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OPTIMAL PORTFOLIO POLICIES UNDER FIXED AND PROPORTIONAL TRANSACTION COSTS

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Abstract

We consider the portfolio optimization problem of maximizing the asymptotic growth rate under a combination of fixed and proportional costs. Expressing the asymptotic growth rate in terms of the risky fraction process, the problem can be transformed to that of controlling a diffusion in one dimension. Then we use the corresponding quasivariational inequalities to obtain the explicit shape together with the existence of an optimal impulse control strategy. This optimal strategy is given by only four parameters: two for the stopping boundaries and two for the new risky fractions the investor chooses at these times.

Keywords: Asymptotic growth rate; impulse control; fixed cost; proportional cost; portfolio optimization; transaction cost

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1. Introduction

Typical transaction costs considered in portfolio theory are constant costs, fixed costs (proportional to the portfolio value), and proportional costs (proportional to the transaction volume). While the latter penalizes the size of the transaction, the first two punish the frequency of trading. So a combination of both types is of interest, from a practical as well as a theoretical point of view. In this case the trading strategies of interest are so-called impulse control strategies consisting of a sequence of stopping times at which trading takes place and the transactions at those times. Optimal impulse control strategies can be described as solutions of quasivariational inequalities.

We use the framework of the Black–Scholes model, which will allow us to obtain an explicit solution, and consider an investor who faces proportional costs and fixed costs. The objective is to maximize the asymptotic growth rate

$$\liminf_{t\to\infty} \frac{1}{t}\,\mathrm{E}[\log V_t \mid V_0 = x,\ \pi_0 = \pi], \tag{1.1}$$

where V_t is the value of the portfolio at time $t \geq 0$ and π_0 the initial fraction of V_0 invested in the stock.

Without transaction costs, the problem of maximizing the expected utility was solved by Merton [12]. Merton showed that, for logarithmic and power utility, the optimal trading strategy

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is given by a constant, $\hat{\eta}$, which is the optimal risky fraction, i.e. the fraction of wealth invested in the stock, to be held at all times. To keep the risky fraction constant involves continuous trading, which under transaction costs is no longer adequate.

A wealth of papers dealing with transaction costs have appeared in the last decade. The first type of costs considered were purely proportional costs for which the optimal solution is given by a cone in which it is optimal not to trade at all, and which corresponds to an interval for the risky fraction. When reaching the boundaries, infinitesimal trading occurs in such a way that the wealth process just stays in the cone. This kind of behaviour was first described in [11]. A rigorous proof of a discounted consumption criterion can be found in [4]. It uses methods of stochastic control theory and shows that the wealth process is a diffusion reflected at the boundaries of the cone. Under weaker assumptions, the authors of [1], [8], and [16] proved the existence and uniqueness of a viscosity solution for the corresponding Hamilton–Jacobi–Bellman equation, and those of [2] and [18] derived similar no-transaction regions for the asymptotic growth rate under somewhat different proportional costs.

Adding a constant component to the transaction costs punishes very frequent trading and so will avoid the occurrence of infinitesimal trading at the boundary. This approach was used in another group of papers which dealt with constant and proportional costs. An investor now has to choose discrete trading times and optimal transactions at these times, so the methodology of optimal impulse control comes into play. The authors of [5] and [9] gave general existence results for finite and infinite horizons and determined optimal strategies for maximizing the utility of terminal wealth both for the identity as utility function and for exponential utility. Maximizing discounted consumption under power utility for an infinite horizon and allowing for continuous consumption, the authors of [14] derived quasivariational Hamilton–Jacobi–Bellman inequalities whose solution yields the optimal strategy. The insight, already obtained in [9], is that there is still some no-transaction region, but reaching the boundary transactions will be done in such a way that the wealth process restarts at some curve between the boundary and the Merton line.

A different approach was provided in [13] for purely fixed costs. There, for the purpose of maximizing the expected asymptotic growth rate (1.1), a factorization of the wealth process into the wealth gained per period was obtained which leads under logarithmic utility to an additive representation. An optimal stopping problem for the risky fraction process with linear costs yields an explicit solution. A general cost structure was treated in [3], where a set of quasivariational inequalities for the optimal trading strategy for (1.1) was derived. From this, the results of [13] were obtained in a different manner. The approaches described above lead to plausible optimal strategies.

In [6], the current authors treated transaction costs which have proportional costs in addition to the fixed costs, which is often the case for private investors. In the spirit of [13], they introduced a class of natural trading strategies which can be described by four parameters, (a, b, α, β): a and b for the stopping boundaries and α and β for the new risky fractions. When the risky fraction process reaches a or b, trading occurs in such a way that the new risky fractions are given by α or β, respectively. Stopping at a corresponds to buying stocks and stopping at b to selling stocks. This class of constant boundary (CB) strategies is motivated by the results discussed above. The cone obtained for proportional costs corresponds to an interval for the risky fraction process. The results of [13] say that for fixed costs a constant, new risky fraction is optimal, and, from the results for combined constant and proportional costs, we may expect that, due to the proportional costs, we now have two different new risky fractions, one after buying and one after selling.

In this class of strategies, the analysis can be simplified to the study of one period between two trading times using renewal-theoretic arguments; see [6]. This yields an explicit functional that, if the aim is to find an optimal CB strategy, has to be maximized with respect to these four parameters, which can easily be done numerically. By adapting the results of [3] and using substantial numerical evidence, it was conjectured in [6] that CB strategies provide optimal strategies within the whole class of impulse control strategies.

Here we shall show that this conjecture is valid. We prove that overall optimal impulse control strategies exist and that they are CB strategies. Our results thus provide explicit solutions, up to a simple numerical optimization, in the case of proportional costs plus fixed costs, thus coming closer to the actual costs a private investor faces.

In a certain average-reward control problem with linear costs, the authors of [7] also arrived at CB strategies and found that such strategies provide optimal solutions. The methodology for solving average-reward control problems used in the above paper also proved useful for the derivations in our problem, but various additional difficulties arose due to the nonlinearity of our costs.

The paper is organized in the following manner. After presenting some notation and the solution to the problem without transaction costs, in Section 2, in Sections 3–7 we shall consider the model with both fixed costs and proportional costs, first using controls given by stopping times and the transactions at those times, described by the amount of money invested in the stock. Then we shall describe how to reformulate the problem in terms of the risky fraction process. In Proposition 3.1 we show that the controls, the costs, and the reward function can all be expressed in terms of these risky fractions; the problem can thus be transformed to that of controlling a diffusion in one dimension.

In Section 4 we give a general formulation of an optimal impulse control problem for maximizing average rewards, which relates our model to [7]. An upper bound on (1.1) based on the standard Hamilton–Jacobi–Bellman approach can be given by a solution to certain quasivariational inequalities. From the proof of this result, Theorem 4.1, it is obvious how we have to proceed to find an optimal solution; see Remark 4.2. In Section 5 we show how to embed the problem of portfolio optimization under fixed and proportional costs, and in Section 6 we derive necessary conditions for the optimality. The solution to the differential equation corresponding to the continuation region, where no controls take place, is given in terms of the speed measure and the scale function of the diffusion.

Based on this representation and our special cost function, we show in Section 7 that a solution to the conditions derived in Section 6 exists and defines a CB strategy. Theorem 7.1 states the main result: an optimal strategy exists and is a CB strategy. These derivations are done under the condition that the Merton fraction $\hat{\eta}$ lies in $(0, 1)$, which implies that short selling and borrowing do not have to be taken into account.

In Section 8 we look at short selling and borrowing, corresponding to $\hat{\eta} < 0$ and $\hat{\eta} > 1$, respectively. There we have to guarantee that the risky fraction process stays in the so-called solvency region, in which the liquidation of the stock holdings is possible with positive wealth remaining. Again we can show that an optimal strategy exists and is a CB strategy.

In Section 9 we provide some examples and discuss the relation of our scenario to the case of purely fixed costs and the case of purely proportional costs.

2. Trading and optimization without transaction costs

We consider one *bond* or *bank account* and one *stock* with price processes $(B_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$, respectively, which evolve according to the Black–Scholes model. Hence, the prices

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are given, for *interest rate* $r \geq 0$, *trend* $\mu \in \mathbb{R}$, and *volatility* $\sigma > 0$, by $B_0 = S_0 = 1$ and

$$\mathrm{d}B_t = B_t r\,\mathrm{d}t, \qquad \mathrm{d}S_t = S_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t),$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion on a suitable probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the augmented filtration generated by $(W_t)_{t\geq 0}$.

Without transaction costs the trading of an investor may be described by the constant *initial capital* $V_0 > 0$ and by the *risky fraction process* $(\pi_t)_{t\geq 0}$, where π_t is the fraction of the total portfolio value (wealth) which the investor chooses to hold in the stock at time t. Given V_0 and $(\pi_t)_{t\geq 0}$, the corresponding *wealth process* $(V_t)_{t\geq 0}$ is defined, to be self-financing, as the continuous solution to

$$\mathrm{d}V_t = (1-\pi_t)V_t r\,\mathrm{d}t + \pi_t V_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t), \qquad t > 0.$$

Our objective is the maximization of the *asymptotic growth rate* (1.1) over all admissible – say bounded and progressively measurable – risky fraction processes $(\pi_t)_{t\geq 0}$. Using

$$\mathrm{E}[\log V_t] = \log V_0 + \mathrm{E}\bigg[\int_0^t \bigg(r + \pi_s(\mu - r) - \frac{1}{2}(\pi_s\sigma)^2\bigg)\,\mathrm{d}s\bigg],$$

a simple pointwise maximization yields as optimal solution $\pi_t = \hat{\eta}$, $t \geq 0$, where

$$\hat{\eta} = \frac{\mu - r}{\sigma^2} \quad \text{and the optimal growth rate is} \quad \hat{R} = r + \frac{1}{2}\bigg(\frac{\mu - r}{\sigma}\bigg)^2.$$

This constant *optimal risky fraction* $\hat{\eta}$ corresponds to the well-known *Merton line*.

Multiplying the prices by e^{-rt}, we see that we may assume from now on that $r = 0$, to simplify notation, whence $B_t = 1$, $t \geq 0$. The solution to (1.1) for general r is then obtained by adding the rate r and using $\mu - r$ instead of μ. So in the following we let

$$\hat{\eta} = \frac{\mu}{\sigma^2} \quad \text{and} \quad \hat{R} = \frac{1}{2}\bigg(\frac{\mu}{\sigma}\bigg)^2.$$

3. Fixed and proportional transaction costs

Let us now assume that an investor faces transaction fees. With current wealth $V_t > 0$, the *transaction costs* for a transaction of volume $\Delta_t \in \mathbb{R}$ are

$$\delta V_t + \gamma|\Delta_t|, \tag{3.1}$$

where $\delta \in (0, 1)$ and $\gamma \in [0, 1-\delta)$. We call δV_t and $\gamma|\Delta_t|$ the *fixed cost* and the *proportional cost*, respectively. Note that the definition of the fixed cost is the same as in [13]. Costs of this type may be interpreted as managing costs.

It is convenient to use two processes to describe the evolution of the wealth. We use the wealth process $(V_t)_{t\geq 0}$ and the risky fraction process $(\pi_t)_{t\geq 0}$.

Since $\delta > 0$, the natural class of strategies to consider is that of impulse control strategies where trading occurs at time points τ_n, $\tau_0 \leq \tau_1 \leq \cdots$; see [3]. In view of (3.1) and [3, Proposition 4.1], we can restrict to trading times which are separated.

Definition 3.1. (i) An *impulse control strategy* $(\tau_n, \Delta_n)_{n\in\mathbb{N}_0}$ consists of stopping times τ_n, $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \infty$, with respect to $(\mathcal{F}_t)_{t\geq 0}$, the *trading times*, which satisfy $\tau_n \to \infty$ almost surely and $\tau_n < \tau_{n+1}$ on $\{\tau_n < \infty\}$; and of $\mathcal{F}_{\tau_n}$-measurable, $\mathbb{R}$-valued random variables Δ_n, $n \in \mathbb{N}_0$, the *transactions*.

