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Existence of Taut Foliations on Seifert Fibered Homology 3-spheres

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Abstract. This paper concerns the problem of existence of taut foliations among 3-manifolds. From the work of David Gabai we know that a closed 3-manifold with non-trivial second homology group admits a taut foliation. The essential part of this paper focuses on Seifert fibered homology 3-spheres. The result is quite different if they are integral or rational but non-integral homology 3-spheres. Concerning integral homology 3-spheres, we can see that all but the 3-sphere and the Poincaré 3-sphere admit a taut foliation. Concerning non-integral homology 3-spheres, we prove there are infinitely many that admit a taut foliation, and infinitely many without a taut foliation. Moreover, we show that the geometries do not determine the existence of taut foliations on non-integral Seifert fibered homology 3-spheres.

1 Introduction

All 3-manifolds are considered compact, connected, and orientable. Taut foliations provide deep information on 3-manifolds, and their contribution to understanding the topology and geometry of 3-manifolds is still in progress. The first result came from S. P. Novikov [16] in 1965, who proved that a 3-manifold that admits a taut foliation has to be irreducible or $\mathbb{S}^2 \times \mathbb{S}^1$. Since then we know from [17] that such manifolds have $\mathbb{R}^3$ for universal cover, that their fundamental group is infinite [16], and Gromov is negatively curved when the manifold is also toroidal [4]. Recently, W. P. Thurston has exhibited an approach with taut foliations towards the geometrization.

In [9], D. Gabai proved that a closed 3-manifold with a non-trivial second homology group admits a taut foliation. Many great works then are concerned with the existence of taut foliations; see for example [1, 2, 5, 12, 13, 19]. The essential part of this paper focuses on this existence problem for Seifert fibered homology 3-spheres. The results for integral homology 3-spheres are quite different from those for rational but non-integral homology 3-spheres.

In this paper, a *non-integral homology* 3*-sphere* means a rational homology 3-sphere, which is not an integral homology 3-sphere. The results are quite different for integral homology 3-spheres or non-integral homology 3-spheres.

Theorem 1.1 *Let M be a Seifert fibered integral homology* 3*-sphere. Then M admits a taut analytic foliation if and only if M is neither homeomorphic to the* 3*-sphere nor to the Poincaré sphere.*

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The final section is devoted to this result. We have to point out here that R. Rustamov has proved in [18] that an integral homology L-space that is the link of an isolated complete intersection surface singularity is either the 3-sphere or the Poincaré sphere (up to orientation). Since every orientable Seifert fibered integral homology 3-sphere is the link of such a singularity, combining this result with P. Lisca and A. I. Stipsicz's result [13, Theorem 1.1], we get Theorem 1.1.

Concerning non-integral homology 3-spheres, the non-existence is not isolated. Of course, the 3-sphere and lens spaces do not admit a taut foliation, but for any choice of the number of exceptional fibers, there exist infinitely many that admit a taut foliation, and infinitely many that do not.

Theorem 1.2 *Let n be a positive integer greater than two. Let $\mathcal{S}_n$ be the set of Seifert fibered 3-manifolds with n exceptional fibers, that are non-integral homology 3-spheres. For each n:*

(i) *there exist infinitely many Seifert fibered manifolds in $\mathcal{S}_n$ that admit a taut analytic foliation;*
(ii) *there exist infinitely many Seifert fibered manifolds in $\mathcal{S}_n$ that do not admit a taut $\mathcal{C}^2$-foliation;*
(iii) *there exist infinitely many Seifert fibered manifolds in $\mathcal{S}_3$ that do not admit a taut $\mathcal{C}^0$-foliation.*

Actually, by considering the normalized Seifert invariant $(0; b_0, b_1/a_1, \ldots, b_n/a_n)$ of a Seifert fibered homology 3-sphere and assuming that b_0 is not equal to -1 (nor to $1-n$), then b_0 determines whether or not M admits a taut $\mathcal{C}^2$-foliation; see Theorem 4.1, which collects the results of [6, 10, 15]. Note that there is a fiber-preserving homeomorphism of M that switches $b_0 = 1-n$ to $b_0 = -1$. Therefore, the problem remains open only for $b_0 = -1$. We will prove (see Theorem 7.1) that even if the 3-manifolds are all equipped with $b_0 = -1$, Theorem 1.2 is still true.

Finally, we will see that the geometries do not determine the existence of taut foliations on Seifert fibered rational homology 3-spheres.

Theorem 1.3 *Let M be a Seifert fibered rational homology 3-sphere. If M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry, then M does not admit a taut $\mathcal{C}^2$-foliation.*

Remark 1.4 There exist infinitely many such manifolds (see Section 7), but the converse is not true (Theorem 7.1). We can give infinitely many such manifolds, which admit the $\widetilde{SL}_2(\mathbb{R})$-geometry (and with $b_0 = -1$) but no taut $\mathcal{C}^2$-foliation.

From Proposition 5.5, if M is a Seifert fibered integral homology 3-sphere that admits the $\widetilde{SL}_2(\mathbb{R})$-geometry, then M is neither homeomorphic to the 3-sphere nor the Poincaré sphere. In particular (Theorem 1.1) M admits a taut analytic foliation.

Schedule of the paper We organize the paper as follows. In Section 2 we recall basic definitions and notations on Seifert fibered 3-manifolds, taut or horizontal foliations and well-known results. Section 3 is devoted to the relationship between taut $\mathcal{C}^2$-foliations and horizontal foliations in Seifert fibered rational homology 3-spheres. Actually, a Seifert fibered rational homology 3-sphere, say M, admits a taut

$\mathcal{C}^2$-foliation if and only if M admits a horizontal foliation (Corollary 3.1). This fact has been proved by P. Lisca and A. I. Stipsicz [13, Theorem 1.1] using contact structures and Ozsvaáth–Szabó invariants. Another way to see this is to note that a taut $\mathcal{C}^2$-foliation of a Seifert fibered homology 3-sphere cannot contain a compact leaf (see Corollary 3.3). Therefore, it can be isotoped to be horizontal, by applying the works on foliations of M. Brittenham, D. Eisenbud, U. Hirsch, G. Levitt, S. Matsumoto, W. Neumann, S. P. Novikov, and W. P. Thurston [1, 6, 11, 14, 16, 24].

