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A Division Algorithm for the Gaussian Integers' Minimal Euclidean Function

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Abstract. **The usual division algorithms on $\mathbb{Z}$ and $\mathbb{Z}[i]$ measure the size of remainders using the algebraic norm. These rings are Euclidean with respect to several functions. The pointwise minimum of all Euclidean functions $f : R \setminus \{0\} \to \mathbb{N}$ on a Euclidean domain R is itself a Euclidean function, called the minimal Euclidean function and denoted by ϕ_R. To the author's knowledge, the integers, $\mathbb{Z}$, and the Gaussians, $\mathbb{Z}[i]$, are the only rings of integers of number fields for which we have a formula to compute their minimal Euclidean functions, $\phi_{\mathbb{Z}}$ and $\phi_{\mathbb{Z}[i]}$. This paper presents the first division algorithm (that the author knows of) for $\mathbb{Z}[i]$ relative to $\phi_{\mathbb{Z}[i]}$, empowering readers to perform the Euclidean algorithm on $\mathbb{Z}[i]$ using its minimal Euclidean function.**

1 Introduction

We call $\mathbb{Z}[i] = \{x + yi : x, y \in \mathbb{Z}, i^2 = -1\}$ the Gaussian integers because Gauss showed the domain has unique factorization. Due to Dirichlet's realization that Euclidean rings have unique factorization, we say the Gaussian integers form a unique factorization domain because they are Euclidean for the algebraic (or field) norm, $\mathrm{Nm}(x + yi) = x^2 + y^2$. The standard proof that $\mathbb{Z}[i]$ is Euclidean uses a division algorithm that, given $a, b \in \mathbb{Z}[i] \setminus \{0\}$, provides $q, r \in \mathbb{Z}[i]$ such that $a = qb + r$ and $\mathrm{Nm}(r) < \frac{\mathrm{Nm}(b)}{2}$ (see Equation 2.2). Accordingly, we call r the Gauss remainder for a and b.

Inspired by Zariski, Motzkin [3] broadened the study of Euclidean domains via Euclidean functions. A domain R is Euclidean if there exists a **Euclidean function** $f : R \setminus \{0\} \to W$, where W is a well-ordered set with $\mathbb{N}$ as an initial segment, such that for all $a, b \in R \setminus \{0\}$, there exist some $q, r \in R$ such that $a = qb + r$, where either $f(r) < f(b)$ or $r = 0$. Using this modern terminology, the algebraic norm is a Euclidean function for $\mathbb{Z}[i]$.

Motzkin [3] further showed that if F is the set of all Euclidean functions on $R \setminus \{0\}$, then $\phi_R(x) = \min_{f \in F} f(x)$ is itself a Euclidean function. For obvious reasons, we call ϕ_R the **minimal Euclidean function** on R. In particular, he showed $\phi_{\mathbb{Z}}(x) = \lfloor \log_2 |x| \rfloor$. Until 2023, this was the only formula the author knew of to compute the minimal Euclidean function for any number field.

Just as every integer has a binary expansion, every Gaussian integer has $(1 + i)$-ary expansions. This is fitting, as $2 = -i(1 + i)^2$ and the quotient $\mathbb{Z}[i]/\langle 1 + i \rangle$ has size 2. We use $(1 + i)$-expansions of the form $x + yi = \sum_{j=0}^{n} u_j (1 + i)^j$ for some n, where

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$u_j \in \{\pm 1, \pm i, 0\}$ and $u_n \neq 0$. These expressions are not unique, as

$$2 + i = -i(1+i)^2 + i = (1+i) + 1. \tag{1.1}$$

Lenstra ([2], section 11) showed that $\phi_{\mathbb{Z}[i]}(x+yi)$ is the minimal degree of all $(1+i)$-ary expansions of $x+yi$, e.g., $\phi_{\mathbb{Z}[i]}(2+i) = \deg_{1+i}((1+i)+1) = 1$. Therefore, if $x+yi \in \mathbb{Z}[i] \setminus \{0\}$, then $\phi_{\mathbb{Z}[i]}((1+i)^n x) = \phi_{\mathbb{Z}[i]}(x) + n$ and $\phi_{\mathbb{Z}[i]}(ux) = \phi_{\mathbb{Z}[i]}(x)$ for all u in $\mathbb{Z}[i]$'s group of multiplicative units, $\mathbb{Z}[i]^\times = \{\pm 1, \pm i\}$. He did not, however, provide methods to calculate $\phi_{\mathbb{Z}[i]}$ for a generic $x+yi \in \mathbb{Z}[i] \setminus \{0\}$; Equation 1.1 shows computing $\phi_{\mathbb{Z}[i]}$ is not straightforward.

The author's recent research gives an explicit formula for $\phi_{\mathbb{Z}[i]}(x+yi)$, using valuations and the sequence $\{w_m\}$ [1]. Figure 1 shows which Gaussian integers map to $\{0,1,2,3\}$ under $\phi_{\mathbb{Z}[i]}$. An explanation of this work requires a great deal of notation.

1.1 Notations and Definitions

1.1.1 Notation

- Every complex number has a real and imaginary part, denoted by $\mathrm{Re}(x+yi) = x$ and $\mathrm{Im}(x+yi) = y$.
- The complex conjugate of z is $\overline{z} = \mathrm{Re}(z) - \mathrm{Im}(z)i$.
- The algebraic norm of $z = x+yi \in \mathbb{Q}(i)$ is $z \cdot \overline{z} = x^2 + y^2$.
- If a divides b, we write $a|b$. Otherwise, $a \nmid b$.
- When $a^c | b$ but $a^{c+1} \nmid b$, we say c is the a-valuation of b, or $v_a(b) = c$
- We use $\gcd(x,y)$ to denote the greatest common divisor of x and y.
- For $x \in \mathbb{R}$, $\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$.
- The ℓ_1-norm is $\ell_1(x+yi) = |x| + |y|$ and the ℓ_∞-norm is $\ell_\infty(x+yi) = \max(|x|, |y|)$.
- The multiplicative units of $\mathbb{Z}[i]$ are $\mathbb{Z}[i]^\times = \{\pm 1, \pm i\}$.
- If x is a real number and n is the integer such that $n \leq x < n+1$, we write $\lfloor x \rfloor = n$ and $\lceil x \rceil = n+1$. We also use the nearest integer function, given by

$$\lfloor x \rceil = \begin{cases} \lfloor x \rfloor & \text{if } 0 \leq x - \lfloor x \rfloor \leq 1/2 \\ \lceil x \rceil & \text{otherwise} \end{cases}.$$

- The minimal Euclidean function on a domain R is ϕ_R. We calculate $\phi_{\mathbb{Z}[i]}$ using the formula in Theorem 1.5; $\phi_{\mathbb{Z}[i]}(x)$ computes the length of x's $(1+i)$-ary expansion.