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(ii) The wealth process V and the risky fraction process π controlled by an impulse control strategy $\tilde{K} = (\tau_n, \Delta_n)_{n\in\mathbb{N}_0}$ are given, for initial values $V_0 > 0$ and $\pi_0 \in [0, 1]$, by

$$\bar{V}_0 = V_0 - \delta V_0 - \gamma|\Delta_0|, \qquad \bar{\pi}_0 = (\pi_0 V_0 + \Delta_0)/\bar{V}_0,$$

and are given, for $n \in \mathbb{N}$ on $\{\tau_n < \infty\}$, by

$$V_t = (1 - \bar{\pi}_{n-1} + \bar{\pi}_{n-1} S_t/S_{\tau_{n-1}})\bar{V}_{n-1}, \qquad t \in (\tau_{n-1}, \tau_n], \tag{3.2}$$

$$\pi_t = \bar{\pi}_{n-1}\bar{V}_{n-1} S_t/(S_{\tau_{n-1}} V_t), \qquad t \in (\tau_{n-1}, \tau_n], \tag{3.3}$$

$$\bar{V}_n = V_{\tau_n} - \delta V_{\tau_n} - \gamma|\Delta_n|, \tag{3.4}$$

$$\bar{\pi}_n = (\pi_{\tau_n} V_{\tau_n} + \Delta_n)/\bar{V}_n. \tag{3.5}$$

(iii) An impulse control strategy is called *admissible* if the corresponding wealth and risky fraction processes satisfy $V_t > 0$ and $\pi_t \in (0, 1)$, respectively, for all $t \geq 0$.

Thus, $(V_t)_{t\geq 0}$ and $(\pi_t)_{t\geq 0}$ are defined to be left continuous with right-hand limits. We call $\bar{V}_n$ the *new wealth* and $\bar{\pi}_n$ the *new risky fraction*. Because $(1 - \bar{\pi}_{n-1})\bar{V}_{n-1}$ is the new value invested in the bond and $\bar{\pi}_{n-1}\bar{V}_{n-1}$ the new value invested in the stock at time τ_{n-1}, these parts evolve without trading according to the dynamics of the bond and of the stock, respectively, yielding (3.2) and (3.3). At τ_n the new wealth $\bar{V}_n$ is the wealth before trading minus the transaction costs to be paid, and the new risky fraction $\bar{\pi}_n$ is the new amount, $\pi_{\tau_n} V_{\tau_n} + \Delta_n$, invested in the stock divided by the new wealth, leading to (3.4) and (3.5). Note that Definition 3.1(iii) implies that short selling and borrowing is not admissible; see also Remark 3.1. The case of short selling and borrowing will be taken up in Section 8.

From Definition 3.1, it follows, for admissible impulse control strategies $(\tau_n, \Delta_n)_{n\in\mathbb{N}}$, that

$$\Delta_n = \frac{(1-\delta)\bar{\pi}_n - \pi_{\tau_n}}{1 + \gamma\bar{\pi}_n A_n} V_{\tau_n}, \tag{3.6}$$

where $A_n = \operatorname{sgn}(\Delta_n) = \operatorname{sgn}((1-\delta)\eta_n - \pi_{\tau_n})$ and

$$\bar{\pi}_n = \frac{\pi_{\tau_n} V_{\tau_n} + \Delta_n}{V_{\tau_n} - \delta V_{\tau_n} - \gamma|\Delta_n|}. \tag{3.7}$$

Using these representations, it was shown in [6, Lemma 2] that we can use a different parametrization of the control. In fact, we have a one-to-one correspondence between admissible impulse control strategies, as defined in Definition 3.1, and *new risky fraction impulse control strategies (NRF strategies)* $(\tau_n, \eta_n)_{n\in\mathbb{N}_0}$ consisting of stopping times $(\tau_n)_{n\in\mathbb{N}_0}$ as defined in Definition 3.1(i) and $\mathcal{F}_{\tau_n}$-measurable random variables η_n with values in $(0, 1)$, the latter corresponding to $\bar{\pi}_n$ in (3.5) and (3.7). We call τ_n the *nth trading time* and η_n the *new risky fraction* at τ_n. We will denote by $\mathcal{K}$ the class of NRF strategies. So by setting $\eta_n = \bar{\pi}_n$ the investor chooses, at τ_n, the new starting value of the risky fraction process as in (3.3) and the wealth process evolves as in (3.2) with new wealth given by (3.4) using (3.6).

Our objective is the maximization of

$$R(K) \equiv R(K, x, \pi) = \liminf_{t\to\infty} \frac{1}{t} \operatorname{E}[\log V_t^K \mid V_0 = x, \ \pi_0 = \pi], \qquad x > 0, \ \pi \in (0, 1), \tag{3.8}$$

over all NRF strategies K. We would like to find an optimal strategy, K^*, for which $R^* = \sup\{R(K)\colon K \text{ is an NRF strategy}\} = R(K^*)$. Note that an NRF strategy automatically corresponds to an admissible impulse control strategy.

To make use of the reformulation of the control problem in terms of the new risky fractions, we have to show that we can also express our objective in terms of the risky fraction process $(\pi_t)_{t\geq 0}$ and new risky fractions $(\eta_n)_{n\in\mathbb{N}_0}$. This was done in [6, Theorem C.1], where a factorization of the wealth process was derived which allows one to proceed by renewal-theoretic arguments. However, in the subsequent stochastic control approach, the following representation is more convenient.

Proposition 3.1. *For any admissible NRF strategy* $(\tau_n, \eta_n)_{n\in\mathbb{N}}$,

$$\log V_t = \log V_0 + \int_0^t \pi_s\Big(\mu - \frac{\sigma^2}{2}\pi_s\Big)\,\mathrm{d}s + \int_0^t \pi_s\sigma\,\mathrm{d}W_s + \sum_{n=0}^{N_t}\bar\Gamma(\pi_{\tau_n}, \eta_n) \tag{3.9}$$

and

$$\pi_t = \int_0^t \pi_s(1-\pi_s)(\mu - \sigma^2\pi_s)\,\mathrm{d}s + \int_0^t \pi_s(1-\pi_s)\sigma\,\mathrm{d}W_s + \sum_{n=0}^{N_t}(\eta_n - \pi_{\tau_n}), \tag{3.10}$$

where $N_t = \sup\{n\in\mathbb{N}_0 : \tau_n < t\}$, $t > 0$, *and*

$$\bar\Gamma(x,y) = \begin{cases} \log\dfrac{1-\delta-\gamma x}{1-\gamma y}, & y < \dfrac{x}{1-\delta}, \\[2ex] \log\dfrac{1-\delta+\gamma x}{1+\gamma y}, & y \geq \dfrac{x}{1-\delta}. \end{cases}$$

Proof. Note that the wealth in the bank account and the wealth in the stock are given by

$$V_t^0 = (1-\pi_0)V_0 + \sum_{k\geq 0}\mathbf{1}_{\{\tau_k<t\}}(-\Delta_k - \gamma|\Delta_k| - \delta V_{\tau_k}),$$
$$V_t^1 = \pi_0 V_0 + \int_0^t V_s^1\mu\,\mathrm{d}s + \int_0^t V_s^1\sigma\,\mathrm{d}W_s + \sum_{k\geq 0}\mathbf{1}_{\{\tau_k<t\}}\,\Delta_k,$$

respectively. Using (3.6) and (3.7), and the definition of $\bar\Gamma$, the representations can be obtained by straightforward applications of Itô's formula for semimartingales with jumps to $V_t = V_t^0 + V_t^1$ and $\pi_t = V_t^1/(V_t^0 + V_t^1)$, respectively.

By (3.9), the asymptotic growth rate does not depend on the initial capital V_0. The investor always has to pay at least the fixed costs when stopping; in particular, fees have to be paid at $\tau_0 = 0$ even if $\Delta_0 = 0$. However, the initial payment does not matter in the maximization of (3.8), so the asymptotic growth rate is also independent of the initial fraction π_0. Using the notation of Proposition 3.1, for all initial values $V_0 > 0$ and $\pi_0 \in (0,1)$ we therefore obtain

$$R(K) = \liminf_{t\to\infty}\frac{1}{t}\,\mathrm{E}\bigg[\int_0^t \pi_s\Big(\mu - \frac{1}{2}\sigma^2\pi_s\Big)\,\mathrm{d}s + \sum_{n=0}^{N_t}\bar\Gamma(\pi_{\tau_n}, \eta_n)\bigg]. \tag{3.11}$$

We will compare these quantities with the growth rates

$$R_0 = 0 \quad\text{and}\quad R_1 = \mu - \sigma^2/2, \tag{3.12}$$

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where R_0 corresponds to the pure-bond portfolio ($\eta_0 = 0$, $\tau_1 = \infty$) and R_1 to the pure-stock portfolio ($\eta_0 = 1$, $\tau_1 = \infty$). For any other buy-and-hold strategy ($\eta_0 \in (0, 1)$, $\tau_1 = \infty$) with growth rate $\tilde{R}$, we can show that $\tilde{R} \le \max\{0, \mu - \sigma^2/2\}$; see [6, Lemma 1]. Hence, within the class of buy-and-hold strategies it is enough to look at the pure-bond and pure-stock strategies.

Advantages of the reformulation of strategies in terms of the new risky fractions η_n are the easier admissibility conditions, the direct use of the representation in Proposition 3.1 leading to the simpler representation (3.11), instead of (3.8), and the fact that we hence only have to control $(\pi_t)_{t\ge 0}$. So we have reduced the control problem to one dimension.

Remark 3.1. Definition 3.1(iii) describes admissible strategies as those which satisfy $V_t > 0$ and $\pi_t \in (0, 1)$ for all $t \ge 0$. Such strategies will be compared with the pure-bond and pure-stock strategies. Now assume that we allow for short selling ($\pi_t < 0$) and borrowing ($\pi_t > 1$), retaining the condition $V_t > 0$ for all t and adding the condition $\mathrm{E} \int_0^t \pi_s \, \mathrm{d}W_s = 0$ for all t, this of course being satisfied for all bounded $(\pi_t)_t$. Then representation (3.11) still holds if $\bar{\Gamma}$ is well defined.

(i) Let us first assume that $0 < \hat{\eta} < 1$. Then

$$\pi(\mu - \tfrac{1}{2}\sigma^2\pi) < \begin{cases} 0 = R_0 & \text{for all } \pi < 0, \\ \mu - \tfrac{1}{2}\sigma^2 = R_1 & \text{for all } \pi > 1; \end{cases}$$

hence, short selling is inferior to pure-bond holding and borrowing is inferior to pure-stock holding. So, for the problem of maximizing $R(K)$, only the admissible strategies of Definition 3.1(iii) have to be considered, and compared with the pure-bond and the pure-stock strategies.

(ii) In the cases $\hat{\eta} = 0$ and $\hat{\eta} = 1$, the pure-bond strategy and the pure-stock strategy are respectively optimal, since these are also optimal in the model without transaction costs and, after the initial trading, no fees have to be paid.

(iii) For $\hat{\eta} < 0$, i.e. negative stock growth rate, we have

$$\pi(\mu - \tfrac{1}{2}\sigma^2\pi) \le 0 \quad \text{for all } \pi > 0,$$

so only strategies with $\pi_t < 0$ (short selling) have to be considered in the maximization of the asymptotic growth rate, and compared with the pure-bond strategy. Consider any strategy starting at some $\pi_0 < 0$ with $V_0 > 0$. Then, with positive probability, the resulting uncontrolled risky fraction process $(\pi_t)_t$ will be explosive. This happens at the first time τ at which $V_\tau = 0$ and, consequently, $\pi_\tau = -\infty$. So the strategy must act before that time.