In Section 4, we give inequalities involving Seifert invariants that will be used for the remainder of the paper. They come from the characterization of M. Jankins, R. Naimi, and W. Neumann [10, 15] for horizontal foliations, because a taut $\mathcal{C}^2$-foliation can be isotoped to be horizontal. We can find such a characterization by combining [12, Theorem 1.3] and [13, Theorem 1.1]. Note that Theorem 1.2(i) and (ii) also follow from this characterization.

Section 5 concerns the geometries of homology 3-spheres. We will prove the following result.

Proposition 1.5 *Let M be a Seifert fibered rational homology 3-sphere with n exceptional fibers. If M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry, then the following statements are all satisfied.*

(i) $n \leq 4$.
(ii) *If $n = 4$, then M admits the $\mathcal{N}il$-geometry and is a non-integral homology 3-sphere.*
(iii) *If M is an integral homology 3-sphere, then M admits the $\mathbb{S}^3$-geometry and is either homeomorphic to the 3-sphere or to the Poincaré sphere.*

We may note that if $n = 2$ then M is a lens space (including $\mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{S}^2$). We combine Proposition 1.5 with the criteria given by the characterization of Section 4, to prove Theorem 1.3.

Sections 6, 7, and 8 are devoted to the proofs of Theorem 1.3, Theorem 7.1, which implies Theorem 1.2, and Theorem 1.1, resepctively.

To prove Theorem 7.1, we first exhibit infinite families of Seifert fibered non-integral homology spheres, which admit the $\widetilde{SL}_2(\mathbb{R})$-geometry (and $b_0 = 1$). Then we prove that they do satisfy (or do not satisfy) the criteria of the characterization described in Section 4.

To prove Theorem 1.1, we need to study the following criteria in more depth.

Perspectives By F. Waldhausen, [25] we know that an incompressible compact surface in a Seifert fibered 3-manifold (not necessarily a homology 3-sphere) can be isotoped to be either horizontal or vertical. This is clearly not the same for foliations.

A vertical leaf is homeomorphic to either a 2-cylinder ($\mathbb{S}^1 \times \mathbb{R}$) or a 2-torus ($\mathbb{S}^1 \times \mathbb{S}^1$). Therefore, taut foliations are not necessarily isotopic to vertical ones and vice-versa, *i.e.*, vertical foliations are not necessarily isotopic to taut foliations, *e.g.*, cylinders that wrap around two tori in a turbulant way; for more details, see [3]. But clearly, horizontal foliations are taut.

By Theorem 3.4, a taut $\mathcal{C}^2$-foliation can be isotoped to a horizontal foliation, if there is no compact leaf.

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One might wonder if a taut $\mathcal{C}^0$-foliation, without compact leaf, of a Seifert fibered 3-manifold can be isotoped to be horizontal and so analytic. By [2] there exist manifolds that admit taut $\mathcal{C}^0$-foliation but not taut $\mathcal{C}^2$-foliation. Therefore, that seems impossible in general, but the question is still open for homology 3-spheres.

Question 1.6 *Let $\mathcal{F}$ be a taut $\mathcal{C}^0$-foliation, without compact leaf, of a Seifert fibered homology 3-sphere. Can $\mathcal{F}$ be isotoped to be horizontal ?*

Brittenham [1] answers the question when the base is $\mathbb{S}^2$ with 3 exceptional fibers; see Remark 3.5 for more details. Gluing Seifert fibered 3-manifolds with boundary components along some of them (or all) give *graph manifolds*. We wonder if we can classify graph manifolds without taut foliations, with their Seifert fibered pieces and gluing homeomorphims.

Question 1.7 *Let M be a graph 3-manifold. What kind of obstructions are there for M not to admit a taut foliation ?*

2 Preliminaries

We may recall here that all 3-manifolds are considered compact, connected, and orientable. This section is devoted to recall basic definitions and notations on Seifert fibered 3-manifolds, taut or horizontal foliations, and well-known results.

Notations Let M be a 3-manifold. If M is an integral homology sphere (resp. a rational homology sphere), we say that M is a $\mathbb{Z}HS$(resp. $\mathbb{Q}HS$). Clearly, a $\mathbb{Z}HS$ is a $\mathbb{Q}HS$. If M is a $\mathbb{Z}HS$(resp. $\mathbb{Q}HS$), and a Seifert fibered 3-manifold, we say that M is *a $\mathbb{Z}HS$(resp. $\mathbb{Q}HS$), Seifert fibered 3-manifold.*

Separating surfaces and non-separating surfaces A properly embedded surface F in a 3-manifold M is said to be *a separating surface* if $M - F$ is not connected; otherwise, F is said to be *a non-separating surface* in M. If F is a separating surface, we call the connected components of $M - F$ the *sides* of F. Note that if M is a $\mathbb{Q}HS$ manifold, then M does not contain any non-separating surface.

A 3-manifold is said to be *reducible* if M contains an *essential* 2*-sphere*, *i.e.*, a 2-sphere that does not bound any 3-ball in M. Then either M is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, or M is a non-trivial connected sum. If M is not a reducible 3-manifold, we say that M is an *irreducible* 3-manifold. We may note that all Seifert fibered 3-manifolds except $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$ are irreducible 3-manifolds.