1.1.2 Definitions

Definition **1.1** We define the sequence $w_m = \begin{cases} 3 \cdot 2^k & \text{if } m = 2k \\ 4 \cdot 2^k & \text{if } m = 2k+1 \end{cases}$.

We repeatedly use that $w_{m+2} = 2w_m$ and that if $2^l < w_n$, then $2^l | w_m$ for all $m > n$. Note that if $2^l < w_n$ and $2^l \nmid w_n$, then $w_n = 3 \cdot 2^{l-1}$.

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Example 1.1 The sequence begins with $w_0 = 3$, $w_1 = 4$, $w_2 = 6$, $w_3 = 8$, $w_4 = 12$, $w_5 = 16$, etc.

***Definition* 1.2** The function $m(z) = \ell_1(z) - \ell_\infty(z)$ is the minimum of the absolute values of the real and imaginary parts of z.

Example 1.2 We see $m(7 - 2i) = 2$, $m(1 + i) = 1$, and $m(7) = m(7i) = 0$.

***Definition* 1.3** If $z \in \mathbb{C} \setminus \mathbb{Z}\{1 \pm i\}$, denote the unique unit such that $\mathrm{Re}(u_z z) = \ell_\infty(z)$ by u_z . If $z \in \mathbb{Z}\{1 \pm i\} \setminus \{0\}$, u_z is the unique unit such that $u_z z = \ell_\infty(z)(1 + i)$.

Example 1.3 Note $u_{2+i} = 1$, $u_{1-i} = i$, and $u_{-3+5i} = -i$.

***Definition* 1.4** For $r \in \mathbb{C}$, we define $s(r) = \mathrm{sgn}(\mathrm{Im}(u_r r))$.

Example 1.4 If $r = -3 + 5i$, $s(r) = \mathrm{sgn}(\mathrm{Im}(-i(-3 + 5i))) = \mathrm{sgn}(\mathrm{Im}(5 + 3i)) = \mathrm{sgn}(3) = 1$.

1.1.3 Background

Complex conjugation is multiplicative, so that $\overline{xy} = \overline{x} \cdot \overline{y}$, and hence the norm is also multiplicative. In other words, if $a, b \in \mathbb{Q}(i)$, then $\mathrm{Nm}(ab) = \mathrm{Nm}(a)\mathrm{Nm}(b)$. The Gaussians' multiplicative units, $\mathbb{Z}[i]^\times = \{\pm 1, \pm i\}$, have norm one and the set is closed under complex conjugation, so both the norm and $\phi_{\mathbb{Z}[i]}$ are invariant under complex conjugation and multiplication by units. If $u \in \mathbb{Z}[i]^\times$ and $z \in \mathbb{Z}[i] \setminus \{0\}$, then $\mathrm{Nm}(z) = \mathrm{Nm}(\overline{z}) = \mathrm{Nm}(uz)$ and $\phi_{\mathbb{Z}[i]}(z) = \phi_{\mathbb{Z}[i]}(\overline{z}) = \phi_{\mathbb{Z}[i]}(uz)$.

1.2 Using the minimal Euclidean function on $\mathbb{Z}[i]$

Here, we present the formula for $\phi_{\mathbb{Z}[i]}$, and a division algorithm on $\mathbb{Z}[i]$ using $\phi_{\mathbb{Z}[i]}$.

***Theorem* 1.5** *(Graves, [1]) Given $x + yi \in \mathbb{Z}[i] \setminus \{0\}$, let $j = v_2(\gcd(x, y))$ and n be the smallest integer such that $\max\left(\frac{|x|}{2^j}, \frac{|y|}{2^j}\right) \leq w_n - 2$. If $\frac{|x|+|y|}{2^j} \leq w_{n+1} - 3$, then $\phi_{Z[i]}(x + yi) = n + 2j$. Otherwise, $\phi_{\mathbb{Z}[i]}(x + yi) = n + 2j + 1$.*

Thus if x and y are N-bit integers, we can compute both $\phi_{\mathbb{Z}[i]}(x + yi)$ and one of $x + yi$'s minimal $(1 + i)$-ary expansions in $O(\log N)$ time, as shown in Section 4 of [1].

***Corollary* 1.6** *(of Theorem 1.5) If $z \in \mathbb{Z}[i] \setminus \{0\}$, if $\ell_\infty(z) \leq w_n - 2^{v_2(z)+1}$, and if $\ell_1(z) \leq w_{n+1} - 3 \cdot 2^{v_2(z)}$, then $\phi_{\mathbb{Z}[i]}(z) \leq n$. Note $\phi_{\mathbb{Z}[i]}(z) > n$ if and only if either $\ell_\infty(z) > w_n - 2^{v_2(z)+1}$ or $\ell_1(z) > w_{n+1} - 3 \cdot 2^{v_2(z)}$.*

Given $a, b \in \mathbb{Z}[i] \setminus \{0\}$, Theorem 1.7 below shows how to adjust the Gauss quotient and the Gauss remainder of a and b to find new quotients and remainders $q, r \in \mathbb{Z}[i]$ such that $a = qb + r$, where either $r = 0$ or $\phi_{\mathbb{Z}[i]}(r) < \phi_{\mathbb{Z}[i]}(b)$. Interestingly, there

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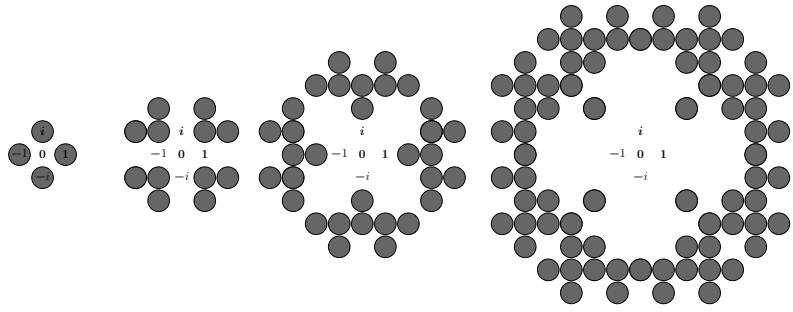

(a) $\phi^{-1}_{\mathbb{Z}[i]}(0)$ (b) $\phi^{-1}_{\mathbb{Z}[i]}(1)$ (c) $\phi^{-1}_{\mathbb{Z}[i]}(2)$ (d) $\phi^{-1}_{\mathbb{Z}[i]}(3)$

Figure 1: Pre-images of $\phi_{\mathbb{Z}[i]}$.

are times we do not have to compute the Gauss remainder to know whether we need to adapt it for $\phi_{\mathbb{Z}[i]}$.