Similarly, for $\hat{\eta} > 1$ only strategies with $\pi_t > 1$ (borrowing) have to be considered. Again we will have an explosion for the uncontrolled risky fraction process with positive probability, at the first time τ at which $V_\tau = 0$ and, consequently, $\pi_\tau = \infty$.

The case $0 < \hat{\eta} < 1$ will be treated in detail in Sections 4–7, and the cases $\hat{\eta} < 0$ and $1 < \hat{\eta}$, in Section 8.

4. A general optimal impulse control problem

We consider a one-dimensional diffusion process $(X_t)_{t\ge 0}$ which is in the regime of an impulse control strategy. The impulse control is given by an adapted, piecewise-constant stochastic process $(Z_t)_{t\ge 0}$ which jumps at times τ_n, $n \in \mathbb{N}$, with jumps of size ΔZ_{τ_n}, $n \in \mathbb{N}$.

We consider $(Z_t)_{t\geq 0}$ to be a càglàd process (one that is left continuous with right limits), so $\Delta Z_t = Z_{t+} - Z_t$, and assume that there are almost surely only finitely many jumps in any finite interval.

We consider a diffusion which is nonexplosive and, uncontrolled, follows the stochastic differential equation

$$\mathrm{d}X_t = \tilde{\mu}(X_t)\,\mathrm{d}t + \tilde{\sigma}(X_t)\,\mathrm{d}W_t, \qquad X_0 = x,$$

on some open interval $I \subseteq \mathbb{R}$, where $\tilde{\mu}, \tilde{\sigma} : I \to \mathbb{R}$. Under the control $(Z_t)_{t\geq 0}$, the controlled process evolves according to

$$\mathrm{d}X_t = \tilde{\mu}(X_t)\,\mathrm{d}t + \mathrm{d}Z_t + \tilde{\sigma}(X_t)\,\mathrm{d}W_t;$$

hence, the controlled process is càglàd. We could also use the càdlàg version of the problem (which is right continuous with left limits), which would, in the following, lead to using Z_t and X_t instead of Z_{t+} and X_{t+}, and Z_{t-} and X_{t-} instead of Z_t and X_t.

There is given a measurable function $h : I \to \mathbb{R}$, and the aim is to maximize the average reward rate

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathrm{E}\biggl[\int_0^T h(X_t)\,\mathrm{d}t + \sum_{t\in[0,T]} C(X_t, \Delta Z_t)\biggr],$$

where $C : \{(x, z) : x \in I,\ z \in I - x\} \to \mathbb{R}$ defines the cost structure. This corresponds to the model treated in [7] for a diffusion on $\mathbb{R}$ with costs $C(x, z) = K_1 z + c_1$ if $z > 0$ and $C(x, z) = -K_2 z + c_2$ if $z < 0$, for constants $K_1, c_1, K_2, c_2 > 0$.

We require of a control $(Z_t)_{t\geq 0}$ that the controlled process not leave I, i.e. that $X_t + \Delta Z_t \in I$ for all t. Note that between jump times the process $(X_t)_{t\geq 0}$ evolves according to

$$\mathrm{d}X_t = \tilde{\mu}(X_t)\,\mathrm{d}t + \tilde{\sigma}(X_t)\,\mathrm{d}W_t. \tag{4.1}$$

To proceed formally, let there be given an open interval $I \subseteq \mathbb{R}$ and functions $\tilde{\mu} : I \to \mathbb{R}$ and $\tilde{\sigma} : I \to \mathbb{R}$ such that the stochastic differential equation (4.1) has a unique nonexplosive weak solution on I for any initial state $x \in I$, where $(W_t)_{t\geq 0}$ is a Wiener process.

Definition 4.1. A controlled stochastic system with initial condition $x \in I$ is given by a Wiener process $(W_t)_{t\geq 0}$, with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$, and an adapted piecewise-constant process $(Z_t)_{t\geq 0}$, having almost surely only finitely many jumps in any finite interval, such that a solution to

$$\mathrm{d}X_t = \tilde{\mu}(X_t)\,\mathrm{d}t + \mathrm{d}Z_t + \tilde{\sigma}(X_t)\,\mathrm{d}W_t, \qquad X_0 = x,$$

exists as a stochastic process on I.

Remark 4.1. We can construct such a solution in the following way. Let τ_n, $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$, denote the jump times of $(Z_t)_{t\geq 0}$. Let $X_0 = x$. Start a diffusion in $x + \Delta Z_0$ according to the uncontrolled stochastic differential equation up to time τ_1. This gives the values of $(X_t)_{t\geq 0}$ on $(0, \tau_1]$. Start a diffusion in $X_{\tau_1} + \Delta Z_{\tau_1}$ according to the uncontrolled stochastic differential equation, giving the values on $(\tau_1, \tau_2]$. Proceed in this manner to obtain the whole controlled process $(X_t)_{t\geq 0}$. Our definition hence implicitly contains the requirement that $X_{\tau_n} + \Delta Z_{\tau_n} \in I$, $n \in \mathbb{N}_0$.

Denote by $\mathcal{K}_x$ the set of all controlled stochastic systems K defined according to Definition 4.1. The aim is to find

$$J^* = \sup_{K \in \mathcal{K}_x} \liminf_{T \to \infty} \frac{1}{T} \mathrm{E}\bigg[\int_0^T h(X_t)\,\mathrm{d}t + \sum_{t \in [0,T]} C(X_t, \Delta Z_t)\bigg] \tag{4.2}$$

and an optimal $(Z_t)_{t \geq 0}$.

As we may switch between initial conditions x and x' by using a suitable Z_0, this quantity J^* does not depend on x. Using Itô's formula and the standard Hamilton–Jacobi–Bellman approach, we first give an upper bound for J^*. Denote by

$$\mathcal{L} = \frac{1}{2}\tilde{\sigma}^2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{\mu}(x)\frac{\mathrm{d}}{\mathrm{d}x}$$

the characteristic operator of the uncontrolled diffusion (4.1). Let $v\colon I \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the following conditions.

(A1) $\mathcal{L}v(x) + h(x) - \lambda \leq 0$ for all $x \in I$.

(A2) $v(x+z) - v(x) + C(x,z) \leq 0$ for all $x \in I$ and $z \in I - x$, $z \neq 0$.

(A3) v is continuously differentiable and piecewise twice continuously differentiable.

Hence, v is an element of the Sobolev space $H^2(0,1)$ and sufficiently smooth that Itô's formula may be applied.

Theorem 4.1. *Assume (A1)–(A3) to hold, assume that, for any controlled system,*

$$\sup_T \mathrm{E}\,|v(X_{T+})| < \infty,$$

and assume that

$$\bigg(\int_0^t \sigma(X_s)v'(X_s)\,\mathrm{d}W_s\bigg)_{t \geq 0} \quad \textit{is a martingale.}$$

Then $J^* \leq \lambda$.

Proof. By Itô's formula,

$$\begin{aligned} v(X_{T+}) &= v(x) + \int_0^T \mathcal{L}v(X_t)\,\mathrm{d}t + \int_0^T v'(X_t)\,\mathrm{d}Z_t + \int_0^T \tilde{\sigma}(X_t)v'(X_t)\,\mathrm{d}W_t \\ &\quad + \sum_{t\in[0,T]} (v(X_{t+}) - v(X_t) - v'(X_t)\Delta X_t) \\ &= v(x) + \int_0^T \mathcal{L}v(X_t)\,\mathrm{d}t + \int_0^T \tilde{\sigma}(X_t)v'(X_t)\,\mathrm{d}W_t + \sum_{t\in[0,T]} (v(X_{t+}) - v(X_t)), \end{aligned}$$

as $\int_0^T v'(X_t)\,\mathrm{d}Z_t = \sum_{t\in[0,T]} v'(X_t)\Delta X_t$. (Here a prime denotes differentiation.)

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This implies that, for any controlled system with initial state x,

$$\begin{aligned}\int_0^T h(X_t)\,\mathrm{d}t + \sum_{t\in[0,T]} C(X_t,\Delta Z_t) &= \lambda T + v(x) - v(X_{T+}) + \int_0^T \tilde{\sigma}(X_t)v'(X_t)\,\mathrm{d}W_t \\ &\quad + \int_0^T (\mathcal{L}v(X_t) + h(X_t) - \lambda)\,\mathrm{d}t \\ &\quad + \sum_{t\in[0,T]} (v(X_t+\Delta Z_t) - v(X_t) + C(X_t,\Delta Z_t)) \\ &\le \lambda T + v(x) - v(X_{T+}) + \int_0^T \tilde{\sigma}(X_t)v'(X_t)\,\mathrm{d}W_t,\end{aligned}$$

using (A1) and (A2). By taking expectations and dividing by T, from the additional assumptions we obtain

$$\limsup_{T\to\infty} \frac{1}{T}\,\mathrm{E}\bigg[\int_0^T h(X_t)\,\mathrm{d}t + \sum_{t\in[0,T]} C(X_t,\Delta Z_t)\bigg] \le \lambda.$$

Remark 4.2. The proof immediately shows how we may proceed to find J^* and an optimal strategy. Find v and λ satisfying (A1)–(A3) and a control strategy such that, at least from some fixed time onwards, $\mathcal{L}v(X_t) + h(X_t) - \lambda = 0$ and $v(X_t+\Delta Z_t) - v(X_t) + C(X_t,\Delta Z_t) = 0$ when controls take place. For such v, λ, and $(Z_t)_{t\ge 0}$, it follows that

$$\lim_{T\to\infty} \frac{1}{T}\,\mathrm{E}\bigg[\int_0^T h(X_t)\,\mathrm{d}t + \sum_{t\in[0,T]} C(X_t,\Delta Z_t)\bigg] = \lambda;$$

hence, $J^* = \lambda$ and $(Z_t)_{t\ge 0}$ defines an optimal control.

Suitable candidates for optimal controls were found in [6] and [7] and, for discrete time, in [15]. These strategies are described by four parameters, a, α, β, and b, with $a < \alpha$ and $\beta < b$, of the state space. Start the control strategy by bringing the process into some state in (a, b). Then let the diffusion run uncontrolled until it hits a or b. From a control it by an upwards jump into state α, and from b, by a downwards jump into state β. Repeat this procedure. This clearly defines a control strategy according to Definition 4.1; see Remark 4.1. Let us call such a strategy a *constant boundary strategy* (*CB strategy*).

5. Specification of the problem

Starting in this section, we show optimality of CB strategies in the problem of portfolio optimization with transaction costs discussed in Section 3. Let $\hat{\eta} \in (0, 1)$.