Seifert fibered 3-manifolds We can find the first definition of Seifert fibered 3-manifolds, called *fibered spaces* by H. Seifert, in [23]. We first consider fibered solid tori. The standard solid torus V is said to be *p/q-fibered* if V is foliated by circles, such that the core is a leaf, and all the other leaves are circles isotopic to the (p, q)-torus knot (*i.e.,* they run p times in the meridional direction and q times in the longitudinal direction), where $q \neq 0$. A solid torus W is *$\mathbb{S}^1$-fibered* if W is foliated by circles such that there exists a leaf-preserving homeomorphism between W and the p/q-fibered standard solid torus V. We may say that W is a *p/q-fibered solid torus.*

A 3-manifold M is said to be *a Seifert fibered* 3-manifold, or a *Seifert fiber space* if M is a disjoint union of simple circles, called *the fibers*, such that the regular neighborhood of each fiber is a $\mathbb{S}^1$-fibered solid torus. Let W be a p/q-fibered solid torus. If $q = 1$, we say that its core is a *regular fiber*; otherwise we say that its core is an *exceptional fiber* and q is the *multiplicity* of the exceptional fiber. By D. B. A. Epstein [8] this is equivalent to saying that M is an $\mathbb{S}^1$-bundle over a 2-orbifold.

Seifert invariants In [22] Seifert developed numerical invariants that give a complete classification of Seifert fibered 3-manifolds. Let M be a closed Seifert manifold based on an orientable surface of genus g, with n exceptional fibers. Let $V_1, \dots, V_n$ be the solid tori that are the regular neighborhoods of exceptional fibers. We do not need to consider non-orientable base surfaces here. If we remove these solid tori, we obtain a trivial $\mathbb{S}^1$-bundle over a genus g compact surface whose boundary is a union of 2-tori: $T_1, \dots, T_n$, where $T_i = \partial V_i$, for $i \in \{1, \dots, n\}$. Gluing back $V_1, \dots, V_n$ consists of assigning a slope b_i/a_i to each of them: we glue V_i along T_i, so that the slope b_i/a_i on V_i bounds a meridian disk of V_i. Formally, if f and s represent a fiber and a section on T_i,respectively, then the boundary of the meridian disk of V_i is attached along the slope represented by $a_i[s] + b_i[f]$ in $H_1(T_i, \mathbb{Z})$.

Clearly, $a_i \geq 2$ is the multiplicity of the core of V_i, and b_i depends on the choice of a section. Removing the regular neighborhood of a regular fiber, we obtain an integer slope b_0. Then the $(n+2)$-tuple $(g, b_0, b_1/a_1, \dots, b_n/a_n)$ completely describes M and is called *the Seifert invariant*. We denote M by $M(g, b_0, b_1/a_1, \dots, b_n/a_n)$.

Seifert normalized invariant and convention New sections are obtained by Dehn twistings along the fiber (along annuli or tori); therefore, a new section does not change b_i modulo a_i. Thus, we can fix b_i so that $0 < b_i < a_i$ for $i \in \{1, \dots, n\}$. This gives rise to the *Seifert normalized invariant* $M(g; b_0, b_1/a_1, \dots, b_n/a_n)$; *i.e.*, $0 < b_i < a_i$ for $i \in \{1, \dots, n\}$.

Seifert [22] showed that $M(g; b_0, b_1/a_1, \dots, b_n/a_n)$ is fiber-preserving homeomorphic to $-M(g, -n - b_0, 1 - b_1/a_1, \dots, 1 - b_n/a_n)$ where $-M$ denotes M with the opposite orientation. In the sequel, we denote this isomorphism by Φ. Therefore, we may assume that $b_0 < 0$, otherwise we switch for $-n - b_0$. For more details, see [22] or [2].

Every $\mathbb{Q}HS$ Seifert fibered 3-manifold M is based on $\mathbb{S}^2$. Indeed, every non-separating curve on the base surface induces a non-separating torus in M, which cannot be in a $\mathbb{Q}HS$. Hence, the base surface of a $\mathbb{Q}HS$ Seifert fibered 3-manifold is a 2-sphere.

From now on, we denote for convenience such M by $M(-b_0, b_1/a_1, \dots, b_n/a_n)$, where $b_0 > 0$ and $0 < b_i < a_i$ for $i \in \{1, \dots, n\}$. We will write

$$M = M(-b_0, b_1/a_1, \dots, b_n/a_n).$$

Euler number When M has a unique fibration, we denote *the Euler number of its fibration* by $e(M)$:

$$e(M) = -b_0 + \sum_{i=1}^{n} b_i/a_i.$$

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Note that, apart from lens spaces, only finitely many closed 3-manifolds have more than one Seifert fibration up to isomorphism; the only rational homology 3-spheres having such a property are lens spaces and $\mathbb{S}^3$.

Taut foliations Let M be a 3-manifold and $\mathcal{F}$ a foliation of M. A simple closed curve γ (respectively, a properly embedded simple arc, when $\partial M \neq \varnothing$) is called *a transverse loop* (respectively a *transverse arc*) if γ is transverse to $\mathcal{F}$, *i.e.,* γ is transverse to every leaf $F \in \mathcal{F}$, such that $\gamma \cap F \neq \varnothing$. We say that a foliation $\mathcal{F}$ is *taut*, if for every leaf of F of $\mathcal{F}$, there exists a transverse loop, or a transverse arc if $\partial M \neq \varnothing$, γ say, such that $\gamma \cap F \neq \varnothing$.

We end this part with the famous theorem of Gabai [9] on the existence of taut foliations, which is stated here for closed 3-manifolds.

Theorem 2.1 (D. Gabai, [9]) *Let M be a closed 3-manifold. If $H_2(M;\mathbb{Q})$ is non-trivial then M admits a taut foliation.*

Horizontal and vertical foliations Let M be a Seifert fibered 3-manifold and $\mathcal{F}$ a foliation of M. We say that $\mathcal{F}$ is *horizontal* if each $\mathbb{S}^1$-fiber is a transverse loop to $\mathcal{F}$. We say that $\mathcal{F}$ is *vertical* if each leaf of $\mathcal{F}$ is $\mathbb{S}^1$-fibered, *i.e.*, a disjoint union of $\mathbb{S}^1$-fibers.