***Theorem* 1.7** *Suppose $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ with $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b) = n$ and $a = qb + r$. Consider the following conditions:*

(1) $2^{v_2(r)} \nmid w_{n-1}$

(2) $Im(u_b b) Im(u_r r) \geq 0$

(3) $m(r) + m(b) \leq \ell_\infty(r)$

(4) $m(b) < \ell_\infty(r) < m(b) + m(r)$ *and* $\ell_\infty(b) - m(r) > w_{n-1} - 2^{v_2(b)+1}$

If any of the conditions hold, then $a = \left(q + \frac{u_b}{u_r}\right) b + \left(r - \frac{u_b}{u_r} b\right)$, *with* $\phi_{\mathbb{Z}[i]}\left(r - \frac{u_b}{u_r} b\right) < \phi_{\mathbb{Z}[i]}(b)$. *If none hold, then* $a = \left(q + \frac{iu_b}{s(r)u_r}\right) b + \left(r - \frac{iu_b}{s(r)u_r} b\right)$, *with* $\phi_{\mathbb{Z}[i]}\left(r - \frac{iu_b}{s(r)u_r} b\right) < \phi_{\mathbb{Z}[i]}(b)$.

The rest of the paper builds the machinery to prove Theorem 1.7. Section 2 presents the standard division algorithm.

Note that while Theorem 1.7 gives an algorithm to find a quotient and remainder using the minimal Euclidean function, the quotient and remainder are not necessarily unique. The two expansions in Equation 1.1 give two different quotient and remainder pairs with respect to the Gaussian's minimal Euclidean function for $2+i$ and $1+i$, $(2, -i)$ and $(1, 1)$.

It also constructs and presents properties of the ensuing (Gauss) remainder for $a, b \in \mathbb{Z}[i]$. Section 3 shows that if $v_2(r) \leq v_2(b)$, then $\phi_{\mathbb{Z}[i]}(r) < \phi_{\mathbb{Z}[i]}(b)$. Section 4 presents two possible alternate remainders for when $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b)$: $r - \frac{u_b}{u_r} b$ and $r - \frac{iu_b}{s(r)u_r} b$. Both sections introduce properties of r and b when $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b)$. Lemma 5.1 shows the absolute values of the real and imaginary parts of the alternate

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remainders are respectively elements of

$$\{|\ell_\infty(b) - \ell_\infty(r)|, m(b) + m(r)\} \text{ and } \{\ell_\infty(b) - m(r), |\ell_\infty(r) - m(b)|\}. \tag{1.2}$$

We therefore use the bounds on $\ell_\infty(r)$, $\ell_1(r)$, $m(r)$ and $\ell_\infty(b) - \ell_\infty(r)$ calculated in Section 4 to bound the sums and differences in (1.2). Lemma 4.6, Proposition 4.12, and Section 5's lemmas each assume one of the conditions in Theorem 1.7. They appropriately bound a subset of Equation (1.2)'s terms to prove which alternative remainder x satisfies $\phi_{\mathbb{Z}[i]}(x) < \phi_{\mathbb{Z}[i]}(b)$. Every Proposition and Lemma cited in Theorem 1.7's proof is followed by an example illustrating our division algorithm in the corresponding scenario.

2 The standard division algorithm

Suppose $a, b \in \mathbb{Z}[i] \setminus \{0\}$, with $a = x + yi$ and $b = c + di$. Then $\frac{a}{b}$ equals

$$\frac{a\overline{b}}{b\overline{b}} = \frac{a\overline{b}}{\mathrm{Nm}(b)} = \frac{(xc + yd) + (yc - xd)i}{\mathrm{Nm}(b)}.$$

If $q_0 = \left\lfloor \frac{xc+yd}{\mathrm{Nm}(b)} \right\rceil$, $q_1 = \left\lfloor \frac{yc-xd}{\mathrm{Nm}(b)} \right\rceil$, $f_0 = \frac{xc+yd}{\mathrm{Nm}(b)} - q_0$, and $f_1 = \frac{yc-xd}{\mathrm{Nm}(b)} - q_1$, then

$$\frac{a\overline{b}}{\mathrm{Nm}(b)} = (q_0 + q_1 i) + (f_0 + f_1 i), \text{ where } |f_0|, |f_1| \leq \frac{1}{2}.$$

Multiplying through by b shows

$$a = (q_0 + q_1 i)b + (f_0 + f_1 i)b, \tag{2.1}$$

where $q_0 + q_1 i, (f_0 + f_1 i)b \in \mathbb{Z}[i]$. The norm is multiplicative, so

$$\mathrm{Nm}(r) = \mathrm{Nm}((f_0 + f_1 i)b) = \mathrm{Nm}(f_0 + f_1 i)\mathrm{Nm}(b) \leq \left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right)\mathrm{Nm}(b) \leq \frac{\mathrm{Nm}(b)}{2}. \tag{2.2}$$

We call $f_0 + f_1 i$ the **Gauss fractional remainder of** a **and** b and $(f_0 + f_1 i)b$ the **Gauss remainder of** a **and** b.

Example 2.1 When $a = 7 + 5i$ and $b = 5 + 3i$,

$$\frac{a\overline{b}}{\mathrm{Nm}(b)} = \frac{(7 + 5i)(5 - 3i)}{5^2 + 3^2} = \frac{50 + 4i}{34} = 1 + \frac{8 + 2i}{17},$$

so the Gauss fractional remainder of a and b is $\frac{8+2i}{17}$. We see $a = 1 \cdot b + r$, where

$$r = \left(\frac{8 + 2i}{17}\right)(5 + 3i) = 2 + 2i.$$

Lemma 2.2 *Suppose $a, b, z \in \mathbb{Z}[i] \setminus \{0\}$. The pair a and b have Gauss remainder r if and only if za and zb have Gauss remainder zr.*

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Proof The fraction $\frac{za}{zb} \cdot \frac{\overline{zb}}{\overline{zb}}$ simplifies to $\frac{a}{b}$, so za and zb have the same Gauss fractional remainder as a and b. This means that the Gauss remainder of za and zb is $za - qzb = zr$, or z times the Gauss remainder of a and b. ∎

Example 2.3 Continuing Example 2.1, $2+12i = (7+5i)(1+i)$ and $2+8i = (5+3i)(1+i)$ have Gauss remainder $4i = (2+2i)(1+i)$.