By Proposition 3.1, the uncontrolled diffusion is given by the risky fraction process $(\pi_t)_{t\ge 0}$, with state space (0, 1), such that

$$\mathrm{d}\pi_t = \pi_t(1-\pi_t)(\mu - \sigma^2\pi_t)\,\mathrm{d}t + \pi_t(1-\pi_t)\sigma\,\mathrm{d}W_t. \tag{5.1}$$

Thus, in the notation of Section 4,

$$\tilde{\mu}(x) = x(1-x)(\mu - \sigma^2 x), \qquad \tilde{\sigma}(x) = \sigma x(1-x), \tag{5.2}$$

and the corresponding operator is given by

$$\mathcal{L}v(x) = x(1-x)(\mu - \sigma^2 x)v'(x) + \tfrac{1}{2}\sigma^2 x^2(1-x)^2 v''(x). \tag{5.3}$$

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Proposition 3.1 and a comparison of (3.11) and (4.2) furthermore show that the function h is given by

$$h(x) = x(\mu - \tfrac{1}{2}\sigma^2 x), \qquad x \in (0, 1). \tag{5.4}$$

For $(\pi_t)_{t\geq 0}$, a scale function ρ_c and a speed measure m_c are given, for some constant $c \in (0, 1)$, by

$$\rho_c(x) = \int_c^x \exp\biggl(-2\int_c^z \frac{\tilde{\mu}(y)}{\tilde{\sigma}^2(y)}\,\mathrm{d}y\biggr)\,\mathrm{d}z, \tag{5.5}$$
$$m_c(\mathrm{d}x) = \frac{2}{\rho_c'(x)\tilde{\sigma}^2(x)}\,\mathrm{d}x.$$

A solution, v_c, to the differential equation $\mathcal{L}v + h - \lambda = 0$, with $v_c'(c) = 0$, is given by

$$v_c'(x) = \rho_c'(x)\int_c^x (\lambda - h(y))m_c(\mathrm{d}y) = \rho_c'(x)\int_c^x (\lambda - h(y))\frac{2}{\rho_c'(y)\tilde{\sigma}^2(y)}\,\mathrm{d}y.$$

Since $\rho_a'(x)$ is of the form $f_\rho(a, b)\rho_b'(x)$, $x \in (0, 1)$, for some function f_ρ, we may insert any scale function $\rho(x)$, to obtain

$$v_c'(x) = \rho'(x)\int_c^x (\lambda - h(y))\frac{2}{\rho'(y)\tilde{\sigma}^2(y)}\,\mathrm{d}y. \tag{5.6}$$

The controls ΔZ_n at τ_n (in Section 4) correspond to the increments of the risky fraction, $\eta_n - \pi_{\tau_n}$. It is easier to use the costs with direct dependence on the new fraction η_n, so we switch to costs depending on x and $y = x + z$ using

$$\bar{\Gamma}(x, y) = C(x, y - x),$$

when controls take place. As discussed earlier, the cost structure is given by

$$\bar{\Gamma}(x, y) = \begin{cases} \log\dfrac{1 - \delta - \gamma x}{1 - \gamma y}, & y < \dfrac{x}{1-\delta}, \\[2ex] \log\dfrac{1 - \delta + \gamma x}{1 + \gamma y}, & y \geq \dfrac{x}{1-\delta}, \end{cases}$$

which is continuous at $y = x/(1-\delta)$ with value $\log(1-\delta)$. For fixed x, the derivative of the cost function $\bar{\Gamma}$ has a discontinuity at $y = x/(1-\delta)$ (equivalently, C has a discontinuity at $z = \delta x/(1-\delta)$) which, if $\bar{\Gamma}$ were used, would lead to problems in our later reasoning. We therefore use the following modification of the cost function:

$$\Gamma(x, y) = \begin{cases} \log\dfrac{1 - \delta - \gamma x}{1 - \gamma y}, & y \leq x, \\[2ex] \log\dfrac{1 - \delta + \gamma x}{1 + \gamma y}, & y > x. \end{cases}$$

We can justify this modification as follows. Our original aim was to find a control strategy K maximizing the asymptotic growth rate (for this argument denoted by $\bar{R}(K)$) with respect to costs $\bar{\Gamma}$. It is easily seen that

$$\log\frac{1 - \delta - \gamma x}{1 - \gamma y} \leq \log\frac{1 - \delta + \gamma x}{1 + \gamma y} \quad \text{for } y \leq \frac{x}{1-\delta};$$

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thus,

$$\bar{\Gamma}(x, y) \leq \Gamma(x, y) \quad \text{for all } x, y \in (0, 1),$$

since we changed the cost function on $(x, x/(1-\delta))$ only. Hence, for any control strategy,

$$\bar{R}(K) \leq R(K),$$

the latter denoting the asymptotic growth rate with respect to costs Γ. Assume now that we have found an optimal strategy K^* with respect to Γ, whence

$$\bar{R}(K) \leq R(K) \leq R(K^*).$$

If $\bar{R}(K^*) = R(K^*)$, K^* is then also optimal with respect to $\bar{\Gamma}$. As introduced at the end of Section 4, a CB strategy is given by four parameters a, α, β, and b, where (a, b) is the continuation region for the uncontrolled $(\pi_t)_{t\geq 0}$ and α and β are the new risky fractions after buying and selling, respectively. For a CB strategy K, $\bar{R}(K) = R(K)$ holds if $\alpha \geq a/(1-\delta)$. So if K^* is an optimal CB strategy with respect to R, it is also optimal with respect to $\bar{R}$ if $\alpha \geq a/(1-\delta)$. Our numerical examples at the end of the paper will show that this condition is usually fulfilled. From now on we shall therefore use the costs Γ.

For later reference, we note that

$$\frac{\partial}{\partial y}\Gamma(x, y) = \begin{cases} \dfrac{\gamma}{1-\gamma y}, & y \leq x, \\ -\dfrac{\gamma}{1+\gamma y}, & y > x, \end{cases} \tag{5.7}$$

$$\frac{\partial}{\partial x}\Gamma(x, y) = \begin{cases} \dfrac{\gamma}{1-\delta+\gamma x}, & x < y, \\ -\dfrac{\gamma}{1-\delta-\gamma x}, & x \geq y. \end{cases} \tag{5.8}$$

6. Conditions for the optimality of CB strategies

According to Section 4 and Section 5 we have to find v and λ and a, α, β, and b, $0 < a < \alpha < \beta < b < 1$, such that

$$\mathcal{L}v(x) + h(x) - \lambda \leq 0, \tag{6.1}$$

$$v(y) - v(x) + \Gamma(x, y) \leq 0, \tag{6.2}$$

for all $x, y \in (0, 1)$. Furthermore, we must also have

$$\mathcal{L}v(x) + h(x) - \lambda = 0, \qquad x \in (a, b), \tag{6.3}$$

$$v(\beta) - v(b) + \Gamma(b, \beta) = 0, \tag{6.4}$$

$$v(\alpha) - v(a) + \Gamma(a, \alpha) = 0. \tag{6.5}$$

As pointed out in Section 5, (6.3) can be satisfied by choosing $v(x) = v_c(x)$ on (a, b), where v_c is given by its derivative in (5.6) and is thus unique up to some constant term. We elaborate on the last two conditions; these will be satisfied if, furthermore,

$$v(\beta) - v(x) + \Gamma(x, \beta) = 0 \quad \text{for all } x \geq b,$$

$$v(\alpha) - v(x) + \Gamma(x, \alpha) = 0 \quad \text{for all } x \leq a.$$

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So, for some $c \in (0, 1)$, we shall choose

$$v(x) = \begin{cases} v_c(\alpha) + \Gamma(x, \alpha), & x \le a, \\ v_c(x), & x \in (a, b), \\ v_c(\beta) + \Gamma(x, \beta), & x \ge b. \end{cases} \tag{6.6}$$

In addition, we look for a solution which has the smooth-fit property

$$v_c'(b) = v'(b-) = v'(b+) = \frac{\partial}{\partial x}\Gamma(x, \beta)\bigg|_{x=b}, \tag{6.7}$$

$$v_c'(a) = v'(a+) = v'(a-) = \frac{\partial}{\partial x}\Gamma(x, \alpha)\bigg|_{x=a}. \tag{6.8}$$

Furthermore, the mappings

$$v(x) - v(b) + \Gamma(b, x), \qquad v(x) - v(a) + \Gamma(a, x)$$

will have local maxima at β and at α, respectively, translating into

$$v_c'(\beta) = -\frac{\partial}{\partial y}\Gamma(b, y)\bigg|_{y=\beta}, \qquad v_c'(\alpha) = -\frac{\partial}{\partial y}\Gamma(a, y)\bigg|_{y=\alpha}. \tag{6.9}$$

Denoting v_c' by g, we shall determine g (i.e. c), λ, a, α, β, and b using the following conditions, where we have written the explicit expressions for the derivatives:

$$g(b) = -\frac{\gamma}{1 - \delta - \gamma b}, \tag{6.10}$$

$$g(\beta) = -\frac{\gamma}{1 - \gamma\beta}, \tag{6.11}$$

$$\int_\beta^b g(x)\,\mathrm{d}x = \Gamma(b, \beta), \tag{6.12}$$

$$g(a) = \frac{\gamma}{1 - \delta + \gamma a}, \tag{6.13}$$

$$g(\alpha) = \frac{\gamma}{1 + \gamma\alpha}, \tag{6.14}$$

$$\int_a^\alpha g(x)\,\mathrm{d}x = -\Gamma(a, \alpha). \tag{6.15}$$

In detail, (6.10) follows from (6.7), (6.11) and (6.14) follow from (6.9), (6.12) follows from (6.4), (6.13) follows from (6.8), and (6.15) follows from (6.5), using (5.7) and (5.8).

Here, according to (5.6), g will be a function of the form

$$g(x) \equiv g(x; c, \lambda) = \rho'(x)\int_c^x (\lambda - h(y))\frac{2}{\rho'(y)\sigma^2(y)}\,\mathrm{d}y,$$

where c and λ will have to be chosen appropriately and ρ is any scale function of the form given in (5.5). We point out that

$$g'(x) = -\frac{2}{\sigma^2(x)}(h(x) + \mu(x)g(x) - \lambda). \tag{6.16}$$

The following arguments will use the properties of g, which in turn depend on the properties of the diffusion.

Different scale functions arise according to whether $\hat{\eta} = \frac{1}{2}$ or $\hat{\eta} \neq \frac{1}{2}$. We shall discuss the case $\hat{\eta} > \frac{1}{2}$ in detail.

7. Existence of optimal CB strategies

We consider the case $\hat{\eta} \in (\frac{1}{2}, 1)$. This implies that $R_1 = \mu - \sigma^2/2 > 0 = R_0$. Hence, by (3.12), we are only interested in an optimal growth rate $\lambda > R_1$ since otherwise the buy-and-hold strategy performs better. On the other hand, due to the transaction costs we expect that $\lambda < \hat{R}$, so we are searching for $\lambda \in (R_1, \hat{R})$. We shall frequently use the relation

$$(2\hat{\eta} - 1)\sigma^2 = 2R_1.$$

According to the argument preceding (5.6), we can choose

$$\rho(x) = -\frac{1}{2\hat{\eta} - 1}\left(\frac{1-x}{x}\right)^{2\hat{\eta}-1}$$

as our scale function. Then

$$\rho'(x) = \frac{1}{x(1-x)}\left(\frac{1-x}{x}\right)^{2\hat{\eta}-1}$$

and, for $c \in (0, 1)$,

$$g(x; c, \lambda) = \frac{((1-x)/x)^{2\hat{\eta}-1}}{x(1-x)R_1}\left\{(\lambda - R_1 x)\left(\frac{x}{1-x}\right)^{2\hat{\eta}-1} - (\lambda - R_1 c)\left(\frac{c}{1-c}\right)^{2\hat{\eta}-1}\right\}. \quad (7.1)$$

To simplify the notation, we do not always note the dependency of g on c and λ, i.e. we may write $g(x)$ instead of $g(x; c, \lambda)$. By (6.16),

$$g'(x) = \frac{\partial}{\partial x} g(x; c, \lambda) = -\frac{x(2\hat{\eta} - x)\sigma^2 + 2x(1-x)(\hat{\eta} - x)g(x)\sigma^2 - 2\lambda}{x^2(1-x)^2\sigma^2}. \quad (7.2)$$

We shall frequently use the mapping

$$\varphi\colon (0, 1) \to \mathbb{R}, \qquad \varphi(x) = x\left(\mu - \frac{\sigma^2}{2}x\right). \quad (7.3)$$

Note that

$$\varphi(0) = 0, \qquad \varphi(\hat{\eta}) = \hat{R} > R_1 = \varphi(2\hat{\eta} - 1) = \varphi(1). \quad (7.4)$$

We shall denote the partial derivatives of $g \equiv g(x; c, \lambda)$ by g', g_c, and g_λ (in an obvious notation).