Note that only Seifert fibered 3-manifolds are concerned with horizontal or vertical foliations. Horizontal foliations are sometimes just called *transverse foliations* to underline the fact that horizontal foliations are transverse to the $\mathbb{S}^1$-fibers. Clearly, horizontal foliations are taut, because any transverse fiber (meeting a leaf) is the required transverse loop, so we have the following result.

Lemma 2.2 *A horizontal foliation is taut.*

3 Horizontal and Taut $\mathcal{C}^2$-foliations in Seifert Fibered Homology 3-spheres

The goal of this section is mainly to see that a Seifert fibered rational homology 3-sphere M admits a taut $\mathcal{C}^2$-foliation if and only if M admits a horizontal foliation.

Lemma 3.1 *Let M be a $\mathbb{Q}HS$ Seifert fibered 3-manifold. Let n be the number of exceptional fibers of M. If $n > 3$ (resp. $n = 3$) then M admits a horizontal foliation if and only if M admits a taut $\mathcal{C}^2$-foliation (resp. a $\mathcal{C}^0$-foliation).*

This fact has been proved by Lisca and Stipsicz using contact structures and Ozsvaáth–Szabó invariants [13, Theorem 1.1]. We underline that the considered taut foliations are actually $\mathcal{C}^2$-foliations, because of the use of contact structure (see [7]). Note that there exists a taut $\mathcal{C}^0$-foliation that is not a taut $\mathcal{C}^2$-foliation; see [2].

A taut foliation $\mathcal{F}$ is said to be *transversely oriented* if there exists a one-dimensional oriented foliation $\mathcal{G}$ transverse to $\mathcal{F}$. This is equivalent to saying that the normal vector field to the tangent planes to the leaves of $\mathcal{F}$ is continuous (and nowhere vanishes). As a consequence, each intersection point between a compact leaf, F say, and a closed transverse loop (or transverse arc) always occurs with the same sign. Therefore, F has a nontrivial homological intersection, and so it cannot be separating. That gives the following fact.

Remark 3.2 A transversely oriented and taut foliation of a closed 3-manifold cannot contain a compact separating leaf.

Another way to see Lemma 3.1 is to note that a taut $\mathcal{C}^2$-foliation of a Seifert fibered homology 3-sphere cannot contain a compact leaf (see Corollary 3.3). Then it can be isotoped to be horizontal by using the works on foliations of Brittenham, Eisenbud, Hirsch, Levitt, Matsumoto, Neumann, Novikov, and Thurston [1, 6, 11, 14, 16, 24]. Horizontal foliations are trivially taut.

Corollary 3.3 is an immediate consequence of Remark 3.2, which concerns all (compact, oriented, and connected) closed 3-manifolds, and can be generalized with some boundary conditions to 3-manifolds with non-empty boundary; see [3] for more details.

Every codimension-1 foliation admits a transverse 1-dimensional foliation (given a Riemann metric, we can construct an orthogonal and integrable 1-dimensional distribution). Passing to a 2-fold covering (if necessary), say $\widetilde{M}$, we can orient the transverse foliation (which is equivalent to orient the tangent space). So if the foliation of M is taut, then the foliation on $\widetilde{M}$ is taut and transversely oriented. Therefore, it cannot admit a compact separating leaf; see Remark 3.2. If M is a $\mathbb{Q}HS$ then a compact leaf has to be an orientable and separating surface (furthermore $\widetilde{M}$ is also a $\mathbb{Q}HS$) Thus we have the following corollary.

Corollary 3.3 *A taut foliation of a $\mathbb{Q}HS$ cannot admit a compact leaf.*

Theorem 3.4 ([1, 6, 11, 14, 16, 24]) *Let M be a $\mathbb{Q}HS$ Seifert fibered 3-manifold, with n exceptional exceptional fibers (where $n \geq 3$). We assume that M admits a taut $\mathcal{C}^0$-foliation $\mathcal{F}$. Moreover, if $n > 3$, we suppose that $\mathcal{F}$ is a $\mathcal{C}^2$-foliation of M. If $\mathcal{F}$ does not have a compact leaf, then $\mathcal{F}$ can be isotoped to be a horizontal foliation.*

Remark 3.5 (History of Theorem 3.4) This theorem has been proved for all Seifert 3-manifolds that are not trivial bundles over the 2-torus. This is a collection of results as follows.

- The case of a circle bundle over an orientable surface that is not a 2-torus is due to Thurston [24]; it was completed and extended to non-orientable base surfaces by Levitt [11].
- In [6] Eisenbud, Hirsch, and Neumann generalized it to Seifert fibered spaces where the base surface is neither $\mathbb{S}^2$ nor the 2-torus with trivial circle bundle.
- Later, in [14], S. Matsumoto focused on the case when the base is $\mathbb{S}^2$ with stricly more than 3 exceptional fibers.
- To this point, the condition of $\mathcal{C}^r$-foliation is necessary, and implies a $\mathcal{C}^r$-isotopy, for each $r \geq 2$.
- The last case (the base is $\mathbb{S}^2$ with 3 exceptional fibers) was solved by Brittenham [1]. The techniques involved are very different, so the author obtained a $\mathcal{C}^0$-isotopy from a $\mathcal{C}^0$-foliation.

- We may recall that when there are one or two exceptional fibers with base $\mathbb{S}^2$, there is no foliation without compact leaves, as was shown by Novikov in [16].