Lemma **2.4** *If $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ and if $\phi_{\mathbb{Z}[i]}(b) = n$, then $\ell_1(r) \leq \ell_\infty(b) < w_n$ and $\ell_\infty(r) \leq \frac{\ell_1(b)}{2} < w_{n-1}$. Equality occurs only if $\frac{r}{b} \in \left\{\pm\frac{1}{2}, \pm\frac{i}{2}, \pm\frac{1}{2} \pm \frac{i}{2}\right\}$.*

Proof For expositional ease, let $b = x + yi$ and denote the Gauss fractional remainder by $f_0 + f_1 i$. Equation 2.1 shows $r = (xf_0 - yf_1) + (xf_1 + yf_0)i$, so

$$\ell_\infty(r) = \max(|xf_0 - yf_1|, |xf_1 + yf_0|) \leq \frac{\ell_1(b)}{2} < \frac{w_{n+1}}{2} = w_{n-1} \tag{2.3}$$

and

$$\begin{aligned} \ell_1(r) &= \max(|xf_0 - yf_1 + xf_1 + yf_0|, |yf_1 - xf_0 + xf_1 + yf_0|) \\ &= \max(|f_0(x+y) + f_1(x-y)|, |f_0(y-x) + f_1(x+y)|) \\ &\leq \frac{\ell_1(b)}{2} + \frac{\ell_\infty(b) - m(b)}{2} = \ell_\infty(b) < w_n. \end{aligned} \tag{2.4}$$

Inequalities (2.3) and (2.4) are strict, unless $f_0, f_1 \in \left\{0, \pm\frac{1}{2}\right\}$. ∎

Corollary **2.5** *If $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ and $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b)$, then $\ell_\infty(r) < \frac{\ell_1(b)}{2}$ and $\ell_1(r) < \ell_\infty(b)$.*

Proof If $f_0, f_1 \in \left\{0, \frac{1}{2}\right\}$, then $\frac{r}{b} \in (\mathbb{Z}[i]^\times)\{(1+i)^{-1}, (1+i)^{-2}\}$ and $\phi_{\mathbb{Z}[i]}(r) \in \{\phi_{\mathbb{Z}[i]}(b) - 1, \phi_{\mathbb{Z}[i]}(b) - 2\}$. Thus $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b)$, our inequalities are strict. ∎

Example 2.6 Staying with Example 2.1, note $\phi_{\mathbb{Z}[i]}(b) = \phi_{\mathbb{Z}[i]}(r) = 3$, $\ell_1(r) = 4 < 5 = \ell_\infty(b) < 8 = w_3$, and $\ell_\infty(r) = 2 < \frac{\ell_1(b)}{2} = \frac{8}{2} = 4 < 6 = w_2$.

3 Gauss remainders and the minimal Euclidean function

Lemma 2.4 shows that if a and b have Gauss remainder r, $\phi_{\mathbb{Z}[i]}(r)$ is often less than $\phi_{\mathbb{Z}[i]}(b)$. Sometimes, we can determine whether $\phi_{\mathbb{Z}[i]}(r) < \phi_{\mathbb{Z}[i]}(b)$ without computing r.

Lemma **3.1** *If $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ and $v_2(r) \leq v_2(b)$, then $\phi_{\mathbb{Z}[i]}(r) < \phi_{\mathbb{Z}[i]}(b) = n$.*

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Proof If we assume either $\ell_1(r) = \ell_\infty(b)$ or $2\ell_\infty(r) = \ell_1(b)$, then Lemma 2.4 shows $\frac{r}{b} \in \pm\frac{1}{2}\{1, i, 1 \pm i\}$, and hence $\phi_{\mathbb{Z}[i]}(r) < \phi_{\mathbb{Z}[i]}(b)$. We therefore assume neither equality holds. Our hypothesis therefore implies

$$2^{v_2(r)} \le \ell_1(r) < \ell_\infty(b) \le w_n - 2^{v_2(b)+1} \le w_n - 2^{v_2(r)+1},$$

which leads to the observation $3 \cdot 2^{v_2(r)} < w_n$. Since $3 \cdot 2^{v_2(r)} \le w_n$, $2^{v_2(r)} | w_m$ for all $m \ge n$ and

$$\ell_1(r) \le w_n - 3 \cdot 2^{v_2(r)}.$$

Our hypothesis also implies

$$\ell_\infty(r) < \frac{\ell_1(b)}{2} \le \frac{w_{n+1} - 3 \cdot 2^{v_2(r)}}{2},$$

so $\ell_\infty(r) \le w_{n-1} - 2^{v_2(r)+1}$ and $\phi_{\mathbb{Z}[i]}(r) \le n - 1 < \phi_{\mathbb{Z}[i]}(b)$. ∎

***Corollary* 3.2** *If $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ and $v_2(a) \le v_2(b)$, then $\phi_{\mathbb{Z}[i]}(r) < \phi_{\mathbb{Z}[i]}(b)$.*

Example 3.3 The Gauss remainder r of $a = 9 + 4i$ and $b = 3 + 5i$ satisfies $\phi_{\mathbb{Z}[i]}(r) < \phi_{\mathbb{Z}[i]}(b)$ because 2 divides neither a nor b.

***Corollary* 3.4** *If $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ with $\phi_{\mathbb{Z}[i]}(r) \ge \phi_{\mathbb{Z}[i]}(b) = n$, then $2^{v_2(b)+1} | w_n$.*

Proof We prove the contrapositive and assume $2^{v_2(b)+1} \nmid w_n$. Jointly, our hypotheses and $2^{v_2(b)} \le \ell_\infty(b) \le w_n - 2^{v_2(b)+1}$ demonstrate $w_n = 3 \cdot 2^{v_2(b)}$. By Lemma 2.4, $2^{v_2(r)} \le \ell_1(r) < \ell_\infty(b) = 2^{v_2(b)}$. Lemma 3.1 then shows $\phi_{\mathbb{Z}[i]}(r) < \phi_{\mathbb{Z}[i]}(b)$. ∎

Example 3.5 Suppose $k = v_2(b)$ and $\phi_{\mathbb{Z}[i]}(b) = n$. The condition $2^{k+1} \nmid w_n$ holds if and only if $\ell_\infty(b) = 2^k$, $n = 2k$, and $m(b) = 0$. For all $a \in \mathbb{Z}[i]$, the Gauss remainder r of a and 2^k satisfies $\phi_{\mathbb{Z}[i]}(r) < \phi_{\mathbb{Z}[i]}(b)$.

4 Constructing Alternate Remainders

Example 4.1 shows that, unlike in Corollary 3.2, computing a remainder for $a, b \in \mathbb{Z}[i] \setminus \{0\}$ and $\phi_{\mathbb{Z}[i]}$ is not so simple when $v_2(r) > v_2(b)$. In this section, we construct alternate remainders, for when $\phi_{\mathbb{Z}[i]}(r) \ge \phi_{\mathbb{Z}[i]}(b)$. The new remainder determines the new quotient. If R is the new remainder, then the associated quotient is the integer $\frac{a-R}{b}$.

Example 4.1 Observe $v_2(4 + i) = 0$, $\phi_{\mathbb{Z}[i]}(2i) = 2 = \phi_{\mathbb{Z}[i]}(4 + i)$, and that the Gauss remainder of 9 and $4 + i$ is $2i$, so $2i$ is not the pair's remainder for $\phi_{\mathbb{Z}[i]}$.