Lemma 7.1. *Suppose that* $c \in (0, 1)$.

(i) $g(c) = 0$, *and* $g'(c) < 0$ *if and only if* $\lambda < \varphi(c)$.

(ii) *For fixed* c, g *is strictly increasing in* λ *on* $(c, 1)$ *and strictly decreasing on* $(0, c)$.

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(iii) *For fixed* λ, g *is strictly increasing in* c *on* $\{c \in (0,1): \lambda < \varphi(c)\}$ *and strictly decreasing in* c *on* $\{c \in (0,1): \lambda > \varphi(c)\}$.

Proof. The definition of g and (7.2) imply (i). For (ii), it suffices to note only that

$$g_\lambda(x; c, \lambda) = \frac{1}{x(1-x)R_1}\left\{1 - \left(\frac{c(1-x)}{x(1-c)}\right)^{2\hat{\eta}-1}\right\} \tag{7.5}$$

and that $x \mapsto ((1-x)/x)^{2\hat{\eta}-1}$ is strictly decreasing on $(0,1)$. Finally, (iii) is obvious from

$$g_c(x; c, \lambda) = \frac{([(1-x)c]/[x(1-c)])^{2\hat{\eta}-1}}{x(1-x)c(1-c)}\frac{2}{\sigma^2}(\varphi(c) - \lambda). \tag{7.6}$$

In Lemma 7.2 we discuss the detailed behaviour of g in the case $\lambda > R_1$, as this will be essential in our arguments.

Lemma 7.2. *Suppose that* $c \in (0,1)$ *and* $\lambda > R_1$.

(i) $\lim_{x\to 0} g(x) = -\infty$ *and* $\lim_{x\to 1} g(x) = \infty$.

(ii) *If* $\lambda < \varphi(c)$ *then* g *has three roots,* $x_1 \equiv x_1(c)$, c, *and* $x_2 \equiv x_2(c)$, *satisfying* $x_1 < c < x_2$, $g'(x_1) > 0$, $g'(c) < 0$, *and* $g'(x_2) > 0$.

(iii) *Suppose that* $\lambda = \varphi(c)$. *Then* $g'(c) = 0$ *and* g *has at most two roots,* $x_1 \equiv x_1(c)$ *and* $x_2 \equiv x_2(c)$.

- *If* $c > \hat{\eta}$ *then* $x_1 < c = x_2$ *and* $g'(x_1) > 0$.
- *If* $c < \hat{\eta}$ *then* $x_1 = c < x_2$ *and* $g'(x_2) > 0$.

Proof. For (i) we look at the representation

$$g(x) = \frac{1}{x(1-x)R_1}\left\{(\lambda - R_1 x) - (\lambda - R_1 c)\left(\frac{c(1-x)}{x(1-c)}\right)^{2\hat{\eta}-1}\right\}.$$

The claim follows because $R_1 > 0$, $\lambda > R_1 x$ for all $x \in (0,1)$, $2\hat{\eta} - 1 > 0$, and

$$\lim_{x\searrow 0}\frac{1-x}{x} = \infty, \qquad \lim_{x\nearrow 1}\frac{1-x}{x} = 0.$$

For (ii) and (iii) we have to look at the behaviour of

$$f(x) := (\lambda - R_1 x)\left(\frac{x}{1-x}\right)^{2\hat{\eta}-1} - (\lambda - R_1 c)\left(\frac{c}{1-c}\right)^{2\hat{\eta}-1}, \qquad x \in (0,1).$$

By (7.1), the signs of g and f are the same. The derivative is

$$\begin{aligned} f'(x) &= \frac{1}{x(1-x)}\left(\frac{x}{1-x}\right)^{2\hat{\eta}-1}((\lambda - R_1 x)(2\hat{\eta} - 1) - R_1 x(1-x)) \\ &= \frac{2R_1}{\sigma^2 x(1-x)}\left(\frac{x}{1-x}\right)^{2\hat{\eta}-1}(\lambda - \varphi(x)). \end{aligned}$$

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Since φ is a polynomial of degree two, f' has at most two roots; hence, f has at most two extrema. By (7.4), f has two extrema if and only if $\lambda \in (R_1, \hat{R})$. Furthermore, for $\lambda > R_1$,

$$\lim_{x\to 0} f(x) = -(\lambda - R_1 c)\left(\frac{c}{1-c}\right)^{2\hat{\eta}-1} < 0, \qquad \lim_{x\to 1} f(x) = \infty.$$

This shows that g has at most three roots. In the case $\lambda < \varphi(c)$, we have $g'(c) < 0$ and g thus has exactly three roots. The other claims in (ii) follow easily from this. In the case $\lambda = \varphi(c)$, we have $f'(c) = 0$. Inspection of the second derivative,

$$f''(x) = \frac{2R_1}{\sigma^2 x^2 (1-x)^2}\left(\frac{x}{1-x}\right)^{2\hat{\eta}-1}(2(\lambda - \varphi(x))(\hat{\eta} - 1 + x) - \sigma^2 x(1-x)(\hat{\eta} - x)),$$

shows that in this case we have a minimum at c if $c > \hat{\eta}$ and a maximum at c if $c < \hat{\eta}$, which proves (iii).

For the moment we keep c fixed and consider g on $[c, 1)$. For $\lambda < \varphi(c)$, according to Lemma 7.2 g will possibly have the right behaviour to satisfy conditions (6.10)–(6.15). So in the following we shall assume that $\lambda \in (R_1, \varphi(c))$. Then the condition $\lambda > R_1$ can only be satisfied if $\varphi(c) > R_1$. Since $\varphi(c) > R_1$ if and only if $c \in (2\hat{\eta} - 1, 1)$, we are led to consider

$$c \in (2\hat{\eta} - 1, 1), \qquad \lambda \in (R_1, \varphi(c)).$$

By Lemma 7.1(ii), g is strictly increasing in λ on $[c, 1)$, and the following lemma will show that

$$\bar{\lambda}_1 \equiv \bar{\lambda}_1(c) = \sup\left\{\lambda > R_1 : \inf_{x\in[c,1)} g(x; c, \lambda) \le -\frac{\gamma}{1-\delta-\gamma x}\right\}$$

is well defined. Note that $g \ge 0$ for $\lambda \ge \hat{R}$.

Lemma 7.3. (i) $\lim_{\lambda\to R_1} \min_{x\in(c,1)} g(x; c, \lambda) = -\infty$; *hence,* $\bar{\lambda}_1 > R_1$.

(ii) *For* $\lambda \in (R_1, \bar{\lambda}_1)$, *unique* b *and* β *exist satisfying* $c < \beta < b < 1$,

$$g(\beta) = -\frac{\gamma}{1-\gamma\beta}, \qquad g(b) = -\frac{\gamma}{1-\delta-\gamma b},$$

and

$$g'(\beta) + \frac{\gamma^2}{(1-\gamma\beta)^2} < 0, \qquad g'(b) + \frac{\gamma^2}{(1-\delta-\gamma\beta)^2} > 0.$$

(iii) *As functions of* λ, $\beta \equiv \beta(\lambda)$ *is strictly increasing and* $b \equiv b(\lambda)$ *is strictly decreasing, and both are continuous and differentiable on* $(R_1, \bar{\lambda}_1)$.

(iv) *For* $\lambda = \bar{\lambda}_1$, *unique* $\bar{b}$ *and* $\bar{\beta}$ *with* $c < \bar{\beta} < \bar{b} < 1$ *exist such that (6.10) and (6.11) hold;*

$$g'(\bar{\beta}) + \frac{\gamma^2}{(1-\gamma\bar{\beta})^2} < 0, \qquad g'(\bar{b}) + \frac{\gamma^2}{(1-\delta-\gamma\bar{b})^2} = 0;$$
$$-\frac{\gamma}{1-\delta-\gamma\bar{b}} \le g(x; c, \bar{\lambda}(c)) \le -\frac{\gamma}{1-\gamma\bar{\beta}}$$

for $x \in (\bar{\beta}, \bar{b})$; *and*

$$\bar{\lambda}_1 = \hat{R} - \frac{\sigma^2}{2}\left(\frac{\gamma}{1-\delta-\gamma\bar{b}}\bar{b}(1-\bar{b}) + \hat{\eta} - \bar{b}\right)^2.$$

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Proof. We write

$$g(x, c, R_1 + \varepsilon) = f_0(x)(\varepsilon f_1(x) + R_1 f_2(x)),$$

where

$$f_0(x) = \frac{((1-x)/x)^{2\hat{\eta}-1}}{x(1-x)R_1},$$
$$f_1(x) = \left(\frac{x}{1-x}\right)^{2\hat{\eta}-1} - \left(\frac{c}{1-c}\right)^{2\hat{\eta}-1},$$
$$f_2(x) = (1-x)\left(\frac{x}{1-x}\right)^{2\hat{\eta}-1} - (1-c)\left(\frac{c}{1-c}\right)^{2\hat{\eta}-1}.$$

Now observe that $f_0(x) \to \infty$, $f_1(x) \to \infty$, and $f_2(x) \to -(1-c)(c/(1-c))^{2\hat{\eta}-1} < 0$ as $x \to 1$. Hence, for any $k < 0$ we can choose an $\varepsilon > 0$ such that $\min_{x\in(c,1)} g(x, c, R_1+\varepsilon) < k$. So (i) follows.

From (i), Lemma 7.1(i), and Lemma 7.2(i), we know that $g(x) = -\gamma/(1-\gamma x)$ has at least two solutions, β_1 and β_2, $\beta_1 < \beta_2$, in $(c, 1)$ with

$$g'(\beta_1) + \frac{\gamma^2}{(1-\gamma\beta_1)^2} \le 0, \qquad g'(\beta_2) + \frac{\gamma^2}{(1-\gamma\beta_2)^2} \ge 0,$$

and at least one solution, β_0, in $(0, c)$ where the derivative is positive. Now consider the function

$$\bar{f}\colon x \mapsto -\frac{x\sigma^2(2\hat{\eta}-x) - 2x(1-x)\sigma^2(\hat{\eta}-x)\gamma/(1-\gamma x) - 2\lambda}{x^2(1-x)^2\sigma^2} + \frac{\gamma^2}{(1-\gamma x)^2},$$

which coincides at $x = \beta_0, \beta_1, \beta_2$, and at any other solution, with the derivative of $g(x) + \gamma/(1-\gamma x)$; see (7.2). A simple computation shows that

$$\bar{f}(x) = \frac{p_2(x)}{x^2(1-x)^2\sigma^2(1-\gamma x)^2},$$

where p_2 is a polynomial of degree two. Note that the terms of degree three and degree four cancel. By arguments similar to those in the proof of Lemma 7.2, this shows that $g(x) = -\gamma/(1-\gamma x)$ has no other solutions, and that the inequalities for the derivatives are strict. So $\beta = \beta_1$ has the desired properties. The same argument applies to b, only we have to take the biggest of the three roots. This proves (ii).

The argument in (ii) also shows that b and β are uniquely determined. By the implicit function theorem, for $\beta \equiv \beta(\lambda)$ and $b \equiv b(\lambda)$ we hence obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\beta(\lambda) = \frac{-g_\lambda(\beta)}{g'(\beta) + \gamma^2/(1-\gamma\beta)^2} > 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda}b(\lambda) = \frac{-g_\lambda(b)}{g'(b) + \gamma^2/(1-\delta-\gamma b)^2} < 0.$$

The first part of (iv) follows similarly to (ii), when we observe that g only touches $x \mapsto -\gamma/(1-\delta-\gamma x)$ and, hence, that the two possible roots greater than c coincide. So we have

$$g'(\bar{b}) = -\frac{\gamma^2}{(1-\delta-\gamma\bar{b})^2} = -g^2(\bar{b}).$$

Using representation (7.2) of g', $2R_1 = \sigma^2(2\hat{\eta}-1)$, and $\hat{R} = \frac{1}{2}\mu^2/\sigma^2$, we solve for $\bar{\lambda}$ to obtain the final equation in (iv).

