\frac{(\lambda\mu\nu\rho/A)_n(\lambda\mu\nu\rho\sigma^2/A)_n}{(\lambda\mu\nu\rho\sigma^2)_n(\lambda\mu\nu\rho)_n}$$

$$\times\,_{10}\phi_9\left[\begin{matrix} \lambda\mu\nu\rho\sigma^2q^{-1},\ q\sqrt{\ },\ -q\sqrt{\ },\ \lambda\sigma,\ \mu\sigma,\ \nu\sigma,\ \rho\sigma,\ A,\ \frac{\lambda^2\mu^2\nu^2\rho^2\sigma^2q^{n-1}}{A},\ q^{-n} \\ \sqrt{\ },\ -\sqrt{\ },\ \mu\nu\rho\sigma,\ \lambda\nu\rho\sigma,\ \lambda\mu\rho\sigma,\ \lambda\mu\nu\sigma,\ \lambda\mu\nu\rho\sigma^2/A,\ \frac{Aq^{1-n}}{\lambda\mu\nu\rho},\ \lambda\mu\nu\rho\sigma^2q^n \end{matrix};q,q\right].$$

The left hand side can be reduced to a real integral if

$$\max(|\lambda|, |\mu|, |\nu|, |\rho|, |\sigma|) < 1:$$

$$(2.11)\quad \frac{1}{2\pi}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}$$
$$\times \frac{h(x;\, 1)h(x;\, -1)h(x;\, \sqrt{q})h(x;\, -\sqrt{q})h(x;\, \lambda\mu\nu\rho\sigma)}{h(x;\, \lambda)h(x;\, \mu)h(x;\, \nu)h(x;\, \rho)h(x;\, \sigma)}$$
$$\times \frac{(\lambda\mu\nu\rho\sigma e^{i\theta}/A)_n(\lambda\mu\nu\rho\sigma e^{-i\theta}/A)_n}{(\lambda\mu\nu\rho\sigma e^{-i\theta})_n(\lambda\mu\nu\rho\sigma e^{i\theta})_n}$$
$$= g_0\frac{(\lambda\mu\nu\rho/A)_n(\lambda\mu\nu\rho\sigma^2/A)_n}{(\lambda\mu\nu\rho\sigma^2)_n(\lambda\mu\nu\rho)_n}$$
$$\times {}_{10}\phi_9\left[\begin{matrix}\lambda\mu\nu\rho\sigma^2q^{-1}, q\sqrt{\ }, -q\sqrt{\ }, \lambda\sigma, \mu\sigma, \nu\sigma, \rho\sigma, \\ \sqrt{\ }, -\sqrt{\ }, \mu\nu\rho\sigma, \lambda\nu\rho\sigma, \lambda\mu\rho\sigma, \lambda\mu\nu\sigma,\end{matrix}\right.$$
$$\left.\begin{matrix}A, & \dfrac{\lambda^2\mu^2\nu^2\rho^2\sigma^2q^{n-1}}{A}, & q^{-n} \\ \lambda\mu\nu\rho\sigma^2/A, & Aq^{1-n}/\lambda\mu\nu\rho, & \lambda\mu\nu\rho\sigma^2q^n\end{matrix};\, q, q\right].$$

This, of course, reduces to (2.1) when $A = 1$ or $n = 0$. However, if we set $A = \mu\nu\rho\sigma$, then (2.11) becomes, via Jackson's summation formula [**5**, (IV.8) p. 247]

$$(2.12)\quad \frac{1}{2\pi}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}$$
$$\times \frac{h(x;\, 1)h(x;\, -1)h(x;\, \sqrt{q})h(x;\, -\sqrt{q})h(x;\, \lambda\mu\nu\rho\sigma q^n)}{h(x;\, \lambda q^n)h(x;\, \mu)h(x;\, \nu)h(x;\, \rho)h(x;\, \sigma)}$$
$$= g_0\frac{(\lambda\mu)_n(\lambda\nu)_n(\lambda\rho)_n(\lambda\sigma)_n}{(\lambda\nu\rho\sigma)_n(\lambda\mu\rho\sigma)_n(\lambda\mu\nu\sigma)_n(\lambda\mu\nu\rho)_n},$$

which is valid provided the numerical values of all the parameters are less than 1, and hence, is a special case of (2.5).