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4.1 Valuations and Preliminaries

Lemma 4.2 *If $\phi_{\mathbb{Z}[i]}(b) = n > 0$, then $m(b) \leq w_{n-1} - 2^{v_2(b)+1}$.*

Proof Since $2^{v_2(b)} \leq w_n - 2^{v_2(b)+1}$, $3 \cdot 2^{v_2(b)} \leq w_n$ and $2^{v_2(b)+1} | w_m$ for all $m \geq n+1$. The bound

$$2m(b) \leq \ell_1(b) \leq w_{n+1} - 3 \cdot 2^{v_2(b)}$$

therefore implies

$$2m(b) \leq w_{n+1} - 4 \cdot 2^{v_2(b)}.$$

Dividing by two finishes the proof. ∎

Example 4.3 If $b = 5+3i$, $\phi_{\mathbb{Z}[i]}(b) = 3$, $v_2(b) = 0$, $w_2 = 6$ and $m(b) = 3 < 6 - 2 = 4$.

Lemma 4.4 *If $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ and $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b) = n$, then $\ell_\infty(b) - m(b) \leq w_n - 3 \cdot 2^{v_2(b)}$.*

Proof If $m(b) > 0$, then

$$\ell_\infty(b) - m(b) \leq w_n - 2^{v_2(b)+1} - 2^{v_2(b)} = w_n - 3 \cdot 2^{v_2(b)}$$

by Theorem 1.5. Corollary 3.4 states $2^{v_2(b)+1} | w_n$, so $2^{v_2(b)+1} | (w_n - 2^{v_2(b)+1})$. When $m(b) = 0$, we see $2^{v_2(b)+1} \nmid \ell_\infty(b)$. Theorem 1.5 therefore implies

$$\ell_\infty(b) - m(b) = \ell_\infty(b) < w_n - 2^{v_2(b)+1}. \tag{4.1}$$

Since $2^{v_2(b)}$ divides all terms in Equation 4.1, our claim follows. ∎

Example 4.5 The pair $a = 21 + 8i$ and $b = 13 + 8i$ have Gauss remainder $r = 8$, and $\phi_{\mathbb{Z}[i]}(8) = 6 > 5 = \phi_{\mathbb{Z}[i]}(13 + 8i)$. Observe $2^{v_2(b)} = 2$, $w_5 = 16$, and

$$\ell_\infty(b) - m(b) = 13 - 8 = 5 < 16 - 3.$$

Lemma 4.6 *Suppose $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ with $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b) = n$. If $2^{v_2(r)} \nmid w_{n-1}$, then $m(r) = 0$. Furthermore, $\phi_{\mathbb{Z}[i]}\left(r - \frac{u_b}{u_r} b\right) < \phi_{\mathbb{Z}[i]}(b)$.*

Proof Since Lemma 2.4 shows $2^{v_2(r)} \leq \ell_\infty(r) \leq \frac{\ell_1(b)}{2} < w_{n-1}$, our hypothesis implies $w_n = 2^{v_2(r)+1}$ and thus

$$2^{v_2(r)} \leq \ell_\infty(r) \leq \ell_1(r) < \ell_\infty(b) < w_n = 2^{v_2(r)+1}.$$

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We infer $\ell_\infty(r) = \ell_1(r)$, $m(r) = 0$, and $u_b b - u_r r \in \{(\ell_\infty(b) - \ell_\infty(r)) \pm m(b)i\}$. Lemma 4.2 shows

$$\begin{aligned}\ell_\infty(u_b b - u_r r) &\le \max(\ell_\infty(b) - \ell_\infty(r), m(b)) \\ &\le (w_n - 2^{v_2(b)+1} - 2^{v_2(r)}, w_{n-1} - 2^{v_2(b)+1}) \\ &\le w_{n-1} - 2^{v_2(b)+1};\end{aligned}$$

we observe

$$\begin{aligned}\ell_1(u_b b - u_r r) &= \ell_1(b) - \ell_\infty(r) \\ &\le w_{n+1} - 3 \cdot 2^{v_2(b)} - 2^{v_2(r)} \\ &= w_n - 3 \cdot 2^{v_2(b)}.\end{aligned}$$

As $v_2(u_b b - u_r r) = v_2(b)$, $\phi_{\mathbb{Z}[i]}\left(r - \frac{u_b}{u_r} b\right) = \phi_{\mathbb{Z}[i]}(u_b b - u_r r) < \phi_{\mathbb{Z}[i]}(b)$. ∎

Example 4.7 Continuing Example 4.5, recall $\phi_{\mathbb{Z}[i]}(b) = 5 < 6 = \phi_{\mathbb{Z}[i]}(r)$. We see $u_b b - u_r r = (13 + 8i) - 8 = 5 + 8i$ and $\phi_{\mathbb{Z}[i]}(5 + 8i) = 4 < 5 = \phi_{\mathbb{Z}[i]}(b)$.

Since we understand what happens when $\phi_{\mathbb{Z}[i]}(r) \ge \phi_{\mathbb{Z}[i]}(b)$ and $2^{v_2(r)} \nmid w_{n-1}$, we study when $2^{v_2(r)} | w_{n-1}$.

***Lemma* 4.8** *Suppose $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$, where $\phi_{\mathbb{Z}[i]}(r) \ge \phi_{\mathbb{Z}[i]}(b) = n$. If $2^{v_2(r)} | w_{n-1}$, then $2^{v_2(r)} | w_m$ for all $m \ge n - 1$ and either*

$$\ell_\infty(r) = w_{n-1} - 2^{v_2(r)} \text{ or } \ell_1(r) \ge w_n - 2^{v_2(r)+1}.$$

In both cases,

$$\ell_\infty(r) \ge w_n - w_{n-1} \text{ and } \ell_1(r) \ge w_{n-1} - 2^{v_2(r)}.$$

If $\ell_\infty(r) \neq w_{n-1} - 2^{v_2(r)}$, then $m(r) \ge w_n - w_{n-1}$.

Proof The first line of Lemma 4.6's proof shows $2^{v_2(r)} \le \ell_\infty(r) < w_{n-1}$, so $\ell_\infty(r) \le w_{n-1} - 2^{v_2(r)}$. Since $2^{v_2(r)+1} \le w_{n-1}$, Definition 1.1 shows $2^{v_2(r)} | w_m$ for all $m \ge n - 1$. Our assumption that $\phi_{\mathbb{Z}[i]}(r) > n - 1$ and Corollary 1.6 show either $\ell_\infty(r) = w_{n-1} - 2^{v_2(r)}$ or $\ell_1(r) \ge w_n - 2^{v_2(r)+1}$. Since $\ell_1(r) \ge \ell_\infty(r)$ and $w_n - 2^{v_2(r)+1} \ge w_{n-1} - 2^{v_2(r)}$, we see that in both cases, $\ell_1(r) \ge w_{n-1} - 2^{v_2(r)}$.