4 Characterization of Taut $\mathcal{C}^2$-foliations in Seifert Fibered Homology 3-spheres

The goal this section is to give inequalities involving Seifert invariants that will be used for the following. They come from the characterization of M. Jankins, R. Naimi, and W. Neumann [10, 15] for horizontal foliations, because a taut $\mathcal{C}^2$-foliation can be isotoped to be horizontal. We can find such a characterization combining [12, Theorem 1.3] and [13, Theorem 1.1].

For this, we define the following *Property* $(*)$:

$$Property\ (*) \begin{cases} \text{(i)} & \dfrac{b_1}{a_1} < \dfrac{m-\alpha}{m} \\[2ex] \text{(ii)} & \dfrac{b_2}{a_2} < \dfrac{\alpha}{m} \\[2ex] \text{(iii)} & \dfrac{b_i}{a_i} < \dfrac{1}{m} \quad \text{for } i \in \{3,\dots,n\}. \end{cases}$$

We say that *m and α satisfy Property $(*)$ for $b_1/a_1, b_2/a_2, \dots, b_n/a_n$*, if the following statements are satisfied:

- m and α are two positive integers such that $\alpha < m$;
- $n \geq 3$ is an integer;
- a_i and b_j are positive integers for all $(i,j) \in \{1,\dots,n\}^2$, such that

$$b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n;$$

- (i), (ii), and (iii) of Property $(*)$ all are satisfied.

When there is no confusion for the b_i/a_i's, we say for short that (m,α) *satisfies Property* $(*)$, or that *the integers α and m satisfy Property* $(*)$.

For convenience, in the following, we denote by (i), (ii), and (iii), respectively, the inequalities (i), (ii), and (iii) of Property $(*)$.

Let M be a Seifert fibered 3-manifold. In the following, we use the previous notations (see Section 2) of Seifert normalized invariant

$$M = M(-b_0, b_1/a_1, \dots, b_n/a_n),$$

where a_i and b_j are positive integers for all $(i,j) \in \{1,\dots,n\} \times \{0,\dots,n\}$, such that $0 < b_i < a_i$. Note that the notations $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ suppose that M contains exactly n exceptional fibers: $a_i \geq 2$, for all $i \in \{1,2,\dots,n\}$.

If $b_0 \notin \{1, n-1\}$, then the existence of a taut $\mathcal{C}^2$-foliation depends uniquely on b_0, as implied by the following theorem.

Theorem 4.1 ([6,10,15]) *Let n be an integer and M be a Seifert manifold based on $\mathbb{S}^2$. We assume that $n \geq 3$ and that $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$, where a_i and b_j are positive integers for all $(i, j) \in \{1, \dots, n\} \times \{0, \dots, n\}$. Then the following statements are satisfied.*

(1) *If $2 \leq b_0 \leq n-2$, then M admits a horizontal foliation.*
(2) *If M admits a horizontal foliation, then $1 \leq b_0 \leq n-1$.*
(3) *If M admits a horizontal $\mathcal{C}^0$-foliation, then M admits a horizontal analytic foliation.*

Corollary 4.2 *Let n be an integer and M be a Seifert manifold based on $\mathbb{S}^2$. We assume that $n \geq 3$ and that $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$, where a_i and b_j are positive integers for all $(i, j) \in \{1, \dots, n\} \times \{0, \dots, n\}$, and $b_0 \notin \{1, n-1\}$. Then M admits an analytic horizontal foliation if and only if $2 \leq b_0 \leq n-2$.*

Therefore, the problem falls on $b_0 = 1$; we recall here from Section 2 that

$$M(-1, b_1/a_1, \dots, b_n/a_n) \cong -M(-(n-1), 1 - b_1/a_1, \dots, 1 - b_n/a_n).$$

The following theorem is a consequence of Lemma 3.1 and the characterization of the existence of horizontal foliations in Seifert-fibered spaces based on $\mathbb{S}^2$ whose formulation can be found in [2, Proposition 6] and [12, Theorem 1.3].

Theorem 4.3 *Let $n > 2$ be an integer and $M = M(-1, b_1/a_1, \dots, b_n/a_n)$ be a $\mathbb{Q}$HS Seifert fibered 3-manifold, where a_i and b_j are positive integers for all $(i, j) \in \{1, \dots, n\}^2$. Assume that $b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n$. If $n > 3$ (resp. $n = 3$), then M admits a taut $\mathcal{C}^2$-foliation (resp. a taut $\mathcal{C}^0$-foliation) if and only if there exist two positive integers m and α such that (m, α) satisfies Property $(*)$.*

We may recall that $\mathcal{P}$ denotes the Poincaré $\mathbb{Z}$HS, *i.e.*, $\mathcal{P} = M(-1, 1/2, 1/3, 1/5)$. Note that Theorem 4.3 implies that $\mathcal{P}$ cannot admit a taut foliation, but this fact was already known; see Novikov [16] (because π_1 is finite in that paper). Note also that if $n \in \{1, 2\}$, then M has to be $\mathbb{S}^3$ or a Lens space, which cannot admit a taut foliation.

Theorem 4.3 has the following corollaries, which will be useful for the next sections.

Corollary 4.4 *Let n be an integer and M be a $\mathbb{Q}$HS Seifert fibered 3-manifold. We assume that $n \geq 3$ and that $M = M(-1, b_1/a_1, \dots, b_n/a_n)$, where a_i and b_j are positive integers for all $(i, j) \in \{1, \dots, n\}^2$. We order the rational coefficients b_i/a_i such that $b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n$. If $b_i/a_i < 1/2$ for all $i \in \{1, \dots, n\}$, then M admits a taut $\mathcal{C}^2$-foliation.*

Proof Using the notation and assumptions of the theorem, if $b_i/a_i < 1/2$ for all $i \in \{1, \dots, n\}$, then Property $(*)$ is satisfied by choosing $m = 2$ and $\alpha = 1$. ■

Corollary 4.5 *Let n be an integer and M be a $\mathbb{Q}$HS Seifert fibered 3-manifold. We assume that $n \geq 3$ and that $M = M(-1, b_1/a_1, \dots, b_n/a_n)$, where a_i and b_j are positive integers for $(i, j) \in \{1, \dots, n\}^2$. We order the rational coefficients b_i/a_i such that $b_1/a_1 \geq b_2/a_2 \geq \cdots \geq b_n/a_n$. If M admits a taut $\mathcal{C}^2$-foliation and $b_1/a_1 \geq 1/2$, then the following two properties are satisfied.*

(i) $b_i/a_i < 1/2$, *for all* $i \geq 2$.
(ii) $b_n/a_n < 1/3$. *In particular,* $a_n \geq 4$.