3. Biorthogonal rational functions. Let us define

$$(3.1)\quad R_n\left(\frac{z+z^{-1}}{2};\, \lambda, \mu, \nu, \rho, \sigma\right)$$

$$= {}_{10}\phi_9\left[\begin{matrix}\lambda^2\mu\nu\rho\sigma q^{-1}, & q\sqrt{\ }, & -q\sqrt{\ }, & \lambda\nu\rho\sigma, & \lambda\mu\rho\sigma, \\ & \sqrt{\ }, & -\sqrt{\ }, & \lambda\mu, & \lambda\nu,\end{matrix}\right.$$
$$\left.\begin{matrix}\lambda\mu\nu\sigma, & \lambda z, & \lambda/z, \lambda\mu\nu\rho q^{n-1}, & q^{-n} \\ \lambda\rho, & \lambda\mu\nu\rho\sigma/z, \lambda\mu\nu\rho\sigma z, & \lambda\sigma q^{1-n}, & \lambda^2\mu\nu\rho\sigma q^n\end{matrix}; q, q\right]$$

and

(3.2) $$S_n\left(\frac{z+z^{-1}}{2}; \lambda, \mu, \nu, \rho, \sigma\right)$$
$$= {}_{10}\phi_9\left[\begin{matrix}\lambda/\sigma, q\sqrt{\ }, -q\sqrt{\ }, q/\mu\sigma, q/\nu\sigma, q/\rho\sigma, & \lambda z, \\ \sqrt{\ }, & -\sqrt{\ }, & \lambda\mu, & \lambda\nu, & \lambda\rho, q/\sigma z,\end{matrix}\right.$$
$$\left.\begin{matrix}\lambda/z, & \lambda\mu\nu\rho q^{n-1}, & q^{-n} \\ qz/\sigma, & q^{2-n}/\mu\nu\rho\sigma, & \lambda q^{n+1}/\sigma\end{matrix}; q, q\right].$$

Note that

(3.3) $$\lim_{\sigma\to 0} R_n\left(\frac{z+z^{-1}}{2}; \lambda, \mu, \nu, \rho, \sigma\right)$$
$$= \lim_{\sigma\to 0} S_n\left(\frac{z+z^{-1}}{2}; \lambda, \mu, \nu, \rho, \sigma\right)$$
$$= p_n\left(\frac{z+z^{-1}}{2}; \lambda, \mu, \nu, \rho\right)$$

defined in (1.7). We shall prove that $R_n\left(\frac{z+z^{-1}}{2}\right)$ and $S_n\left(\frac{z+z^{-1}}{2}\right)$ are biorthogonal with respect to the complex weight function $g(z)$, i.e.,

(3.4) $$\frac{1}{2\pi i}\int_C R_n\left(\frac{z+z^{-1}}{2}\right)S_m\left(\frac{z+z^{-1}}{2}\right)g(z)dz = 0, \quad \text{if } m \neq n.$$

By Jackson's transformation [**2**, p. 68]

(3.5) $$S_m\left(\frac{z+z^{-1}}{2}; \lambda, \mu, \nu, \rho, \sigma\right)$$
$$= \frac{(\lambda q/\sigma)_m(\lambda\nu\rho\sigma/q)_m(\mu\rho)_m(\mu\nu)_m}{(\mu q/\sigma)_m(\mu\nu\rho\sigma/q)_m(\lambda\rho)_m(\lambda\nu)_m}\,{}_{10}\phi_9\left[\begin{matrix}\mu/\sigma, q\sqrt{\ }, -q\sqrt{\ }, q/\lambda\sigma, \\ \sqrt{\ }, & -\sqrt{\ }, & \lambda\mu,\end{matrix}\right.$$
$$\left.\begin{matrix}q/\nu\sigma, q/\rho\sigma, & \mu z, & \mu/z, & \lambda\mu\nu\rho q^{m-1}, & q^{-m} \\ \lambda\nu, & \mu\rho, q/\sigma z, & qz/\sigma, & q^{2-m}/\lambda\nu\rho\sigma, & \mu q^{m+1}/\sigma\end{matrix}; q, q\right].$$

