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A PRESENTATION OF $PGL(2,p)$ WITH THREE DEFINING RELATIONS

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Let p be an odd prime and let $GL(2,p)$ denote the general linear group of invertible 2×2 matrices with entries in the field of p elements. The group $PGL(2,p)$ is the factor of $GL(2,p)$ by its centre and has derived group $PSL(2,p)$ with derived factor C_2, the cyclic group of order 2.

For any group G let G' and $Z(G)$ denote the derived group and the centre of G respectively. We shall let G^{ab} denote G/G' and $M(G)$ denote the Schur multiplier of G. It is well known that $M(PGL(2,p))=C_2$ and this imposes a bound on the minimum number of relations required to define $PGL(2,p)$. We show that this bound is attained and so $PGL(2,p)$ is efficient in the following sense. A finite group G is called *efficient*, see **[2]** or **[6]**, if it has a presentation with d generators and r relations while $M(G)$ requires $r-d$ generators.

A group C is called a *covering group* of the finite group G if $M(G)\cong A\leqq Z(C)\cap C'$ with $C/A\cong G$. We find the minimum number of relations required for a covering group of $PGL(2,p)$ and show that this covering group has a deficiency zero presentation, that is a presentation with an equal number of generators and relations. See **[4]** for a survey of finite groups of deficiency zero.

We shall prove the following results:

Theorem A. *If p is an odd prime*

$$PGL(2,p)=\langle a,b \mid a^2b^p=(ab^2)^4=(abab^2)^3b^p=1\rangle.$$

Theorem B. *If p is an odd prime*

$$\langle a,b \mid a^2b^p=1,\ (ab^2)^2=b^{p-1}(ab^2ab)^2\rangle$$

is a covering group of $PGL(2,p)$.

We introduce the class of groups

$$G(p,q)=\langle a,b,c \mid a^2=b^p=(ac)^2=(abc)^3=1,\ cbc^{-1}=b^q\rangle$$

where p is an odd prime and $1<q<p$. Now $q-1$ must be coprime to p so, abelianising the relations of $G(p,q)$, we have $G(p,q)^{\mathrm{ab}}\cong C_2$.

Lemma 1. *Suppose p is an odd prime and $1<q<p$. Then*

$$G'(p,q)=\langle w,y,z \mid y^2=(yz)^2=w^p=(wy)^3=(w^qzy)^3=1, zwz^{-1}=w^{q^2}\rangle.$$

Proof. We show that $K=\langle b,ac,c^2\rangle=G'(p,q)$ and find a presentation for K. It is clear that $K\leqq G'(p,q)$. Defining cosets 1 and 2 of K in $G(p,q)$ by $1=K$, $2=Kc$ we see, using the Todd Coxeter coset enumeration algorithm, that the generators a,b,c of $G(p,q)$ act as the permutations (12), (1), (12), respectively, on the cosets so K has index 2 in $G(p,q)$. Let $x=b$, $y=ac$, $z=c^2$ and choose the transversal $T=\{1,c\}$ of K in $G(p,q)$. Then rewriting the Schreier generators $s_{t,g}$, $t\in T$, $g\in\{a,b,c\}$ in terms of x,y,z gives:

$$s_{1,a}=yz^{-1},\ s_{1,b}=x,\ s_{1,c}=1,\ s_{c,a}=zy^{-1},\ s_{c,b}=x^q,\ s_{c,c}=z$$

and the following presentation for K on the generators x,y,z is obtained by the method described in [5]

$$\langle x,y,z \mid y^2=(yz)^2=x^p=(xzy)^3=(x^qzyz^{-1})^3=1,\ zxz^{-1}=x^{q^2}\rangle. \tag{1}$$

Now letting $w=x^r$, where $rq\equiv 1\,(\mathrm{mod}\,p)$, (1) may be transformed to

$$\langle w,y,z \mid y^2=(yz)^2=w^p=(w^qzy)^3=(w^{q^2}zyz^{-1})^3=1,\ zw^qz^{-1}=w^{q^3}\rangle.$$

Clearly we may replace the relation $zw^qz^{-1}=w^{q^3}$ by $zwz^{-1}=w^{q^2}$ and, substituting for w^{q^2}, the relation $(w^{q^2}zyz^{-1})^3=1$ simplifies to $(wy)^3=1$. This completes the proof.

Lemma 2. *When* $q=2$, or $q=(p+1)/2$, *or q is a primitive element of $GF(p)$, then* $G'(p,q)\cong PSL(2,p)$.

Proof. Let $rq\equiv 1\ (\mathrm{mod}\,p)$ and consider the presentation for $G'(p,q)$ given in Lemma 1. Now if s is a non-negative integer $z^sw=w^{q^s}z^s$ while $z^{-s}w=w^{r^s}z^{-s}$. Using these results together with $z^ny=yz^{-n}$ for any integer n we obtain from the relation $z^s(wy)^3=z^s$

$$z^{2s}=w^{q^{2s}}yw^{r^{2s}}yw^{q^{2s}}y. \tag{2}$$

Similarly from $z^s(w^qzy)^3=z^s$ we obtain

$$z^{2s+1}=w^{q^{2s+1}}yw^{r^{2s+1}}yw^{q^{2s+1}}y. \tag{3}$$

From (2) and (3) we deduce

$$z^s=w^{q^s}yw^{r^s}yw^{q^s}y. \tag{4}$$

Suppose t is such that $q^t\equiv\pm 1\,(\mathrm{mod}\,p)$. Then from (4) we obtain $z^t=1$. Putting $S=w$, $T=y$, $V=z^{-1}$ we now have

$$G'(p,q)=\langle S,T,V \mid S^p=V^t=T^2=(ST)^3=(TV)^2=(S^qTV)^3=1,\ V^{-1}SV=S^{q^2}\rangle.$$

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If q is primitive and $t=(p-1)/2$ this is Frasch's presentation for $PSL(2,p)$, see [3].