If $\ell_{\infty(r)} \le w_{n-1} - 2^{v_2(r)+1}$, then $\ell_1(r) = \ell_\infty(r) + m(r) \ge w_n - 2^{v_2(r)+1}$ implies

$$m(r) \ge w_n - 2^{v_2(r)+1} - (w_{n-1} - 2^{v_2(r)+1}) = w_n - w_{n-1}.$$

In both cases, $\ell_\infty(r) \ge w_n - w_{n-1}$, as $\ell_\infty(r) \ge \min(m(r), w_{n-1} - 2^{v_2(r)})$ and

$$2(w_{n-1}) - 2^{v_2(r)} = w_{n+1} - 2^{v_2(r)} \ge w_n.$$

∎

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Example 4.9 In Example 2.1, $\phi_{\mathbb{Z}[i]}(r) = \phi_{\mathbb{Z}[i]}(b) = 3$. Note $2^{v_2(r)} = 2$ divides $w_2 = 6$ and all w_m, $m \geq 2$. We see $\ell_\infty(r) = 2 = w_2 - 2^{v_2(r)}$ and $\ell_1(r) = 4 > w_3 - 2^{v_2(r)+1} = 8 - 8 = 0$, so our example is in both of Lemma 4.8's (non-exclusive) scenarios. We see $\ell_\infty(r) = m(r) = 2 = w_3 - w_2$ and $\ell_1(r) = 4 > w_2 - 2^{v_2(r)} = 6 - 4 = 2$.

***Corollary* 4.10** *Suppose $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$, with $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b) = n$. If $2^{v_2(r)} | w_{n-1}$, then $m(r)$, $\ell_\infty(r)$, and $\ell_\infty(b) - \ell_\infty(r) \leq w_{n-1} - 2^{v_2(b)+1}$.*

Proof Lemma 3.1 shows $v_2(r) > v_2(b)$, so

$$m(r) \leq \ell_\infty(r) \leq w_{n-1} - 2^{v_2(r)} \leq w_{n-1} - 2^{v_2(b)+1}.$$

By Lemma 4.8,

$$\ell_\infty(b) - \ell_\infty(r) \leq (w_n - 2^{v_2(b)+1}) - (w_n - w_{n-1}) = w_{n-1} - 2^{v_2(b)+1}.$$

■

Example 4.11 Continuing Example 2.1,we see $m(r) = \ell_\infty(r) = 2$ and $\ell_\infty(b) - \ell_\infty(r) = 5 - 2 = 3$ are both less than $w_2 - 2^{v_2(b)+1} = 6 - 2 = 4$.

4.2 When imaginary parts align

Determining an alternate remainder is fairly straightforward when $\mathrm{Im}(u_b b)\mathrm{Im}(u_r r) \geq 0$, i.e., when $u_b b$ and $u_r r$ lie in the same quadrant. They lie in the same quadrant, and not just the same half-plane, because Definition 1.3 ensures $\mathrm{Re}(u_b b), \mathrm{Re}(u_r r) \geq 0$. The next section shows things become much more complicated when $u_b b$ and $u_r r$ lie in different quadrants.

Proposition 4.12 Suppose $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ with $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b) = n$. If $\mathrm{Im}(u_b b)\mathrm{Im}(u_r r) \geq 0$ then $\phi_{\mathbb{Z}[i]}\left(r - \frac{u_b}{u_r} b\right) < \phi_{\mathbb{Z}[i]}(b)$.

Proof Lemma 4.6 lets us assume $2^{v_2(r)} | w_m$ for all $m \geq n - 1$. Since

$$u_b b - u_r r \in \{(\ell_\infty(b) - \ell_\infty(r)) \pm (m(b) - m(r))i\},$$

Corollary 4.10 shows

$$\ell_\infty(u_b b - u_r r) \leq \max(\ell_\infty(b) - \ell_\infty(r), m(b), m(r)) \leq w_{n-1} - 2^{v_2(b)+1}.$$

Observe that, due to Lemmas 4.4 and 4.8,

$$\begin{aligned}
\ell_1(u_b b - u_r r) &= \max(\ell_1(b) - \ell_1(r), \ell_\infty(b) - m(b) + m(r) - \ell_\infty(r)) \\
&\leq \max(w_{n+1} - 3 \cdot 2^{v_2(b)} - (w_{n-1} - 2^{v_2(r)}), \ell_\infty(b) - m(b)) \\
&\leq \max(w_{n-1} + 2^{v_2(r)} - 3 \cdot 2^{v_2(b)}, w_n - 3 \cdot 2^{v_2(b)}) \\
&\leq w_n - 3 \cdot 2^{v_2(b)}.
\end{aligned}$$

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Lemma 3.1 shows $v_2(u_b b - u_r r) = v_2(b)$ and thus $\phi_{\mathbb{Z}[i]}\left(r - \frac{u_b}{u_r} b\right) = \phi_{\mathbb{Z}[i]}(u_b b - u_r r) < \phi_{\mathbb{Z}[i]}(b)$. ∎

Example 4.13 Continuing Example 2.1, note $\mathrm{Im}(u_b b)\mathrm{Im}(u_r r) = 3 \cdot 2 = 6 > 0$ and $\phi_{\mathbb{Z}[i]}(u_b b - u_r r) = \phi_{\mathbb{Z}[i]}(3 + i) = 2 < 3 = \phi_{\mathbb{Z}[i]}(b) = \phi_{\mathbb{Z}[i]}(r)$.

5 Proving our Main Result

When $\mathrm{Im}(u_b b)$ and $\mathrm{Im}(u_r r)$ have opposite signs, finding alternate remainders becomes more complicated. It only partially depends on how r relates to $m(b)$. In this section, we construct alternate remainders for when $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b)$ and $\mathrm{Im}(u_b b)\mathrm{Im}(u_r r) < 0$, allowing us to prove Theorem 1.7.