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For fixed c we have to solve $\psi(c,\lambda)=0$, where

$$\psi(c,\lambda) := \int_{\beta(\lambda)}^{b(\lambda)} g(x;c,\lambda)\,\mathrm{d}x - \log\frac{1-\delta-\gamma b(\lambda)}{1-\gamma\beta(\lambda)}.$$

Lemma 7.4. *There exists a unique* $c_1 \in (2\hat{\eta}-1,\hat{\eta})$ *such that, for each* $c \in [c_1,1)$*, the equation* $\psi(c,\lambda)=0$ *has a unique solution,* $\lambda_1(c) \in (R_1,\bar{\lambda}_1(c))$*, with* $\lambda_1(c) < \varphi(c)$ *for* $c > c_1$ *and* $\lambda_1(c_1)=\varphi(c_1)$*. As a function of* c*,* λ_1 *is continuous and strictly decreasing on* $(c_1,1)$*.*

Proof. From Lemma 7.3(i), $\psi(c,\lambda)\to-\infty$ as $\lambda\to R_1$, and, from Lemma 7.3(iv),

$$\psi(c,\lambda)\to\psi(c,\bar{\lambda}_1)=\int_{\bar{\beta}}^{\bar{b}} g(x;c,\bar{\lambda}_1)\,\mathrm{d}x-\log\frac{1-\delta-\gamma\bar{b}}{1-\gamma\bar{\beta}}$$

as $\lambda\to\bar{\lambda}_1$. However, $-\gamma/(1-\delta-\gamma x)\le g(x;c,\bar{\lambda}_1)\le-\gamma/(1-\gamma x)$ on $(\bar{\beta},\bar{b})$; hence,

$$\log\frac{1-\delta-\gamma\bar{b}}{1-\delta-\gamma\bar{\beta}}\le\int_{\bar{\beta}}^{\bar{b}} g(x;c,\bar{\lambda}_1)\,\mathrm{d}x\le\log\frac{1-\gamma\bar{b}}{1-\gamma\bar{\beta}}.$$

Therefore,

$$\psi(c,\bar{\lambda}_1)\ge-\log\frac{1-\delta-\gamma\bar{\beta}}{1-\gamma\bar{\beta}}>0.$$

So a solution $\lambda_1\equiv\lambda_1(c)$ exists. For some $x_0\in(0,1)$, write

$$G(x,c,\lambda)=\int_{x_0}^{x} g(y;c,\lambda)\,\mathrm{d}y,$$

and for its derivatives write G_x, G_c, and G_λ. Then

$$\psi(c,\lambda)=G(b(\lambda),c,\lambda)-G(\beta(\lambda),c,\lambda)-\log\frac{1-\delta-\gamma b(\lambda)}{1-\gamma\beta(\lambda)}$$

and, using Lemma 7.3(iii),

$$\begin{aligned}\psi_\lambda(c,\lambda):=\frac{\partial}{\partial\lambda}\psi(c,\lambda)&=\bigg(g(b(\lambda);c,\lambda)-\frac{\gamma}{1-\delta-\gamma b(\lambda)}\bigg)b'(\lambda)\\&\quad-\bigg(g(\beta(\lambda);c,\lambda)-\frac{\gamma}{1-\gamma\beta(\lambda)}\bigg)\beta'(\lambda)\\&\quad+G_\lambda(b(\lambda),c,\lambda)-G_\lambda(\beta(\lambda),c,\lambda)\\&=\int_{\beta(\lambda)}^{b(\lambda)} g_\lambda(x;c,\lambda)\,\mathrm{d}x>0,\end{aligned}$$

the latter due to Lemma 7.1(ii). Hence, ψ is strictly increasing in λ, so the solution λ_1 is unique.

Let us now consider b and β also to be functions of c. Suppose that some c and λ are given with $\lambda<\varphi(c)$. As in the proof of Lemma 7.3, we can show that $\beta\equiv\beta(c,\lambda)$ and $b\equiv b(c,\lambda)$ are differentiable, with

$$\beta_c(c,\lambda)=\frac{-g_c(\beta;c,\lambda)}{g'(\beta;c,\lambda)+\gamma^2/(1-\gamma\beta(c,\lambda))^2}>0,$$

$$b_c(c,\lambda)=\frac{-g_c(b;c,\lambda)}{g'(b;c,\lambda)+\gamma^2/(1-\delta-\gamma b(c,\lambda))^2}<0,$$

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since, by Lemma 7.1(iii), $g_c(x; c, \lambda) > 0$ if $\lambda < \varphi(c)$. Therefore,

$$\begin{aligned}\psi_c(c, \lambda) := \frac{\partial}{\partial c}\psi(c, \lambda) &= \left(g(b(c, \lambda); c, \lambda) - \frac{\gamma}{1 - \delta - \gamma b(c, \lambda)}\right) b_c(c, \lambda) \\ &\quad - \left(g(\beta(c, \lambda); c, \lambda) - \frac{\gamma}{1 - \gamma\beta(c, \lambda)}\right)\beta_c(c, \lambda) \\ &\quad + G_c(b(c, \lambda), c, \lambda) - G_c(\beta(c, \lambda), c, \lambda) \\ &= \int_{\beta(c,\lambda)}^{b(c,\lambda)} g_c(x; c, \lambda)\, \mathrm{d}x > 0.\end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}c}\lambda_1(c) = -\frac{\psi_c(c, \lambda)}{\psi_\lambda(c, \lambda)} < 0$$

if $\lambda_1(c) < \varphi(c)$. By uniqueness, λ_1 is also continuous in c. So we have shown that λ_1 is strictly decreasing on $\{c\colon \lambda_1(c) < \varphi(c)\}$. However, by Lemma 7.2(iii), we know that for $c > \hat{\eta}$ no solution to (6.12) exists for $\lambda = \varphi(c)$; hence, $\bar{\lambda}_1(c) < \varphi(c)$, implying that $\lambda_1(c) < \varphi(c)$. Furthermore, for $c = \hat{\eta}$, the expression for $\bar{\lambda}_1$ in Lemma 7.3 shows that $\bar{\lambda}_1(\hat{\eta}) < \hat{R}$. So in this case $\lambda_1(\hat{\eta}) < \varphi(\hat{\eta})$ also, since $\varphi(\hat{\eta}) = \hat{R}$. Thus, $[\hat{\eta}, 1) \subset \{c\colon \lambda_1(c) < \varphi(c)\}$. We proved above that $\lambda_1(c)$ is strictly decreasing as long as $\lambda_1(c) < \varphi(c)$. On the other hand, φ is strictly increasing on $(2\hat{\eta} - 1, \hat{\eta}]$, with $\varphi(2\hat{\eta} - 1) = R_1$. Hence, λ_1 and φ intersect at some $c_1 \in (2\hat{\eta} - 1, \hat{\eta})$.

Bringing together Lemma 7.3 and Lemma 7.4, we see that for $c \in (c_1, 1)$ we have found a $\lambda_1 \equiv \lambda_1(c)$ such that conditions (6.10)–(6.12) hold for $\beta = \beta(c, \lambda_1(c))$ and $b = b(c, \lambda_1(c))$.

We can now proceed in the same way to find α and a such that (6.13)–(6.15) hold. We thus consider the interval $(0, c)$ and define

$$\bar{\lambda}_2(c) = \sup\left\{\lambda > 0\colon \max_{x \in (0,c]} g(x; c, \lambda) \geq \frac{\gamma}{1 - \delta + \gamma x}\right\}.$$

We obtain the following result.

Lemma 7.5. (i) *For each* $c \in (2\hat{\eta} - 1, 1)$ *and* $\lambda \in (0, \bar{\lambda}_2(c))$, *there exist unique* $a \equiv a(c, \lambda)$ *and* $\alpha \equiv \alpha(c, \lambda)$ *satisfying* $2\hat{\eta} - 1 < a < \alpha < c$ *and*

$$g(\alpha) = \frac{\gamma}{1 + \gamma\alpha}, \qquad g(a) = \frac{\gamma}{1 - \delta + \gamma a}.$$

(ii) *There exists a unique* $c_2 \in (\hat{\eta}, 1)$ *such that, for each* $c \in (2\hat{\eta} - 1, c_2]$, *the equation*

$$\tilde{\psi}(c, \lambda) := \int_{a(c,\lambda)}^{\alpha(c,\lambda)} g(x; c, \lambda)\, \mathrm{d}x + \Gamma(a(c, \lambda), \alpha(c, \lambda)) = 0$$

has a unique solution $\lambda_2(c) \in (0, \bar{\lambda}_2(c))$ *with* $\lambda_2(c) < \varphi(c)$ *for* $c < c_2$ *and* $\lambda_2(c_2) = \varphi(c_2)$. *As a function of c,* λ_2 *is continuous and strictly increasing on* $(2\hat{\eta} - 1, c_2)$. *Furthermore,* $\lambda_2(c_2) > R_1$.

To prove Lemma 7.5, we can proceed exactly as in Lemmas 7.3 and 7.4; we omit the details.

Proposition 7.1. *There exist unique* λ, a, α, b, *and* β *which satisfy (6.10)–(6.15).*

Proof. We have a solution to (6.10)–(6.15) as soon as we find a c^* with $\lambda_1(c^*) = \lambda_2(c^*)$. By Lemmas 7.4 and 7.5(ii), there exist $c_1 \in (2\hat{\eta} - 1, \hat{\eta})$ and $c_2 \in (\hat{\eta}, 1)$ with $\lambda_1(c_1) = \varphi(c_1)$ and $\lambda_2(c_2) = \varphi(c_2)$. Furthermore, λ_1 is strictly decreasing on $(c_1, 1)$ and λ_2 is strictly increasing on $(2\hat{\eta} - 1, c_2)$. Since $\lambda_2(c_1) < \varphi(c_1) = \lambda_1(c_1)$ and $\lambda_1(c_2) < \varphi(c_2) = \lambda_2(c_2)$, there exists a unique $c^* \in (c_1, c_2)$ satisfying $\lambda_1(c^*) = \lambda_2(c^*)$. Uniqueness of c^* follows from uniqueness in each step.

We shall denote the optimal values corresponding to c^* simply by λ, a, α, b, and β. They provide our solutions to (6.10)–(6.15).

We define v by choosing a constant k freely and setting

$$v(x) = \begin{cases} k + \Gamma(x, \alpha), & x \le a, \\ v(a) + \displaystyle\int_a^x g(y; c^*, \lambda)\,\mathrm{d}y, & a < x \le b, \\ v(\beta) + \Gamma(x, \beta), & x > b. \end{cases}$$

By (6.15), we have $k = v(\alpha)$. Obviously $v'(x)$ is greater than, equal to, or less than 0 for x less than, equal to, or greater than c^*, respectively. Furthermore, this defines a bounded function in $H^2(0, 1)$ with bounded derivatives. Hence, (A3) and the finiteness and martingale conditions in Theorem 4.1 are fulfilled.

To derive the conditions (6.10)–(6.15) we used some local arguments. To prove optimality we have to show that the inequalities (A1) and (A2) hold globally.

Theorem 7.1. *For the parameters λ, a, α, b, and β in Proposition 7.1, the CB strategy with parameters a, α, b, and β is an optimal impulse control strategy with asymptotic growth rate λ for the costs Γ. If $\alpha > a/(1-\delta)$ then the solution is optimal for the original problem with costs $\bar{\Gamma}$.*

Proof. To prove optimality it only remains to show that

$$\begin{aligned} \mathcal{L}v(x) + h(x) - \lambda &\le 0, & x &< a, \\ \mathcal{L}v(x) + h(x) - \lambda &\le 0, & x &> b, & (7.7) \\ v(y) - v(x) + \Gamma(x, y) &\le 0, & y &\le x, & (7.8) \\ v(y) - v(x) + \Gamma(x, y) &\le 0, & y &> x. \end{aligned}$$

We shall only show (7.7) and (7.8), the other two following similarly.

