Can. J. Math.Vol. 45 (5), 1993 pp. 930–943

ORTHOGONAL POLYNOMIALS AND RATIONAL MODIFICATIONS OF MEASURES

E. GODOY AND F. MARCELLAN

ABSTRACT Given a finite positive measure on the Borel subsets of the complex plane with compact support containing infinitely many points, we deduce some formulas for the sequence of monic orthogonal polynomials associated to a rational modification of the measure These expressions depend on so called functions of the second kind Some examples for particular Jordan curves are given

1. **Introduction.** The study of rational modifications for a measure has been introduced by Szegö in the case of the unit circle $T = \{z \in \mathbb{C} / |z| = 1\}$. In fact, if

$$d\mu = \frac{d\theta}{|h(e^{i\theta})|^2}$$

where $h(z)$ is a polynomial of degree k, the system $\varphi_n(z)$ of orthonormal polynomials associated to μ can be calculated explicitly (see [15], pp. 289–290), except for a finite number of terms, by means of

$$\varphi_n(z) = z^n \bar{h}(z^{-1}) \text{ with } n \geq k.$$

On the real line, given a real or complex measure μ, a determinant representation is known (the Christoffel formula) [15, pp. 29–30] that gives orthogonal polynomials relative to the rational modification of the measure in the interval of orthogonality $[a, b]$ in terms of orthogonal polynomials with respect to $d\mu$ and the functions of the second kind ([3], [16]).

In a similar way, Paszkowski ([14]) obtains an alternative expression in determinantal form, in terms of orthogonal polynomials with respect to the initial measure, their derivatives and their associated polynomials of first kind.

Also, in ([2]) a study of the rational modification of a linear regular functional is presented. This corresponds to a new approach to the subject in the framework of general orthogonal polynomials.

In this paper, we consider some analogous problems for rational modifications of finite and positive measures supported on a compact set of the complex plane with infinitely many points as the next step in the study of a more general problem started in [6], [7], [8].

The second author was partially supported by the Comisión Interministerial de Ciencia y Tecnologia (CICYT) of Spain under grant PB89-0181-C02-01

Received by the editors January 14, 1992, revised July 10, 1992

AMS subject classification 42C05

Key words and phrases Orthogonal polynomials, measures, rational modifications

Rational modifications of measures have been used (see [12]) to establish a set of quadrature formulas on $[-1,1]$, that are valid for analytic functions, as well as to show that orthogonal polynomials on $[-1,1]$ with respect to $w(t)/(t-x)$ are linear combinations of orthogonal polynomials on $[-1,1]$ with respect to $w(t)$.

The aim of this paper is to present some results concerning the calculus of the monic orthogonal polynomials defined by a rational modification of a finite and positive Borel measure on a compact set (in particular, Jordan arc/curve are included) and to give as example some particular modifications on the unit circle and Bernoulli lemniscate.

Recently, we have received a preprint of a paper concerning this question ([11]). The methods used by Ismail and Ruedemann are essentially different from ours and they apply only to the unit circle, while ours apply to any compact set. Moreover an extension to complex measures can be studied.

The structure of the paper is the following:

In Section 2, we give the basic definitions and we present the main results, using functions of the second kind.

In Section 3, we consider an exhaustive study of some rational modifications of Borel measures on the unit circle T and the Bernoulli lemniscate BL respectively.

2. **Main results.** Let μ be a finite and positive Borel measure such that $C = \operatorname{supp}\mu$ is a compact set of the complex plane with infinitely many points and we assume, for simplicity, that $A(z)$ is a fixed monic polynomial of degree m whose roots are simple and not on C

$$A(z) = \prod_{i=1}^{m}(z-\alpha_i) \text{ with } \alpha_i \neq \alpha_j \text{ if } i \neq j \text{ and } \alpha_i \notin C, \quad 1 \leq i \leq m.$$

We define

$$d\mu_1 = \frac{d\mu}{|A(z)|^2}$$

such that

$$|c_{jk}(\mu_1)| < \infty \text{ where } c_{jk}(\mu_1) = \int_C z^j \bar{z}^k \, d\mu_1 = \langle z^i, z^j \rangle_{\mu_1}.$$

Also, let $\{\Phi_n(z;\mu)\}$ and $\{\Phi_n(z;\mu_1)\}$ be the monic orthogonal polynomial sequences (MOPS) on C associated to μ and μ_1 respectively and let $\{\varphi_n(z;\mu)\}$ and $\{\varphi_n(z;\mu_1)\}$ be the corresponding orthonormal polynomial sequences. Then, we have

$$\Phi_n(z;\mu) = \sqrt{e_n(\mu)}\varphi_n(z;\mu)$$

where

$$e_n(\mu) = \frac{\Delta_n(\mu)}{\Delta_{n-1}(\mu)}, \quad \forall n \geq 1 \text{ with } e_0(\mu) = \Delta_0(\mu) = c_{00}(\mu)$$

and

$$\Delta_n(\mu) = \det\bigl(c_{ij}(\mu)\bigr)_{i,j=0}^{n} \text{ with the convention } \Delta_{-1}(\mu) = 1.$$

The reproducing kernels associated to $\{\varphi_n(z;\mu)\}$ are defined by

$$K_n(z,y;\mu)=\sum_{j=0}^{n}\overline{\varphi_j(y;\mu)}\varphi_j(z;\mu).$$

The functions

$$q_k(t)=\int_C \frac{\overline{\varphi_k(z;\mu)}}{t-z}\,d\mu(z),\qquad t \text{ exterior to } C$$

are called *functions of the second kind* ([17, p. 137] and [4, p. 185, equation 1.24] for the real line). From this definition, it is clear that $q_k(t)$ is analytic everywhere in the extended plane exterior to C, and vanishes at infinity. We denote also

$$Q_k(t)=\int_C \frac{\overline{\Phi_k(z;\mu)}}{t-z}\,d\mu(z)=\sqrt{e_k(\mu)}\,q_k(t),\qquad t \text{ exterior to } C$$

Following Gautschi [3], we can make the transition from $d\mu$ to $d\mu_1$ following a sequence of elementary steps of the form

$$d\mu_2=\frac{1}{|z-\alpha|^2}\,d\mu \text{ where } \alpha\notin C$$

since the general case can be solved by an iterated application.