Lemma 5.1 *Suppose $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ with $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b) = n$. If $Im(u_b b)Im(u_r r) < 0$, then*

$$u_b b - u_r r \in \{(\ell_\infty(b) - \ell_\infty(r)) \pm (m(b) + m(r))i\}$$

and

$$u_b b + s(r) i u_r r \in \{(\ell_\infty(b) - m(r)) \pm (m(b) - \ell_\infty(r))i\}.$$

Proof The first equation follows from the definitions. The assumption $\mathrm{Im}(u_b b)\mathrm{Im}(u_r r) < 0$ implies that if $u_b b = \ell_\infty(b) \pm m(b)i$, then $u_r r = \ell_\infty(r) \mp m(r)i$ and $i u_r r = \pm m(r) + \ell_\infty(r)i$. Since $s(r) = \mathrm{sgn}(\mathrm{Im}(u_r r))$, $s(r) i u_r r = -m(r) \mp \ell_\infty(r)i$. We conclude that $u_b b + s(r) i u_r r = (\ell_\infty(b) - m(r)) \pm (m(b) - \ell_\infty(r))i$. ∎

Example 5.2 The Gauss remainder of $a = 16 + i$ and $b = 10 + 3i$ is $r = 6 - 2i$, with $\phi_{\mathbb{Z}[i]}(b) = \phi_{\mathbb{Z}[i]}(r) = 4$ and $\mathrm{Im}(b)\mathrm{Im}(r) = -6 < 0$. We see

$$u_b b - u_r r = (10 + 3i) - (6 - 2i) = 4 + 5i = (\ell_\infty(b) - \ell_\infty(r)) + (m(b) + m(r))i$$

and

$$u_b b + s(r) i u_r r = (10 + 3i) - i(6 - 2i) = 8 - 3i = (\ell_\infty(b) - m(r)) + (m(b) - \ell_\infty(r))i.$$

Lemma 5.3 *Suppose $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ with $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b) = n$. If $Im(u_b b)Im(u_r r) < 0$ and $m(b) \geq \ell_\infty(r)$, then $\phi_{\mathbb{Z}[i]}\left(r - \frac{u_b i}{s(r) u_r} b\right) < \phi_{\mathbb{Z}[i]}(b)$.*

Proof Harken back to Lemma 3.1 and observe $v_2(b) = v_2(u_b b + s(r) i u_r r)$. Since $m(b) \geq \ell_\infty(r)$, Lemma 5.1 shows

$$\ell_1(u_b b + s(r) i u_r r) = \ell_1(b) - \ell_1(r). \tag{5.1}$$

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Our assumption $\mathrm{Im}(u_b b)\mathrm{Im}(u_r r) < 0$ implies $m(r) \neq 0$. If $\ell_\infty(r) = w_{n-1} - 2^{v_2(r)}$, then

$$\ell_1(r) = \ell_\infty(r) + m(r) \geq (w_{n-1} - 2^{v_2(r)}) + 2^{v_2(r)} = w_{n-1}.$$

Equation 5.1 then shows

$$\ell_1(u_b b + s(r) i u_r r) \leq w_{n+1} - 3 \cdot 2^{v_2(b)} - w_{n-1} = w_{n-1} - 3 \cdot 2^{v_2(b)},$$

demonstrating $\phi_{\mathbb{Z}[i]}(u_b b + s(r) i u_r r) \leq n - 1$.

If $\ell_\infty(r) \neq w_{n-1} - 2^{v_2(r)}$, Lemmas 4.6 and 4.8 show $\ell_1(r) \geq w_n - 2^{v_2(r)+1}$ and $2^{v_2(r)+1}$ divides both w_{n+1} and w_{n+2}. This means $2^{v_2(r)+1} \leq w_{n+2} - w_{n+1}$, so $\ell_1(r) \geq w_{n+1} - w_n$. We deduce from Equation 5.1 that

$$\ell_1(u_b b + s(r) i u_r r) \leq (w_{n+1} - 3 \cdot 2^{v_2(b)}) - (w_{n+1} - w_n) = w_n - 3 \cdot 2^{v_2(b)}.$$

Corollary 4.10 and Lemma 5.1 show

$$\ell_\infty(u_b b + s(r) i u_r r) \leq \max(\ell_\infty(b) - m(r), m(b), \ell_\infty(r)) \leq w_{n-1} - 2^{v_2(b)+1},$$

proving $\phi_{\mathbb{Z}[i]}\left(r - \frac{i u_b}{s(r) u_r} b\right) = \phi_{\mathbb{Z}[i]}(u_b b + s(r) i u_r r) < \phi_{\mathbb{Z}[i]}(b)$. ∎

Example 5.4 The Gauss remainder of $a = 8 + 8i$ and $b = 30 - 9i$ is $r = 8 + 8i$, and thus $u_b b + s(r) i u_r r = (30 - 9i) + i(8 + 8i) = 22 - i$. Note $\phi_{\mathbb{Z}[i]}(b) = \phi_{\mathbb{Z}[i]}(r) = 7$, $\mathrm{Im}(u_b b)\mathrm{Im}(u_r r) = -72 < 0$, and $\phi_{\mathbb{Z}[i]}(22 - i) = 6 < \phi_{\mathbb{Z}[i]}(b)$.

***Lemma* 5.5** *Suppose $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ with $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b) = n$. If $\mathrm{Im}(u_b b)\mathrm{Im}(u_r r) < 0$ and $m(r) + m(b) \leq \ell_\infty(r)$, then $\phi_{\mathbb{Z}[i]}\left(r - \frac{u_b}{u_r} b\right) < \phi_{\mathbb{Z}[i]}(b)$.*

Proof Lemmas 3.1 and 4.8 show $v_2(r) > v_2(b) = v_2(u_b b - u_r r)$ and $2^{v_2(b)+1}$ divides w_m for all $m \geq n + 1$. By Lemma 5.1 and Corollary 4.10,

$$\begin{aligned} \ell_\infty(u_b b - u_r r) &\leq \max(\ell_\infty(b) - \ell_\infty(r), m(b) + m(r)) \\ &\leq \max(\ell_\infty(b) - \ell_\infty(r), \ell_\infty(r)) \\ &\leq w_{n-1} - 2^{v_2(b)+1} \end{aligned}$$

and

$$\ell_1(u_b b - u_r r) = \ell_\infty(b) - \ell_\infty(r) + m(b) + m(r).$$

If $\ell_\infty(r) > m(b) + m(r)$,

$$\ell_1(u_b b - u_r r) \leq \ell_\infty(b) - 2^{v_2(r)} \leq w_n - 3 \cdot 2^{v_2(r)}.$$

When $\ell_\infty(r) = m(r) + m(b)$, $2^{v_2(r)} | m(b)$ and thus $2^{v_2(b)+1} | m(b)$. Hence $2^{v_2(b)+1} \nmid \ell_\infty(b)$ and, as $2^{v_2(b)+1} | (w_n - 2^{v_2(b)+1})$, $\ell_1(u_b b - u_r r) = \ell_\infty(b) \leq w_n - 3 \cdot 2^{v_2(b)}$. We conclude $\phi_{\mathbb{Z}[i]}\left(r - \frac{u_b}{u_r} b\right) = \phi_{\mathbb{Z}[i]}(u_b b - u_r r) < \phi_{\mathbb{Z}[i]}(b)$. ∎

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Example 5.6 In Example 5.2, we see $m(r) + m(b) = 5 < 6 = \ell_\infty(r)$, so

$$\phi_{\mathbb{Z}[i]}(u_b b - u_r r) = \phi_{\mathbb{Z}[i]}(4 + 5i) = 3 < \phi_{\mathbb{Z}[i]}(b) = 4.$$

We found alternate remainders when $\ell_\infty(r) \leq m(b)$ and when $\ell_\infty(r) \geq m(b) + m(r)$, so we examine when $m(b) + m(r) > \ell_\infty(r) > m(b)$.