Proof Using the notations and assumptions of Theorem 4.3, if M admits a taut $\mathcal{C}^2$-foliation, then we can find positive integers m, α such that $\alpha < m$ and Property $(*)$ is satisfied.

First, note that if $m = 2$, then $\alpha = 1$ and $b_1/a_1 < 1/2$, which is a contradiction to the hypothesis. Thus, $m \geq 3$. Now if $\frac{m-\alpha}{m} > \frac{1}{2}$ then $\frac{\alpha}{m} < \frac{1}{2}$, hence Property $(*)$ implies $\frac{b_i}{a_i} < \frac{1}{2}$ for $i \in \{2, \dots, n\}$, which proves (i). Finally, assume that

$$\frac{b_1}{a_1} \geq \frac{1}{2} \geq \frac{b_2}{a_2} \geq \frac{b_3}{a_3} \geq \frac{1}{3}.$$

Then $b_3/a_3 \geq 1/m$ for all $m \geq 3$, so (iii) of Property $(*)$ cannot be satisfied. ∎

5 Geometries of Seifert Fibered Homology 3-spheres

The goal of this section is to recall general results on the geometries of Seifert fibered homology 3-spheres, and prove Proposition 1.5.

Let n be a positive integer and $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ be a $\mathbb{Q}$HS Seifert fibered 3-manifold. Recall that $e(M)$ denotes the Euler number of M; see Section 2. The following lemma is a well-known result; see [20] for more details.

Lemma 5.1 *Let* $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ *be a Seifert fibered 3-manifold. Then*

(i) *M is a $\mathbb{Z}$HS if and only if* $a_1a_2 \dots a_n e(M) = \varepsilon$, *where* $\varepsilon \in \{-1, +1\}$;
(ii) *M is a $\mathbb{Q}$HS if and only if* $e(M) \neq 0$.

Remark 5.2 Note that (i) implies that the a_i's are pairwise relatively prime integers, and therefore they are different.

Then we define the rational number χ_M as follows:

$$\chi_M = 2 - \sum_{i=1}^{n} \left(1 - \frac{1}{a_i}\right) = 2 - n + \sum_{i=1}^{n} \frac{1}{a_i}.$$

We have the following well-known result (which can be found in [21], for example).

Proposition 5.3 *Let n be a positive integer and* $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ *be a $\mathbb{Q}$HS Seifert fibered 3-manifold. Then the following properties are all satisfied.*

(i) $\chi_M > 0$ *if and only if M admits the $\mathbb{S}^3$-geometry.*
(ii) $\chi_M < 0$ *if and only if M admits the $\widetilde{SL}_2(\mathbb{R})$-geometry.*
(iii) $\chi_M = 0$ *if and only if M admits the $\mathcal{N}il$-geometry.*

Proposition 5.4 *Let n be a positive integer and* $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ *be a $\mathbb{Q}$HS Seifert fibered 3-manifold. If M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry, then* $n \leq 4$. *Furthermore, if* $n = 4$, *then* $M = M(-b_0, 1/2, 1/2, 1/2, 1/2)$ *with* $b_0 \neq 2$, *so M admits the $\mathcal{N}il$-geometry and is a non-integral $\mathbb{Q}$HS.*

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Proof Let n be a positive integer and $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$ be a $\mathbb{Q}HS$. Assume that M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry. Then, by Proposition 5.3, $\chi_M \geq 0$. Therefore, $n-2 \leq \sum_{i=1}^{n} \frac{1}{a_i}$. Since $a_i \geq 2$ for all $i \in \{1, \dots, n\}$,

$$n-2 \leq \sum_{i=1}^{n} \frac{1}{a_i} \leq n/2 \Rightarrow n \leq 4.$$

Assume first that $n = 4$. Then $\sum_{i=1}^{4} \frac{1}{a_i} \geq 2$. On the other hand, if $a_i \geq 2$ for all $i \in \{1, \dots, 4\}$, then $\sum_{i=1}^{4} \frac{1}{a_i} \leq 2$, and if one $a_i > 2$, then $\sum_{i=1}^{4} \frac{1}{a_i} < 2$. Therefore, $a_i = 2$ for all $i \in \{1, \dots, 4\}$. Thus, $\chi_M = 0$, which means that M admits the $\mathcal{N}$il-geometry. Moreover, Lemma 5.1(ii) implies that $b_0 \neq 2$. Note that such M cannot be a $\mathbb{Z}HS$by Remark 5.2. ∎

Corollary 5.5 *Let M be a $\mathbb{Z}HS$ Seifert fibered 3-manifold. Then M has the $\widetilde{SL}_2(\mathbb{R})$-geometry or the $\mathbb{S}^3$-geometry. Furthermore, if M has the $\mathbb{S}^3$-geometry, then M is either homeomorphic to $\mathbb{S}^3$ or to the Poincaré sphere $\mathcal{P}$.*

Proof Let M be a $\mathbb{Z}HS$ Seifert fibered 3-manifold. Assume that M does not have the $\widetilde{SL}_2(\mathbb{R})$-geometry. Note that if $n \leq 2$, then M has to be homeomorphic to $\mathbb{S}^3$. By Proposition 5.4, we may assume that $n = 3$ and that $a_3 > a_2 > a_1 \geq 2$ (by Remark 5.2).