Therefore, we are interested in constructing the orthogonal polynomial sequence $\{\Phi_n(z;\mu_2)\}$ associated with μ_2, so that we will find some expressions to compute the elements of the above mentioned family.

PROPOSITION 1.

$$\frac{\Delta_n(\mu_2)}{\Delta_{n-1}(\mu)}=\|\mu_2\|-\sum_{j=0}^{n-1}|q_j(\alpha)|^2>0 \tag{1}$$

PROOF. If we consider the family $\{1,z-\alpha,(z-\alpha)^2,\ldots,(z-\alpha)^n\}$, we define

$$d_{ij}(\mu)=\int_C (z-\alpha)^i\overline{(z-\alpha)}^j\,d\mu=\int_C (z-\alpha)^{i+1}\overline{(z-\alpha)}^{j+1}\,d\mu_2=d_{i+1,j+1}(\mu_2)$$

The monic polynomials $\Phi_n(z;\mu)$ are then given by

$$\Phi_n(z;\mu)=\frac{1}{\Delta_{n-1}(\mu)}\begin{vmatrix} d_{00}(\mu) & d_{10}(\mu) & \cdots & d_{n0}(\mu)\\ \vdots & \vdots & \vdots & \vdots\\ d_{0,n-1}(\mu) & d_{1,n-1}(\mu) & \cdots & d_{n,n-1}(\mu)\\ 1 & z-\alpha & \cdots & (z-\alpha)^n\end{vmatrix}$$

and

$$\Delta_n(\mu_2)=\begin{vmatrix} d_{00}(\mu_2) & \cdots & d_{n,0}(\mu_2)\\ \vdots & \vdots & \vdots\\ d_{0,n}(\mu_2) & \cdots & d_{n,n}(\mu_2)\end{vmatrix}=\begin{vmatrix} d_{00}(\mu_2) & d_{10}(\mu_2) & \cdots & d_{n0}(\mu_2)\\ d_{01}(\mu_2) & d_{00}(\mu) & \cdots & d_{n-1,0}(\mu)\\ \vdots & \vdots & \vdots & \vdots\\ d_{0n}(\mu_2) & d_{0,n-1}(\mu) & \cdots & d_{n-1\,n-1}(\mu)\end{vmatrix}.$$

By application of Sylvester's identity [10, p. 22, (0.8.6)], we get

$$\Delta_n(\mu_2)\,\Delta_{n-2}(\mu) = \Delta_{n-1}(\mu_2)\,\Delta_{n-1}(\mu) - |R|^2$$

where

$$R = \begin{vmatrix} \langle z-\alpha, 1\rangle_{\mu_2} & \cdots & \langle (z-\alpha)^n, 1\rangle_{\mu_2} \\ d_{00}(\mu) & \cdots & d_{n-1,0}(\mu) \\ \vdots & \vdots & \vdots \\ d_{0,n-2}(\mu) & \cdots & d_{n-1,n-2}(\mu) \end{vmatrix} = (-1)^{n-1}\Delta_{n-2}(\mu)\Big\{\Big\langle \Phi_{n-1}(z,\mu), \frac{1}{z-\alpha}\Big\rangle_\mu\Big\}$$
$$= (-1)^{n-1}\sqrt{e_{n-1}(\mu)}\,\Delta_{n-2}(\mu)\,\overline{q_{n-1}(\alpha)} = (-1)^{n-1}\Delta_{n-2}(\mu)\,\overline{Q_{n-1}(\alpha)}.$$

Therefore

$$\Delta_n(\mu_2)\,\Delta_{n-2}(\mu) = \Delta_{n-1}(\mu_2)\,\Delta_{n-1}(\mu) - e_{n-1}(\mu)\,\Delta^2_{n-2}(\mu)\,|q_{n-1}(\alpha)|^2$$

From this we obtain

$$\frac{\Delta_n(\mu_2)}{\Delta_{n-1}(\mu)} = \frac{\Delta_{n-1}(\mu_2)}{\Delta_{n-2}(\mu)} - |q_{n-1}(\alpha)|^2$$

which by $d_{00}(\mu_2) = \|\mu_2\|$ leads to

$$\frac{\Delta_n(\mu_2)}{\Delta_{n-1}(\mu)} = \|\mu_2\| - \sum_{j=0}^{n-1}|q_j(\alpha)|^2. \qquad \blacksquare$$

Note that, in fact, we obtain

COROLLARY 2.

$$\|\mu_2\| - \sum_{j=0}^{n}|q_j(\alpha)|^2 = \frac{e_{n+1}(\mu_2)}{e_n(\mu)}\Big(\|\mu_2\| - \sum_{j=0}^{n-1}|q_j(\alpha)|^2\Big) \tag{2}$$

PROOF. It follows from the definition of $e_n(\mu)$ and from the result of Proposition 1. ∎

The main result of this section is the following.