Lemma 5.7 *Suppose $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ with $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b) = n$. If*

(1) $Im(u_b b)Im(u_r r) < 0$,
(2) $m(b) < \ell_\infty(r) < m(b) + m(r)$, and
(3) $\ell_\infty(b) - m(r) \leq w_{n-1} - 2^{v_2(b)+1}$,

then $\phi_{\mathbb{Z}[i]}\left(r - \frac{iu_b}{s(r)u_r} b\right) < \phi_{\mathbb{Z}[i]}(b)$.

Proof Lemma 3.1 shows $v_2(r) > v_2(b) = v_2(u_b b + s(r)iu_r r)$. By Lemma 5.1 and Corollary 4.10, we see

$$\ell_\infty(u_b b + s(r)iu_r r) \leq \max(\ell_\infty(b) - m(r), \ell_\infty(r)) \leq w_{n-1} - 2^{v_2(b)+1}$$

and

$$\ell_1(u_b b + s(r)iu_r r) = \ell_\infty(b) + (\ell_\infty(r) - (m(b) + m(r))) < w_n - 2^{v_2(b)+1}.$$

Lemma 4.8 shows $2^{v_2(b)} | w_n$, so $\ell_1(u_b b + s(r)iu_r r) \leq w_n - 3 \cdot 2^{v_2(b)}$. We conclude that $\phi_{\mathbb{Z}[i]}\left(r - \frac{iu_b}{s(r)u_r} b\right) = \phi_{\mathbb{Z}[i]}(u_b b + s(r)iu_r r) < \phi_{\mathbb{Z}[i]}(b)$. ∎

Example 5.8 The Gauss remainder of $a = 16 - 8i$ and $b = 30 + 15i$ is $r = a$. Observe that $\phi_{\mathbb{Z}[i]}(16 - 8i) = \phi_{\mathbb{Z}[i]}(30 + 15i) = 7$, that $\mathrm{Im}(16 - 8i)\mathrm{Im}(30 + 15i) < 0$, that $15 < 16 < 8 + 15$, and that $30 - 8 = 22 = w_6 - 2$. As expected,

$$\phi_{\mathbb{Z}[i]}(u_b b + s(r)iu_r r) = \phi_{\mathbb{Z}[i]}(30 + 15i - (8 + 16i)) = \phi_{\mathbb{Z}[i]}(22 - i) = 6 < 7.$$

Lemma 5.9 *Suppose $a, b \in \mathbb{Z}[i] \setminus \{0\}$ have Gauss remainder $r \neq 0$ with $\phi_{\mathbb{Z}[i]}(r) \geq \phi_{\mathbb{Z}[i]}(b) = n$. If*

(1) $Im(u_b b)Im(u_r r) < 0$,
(2) $m(b) < \ell_\infty(r) < m(r) + m(b)$, and
(3) $\ell_\infty(b) - m(r) > w_{n-1} - 2^{v_2(b)+1}$,

then $\phi_{\mathbb{Z}[i]}\left(r - \frac{u_b}{u_r} b\right) < \phi_{\mathbb{Z}[i]}(b)$.

Proof For the last time, Lemmas 3.1 and 4.8 show that $2^{v_2(r)} | w_m$ for all $m \geq n - 1$ and $v_2(r) > v_2(b) = v_2(u_b b - u_r r)$. Condition (3) is equivalent to

$$\ell_\infty(b) - w_{n-1} + 2^{v_2(b)+1} > m(r). \tag{5.2}$$

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Using $w_n - 2^{v_2(b)+1} \geq \ell_\infty(b)$ yields

$$w_n - w_{n-1} - 2^{v_2(r)} \geq m(r). \tag{5.3}$$

Adding $m(b)$ to both sides of Equation 5.2 and recalling $w_{n+1} - 3 \cdot 2^{v_2(b)} \geq \ell_1(b)$, we realize

$$w_{n-1} - 2^{v_2(b)+1} \geq m(b) + m(r). \tag{5.4}$$

Together, Equation 5.3 and Lemma 4.8 show that, as $m(r) < w_n - w_{n-1}$, $\ell_\infty(r) = w_{n-1} - 2^{v_2(r)}$. This equality, along with Lemma 5.1, Corollary 4.10, Equation 5.3, and Equation 5.4 demonstrate

$$\ell_\infty(u_b b - u_r r) \leq \max(\ell_\infty(b) - \ell_\infty(r), m(b) + m(r)) \leq w_{n-1} - 2^{v_2(b)+1}$$

and

$$\begin{aligned}\ell_1(u_b b - u_r r) &= \ell_\infty(b) + m(b) - \ell_\infty(r) + m(r) \\ &\leq (w_{n+1} - 3 \cdot 2^{v_2(b)}) - (w_{n-1} - 2^{v_2(r)}) + (w_n - w_{n-1} - 2^{v_2(r)}) \\ &= w_n - 3 \cdot 2^{v_2(b)}.\end{aligned}$$

In summary, $\phi_{\mathbb{Z}[i]}\left(r - \frac{u_b}{u_r} b\right) = \phi_{\mathbb{Z}[i]}(u_b b - u_r r) < \phi_{\mathbb{Z}[i]}(b)$. ∎

Example 5.10 The Gauss remainder of $a = 20 - 4i$ and $b = 28 + 17i$ is $r = a$. We see $\phi_{\mathbb{Z}[i]}(20 - 4i) = \phi_{\mathbb{Z}[i]}(28 + 17i) = 7$, $\mathrm{Im}(20 - 4i)\mathrm{Im}(28 + 17i) < 0$, $17 < 20 < 17 + 4$, and $28 - 4 = 24 > w_6 - 2$. As claimed,

$$\phi_{\mathbb{Z}[i]}(u_b b - u_r r) = \phi_{\mathbb{Z}[i]}(28 + 17i - (20 - 4i)) = \phi_{\mathbb{Z}[i]}(8 + 21i) = 6 < 7.$$

We now assemble our lemmas to prove Theorem 1.7.

Proof Proposition 4.12 proves our claim when condition 1 holds, Lemma 5.5 proves it when condition 2 holds, and Lemma 5.9 proves it when condition 3 holds.

If $\mathrm{Im}(u_b b)\mathrm{Im}(u_r r) < 0$ and neither condition 2 nor condition 3 hold, then either $m(b) \geq \ell_\infty(r)$ or $m(b) < \ell_\infty(r) < m(b) + m(r)$ and $\ell_\infty(b) - m(r) \leq w_{n-1} - 2^{v_2(b)+1}$. Lemmas 5.3 proves our theorem in the first situation, and Lemma 5.7 proves it in the second. ∎

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