Since $\chi_M \geq 0$, $\sum_{i=1}^{3} \frac{1}{a_i} \geq 1$. If $a_1 \geq 3$, then

$$\sum_{i=1}^{3} \frac{1}{a_i} \leq 1/3 + 1/4 + 1/5 < 1,$$

which is a contradiction. Then $a_1 = 2$. If $a_2 \neq 3$ then $a_2 \geq 5$ by Remark 5.2. Hence,

$$\sum_{i=1}^{3} \frac{1}{a_i} \leq 1/2 + 1/5 + 1/7 < 1,$$

which is a contradiction. Therefore, $a_1 = 2$ and $a_2 = 3$. Similarly $a_3 = 5$. Since $n = 3$ and $(a_1, a_2, a_3) = (2, 3, 5)$, M has to be homeomorphic to the Poincaré sphere, which satisfies $\chi_M > 0$, so $\mathcal{P}$ has the $\mathbb{S}^3$-geometry. ∎

To end this section, we simply note that Proposition 5.4 together with Corollary 5.5 clearly imply Proposition 1.5.

6 Proof of Theorem 1.3

We keep the previous notation. Let n be a positive integer and M be a $\mathbb{Q}$HS Seifert fibered 3-manifold, with n exceptional fibers: $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$. Assume that M does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry. We argue by contradiction. Suppose that M admits a taut $\mathcal{C}^2$-foliation. We may recall that if $n \in \{1, 2\}$, then M has a finite π_1, hence M cannot admit a taut $\mathcal{C}^2$-foliation. Therefore, by Proposition 5.4, we have $n \in \{3, 4\}$.

Assume that $n = 4$. By Theorem 4.1 and Lemma 3.1, since M admits a taut $\mathcal{C}^2$-foliation, $b_0 \in \{1, 2, 3\}$. Moreover the cases $b_0 = 1$ and $b_0 = 3$ are equivalent (see the fiber-preserving homeomorphism Φ in Section 2).

On the other hand, Proposition 5.4 implies that $M = M(-b_0, 1/2, 1/2, 1/2, 1/2)$ with $b_0 \neq 2$ and Corollary 4.5(i) implies that $b_0 \neq 1$.

Therefore, we may assume that $n = 3$. Similarly $b_0 \in \{1, 2\}$ and $b_0 = 1$ and $b_0 = 2$ are equivalent cases, by considering the fiber-preserving homeomorphism Φ.

So, we may assume that $b_0 = 1$. Let $M = M(-1, b_1/a_1, b_2/a_2, b_3/a_3)$.

Since M is a $\mathbb{Q}$HS Seifert fibered 3-manifold that does not admit the $\widetilde{SL}_2(\mathbb{R})$-geometry, Proposition 5.3 and Lemma 5.1(ii) give respectively:

$$\sum_{i=1}^{3} \frac{1}{a_i} \geq 1, \tag{6.1}$$

$$\sum_{i=1}^{3} \frac{b_i}{a_i} \neq 1. \tag{6.2}$$

Without loss of generality, we may assume that $b_1/a_1 \geq b_2/a_2 \geq b_3/a_3$. Let $a_{i_0} = \min(a_1, a_2, a_3)$. By (6.1), $a_{i_0} \in \{2, 3\}$. First, we prove that a_{i_0} cannot be 3. We argue by contradiction. Assume that $a_{i_0} = 3$, then (6.1) implies that $a_i = 3$ for all $i \in \{1, 2, 3\}$. Now, for all $i \in \{1, 2, 3\}$, $b_i < a_i$, so $b_i \leq 2$. If there exists $i \in \{1, 2, 3\}$ such that $b_i = 2$, then $b_i/a_i = 2/3 > 1/2$. But for $j \neq i$, $b_j/a_j \geq 1/3$, which is a contradiction to Corollary 4.5(ii). Therefore, $b_i/a_i = 1/3$, for all $i \in \{1, 2, 3\}$, which contradicts (6.2).

Hence, we may assume that $a_{i_0} = 2$. Then $b_{i_0}/a_{i_0} = 1/2$. By Corollary 4.5(i), $b_{i_0}/a_{i_0} = b_1/a_1$. Then Corollary 4.5(i) and (ii) imply respectively that $a_3 \geq 4$ and $a_2 \geq 3$. Now (6.1) implies that $\{a_1, a_2, a_3\}$ is one of the following sets:

$$\{2, 3, 4\}, \quad \{2, 3, 5\}, \quad \{2, 3, 6\}, \quad \text{or} \quad \{2, 4, 4\}.$$

We distinguish the cases $a_2 = 3$ and $a_2 = 4$.

Case 1: $a_2 = 3$.

Then Corollary 4.5(i) implies that $b_2/a_2 = 1/3$. Now, by Theorem 4.3, there exist positive integers α and m that satisfy Property $(*)$. Now Corollary 4.5(ii) implies that $\frac{b_3}{a_3} \in \left\{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}\right\}$. Hence, $m \leq 5$ by $(*)$(iii). Since $b_1/a_1 = 1/2$, $m > 2$. If $m = 3$, then $\alpha \in \{1, 2\}$, but in both cases $(*)$(i) or $(*)$(ii) cannot be satisfied. Similarly, if $m = 4$, then $\alpha \in \{1, 2, 3\}$, but in all cases $(*)$(i) or $(*)$(ii) cannot be satisfied. If $m = 5$, then $a_3 = 6$ and $b_3 = 1$; otherwise $(*)$ (iii) cannot be satisfied.

Thus, $b_1/a_1 + b_2/a_2 + b_3/a_3 = 1/2 + 1/3 + 1/6 = 1$, which is in contradiction to (6.2), *i.e.*, M cannot be a $\mathbb{Q}HS$.