PROPOSITION 3. *The sequence of monic orthogonal polynomials* $\big(\Phi_n(z;\mu_2)\big)$ *can be obtained from*

$$\Phi_{n+1}(z;\mu_2) = (z-\alpha)\,\Phi_n(z;\mu) + \beta_{n+1}\big[(z-\alpha)\,R_{n-1}(z,\alpha;\mu) + 1\big], \quad \forall n \geq 1 \tag{3}$$

where

$$\Phi_0(z;\mu_2) = 1, \quad \Phi_1(z;\mu_2) = z - \alpha + \frac{\overline{Q_0(\alpha)}}{\|\mu_2\|}$$

and

$$R_{n-1}(z,\alpha;\mu) = \sum_{j=0}^{n-1} q_j(\alpha)\,\varphi_j(z;\mu) = \sum_{j=0}^{n-1}\frac{Q_j(\alpha)}{e_j(\mu)}\Phi_j(z;\mu), \quad \forall n \geq 1$$

$$\beta_{n+1} = \frac{\overline{Q_n(\alpha)}}{\|\mu_2\| - \sum_{j=0}^{n-1}|q_j(\alpha)|^2} \text{ with } \|\mu_2\| = \int_C d\mu_2.$$

https://doi.org/10.4153/CJM-1993-052-1 Published online by Cambridge University Press

PROOF. Since $\{(z-\alpha)\,\varphi_j(z;\mu)\}_{0\le j\le n}$ is an orthonormal family in $(z-\alpha)\mathbb{P}_n$ with respect to the inner product

$$\langle f(z), g(z)\rangle_{\mu_2} = \int_C f(z)\overline{g(z)}\,d\mu_2$$

we can consider

$$\varphi_{n+1}(z;\mu_2) - \varphi_{n+1}(\alpha;\mu_2) = (z-\alpha)\Big[\sum_{j=0}^{n}\lambda_j\varphi_j(z;\mu)\Big], \quad \forall n \ge 0.$$

Then

$$\begin{aligned}\lambda_j &= \int_C [\varphi_{n+1}(z;\mu_2) - \varphi_{n+1}(\alpha;\mu_2)]\overline{(z-\alpha)\,\varphi_j(z;\mu)}\,d\mu_2\\ &= \int_C \varphi_{n+1}(z;\mu_2)\overline{(z-\alpha)\,\varphi_j(z;\mu)}\,d\mu_2 - \varphi_{n+1}(\alpha;\mu_2)\int_C \overline{(z-\alpha)\,\varphi_j(z;\mu)}\,d\mu_2\\ &= \sqrt{\frac{e_{n+1}(\mu_2)}{e_n(\mu)}}\delta_{jn} + q_j(\alpha)\,\varphi_{n+1}(\alpha;\mu_2) \text{ for } j = 0,1,2,\ldots,n.\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi_{n+1}(z;\mu_2) &= \varphi_{n+1}(\alpha;\mu_2)\Big[1 + (z-\alpha)\sum_{j=0}^{n-1} q_j(\alpha)\varphi_j(z;\mu)\Big]\\ &\quad + \Big[\sqrt{\frac{e_{n+1}(\mu_2)}{e_n(\mu)}} + q_n(\alpha)\,\varphi_{n+1}(\alpha;\mu_2)\Big](z-\alpha)\,\varphi_n(z;\mu).\end{aligned} \tag{4}$$

For the leading coefficients

$$\frac{1}{\sqrt{e_{n+1}(\mu_2)}} = \frac{1}{\sqrt{e_n(\mu)}}\Big[\sqrt{\frac{e_{n+1}(\mu_2)}{e_n(\mu)}} + q_n(\alpha)\,\varphi_{n+1}(\alpha;\mu)\Big]$$

i.e.

$$e_n(\mu) - e_{n+1}(\mu_2) = Q_n(\alpha)\,\Phi_{n+1}(\alpha;\mu_2), \quad \forall n \ge 0. \tag{5}$$

In particular, for the monic polynomials, the equation (4) becomes

$$\Phi_{n+1}(z;\mu_2) = \Phi_{n+1}(\alpha;\mu_2)[1 + (z-\alpha)\,R_{n-1}(z,\alpha;\mu)] + (z-\alpha)\,\Phi_n(z;\mu)$$

Moreover, taking into account the norms, we get

$$1 + \|\mu_2\|\,|\varphi_{n+1}(\alpha;\mu_2)|^2 = \sum_{j=0}^{n}|\lambda_j|^2 = |\varphi_{n+1}(\alpha;\mu_2)|^2\cdot\Big(\sum_{j=0}^{n-1}|q_j(\alpha)|^2\Big) + \frac{e_n(\mu)}{e_{n+1}(\mu_2)}$$

i.e.

$$|\varphi_{n+1}(\alpha;\mu_2)|^2\cdot\Big(\|\mu_2\| - \sum_{j=0}^{n-1}|q_j(\alpha)|^2\Big) = \frac{e_n(\mu) - e_{n+1}(\mu_2)}{e_{n+1}(\mu_2)} = \frac{\Phi_{n+1}(\alpha;\mu_2)Q_n(\alpha)}{e_{n+1}(\mu_2)}$$

so that

$$\Phi_{n+1}(\alpha;\mu_2) = \frac{\overline{Q_n(\alpha)}}{\|\mu_2\| - \sum_{j=0}^{n-1}|q_j(\alpha)|^2} = \beta_{n+1}, \quad \forall n \geq 1. \tag{6}$$