The cases $q=(p+1)/2$ and $q=2$ proceed by eliminating z from the presentation of Lemma 1 using (4) with $s=1$. Further Tietze transformations then reduce the presentation to the Behr–Mennicke presentation for $PSL(2,p)$, see [1]. Details may be found in [7].

The results of Lemma 2 may fail for other values of q. For example in [7] it is shown that $G'(29,12)\ncong PSL(2,29)$ and $G'(89,34)\ncong PSL(2,89)$.

Lemma 3. *When $q=2$, or $q=(p+1)/2$, or q is a primitive element of $GF(p)$, then $G(p,q)\cong PGL(2,p)$.*

Proof. Using Lemma 2 together with $G(p,q)^{\mathrm{ab}}=C_2$ we see that

$$|G(p,q)|=|PGL(2,p)|.$$

However $PGL(2,p)$ is easily seen to be a homomorphic image of $G(p,q)$ using the map induced by

$$a\mapsto\begin{pmatrix}0 & -r\\ 1 & 0\end{pmatrix},\qquad b\mapsto\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\qquad c\mapsto\begin{pmatrix}q & 0\\ 0 & 1\end{pmatrix}$$

where $rq\equiv 1 \pmod p$.

Theorem 4. *$PGL(2,p)$ may be presented by*

$$\langle a,b\,|\,a^2=b^p=(ab^2ab^r)^2=(abab^r)^3=1\rangle$$

where $r=2$, or $r=(p+1)/2$, or r is a primitive element of $GF(p)$.

Proof. Let $rq\equiv 1 \pmod p$. Then notice that the conditions given on r imply that q satisfies the conditions in Lemma 3 so $G(p,q)\cong PGL(2,p)$. We show that c is a redundant generator of $G(p,q)$ as follows. The relation $(abc)^3=1$ becomes, on putting $ca=ac^{-1}$

$$bac^{-1}bcaba=c$$

that is

$$c=bab^raba. \tag{5}$$

Using (5) to eliminate c the presentation for $G(p,q)$ becomes

$$\langle a,b\,|\,a^2=b^p=(ab^2ab^r)^2=(b^2ab^rab)^3=1,\ ab^rababab^{-1}ab^{-r}a=b^q\rangle. \tag{6}$$

Now it is easy to see that the relation $(b^2ab^rab)^3=1$ simplifies to $(abab^r)^3=1$ on substituting $b^2ab^ra=ab^{-r}ab^{-2}$. Further the final relation in the presentation (6) is redundant since it may be deduced from the first four relations as follows:

$$\begin{aligned} ab^r ababab^{-1}ab^{-r}a &= (ab^r abab^r ab^{-1}ab^{-r}a)^q \\ &= (b^{-1}ab^{-r}ab^{-2}ab^{-r}a)^q \\ &= (b^{-1}ab^{-r}a\,.\,ab^r ab^2)^q \\ &= b^q. \end{aligned}$$

We restate Theorem 4 in the case $r=2$.

Corollary 5. $PGL(2,p)=\langle a,b \mid a^2=b^p=(ab^2)^4=(abab^2)^3=1\rangle$.

It may be of interest to note that this presentation may be rewritten, on the same generators a and b, as

$$PGL(2,p)=\langle a,b \mid a^2=b^p=(ab^2)^4=(abab^4)^2=1\rangle.$$

We now prove Theorem A. Let G be the group with presentation

$$\langle a,b \mid a^2b^p=(ab^2)^4=(abab^2)^3b^p=1\rangle.$$

Clearly, in view of Corollary 5, it suffices to prove that $b^p=1$ in G. Certainly $b^p\in Z(G)$ since $b^p=a^{-2}$. Now$(\mathrm{ab}^2)^4=1$ gives

$$bab^2ab=b^{-1}a^{-1}b^{-2}a^{-1}b^{-1}$$

and substituting this into $(abab^2)^3b^p=1$, using the fact that $a^2\in Z(G)$, gives

$$(abab^2a)^{-1}b^2(abab^2a)=b^{1-p}a^{-2}=b. \tag{7}$$

Raising (7) to the power p and using the fact that $b^p\in Z(G)$ gives $b^{2p}=b^p$ so $b^p=1$ as required.

Finally we give a proof of Theorem B. Let $\tilde{G}$ denote the group with presentation given in the theorem. Notice that the relations of $\tilde{G}$ can be written as

$$a^2b^p=1,\ (ab^2)^4=b^p(abab^2)^3.$$

Now $\langle ab,ab^2\rangle=\tilde{G}$ since $b=(ab)^{-1}ab^2$, $a=ab(ab^2)^{-1}ab$. Let $H=\langle(ab^2)^4\rangle$. Now $b^p\in Z(\tilde{G})$, since $b^p=a^{-2}$, so $abab^2$ commutes with $b^p(abab^2)^3$ and so commutes with $(ab^2)^4$. Therefore $(ab^2)^4\in Z(\tilde{G})$. Now in $\tilde{G}^{\mathrm{ab}}$ we have $a^2=b=1$ so $(ab^2)^4\in\tilde{G}'$. Hence $H\leqq Z(\tilde{G})\cap\tilde{G}'$ and $\tilde{G}/H\cong PGL(2,p)$ by Theorem A.

Now $\tilde{G}$ cannot be $PGL(2,p)$ since $\tilde{G}$, having deficiency zero, must have trivial Schur multiplier. Therefore $\tilde{G}$ is a covering group of $PGL(2,p)$ and the proof is complete.

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