Let

$$(3.6)\qquad \xi_{m,n} \equiv \frac{1}{2\pi i}\int_C R_n\left(\frac{z+z^{-1}}{2}\right)S_m\left(\frac{z+z^{-1}}{2}\right)g(z)dz.$$

Using the series in (3.1) and (3.5) we get

$$(3.7)\qquad \frac{(\mu q/\sigma)_m(\mu\nu\rho\sigma/q)_m(\lambda\rho)_m(\lambda\nu)_m}{(\lambda q/\sigma)_m(\lambda\nu\rho\sigma/q)_m(\mu\rho)_m(\mu\nu)_m}\xi_{m,n}$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{m}\frac{(\lambda^2\mu\nu\rho\sigma/q)_k(q\sqrt{\ })_k(-q\sqrt{\ })_k}{(q)_k(\sqrt{\ })_k(-\sqrt{\ })_k}$$

$$\times \frac{(\lambda\nu\rho\sigma)_k(\lambda\mu\rho\sigma)_k(\lambda\mu\nu\sigma)_k(\lambda\mu\nu\rho q^{n-1})_k(q^{-n})_k}{(\lambda\mu)_k(\lambda\nu)_k(\lambda\rho)_k(\lambda\sigma q^{1-n})_k(\lambda^2\mu\nu\rho\sigma q^n)_k}$$

$$\times q^k\frac{(\mu/\sigma)_l(q\sqrt{\ })_l(-q\sqrt{\ })_l}{(q)_l(\sqrt{\ })_l(-\sqrt{\ })_l}$$

$$\times \frac{(q/\lambda\sigma)_l(q/\nu\sigma)_l(q/\rho\sigma)_l(\lambda\mu\nu\rho q^{m-1})_l(q^{-m})_l}{(\lambda\mu)_l(\mu\nu)_l(\mu\rho)_l(q^{2-m}/\lambda\nu\rho\sigma)_l(\mu q^{m+1}/\sigma)_l}q^l I_{k,l},$$

where

$$(3.8)\qquad I_{k,l} = \frac{1}{2\pi i}\int_C \frac{dz}{z}\frac{(z^2)_\infty(z^{-2})_\infty(\lambda\mu\nu\rho\sigma z q^k)_\infty}{(\lambda z q^k)_\infty(\lambda q^k/z)_\infty(\mu z q^l)_\infty(\mu q^l/z)_\infty}$$

$$\times \frac{(\lambda\mu\nu\rho\sigma q^k/z)_\infty}{(\nu z)_\infty(\nu/z)_\infty(\rho z)_\infty(\rho/z)_\infty}$$

$$\times \frac{1}{(\sigma z)_\infty(\sigma/z)_\infty(qz/\sigma)_l(q/\sigma z)_l}.$$

Since

$$(3.9)\qquad (\sigma z q^{-l})_\infty(\sigma q^{-l}/z)_\infty$$

$$= (\sigma z)_\infty(\sigma/z)_\infty(\sigma z q^{-l})_l(\sigma q^{-l}/z)_l$$

$$= (\sigma z)_\infty(\sigma/z)_\infty(q/\sigma z)_l(qz/\sigma)_l q^{-l(l+1)}\sigma^{2l},$$

we get

$$(3.10)\qquad I_{k,l} = q^{-l(l+1)}\sigma^{2l}$$

$$\times \frac{1}{2\pi i}\int_C \frac{dz}{z}\frac{(z^2)_\infty(z^{-2})_\infty(\lambda\mu\nu\rho\sigma z q^k)_\infty(\lambda\mu\nu\rho\sigma q^k/z)_\infty}{(\lambda z q^k)_\infty(\lambda q^k/z)_\infty(\mu z q^l)_\infty(\mu q^l/z)_\infty}$$

$$\times \frac{1}{(\nu z)_\infty(\nu/z)_\infty(\rho z)_\infty(\rho/z)_\infty(\sigma z q^{-l})_\infty(\sigma q^{-l}/z)_\infty}.$$