This completes the proof. ∎

REMARKS. 1) Since

$$\frac{1}{\alpha - z} = \sum_{j=0}^{\infty} q_j(\alpha)\,\varphi_j(z;\mu)$$

is valid uniformly for z on C and for α on compact sets exterior to C, it follows that

$$\|\mu_2\| = \sum_{j=0}^{\infty} |q_j(\alpha)|^2.$$

2) From

$$(z-\alpha)R_{n-1}(z,\alpha;\mu) + 1 = 1 + \int_C \frac{z-\alpha}{\alpha - t} K_{n-1}(z,t;\mu)\,d\mu(t) = \int_C \frac{z-t}{\alpha-t} K_{n-1}(z,t;\mu)\,d\mu(t)$$

equation (3) in Proposition 3 can be written as

$$\Phi_{n+1}(z;\mu_2) = (z-\alpha)\,\Phi_n(z;\mu) + \beta_{n+1}\int_C \frac{z-t}{\alpha-t} K_{n-1}(z,t;\mu)\,d\mu(t), \quad \forall n \geq 1 \tag{7}$$

where β_{n+1} is given by (6).

PROPOSITION 4. *a) If $\beta_n \neq 0$, then the sequence of monic orthogonal polynomials associated to μ_2 satisfies the following recurrence relation*

$$\begin{vmatrix} \Phi_n(z;\mu_2) & \Phi_{n+1}(z;\mu_2) \\ \varepsilon_{n-1}\overline{Q_{n-1}(\alpha)} & \varepsilon_n\overline{Q_n(\alpha)} \end{vmatrix} = \varepsilon_{n-1}(z-\alpha)\begin{vmatrix} \Phi_{n-1}(z;\mu) & \Phi_n(z;\mu) \\ \overline{Q_{n-1}(\alpha)} & \overline{Q_n(\alpha)} \end{vmatrix}, \quad \forall n \geq 1 \tag{8}$$

where

$$\varepsilon_n = \frac{1}{\|\mu_2\| - \sum_{j=0}^{n-1}|q_j(\alpha)|^2}. \tag{9}$$

b) If $\beta_n = 0$, then: $\Phi_n(z;\mu_2) = (z-\alpha)\,\Phi_{n-1}(z;\mu)$.

PROOF. a) By Proposition 3 we have

$$\frac{\Phi_n(z;\mu_2) - \Phi_n(\alpha;\mu_2)}{z-\alpha} = \Phi_{n-1}(z;\mu) + \beta_n R_{n-2}(z,\alpha;\mu)$$

and

$$\frac{\Phi_{n+1}(z;\mu_2) - \Phi_{n+1}(\alpha;\mu_2)}{z-\alpha} = \Phi_n(z;\mu) + \beta_{n+1}R_{n-1}(z,\alpha;\mu).$$

If we multiply the first equation by β_{n+1} and the second one by β_n, we obtain by subtraction

$$\beta_n[\Phi_{n+1}(z;\mu_2) - \Phi_{n+1}(\alpha;\mu_2)] - \beta_{n+1}[\Phi_n(z;\mu_2) - \Phi_n(z;\mu_2)]$$
$$= (z-\alpha)\left[\beta_n\Phi_n(z;\mu) - \beta_{n+1}\left[1 - \frac{\beta_n Q_{n-1}(\alpha)}{e_{n-1}}\right]\Phi_{n-1}(z;\mu)\right]$$

i.e.

$$\beta_n\Phi_{n+1}(z;\mu_2)-\beta_{n+1}\Phi_n(z;\mu_2) = (z-\alpha)\left[\beta_n\Phi_n(z;\mu)-\beta_{n+1}\left[1-\frac{\beta_n Q_{n-1}(\alpha)}{e_{n-1}}\right]\Phi_{n-1}(z;\mu)\right].$$

Since

$$\beta_{n+1}\left[1-\frac{\beta_n Q_{n-1}(\alpha)}{e_{n-1}}\right] = \varepsilon_{n-1}\overline{Q_n(\alpha)}$$

it follows that

$$\beta_{n+1}\Phi_n(z;\mu_2)-\beta_n\Phi_{n+1}(z;\mu_2) = (z-\alpha)\left[\varepsilon_{n-1}\overline{Q_n(\alpha)}\,\Phi_{n-1}(z;\mu)-\beta_n\Phi_n(z;\mu)\right].$$

and

$$\begin{vmatrix}\Phi_n(z;\mu_2) & \Phi_{n+1}(z;\mu_2)\\ \beta_n & \beta_{n+1}\end{vmatrix} = (z-\alpha)\begin{vmatrix}\Phi_{n-1}(z;\mu) & \Phi_n(z;\mu)\\ \beta_n & \varepsilon_{n-1}\overline{Q_n(\alpha)}\end{vmatrix}$$

b) If $Q_{n-1}(\alpha)=0$, then $\beta_n=0$ and using (3), we can deduce

$$\Phi_n(z;\mu_2) = (z-a)\,\Phi_{n-1}(z;\mu). \qquad \blacksquare$$

For the reproducing kernels, we have

PROPOSITION 5. *The sequence of reproducing kernels $\big(K_n(z,y;\mu_2)\big)$ can be obtained from*

$$\begin{aligned}K_n(z,y;\mu_2) &= \overline{(y-\alpha)}(z-\alpha)\,K_{n-1}(z,y;\mu)\\ &\quad+\varepsilon_n[1+(z-\alpha)\,R_{n-1}(z,\alpha;\mu)]\left[1+\overline{(y-\alpha)\,R_{n-1}(y,\alpha;\mu)}\right]\end{aligned}$$

where ε_n is given by (9) and

$$R_{n-1}(z,\alpha;\mu) = \sum_{j=0}^{n-1} q_j(\alpha)\,\varphi_j(z;\mu) = \sum_{j=0}^{n-1}\frac{Q_j(\alpha)}{e_j(\mu)}\Phi_j(z;\mu), \quad \forall n\geq 1.$$