According to our construction,

$$g'(b) = -\frac{2}{(1-b)^2 b^2}[h(b) + \mu(b)g(b) - \lambda] > 0,$$

which implies that

$$h(x) + \mu(x)v'(x) - \lambda < 0 \quad \text{for } x > b$$

and, thus, that

$$\tfrac{1}{2}\sigma^2(x)v''(x) + \mu(x)v'(x) + h(x) - \lambda < 0 \quad \text{for } x > b$$

(which is (7.7)), since

$$v''(x) = -\frac{\gamma^2}{(1-\delta-\gamma x)^2} < 0, \qquad x > b.$$

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We write (7.8) equivalently as

$$\int_y^x \left(v'(z) - \frac{-\gamma}{1-\delta-\gamma z}\right) \mathrm{d}z + \log \frac{1-\gamma y}{1-\delta-\gamma y} \geq 0$$

for $y \leq x$. Note that, by the discussion in Lemma 7.2,

$$v'(x) - \frac{-\gamma}{1-\delta-\gamma x} \begin{cases} = 0, & x \geq b, \\ < 0, & b_1 < x < b, \\ \geq 0, & x \leq b_1, \end{cases}$$

and that $\log[(1-\gamma x)/(1-\delta-\gamma x)]$ is increasing in x. Here b_1 denotes the first root of $g(x) = -\gamma/(1-\delta-\gamma x)$ greater than c; cf. the proof of Lemma 7.3. Thus, it is sufficient to show that

$$\tilde{f}(x) := \int_x^b \left(g(z) - \frac{-\gamma}{1-\delta-\gamma z}\right) \mathrm{d}z + \log \frac{1-\gamma x}{1-\delta-\gamma x} \geq 0$$

for all x, $a \leq x \leq b$. We compute

$$\begin{aligned} \tilde{f}'(x) &= -\left(g(x) - \frac{-\gamma}{1-\delta-\gamma x}\right) + \left(\frac{\gamma}{1-\delta-\gamma x} - \frac{\gamma}{1-\gamma x}\right) \\ &= -\left(g(x) - \frac{-\gamma}{1-\gamma x}\right). \end{aligned}$$

Hence, $\tilde{f}'(x) < 0$ for $x < \beta$ and $\tilde{f}'(x) > 0$ for $x > \beta$. Since $\tilde{f}(\beta) = 0$ due to our construction, the above inequality follows.

Remark 7.1. The case $\hat{\eta} < \frac{1}{2}$ can be solved analogously using the same scale function as in the case $\hat{\eta} > \frac{1}{2}$. In the case $\hat{\eta} = \frac{1}{2}$, we have

$$\mu(x) = x(1-x)(\tfrac{1}{2} - x)\sigma^2, \qquad \sigma(x) = x(1-x)\sigma, \qquad h(x) = \tfrac{1}{2}x(1-x)\sigma^2.$$

We can use the scale function

$$\rho(x) = \log \frac{x}{1-x} \quad \text{with} \quad \rho'(x) = \frac{1}{x(1-x)}$$

and, hence, consider the function

$$g(x) = \frac{1}{x(1-x)} \int_c^x \left(\frac{\lambda}{\sigma^2} - \frac{1}{2} y(1-y)\right) \frac{2}{y(1-y)} \, \mathrm{d}y$$

with derivative

$$g'(x) = -\frac{2}{x^2(1-x)^2} \left(\frac{1}{2}x(1-x) + x(1-x)\left(\frac{1}{2} - x\right) g(x) - \frac{\lambda}{\sigma^2}\right).$$

From this point, this case can be treated in a manner similar to the case $n > \frac{1}{2}$.

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8. Short selling and borrowing

We shall now look at the cases $\hat{\eta} < 0$ and $\hat{\eta} > 1$, respectively corresponding to $\mu < 0$ (short selling) and $\mu > \sigma^2$ (borrowing). As pointed out in Remark 3.1, in the first case we consider strategies with $\pi_t < 0$, to be compared with the pure-bond strategy, and in the second case we consider strategies with $\pi_t > 1$, to be compared with the pure-stock strategy.

Looking at the costs $\bar{\Gamma}$ in (3.10), we see that these costs are well defined and that we can always liquidate the position in the stock with strictly positive new wealth as long as the risky fraction process $(\pi_t)_{t\geq 0}$ stays in the *solvency region*

$$\mathcal{S} = \left(-\frac{1-\delta}{\gamma}, \frac{1-\delta}{\gamma}\right).$$

For an *admissible* NRF strategy $(\tau_n, \eta_n)_{n\in\mathbb{N}_0}$, we now require $\eta_n \in \mathcal{S}$ and, for the controlled processes, $V_t > 0$ and $\pi_t \in \mathcal{S}$ for all $t \geq 0$, the latter translating to a condition on the stopping times.

As noted in Remark 3.1, the uncontrolled risky fraction process is explosive, which leads to the necessity of some minor modifications in the arguments corresponding to those in Section 4. This section is structured as follows. First, we present these modifications, in Remark 8.1. Then we show that the results of Section 7 may be carried over to the respective cases of short selling and borrowing, the first case being treated in some detail and the second in condensed form, in Remark 8.2.

Remark 8.1. Consider the general situation of Section 4, but allow for an explosion. Fix an open interval J of the state space such that, for any starting point in J, the first exit time from J is strictly less than the explosion time. We consider control strategies $(Z_t)_{t\geq 0}$ as in Section 4, but for an admissible control strategy we additionally require that the controlled process not leave J. Hence, $X_t + \Delta Z_t \in J$ for all $t \geq 0$. So such a control strategy has to satisfy $X_{\tau_n} + \Delta Z_{\tau_n} \in J$, and the next control has to take place before the process X_t, $t > \tau_n$, leaves J.

With I replaced by J in (A1)–(A3) of Section 4, Theorem 4.1 remains valid for admissible control strategies, as does Remark 4.2. We may thus use these results to derive optimal admissible strategies.

8.1. Short selling

We now consider the case $\hat{\eta} < 0$ in some detail. Starting with $\pi_0 < 0$, the uncontrolled diffusion (5.1) stays in $(-\infty, 0)$ up to its explosion time and we require that the controlled diffusion stay in $(-(1-\delta)/\gamma, 0)$. We thus follow the procedure in Remark 8.1 with

$$J = \left(-\frac{1-\delta}{\gamma}, 0\right).$$

A good candidate for an optimal admissible strategy is a CB strategy with parameters satisfying

$$-\frac{1-\delta}{\gamma} < a < \alpha < \beta < b < 0,$$

and we now show that this may be verified using arguments similar to those in the earlier sections.

As in Section 5, we may use the modification Γ of the cost function. The corresponding inequality, $\bar{\Gamma}(x, y) \leq \Gamma(x, y)$, holds for all $x, y \in J$. For the uncontrolled diffusion before

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explosion, the coefficients $\tilde{\mu}$ and $\tilde{\sigma}$, the operator $\mathcal{L}$, and the function h are respectively the same as in (5.2), (5.3), and (5.4). These functions are now considered on $(-\infty, 0)$. As in Section 6, we use the approach of (6.1)–(6.5) with a suitable v of the form in (6.6), constructed via the speed measure and the scale function, whose derivative equals 0 at $c < 0$. Again, g denotes this derivative. Since the variable x is less than 0 we write $g(x) \equiv g(x; c, \lambda)$ (see (7.1)) in the form

$$g(x) = \frac{1}{R_1 x(1-x)} \left(\frac{1-x}{-x} \right)^{2\hat{\eta}-1} \tilde{f}(x), \tag{8.1}$$

where

$$\tilde{f}(x) = (\lambda - R_1 x) \left(\frac{-x}{1-x} \right)^{2\hat{\eta}-1} - (\lambda - R_1 c) \left(\frac{-c}{1-c} \right)^{2\hat{\eta}-1}$$

and $R_1, 2\hat{\eta} - 1 < -1$. The representations of g in the proof of Lemma 7.2 and those of the derivatives g', g_λ, and g_c in (7.2), (7.5), and (7.6), respectively, are still valid. Again it is convenient to use φ as defined in (7.3). Since $\varphi(0) = \varphi(2\hat{\eta}) = 0 = R_0$ and $\varphi(c) > 0$ for $c \in (2\hat{\eta}, 0)$, we expect from the argument in Section 7 that we can find a solution with $c \in (2\hat{\eta}, 0)$ and $\lambda \in (0, \varphi(c))$.

Lemma 8.1. *Suppose that* $c \in (2\hat{\eta}, 0)$.

(i) $\lim_{x \to -\infty} g(x) = 0$ *and*

$$\lim_{x \nearrow 0} g(x) = \begin{cases} \infty & \text{if } \lambda > 0, \\ -1 & \text{if } \lambda = 0, \\ -\infty & \text{if } \lambda < 0. \end{cases}$$

(ii) *If* $\lambda \in (0, \varphi(c))$ *then* g *has three roots,* x_1, c, *and* x_2, $x_1 < c < x_2 < 0$, *satisfying* $g'(x_1) > 0$, $g'(c) < 0$, *and* $g'(x_2) > 0$.

(iii) *If* $\lambda \in (0, \varphi(c))$ *then* $g(x)$ *has two extrema: a maximum attained at* $\bar{x} \in (2\hat{\eta}, c)$ *and a minimum attained at* $\underline{x} \in (c, 0)$.

(iv) *For each* $c \in (2\hat{\eta}, 0)$, *we have* $\inf_{\lambda \in (0, \varphi(c))} \min_{x \in (c,0)} g(x; c, \lambda) \leq -1$.

(v) *For each* $x < 0$, *we have* $\lim_{c \nearrow 0} g(x; c, 0) = \infty$.

Proof. Most of the properties can be verified as in Lemmas 7.1 and 7.2. One difference is that $g(x) \to 0$ as $x \to -\infty$. By looking at the derivative of $\tilde{f}$ (see (8.1)), we can again show that $\tilde{f}$ has three roots, yielding (ii), and two extrema. The main difference in the argument is that we have to show that the maximum is attained at $\bar{x} > 2\hat{\eta}$. From the properties of g derived above we know that a unique $\bar{x} \in (x_1, c)$ exists. Furthermore, from (7.2), we obtain

$$g'(2\hat{\eta}) = \frac{-4\hat{\eta}(1 - 2\hat{\eta})(-\hat{\eta}) g(2\hat{\eta}) + 2\lambda}{4\hat{\eta}^2 (1 - 2\hat{\eta})^2 \sigma^2}.$$

In the case $g(2\hat{\eta}) \geq 0$, we have $g'(2\hat{\eta}) > 0$ since $\lambda > 0$. Therefore, $x_1 \leq 2\hat{\eta} < \bar{x}$. In the case $g(2\hat{\eta}) < 0$, it follows that $2\hat{\eta} < x_1 < \bar{x}$ from (ii) and (iii).