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But the integral on the right is precisely the same as that in (2.4) with λ, μ and σ replaced by λq^k, μq^l and σq^{-l}, respectively. Hence

$$(3.11)\quad I_{k,l} = \frac{q^{-l(l+1)}\sigma^{2l}\cdot 2(\lambda\mu\nu\rho q^{k+1})_\infty(\lambda\mu\nu\sigma q^k)_\infty}{(q)_\infty(\lambda\mu q^{k+l})_\infty(\lambda\nu q^k)_\infty(\lambda\rho q^k)_\infty(\lambda\sigma q^{k-l})_\infty}$$
$$\times\frac{(\lambda\mu\rho\sigma q^k)_\infty(\lambda\nu\rho\sigma q^{k-l})_\infty(\mu\nu\rho\sigma)_\infty}{(\mu\nu q^l)_\infty(\mu\rho q^l)_\infty(\mu\sigma)_\infty(\nu\rho)_\infty(\nu\sigma q^{-l})_\infty(\rho\sigma q^{-l})}$$
$$= 2g_0\frac{(\lambda\nu)_k(\lambda\rho)_k(\lambda\sigma)_k(\lambda\mu)_{k+l}}{(\lambda\mu\nu\sigma)_k(\lambda\mu\rho\sigma)_k(\lambda\nu\rho\sigma)_k(\lambda\mu\nu\rho)_{k+l}}$$
$$\times\frac{(\mu\nu)_l(\mu\rho)_l(q^{1-k}/\lambda\nu\rho\sigma)_l}{(q/\rho\sigma)_l(q/\nu\sigma)_l(q^{1-k}/\lambda\sigma)_l}.$$

Using this in (3.7) we get

$$(3.12)\quad \xi_{m,n} = 2g_0\frac{(\mu\nu)_m(\mu\rho)_m(\lambda q/\sigma)_m(\lambda\nu\rho\sigma/q)_m}{(\lambda\rho)_m(\lambda\nu)_m(\mu q/\sigma)_m(\mu\nu\rho\sigma/q)_m}$$
$$\times\sum_{k=0}^{n}\frac{(\lambda^2\mu\nu\rho\sigma/q)_k(q\sqrt{\ })_k(-q\sqrt{\ })_k}{(q)_k(\sqrt{\ })_k(-\sqrt{\ })_k}$$
$$\times\frac{(\lambda\sigma)_k(\lambda\mu\nu\rho q^{n-1})_k(q^{-n})_k}{(\lambda\mu\nu\rho)_k(\lambda\sigma q^{1-n})_k(\lambda^2\mu\nu\rho\sigma q^n)_k}q^k$$
$$\times {}_8\phi_7\left[\begin{matrix}\mu/\sigma,\ q\sqrt{\ },\ -q\sqrt{\ },\ q/\lambda\sigma,\quad \lambda\mu q^k,\\ \sqrt{\ },\ -\sqrt{\ },\quad \lambda\mu,\ q^{1-k}/\lambda\sigma,\end{matrix}\right.$$
$$\left.\begin{matrix}q^{1-k}/\lambda\nu\rho\sigma,\ \lambda\mu\nu\rho q^{m-1},\quad q^{-m}\\ \lambda\mu\nu\rho q^k,\ q^{2-m}/\lambda\nu\rho\sigma,\ \mu q^{m+1}/\sigma\end{matrix};q,q\right].$$