SKETCH OF PROOF. The proof is a straightforward calculation analogous to that used to prove Proposition 1. Here, the Sylvester identity is applied to the determinant expression of the reproducing kernels $K_n(z,y;\mu_2)$ in the same way as it is applied to the minors $\Delta_n(\mu_2)$ in Proposition 1. For more details the reader is referred to [6]. ■

REMARK. From Proposition 5, if $z=y=\alpha$ we get

$$\varepsilon_n = K_n(\alpha,\alpha;\mu_2), \quad \forall n\geq 1.$$

3. **Applications.** This section is devoted to the study of rational modifications of Borel measures on the unit circle T and the lemniscate of Bernoulli BL, as an application of the general theory mentioned above.

3.1 *Orthogonal polynomials on the unit circle.* Let μ be a finite positive Borel measure on $T = \{z \in \mathbb{C} \mid |z| = 1\}$ and

$$d\mu_2 = \frac{1}{|z-\alpha|^2}\, d\mu \text{ where } \alpha \notin T$$

If $\{\Phi_n(z;\mu)\}$ and $\{\Phi_n(z;\mu_2)\}$ are the monic orthogonal polynomial sequences on T associated to μ and μ_2 respectively and we denote

$$\Phi_n^*(z;\mu) = z^n \overline{\Phi_n}\Big(\frac{1}{z}\Big)$$

we get

PROPOSITION 6. *The monic orthogonal polynomials with respect to μ_2 on the unit circle T satisfy*

$$\Phi_{n+1}(z;\mu_2) = \big(z - A_n(\alpha)\big)\,\Phi_n(z;\mu) + B_n(\alpha)\,\Phi_n^*(z;\mu), \quad \forall n \geq 1 \tag{10}$$

and

$$\Phi_0(z;\mu_2) = 1, \quad \Phi_1(z;\mu_2) = z - \alpha + \frac{\overline{Q_0(\alpha)}}{\|\mu_2\|}$$

where

$$A_n(\alpha) = \alpha\frac{e_{n+1}(\mu_2)}{e_n(\mu)} = \alpha\left[\frac{\|\mu_2\| - \sum_{j=0}^{n} |q_j(\alpha)|^2}{\|\mu_2\| - \sum_{j=0}^{n-1} |q_j(\alpha)|^2}\right]$$

and

$$B_n(\alpha) = \frac{1}{\alpha^{n+1}}\left[\frac{\overline{q_n(\alpha)}\,\overline{q_n(\frac{1}{\alpha})}}{\|\mu_2\| - \sum_{j=0}^{n-1} |q_j(\alpha)|^2}\right]$$

with

$$q_k(t) = \frac{1}{\sqrt{e_k(\mu)}}\int_T \frac{\overline{\Phi_k(z;\mu)}}{t-z}\, d\mu(z) \text{ and } \|\mu_2\| = \int_T d\mu_2.$$

PROOF. According to formula (7)

$$\Phi_{n+1}(z;\mu_2) = (z-\alpha)\,\Phi_n(z;\mu) + \beta_{n+1}\int_T \frac{z-t}{\alpha-t} K_{n-1}(z,t;\mu)\, d\mu(t)$$

and from the Christoffel-Darboux formula for the unit circle (see [15], p. 293)

$$K_{n-1}(z,t;\mu) = \frac{\Phi_n^*(z;\mu)\,\overline{\Phi_n^*(t;\mu)} - \Phi_n(z;\mu)\,\overline{\Phi_n(t;\mu)}}{e_n(\mu)[1 - z\bar{t}]}$$

we have

$$\Phi_{n+1}(z,\mu_2)$$
$$= (z-\alpha)\,\Phi_n(z,\mu) + \frac{\beta_{n+1}}{e_n(\mu)}\int_T \frac{(-t)}{\alpha - t}\Big[\Phi_n^*(z,\mu)\,\overline{\Phi_n^*(t,\mu)} - \Phi_n(z,\mu)\,\overline{\Phi_n(t,\mu)}\Big]\,d\mu(t)$$
$$= \big(z - A_n(\alpha)\big)\Phi_n(z,\mu) + B_n(\alpha)\,\Phi_n^*(z,\mu), \quad \forall n \geq 1$$

where

$$A_n(\alpha) = \alpha - \frac{\beta_{n+1}}{e_n(\mu)}\int_T \frac{t}{\alpha - t}\overline{\Phi_n(t,\mu)}\,d\mu(t) = \alpha - \frac{\alpha\beta_{n+1}}{e_n(\mu)}\int_T \frac{\overline{\Phi_n(t,\mu)}}{\alpha - t}\,d\mu(t)$$
$$= \alpha\left[1 - \frac{\beta_{n+1}}{e_n(\mu)}Q_n(\alpha)\right]$$

In the same way

$$B_n(\alpha) = \frac{\beta_{n+1}}{e_n(\mu)}\int_T \frac{(-t)}{\alpha - t}\overline{\Phi_n^*(t,\mu)}\,d\mu(t) = -\frac{\beta_{n+1}}{e_n(\mu)}\int_T \frac{\Phi_n(t,\mu)}{t^{n-1}(\alpha - t)}\,d\mu(t)$$
$$= -\frac{\beta_{n+1}}{\alpha^{n-1}e_n(\mu)}\int_T \frac{\Phi_n(t,\mu)}{\alpha - t}\,d\mu(t) = \frac{\beta_{n+1}}{\alpha^{n+1}e_n(\mu)}\overline{Q_n\Big(\frac{1}{\bar{\alpha}}\Big)} \qquad \blacksquare$$