The results of Lemma 8.1 allow us to proceed essentially as in Section 7. From Lemma 8.1(iv) and $-\gamma/(1-\delta) > -1$, we can find a $c_1 \in (2\hat{\eta}, \hat{\eta})$ such that, for all $c \in (c_1, 0)$, unique solutions $\lambda_1(c) \in (0, \varphi(c_1))$, $b = b(c, \lambda_1(c))$, and $\beta = \beta(c, \lambda_1(c))$ exist to (6.10), (6.11), and (6.12). Furthermore, λ_1 is strictly decreasing with $\lambda_1(c_1) = \varphi(c_1)$. On the other hand, mainly owing to Lemma 8.1(v), we can find $c_2 \in (\hat{\eta}, 0)$ and $\underline{c} \in (2\hat{\eta}, c_2)$ such that, for $c \in (\underline{c}, c_2)$, solutions

$\lambda_2(c) \in (0, \varphi(c_2))$, $a = a(c, \lambda_2(c))$, and $\alpha = \alpha(c, \lambda_2(c))$ can be found to (6.13), (6.14), and (6.15). Here λ_2 is strictly increasing with $\lambda_2(c_2) = \varphi(c_2) > 0$. The main difference from Section 7 is that, for the existence of a, we need the condition $2\hat{\eta} > -(1-\delta)/\gamma$. In Lemma 8.1(iii) we proved that the maximum lies at $\bar{x} > 2\hat{\eta}$. Therefore, a solution to (6.13) with $a > -(1-\delta)/\gamma$ exists since $\gamma/(1-\delta+\gamma x) \to \infty$ as $x \searrow -(1-\delta)/\gamma$.

Thus, a unique solution, c^*, can be found such that $\lambda = \lambda_1(c^*) = \lambda_2(c^*)$ and the corresponding parameters, a, α, β, and b, solve (6.10)–(6.15). This implies the existence of c, λ, a, α, β, and b in part (i) of Theorem 8.1, below.

Remark 8.2. We briefly comment on the case $\hat{\eta} > 1$. Starting with $\pi_0 > 1$, the uncontrolled diffusion (5.1) stays in $(1, \infty)$ up to its explosion time, and we require that the controlled diffusion stay in $(1, (1-\delta)/\gamma)$. Thus, Remark 8.1 applies with

$$J = \left(1, \frac{1-\delta}{\gamma}\right).$$

Our candidate for an optimal admissible strategy here is a CB strategy with parameters satisfying

$$1 < a < \alpha < \beta < b < \frac{1-\delta}{\gamma}.$$

Again we may use the modification Γ of the cost function, and, for the uncontrolled diffusion before explosion, the coefficients $\tilde{\mu}$ and $\tilde{\sigma}$, the operator $\mathcal{L}$, and the function h are the same as before. The same derivations as for $\hat{\eta} < 0$ can be carried out, now working on $(1, \infty)$ and considering $c \in (1, 2\hat{\eta} - 1)$ and $\lambda \in (R_1, \varphi(c))$. As the upper bound of the solvency region is given by $(1-\delta)/\gamma$, we can find a solution if $2\hat{\eta} - 1 < (1-\delta)/\gamma$. This leads to the existence of c, λ, a, α, β, and b in part (ii) of Theorem 8.1.

From the preceding comments, it follows that the optimality in parts (i) and (ii) of the following theorem can be proved as in Theorem 7.1, the corresponding result.

Theorem 8.1. (i) *Suppose that $\mu < 0$ and $2\hat{\eta} > -(1-\delta)/\gamma$. Then unique parameters satisfying $-(1-\delta)/\gamma < a < \alpha < c < \beta < b < 0$, $\lambda \in (0, \varphi(c))$, and (6.10)–(6.15) exist. The CB strategy with parameters a, α, b, and β is an optimal impulse control strategy with asymptotic growth rate λ for the costs Γ, and is optimal for the original problem with costs $\bar{\Gamma}$ if $\beta < b/(1-\delta)$.*

(ii) *Suppose that $\mu > \sigma^2$ and $2\hat{\eta} - 1 < (1-\delta)/\gamma$. Then unique parameters satisfying $1 < a < \alpha < c < \beta < b < (1-\delta)/\gamma$, $\lambda \in (0, \varphi(c))$, and (6.10)–(6.15) exist. The CB strategy with parameters a, α, b, and β is an optimal impulse control strategy with asymptotic growth rate λ for the costs Γ, and is optimal for the original problem if $\alpha > a/(1-\delta)$.*

Recall that in the cases $\hat{\eta} = 0$ and $\hat{\eta} = 1$, the pure-bond strategy ($\pi_t = 0$) and the pure-stock strategy ($\pi_t = 1$) are optimal, respectively. Thus, by Theorems 7.1 and 8.1, and formulating the conditions in terms of μ, we have found optimal strategies for all

$$\mu \in \left(-\frac{\sigma^2}{2}\frac{1-\delta}{\gamma}, \frac{\sigma^2}{2}\frac{1-\delta+\gamma}{\gamma}\right).$$

Note that, for reasonable costs and parameters μ and σ, this condition will be fulfilled.

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9. Numerical results

We first give numerical examples for fixed and proportional costs and then compare our results with purely fixed costs and purely proportional costs.

Example 9.1. For volatility parameter $\sigma = 0.4$, we look at different scenarios depending on the choice of μ and the transaction cost parameters δ and γ. In Table 1 we gather the results. For each scenario, we give the optimal risky fraction and the optimal growth rate without costs, respectively $\hat{\eta}$ and $\hat{R}$, and we compute the optimal parameters $\hat{a}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{b}$, and $\hat{\lambda}$. They are computed with the auxiliary parameter c as roots of (6.10)–(6.15).

The first three scenarios use realistic costs given by $\delta = 0.0001$ and $\gamma = 0.003$. Scenario I is the same as that in Example 1 of [6], where renewal-theoretic arguments were used to evaluate the performance of CB strategies. We obtain the same solution. For graphs showing the dependency of the boundaries on the transaction cost parameters, we refer the reader to Figure 2 of [6]. Here we shall look at some more extreme settings.

Scenarios II and III provide cases where the optimal risky fraction without costs lies close to the boundaries 0 and 1, respectively. However, we can still determine the optimal parameters. We note that scenario III gives the correct answer to Example 2 of [6], where, based on numerical results which were not sufficiently precise, it was concluded that a solution might not exist. In our computations for Table 1 we used a working precision of 64 digits. Furthermore, a good choice of the initial parameters is vital, in particular to compute the boundaries for the extreme costs in scenario IV, where $\gamma = 0.99$, and scenario V, where $\delta = 0.65$. These costs are of course unreasonably high, but they show that the optimal solutions can still be computed in extreme cases, which is quite surprising; cf. [10]. That we cannot use much higher values of δ in scenario V is only due to the fact that the computer algebra system we used identified 10^{-17} with 0, which for $\delta > 0.7$ yields approximately $\hat{b} = 1$ and, thus, leads to a singular Jacobian in the algorithm. Nevertheless, a solution should still exist.

We point out that all solutions in Table 1 indeed satisfy the condition $\alpha \geq a/(1-\delta)$ (see Theorem 7.1); hence, all solutions provide optimal CB strategies for the original problem.

Example 9.2. For the costs of scenario I in Table 1 and using $\sigma = 0.4$, in Table 2 we present some examples for short selling ($\mu < 0$) and borrowing ($\mu > \sigma^2 = 0.16$), based on the results of Section 8.

Table 1: Optimal parameters for Example 9.1.

Scenario	μ	δ	γ	$\hat{\eta}$	$\hat{a}$
I	0.096	0.0001	0.003	0.6	0.4876
II	0.01	0.0001	0.003	0.0625	0.0213
III	0.159	0.0001	0.003	0.9938	0.9610
IV	0.0001	0.99	0.096	0.6	0.1136
V	0.096	0.65	0.003	0.6	0.0091
Scenario	$\hat{\alpha}$	$\hat{\beta}$	$\hat{b}$	$\hat{R}$	$\hat{\lambda}$
I	0.5680	0.6338	0.708 1	0.028 8	0.028 4
II	0.0498	0.0691	0.121 1	0.000 3	0.000 2
III	0.9932	0.9987	0.999 999 97	0.079 003	0.079 000 0004
IV	0.1356	0.9995	0.999 9	0.028 8	0.017 3
V	0.6702	0.6721	0.999 999 99	0.028 8	0.016 2

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TABLE 2: Optimal parameters for Example 9.2.

μ	$\hat{\lambda}$	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{b}$	$\hat{\eta}$	$\hat{R}$
−0.100	0.029	−0.899	−0.708	−0.513	−0.399	−0.625	0.031
0.200	0.124	1.138	1.204	1.281	1.393	1.25	0.125
26.0	1096.0	56.4	60.6	145.3	156.7	162.5	2113.0

TABLE 3: Optimal parameters as $\delta \to 0$ (see Remark 9.2).

δ	$\hat{\lambda}$	$\hat{a}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{b}$
10^{-8}	0.028 479	0.534 000	0.537 882	0.662 176	0.665 790
10^{-12}	0.028 479	0.535 870	0.536 049	0.663 882	0.664 050
0	0.028 5	—	0.535 960	0.663 971	—

The last example in Table 2 shows that we can find a solution even in extreme cases as long as $2\hat{\eta} - 1 < (1-\delta)/\gamma$, as proved in Theorem 8.1. We have $2\hat{\eta} - 1 = 324$ and $(1-\delta)/\gamma = 333.3$ in this example, which of course is of a theoretical nature due to the large size of μ. However, it shows that for large μ the Merton fraction $\hat{\eta}$ might no longer lie in the no-trading region (a, b); for a similar effect see, e.g. [16].

Remark 9.1. The special case of purely fixed costs ($\gamma = 0$) can be included in the model. In that case we still have to look at impulse control strategies, so the argument is still valid but on the right-hand side of each of the equations (6.10), (6.11), (6.13), and (6.14) we have a 0. This shows that the optimal parameters coincide with the roots x_1, c, and x_2, $x_1 < c < x_2$, of g: in fact $a = x_1$, $\alpha = c = \beta$, and $b = x_2$. For parameters $\sigma = 0.4$, $\mu = 0.175$, $\gamma = 0$, and $\delta = 0.001$, a computation yields the optimal values

$$\hat{\lambda} = 0.0386, \qquad a = 0.540, \qquad b = 0.837, \qquad \alpha = \beta = 0.702,$$

compared to $\hat{\eta} = 0.700$ and $\hat{R} = 0.392$ without costs. These are the same values as in [13] if we add the interest rate of 0.07 used there.

Remark 9.2. The case of purely proportional costs ($\delta = 0$) cannot be included since the optimal strategies are no longer of impulsive form. Following an approach like that in, e.g. [2], it can be shown that the Hamilton–Jacobi–Bellman equation for the maximization of the asymptotic growth rate for purely proportional costs is of the form

$$\max\left\{\mathcal{L}v(x) + \left(\mu - \frac{\sigma^2}{2}x\right)x - \lambda,\, v'(x) - \frac{\gamma}{1+\gamma x},\, -v'(x) - \frac{\gamma}{1-\gamma x}\right\} = 0, \qquad (9.1)$$

where the equality holds for the first argument if no trading is optimal, for the second argument if buying is optimal, and for the third argument if selling is optimal. Note that the authors of [2], [17], and [18] used risky fractions adjusted for the liquidation costs. However, (9.1) is the same as the expression provided after, e.g. [17, Equation (3.4)], if we apply a change of variables to our risky fractions.

For $\mu = 0.096$, $\sigma = 0.4$, and $\gamma = 0.003$, in Table 3 we look at different fixed costs δ and do a comparison with the solution to (9.1). For $\delta = 10^{-4}$, the parameters correspond to scenario I of Table 1.

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The solution for $\delta = 0$ is obtained by an approach to solving (9.1), as in (6.6), depending on the four parameters λ, c, $\alpha = a$, and $\beta = b$. This leads to (6.11), (6.14), and two further equations which guarantee that (6.1) holds with equality at the boundaries α and β. In fact, the results in Table 3 show that we can expect convergence of the parameters. Furthermore, the condition $\alpha \geq a/(1-\delta)$ is always satisfied in the numerical examples. The case of purely proportional costs may thus be obtained as a limiting case of our model. To make these arguments rigorous, including the rather technical analysis of the convergence of the optimal strategies, will be the objective of a future publication.

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