By Jackson's summation formula,

$${}_8\phi_7[\ ] = \frac{(\mu q/\sigma)_m(\lambda^2\mu\nu\rho\sigma q^{k-1})_m(\nu\rho)_m(q^{-k})_m}{(\lambda\mu)_m(\lambda\mu\nu\rho q^k)_m(q^{1-k}/\lambda\sigma)_m(\lambda\nu\rho\sigma/q)_m}$$

which vanishes unless $k \geqq m$. A little simplification yields

$$(3.13)\quad \xi_{m,n} = 0 \text{ if } n < m$$

and

$$(3.14)\quad \xi_{m,n} = 2g_0\frac{(\mu\nu)_m(\mu\rho)_m(\lambda q/\sigma)_m(\lambda\nu\rho\sigma/q)_m}{(\lambda\nu)_m(\lambda\rho)_m(\mu q/\sigma)_m(\mu\nu\rho\sigma/q)_m}$$
$$\times\frac{(\lambda^2\mu\nu\rho\sigma)_{2m}(\lambda\mu\nu\rho q^{n-1})_m}{(\lambda\mu\nu\rho)_{2m}(\lambda\sigma q^{1-n})_m}$$

$$\times \frac{(\mu q/\sigma)_m(\nu\rho)_m(q^{-n})_m}{(\lambda\mu)_m(\lambda\nu\rho\sigma/q)_m(\lambda^2\mu\nu\rho\sigma q^n)_m}(\lambda\sigma)^m$$

$$\times {}_6\phi_5\left[\begin{matrix}\lambda^2\mu\nu\rho\sigma q^{2m-1}, q\sqrt{\ }, -q\sqrt{\ }, & \lambda\sigma, \lambda\mu\nu\rho q^{m+n-1}, \\ \sqrt{\ }, -\sqrt{\ }, q^{2m}\lambda\mu\nu\rho, & \lambda\sigma q^{1-n+m}, \end{matrix}\right.$$

$$\left.\begin{matrix} q^{m-n} \\ \lambda^2\mu\nu\rho\sigma q^{n+m} \end{matrix}; q, q\right], \text{ if } n \geqq m.$$

Summing the last ${}_6\phi_5$ series as a special case of Jackson's formula we find that, for $n \geqq m$

$$(3.15)\quad \xi_{m,n} = 2g_0\frac{(\mu\nu)_m(\mu\rho)_m(\nu\rho)_m(\lambda q/\sigma)_m(q^{-n})_m}{(\lambda\nu)_m(\lambda\mu)_m(\lambda\rho)_m(\mu\nu\rho\sigma/q)_m(\lambda^2\mu\nu\rho\sigma q^n)_m}(\lambda\sigma)^m$$

$$\times \frac{(\lambda^2\mu\nu\rho\sigma)_{2m}(\lambda\mu\nu\rho q^{n-1})_m}{(\lambda\mu\nu\rho)_{2m}(\lambda\sigma q^{1-n})_m}$$

$$\times \frac{(\lambda^2\mu\nu\rho\sigma q^{2m})_{n-m}(q^{1+m-n})_{n-m}}{(\lambda\mu\nu\rho q^{2m})_{n-m}(\lambda\sigma q^{1-n+m})_{n-m}}$$

which, because of the factor $(q^{1+m-n})_{n-m}$, vanishes unless $m \geqq n$. Hence we have

$$(3.16)\quad \xi_{m,n} = 2g_0\delta_{m,n}/N_n,$$

where

$$(3.17)\quad N_n = \frac{(\lambda\mu\nu\rho q^{-1})_n(q\sqrt{\ })_n(-q\sqrt{\ })_n(\lambda\mu)_n(\lambda\nu)_n}{(q)_n(\sqrt{\ })_n(-\sqrt{\ })_n(\nu\rho)_n(\mu\rho)_n(\mu\nu)_n}$$

$$\times \frac{(\lambda\rho)_n(\mu\nu\rho\sigma/q)_n(\lambda^{-1}\sigma^{-1})_n}{(\lambda q/\sigma)_n(\lambda^2\mu\nu\rho\sigma)_n}q^n.$$

Note that (3.6) cannot be reduced to orthogonality on the real line because of the factors $(\sigma z q^{-l})_\infty(\sigma q q^{-l}/z)_\infty$ in the denominator of the integrand in (3.10).

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Carleton University,
Ottawa, Ontario