REMARK A relation between the parameters $A_n(\alpha)$ and $B_n(\alpha)$ is given by

(11) $$\alpha^{n+2}Q_n(\alpha)\,B_n(\alpha) = \big(\alpha - A_n(\alpha)\big)\overline{Q_n\Big(\frac{1}{\bar{\alpha}}\Big)}, \quad \forall n \geq 1 \qquad \blacksquare$$

If we consider the functions

$$Q_k(t) = \int_T \frac{\overline{\Phi_k(z,\mu)}}{t - z}\,d\mu(z)$$

where $\{\Phi_n(z,\mu)\}$ is the monic orthogonal polynomial sequence on T associated to μ, we can derive the recurrence formulas satisfied by these functions by noting that

$$\Phi_k(z,\mu) = z\Phi_{k-1}(z,\mu) + \Phi_k(0)z^{k-1}\overline{\Phi_{k-1}\Big(\frac{1}{\bar z},\mu\Big)}$$

and

(12) $$Q_k(t) = \int_T \frac{\overline{\Phi_{k-1}(z,\mu)}}{z(t-z)}\,d\mu(z) + \overline{\Phi_k(0)}\int_T \frac{\Phi_{k-1}(z,\mu)}{z^{k-1}(t-z)}\,d\mu(z)$$
$$= \frac{1}{t}Q_{k-1}(t) - \frac{\overline{\Phi_k(0)}}{t^{k+1}}\overline{Q_{k-1}\Big(\frac{1}{\bar t}\Big)}, \quad \forall k \geq 1$$

The dual relation

(13) $$\overline{Q_k\Big(\frac{1}{\bar t}\Big)} = t\overline{Q_{k-1}\Big(\frac{1}{\bar t}\Big)} - t^{k+1}\Phi_k(0)\,Q_{k-1}(t), \quad \forall k \geq 1$$

follows easily and the above equations can be written in the compact form

(14) $$\mathcal{G}(t,k) = \mathcal{T}(t,k)\,\mathcal{G}(t,k-1), \quad \forall k \geq 1$$

https://doi.org/10.4153/CJM-1993-052-1 Published online by Cambridge University Press

with

$$\mathcal{G}(t,k-1) = \begin{bmatrix} \dfrac{Q_{k-1}(t)}{\overline{Q_{k-1}(\frac{1}{\bar{t}})}} \end{bmatrix}$$

and

$$\mathcal{T}(t,k) = \begin{bmatrix} \frac{1}{t} & -\frac{\overline{\Phi_k(0)}}{t^{k+1}} \\ -t^{k+1}\Phi_k(0) & t \end{bmatrix}$$

with the initial conditions

$$Q_0(t) = \int_T \frac{d\mu(z)}{t-z} = \bar{t}c_{00}(\mu_2) - c_{01}(\mu_2)$$

and

$$\overline{Q_0\Big(\frac{1}{\bar{t}}\Big)} = tc_{00}(\mu) - t^2 Q_0(t). \qquad \blacksquare$$

EXAMPLES. Let $d\theta$ be the Lebesgue measure. In $\mathbb{P}$ an inner product is defined as

$$\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta.$$

The sequence of monic orthogonal polynomials associated to $d\theta$ on T is

$$\forall n \geq 0 \quad \Phi_n(z;\theta) = z^n = \varphi_n(z;\theta).$$

EXAMPLE 3.1.1. If we consider on T

$$d\mu_2 = \frac{d\theta}{|z-\alpha|^2} \text{ with } |\alpha| > 1$$

and using (10), the monic orthogonal polynomial sequence on T with respect to μ_2 is given by

$$\Phi_{n+1}(z;\mu_2) = z^n\Big(z - \frac{1}{\bar{\alpha}}\Big), \quad \forall n \geq 0 \text{ and } \Phi_0(z;\mu_2) = 1.$$

EXAMPLE 3.1.2. If we consider on T

$$d\mu_2 = \frac{|z-1|^2}{|z-\alpha|^2}\,d\theta \text{ with } |\alpha| > 1$$

it follows (see [8]) that if we define

$$d\mu_1 = |z-1|^2\,d\theta$$

then

$$\Phi_n(z;\mu_1) = \sum_{k=0}^{n} \frac{k+1}{n+1} z^k, \quad \forall n \geq 0$$

The measure $d\mu_1$ gives a special case of ultraspherical polynomials on the unit circle. Ultraspherical polynomials on the unit circle have been considered earlier by Delsarte and Genin [1, 5], but also by Golinskii [9]. We use again Proposition 5. By means of (10)

$$\Phi_n(z;\mu_2) = z^n + \frac{[1-\frac{1}{\bar{\alpha}}]}{[n+\frac{1}{\bar{\alpha}-1}][1-\frac{1}{\alpha}]+1}\Big[\sum_{k=0}^{n-1}\Big(k+1-\frac{k}{\alpha}\Big)z^k\Big], \quad \forall n \geq 1$$

and $\Phi_0(z, \mu_2) = 1$ holds ∎

3 2 *Example on the Bernoulli lemniscate* Another important case study is Bernoulli lemniscate BL given by

$$\mathrm{BL} = \{z \in \mathbb{C} \mid |z^2 - 1| = 1\} = \{z \in \mathrm{BL} \mid \operatorname{Re} z \geq 0\} \cup \{z \in \mathrm{BL} \mid \operatorname{Re} z \leq 0\}$$

where we can consider the measure

$$\mu(z) = \arg(z)$$

so that

$$\mu(z) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$$

and (see [12])

$$\Phi_0(z, \mu) = 1 \quad \Phi_1(z, \mu) = z$$

$$\Phi_{2n}(z, \mu) = (z^2 - 1)^n \quad \Phi_{2n+1}(z, \mu) = \frac{z}{\Delta_{n-1}} \begin{vmatrix} d_{00} & \cdots & d_{n0} \\ \vdots & & \vdots \\ d_{0\,n-1} & \cdots & d_{n\,n-1} \\ 1 & \cdots & (z^2 - 1)^n \end{vmatrix}$$

with

$$d_{kj} = \frac{(-1)^{k-j}\, 4}{1 - 4(k - j)^2}$$

Since $\mathbb{P} = \mathbb{P}(z^2 - 1) \oplus z\mathbb{P}(z^2 - 1)$ where $\mathbb{P}(z^2 - 1)$ is the linear subspace of the complex polynomials in $z^2 - 1$, from the above monic orthogonal polynomials on BL, we can define two sequences of monic orthogonal polynomials on T given by

$$\Phi_{2n}(z, \mu) = w^n \text{ and } \Phi_{2n+1}(z, \mu) = z\Psi_n(w, \sigma), \quad \forall n \geq 0$$

where $w = z^2 - 1$ and $\{\Psi_n(w, \sigma)\}$ is a MOPS with respect to the measure $\sigma(\theta)$ on T with

$$d\sigma(\theta) = \cos\frac{\theta}{2}\, d\theta$$

This absolutely continuous measure σ on T is induced by

$$\tau(x) = \frac{1}{\sqrt{1 - x}}$$

on the interval $-1 \leq x \leq +1$ and again gives ultraspherical polynomials on the unit circle Then, $\Psi_n(w, \sigma)$ is related with monic Jacobi polynomials $P_n^{(-1/2,0)}(x)$ as follows

$$\Psi_{2n}(w, \sigma) = 2^{n-1} w^n \left[\left(1 + \frac{1}{4n+1}\right) P_n^{(-1/2,0)}(x) + \frac{w - w^{-1}}{2}\left(1 - \frac{1}{4n+1}\right) P_{n-1}^{(1/2,1)}(x)\right]$$

and

$$\Psi_{2n-1}(w, \sigma) = (2w)^{n-1}\left[P_n^{(-1/2,0)}(x) + \frac{w - w^{-1}}{2} P_{n-1}^{(1/2,1)}(x)\right]$$

where

$$x = \frac{w + w^{-1}}{2}$$

with

$$\Psi_n(0; \sigma) = \frac{(-1)^n}{2n + 1}$$

(see [15], pp. 294–295, (11.5.4.)).

EXAMPLE 3.2.1. If we define on BL

$$d\mu_2 = \frac{d\mu}{|z - \alpha|^2}, \text{ with } \alpha \in \mathbb{R} \text{ and } \alpha \in \operatorname{Ext} \mathrm{BL}$$

then

$$\|\mu_2\| = \int_{\mathrm{BL}} \frac{d\mu}{|z - \alpha|^2} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{d\mu}{|z - \alpha|^2} + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{d\mu}{|z - \alpha|^2} = \frac{1}{4} \int_{-\pi}^{\pi} \frac{4 \cos \frac{\theta}{2} + 2\alpha^2}{1 + \beta^2 - 2\beta \cos \theta} \, d\theta$$

where $\beta = \alpha^2 - 1$, so that

$$\|\mu_2\| = \frac{\pi}{\alpha^2 - 2} + \frac{2}{(\alpha^2 - 2)\sqrt{\alpha^2 - 1}} \operatorname{arctg} \frac{2\sqrt{\alpha^2 - 1}}{\alpha^2 - 2}.$$

Moreover,

$$\overline{Q_{2k}(\alpha)} = \int_{\mathrm{BL}} \frac{(z^2 - 1)^k}{\alpha - \bar{z}} \, d\mu = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(z^2 - 1)^k}{\alpha - \bar{z}} \, d\mu + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \frac{(z^2 - 1)^k}{\alpha - \bar{z}} \, d\mu = \frac{\pi\alpha}{2\pi i \beta} \int_{-\pi}^{\pi} \frac{w^k \, dw}{w - \frac{1}{\beta}}.$$

By application of the residue formula, we get

$$Q_{2k}(\alpha) = \frac{\pi\alpha}{(\alpha^2 - 1)^{k+1}}.$$

Also,

$$\overline{Q_{2k+1}(\alpha)} = \int_{\mathrm{BL}} \frac{z\Psi_k(z^2 - 1)}{\alpha - \bar{z}} \, d\mu = \int_{-\pi}^{\pi} \frac{\Psi_k(e^{i\theta}) \cos \frac{\theta}{2}}{\beta - e^{-i\theta}} \, d\theta = \frac{1}{i\beta} \int_T \frac{\Psi_k(z)}{z - \frac{1}{\beta}} \cos \frac{\theta}{2} \, dz = I_1 + I_2$$

where

$$I_1 = \frac{1}{i\beta} \int_T \frac{\Psi_k(z) - \Psi_k(\frac{1}{\beta})}{z - \frac{1}{\beta}} \cos \frac{\theta}{2} \, dz = \frac{1}{2i\beta} \int_T g(z) z^{-1/2} \, dz$$

with

$$g(z) = (z + 1) \frac{\Psi_k(z) - \Psi_k(\frac{1}{\beta})}{z - \frac{1}{\beta}} = \sum_{j=0}^{k} a_{kj} z^j$$

a polynomial of degree k. Thus,

$$I_1 = \frac{1}{2\beta} \int_{-\pi}^{\pi} g(e^{i\theta}) e^{i\theta/2} \, d\theta = \frac{2}{\beta} \sum_{j=0}^{k} (-1)^j \frac{a_{kj}}{2j + 1}.$$

Similarly, one obtains for the second integral,

$$I_2 = \frac{\Psi_k(\frac{1}{\beta})}{i\beta}\int_T \frac{\cos\frac{\theta}{2}}{z-\frac{1}{\beta}}\,dz = \frac{\Psi_k(\frac{1}{\beta})}{\beta}\int_{-\pi}^{\pi}\frac{e^{i\theta}}{e^{i\theta}-\frac{1}{\beta}}\cos\frac{\theta}{2}\,d\theta = \Psi_k\Big(\frac{1}{\beta}\Big)\sum_{j=0}^{\infty}\frac{(-\frac{1}{\beta})^{j+1}}{j^2-\frac{1}{4}}$$

The result is

$$Q_{2k+1}(\alpha) = \frac{2}{\alpha^2-1}\sum_{j=0}^{k}(-1)^j\frac{a_{kj}}{2j+1} + \Psi_k\Big(\frac{1}{\alpha^2-1}\Big)\sum_{j=0}^{\infty}\frac{[-\frac{1}{\alpha^2-1}]^{j+1}}{j^2-\frac{1}{4}}$$

where

$$(z+1)\frac{\Psi_k(z,\sigma)-\Psi_k(\frac{1}{\alpha^2-1},\sigma)}{z-\frac{1}{\alpha^2-1}} = \sum_{j=0}^{k} a_{kj}z^j$$

and the polynomials $\Phi_n(z,\mu_2)$ can be computed by using (3)

ACKNOWLEDGEMENTS We are grateful to the referee for suggestions and remarks

REFERENCES

1. P Delsarte and Y Genin, *On the role of orthogonal polynomials on the unit circle in digital signal processing applications*, Orthogonal Polynomials Theory and Practice, (ed P Nevai), NATO ASI (C) **294** Kluwer Dordrecht, 1990, 115–133
2. J Dini and P Maroni, *The product of a linear form by a rational fraction Application to Laguerre Hahn forms*, Lecture Notes in Pure and Applied Mathematics **117**, Orthogonal Polynomials and their Applications, (ed J Vinuesa), Marcel Dekker, 1989, New York, 131–138
3. W Gautschi, *An algorithmic implementation of the Christoffel theorem* In Numerical Integration (ed G Hammerlin), Birkhauser Basel, 1982, 89–106
4. ———, *Computational aspects of orthogonal polynomials*, Orthogonal Polynomials Theory and Practice (ed P Nevai), NATO ASI (C) **294**, Kluwer, Dordrecht, 1990, 181–216
5. Y Genin, *On a duality in the theory of orthogonal polynomials and its application in signal processing* ICIAM '87, Proceedings of the first international conference on industrial and applied mathematics, SIAM Philadelphia, (1988), 102–113
6. E Godoy, *Polinomios ortogonales asociados a modificaciones de medidas*, Doctoral Dissertation Univ Santiago de Compostela, 1987
7. E Godoy and F Marcellan, *Rational modifications of Borel measures* Actas XII Jornadas Luso Espanholas de Matematica **II**(1987), 326–330
8. ———, *An analog of the Christoffel formula for polynomial modification of a measure on the unit circle* Boll Un Mat Ital (7) **5-A**(1991), 1–12
9. B L Golinskii, *On the asymptotic behavior of the prediction error*, Theory Prob Appl **19**(1974) 693–709
10. R A Horn and C R Johnson, *Matrix Analysis*, Cambridge University Press, 1985
11. M E H Ismail and R W Ruedemann, *Relation between polynomials orthogonal on the unit circle with respect to different weights*, J Approx Theory **71**(1992), 39–60
12. R Kumar, *A class of quadrature formulas*, Math Comp (127) **28**(1974), 769–778
13. F Marcellan and L Moral, *Propiedades asintoticas de los polinomios ortogonales sobre la lemniscata de Bernouilli*, Actas del III Simposium sobre *Polinomios Ortogonales y Aplicaciones*, F Marcellan Editor Segovia (1985), 123–133
14. S Paszkowski, *Sur des Transformations d une fonction de poids*, Lecture Notes in Mathematics (1171) Polynômes Orthogonaux et Applications, (eds C Brezinski, et al), Proceedings, Bar le Duc, 1984 Springer-Verlag, 1985, 239–246
15. G Szego, *Orthogonal Polynomials*, A M S , Colloquium Publications **XXIII**, fourth edition second printing, Providence, Rhode Island, 1978

16. V B Uvarov, *The connection between systems of polynomials orthogonal with respect to different distri bution functions*, USSR Comp Math and Phys (6) **9**(1969), 25–36
17. J Walsh, *Interpolation and Approximation by Rational Functions in the Complex Plane*, A M S , Colloquium Publications, Providence, Rhode Island, Fifth edition, 1969

Universidad de Vigo
Departamento de Matematica Aplicada
Escuela Tecnica Superior de Ingenieros Industriales
Apartado de Correos No 62
36280 VIGO
Spain

Universidad Carlos III de Madrid
Escuela Politecnica Superior
Avda del Mediterraneo s/n
28913 LEGANES
Madrid
Spain