Case 2: $a_2 = 4$.

Then $a_2 = a_3 = 4$. Therefore Corollary 4.5(i) implies that $\frac{b_2}{a_2} = \frac{b_3}{a_3} = \frac{1}{4}$. Therefore (6.2) is not satisfied, which is the final contradiction.

This ends the proof of Theorem 1.3. ∎

7 Proof of Theorem 1.2

Let n be a positive integer greater than two. We keep the previous conventions and notations and denote any $\mathbb{Q}HS$ Seifert fibered 3-manifolds M with its normalized Seifert invariant by $M = M(-b_0, b_1/a_1, b_2/a_2, \ldots, b_n/a_n)$.

Let $\mathcal{SF}_1$ be the set of all Seifert fibered 3-manifolds for which $b_0 = 1$ and that admit the $\widetilde{SL}_2(\mathbb{R})$-geometry.

Let $\mathcal{Q}_n = \mathcal{S}_n \cap \mathcal{SF}_1$. Then $\mathcal{Q}_n$ is the set of non-integral $\mathbb{Q}HS$ Seifert fibered 3-manifolds M with n exceptional fibers, which admit the $\widetilde{SL}_2(\mathbb{R})$-geometry and

$$M = M(-1, b_1/a_1, b_2/a_2, \ldots, b_n/a_n).$$

This section is devoted to prove the following result, which clearly implies Theorem 1.2.

Theorem 7.1 *Let n be a positive integer greater than two.*

(i) *There exist infinitely many Seifert fibered manifolds in $\mathcal{Q}_n$ that admit a taut analytic foliation.*

(ii) *There exist infinitely many Seifert fibered manifolds in $\mathcal{Q}_n$ that do not admit a taut $\mathcal{C}^2$-foliation.*

(iii) *There exist infinitely many Seifert fibered manifolds in $\mathcal{Q}_3$ that do not admit a taut $\mathcal{C}^0$-foliation.*

Proof The proof of Theorem 7.1 is an immediate consequence of the two following lemmata. Let n be a positive integer greater than two. Let $\mathcal{M}(n)$ be the family of Seifert fibered 3-manifolds M with n exceptional fibers such that

$$M = M\Big(-1, \frac{1}{2}, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \ldots, \frac{b_n}{a_n}\Big),$$

and the exceptional slopes are ordered in the following way: $\frac{1}{2} > \frac{b_2}{a_2} \geq \frac{b_3}{a_3} \geq \cdots \geq \frac{b_n}{a_n}$.

Lemma 7.2 *Let n be a positive integer greater than two. We consider the following families of infinite Seifert fibered 3-manifolds:*

$$\mathcal{M}_1(n) = \Big\{ M \in \mathcal{M}(n), \ \textit{with}\ \frac{b_2}{a_2} = \frac{2}{5}, \ \frac{b_3}{a_3} > \frac{1}{5}, n > 3 \Big\},$$
$$\mathcal{M}_2 = \Big\{ M \in \mathcal{M}(3), \ \textit{with}\ \frac{b_2}{a_2} = \frac{2}{5}, \ \frac{b_3}{a_3} > \frac{1}{5}, \ \textit{and}\ a_3 \geq 4 \Big\}.$$

If $M \in \mathcal{M}_1(n)$, then $M \in \mathcal{Q}_n$. In particular, M is a non-integral homology 3-sphere that admits the $\widetilde{SL}_2(\mathbb{R})$-geometry, and M does not admit a taut $\mathcal{C}^2$-foliation. Furthermore, if $M \in \mathcal{M}_2$, then $M \in \mathcal{Q}_3$, and M does not admit a taut $\mathcal{C}^0$-folitation.

Proof First, considering Lemma 5.1, we may easily check that when $M \in \mathcal{M}_1(n)$, M is a $\mathbb{Q}HS$ but not a $\mathbb{Z}HS$. Indeed if $M \in \mathcal{M}_1(n)$, then $e(M) > -1 + 1/2 + 2/5 + 1/5$. So, $e(M) > 1/10$, so $e(M) \neq 0$; hence, M is a $\mathbb{Q}HS$.

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On the other hand, if $e(M) = \frac{\varepsilon}{a_1 a_2 \dots a_n}$ (where $\varepsilon = \pm 1$) then $e(M) < \frac{1}{10 a_3}$; which is a contradiction. Then M is not a $\mathbb{Z}HS$. Now, we check that they all have the $\widetilde{SL}_2(\mathbb{R})$-geometry. If $n \geq 4$, then it is a direct consequence of Proposition 5.4. If $n = 3$, that follows from $\sum_{i=1}^{3} \frac{1}{a_i} < 1$ (here, we need that $a_3 \geq 4$). In conclusion, $\mathcal{M}_1(n) \subset \mathcal{Q}_n$ (for $n \geq 3$). Finally, we check that they do not admit a taut $\mathcal{C}^2$-foliation.

If $M \in \mathcal{M}_1(n)$, then $\frac{b_2}{a_2}$ and $\frac{b_3}{a_3}$ both are greater than $1/5$; therefore, Property $(*)$(iii) implies that $m \leq 4$. Thus, $\alpha \in \{1, 2, 3\}$. In all cases, Property $(*)$(i) or Property $(*)$(ii) cannot be satisfied. Furthermore, by Lemma 3.1, if $M \in \mathcal{M}_2$, then M cannot admit a taut $\mathcal{C}^0$-folitation. ∎

Lemma 7.3 *Let n be a positive integer greater than two. Let $\mathcal{M}_3$ and $\mathcal{M}_4(n)$ be the two following families of infinite Seifert fibered 3-manifolds:*

$$\mathcal{M}_3 = \Big\{ M\Big(-1, \frac{1}{2}, \frac{2}{5}, \frac{k}{7k+1}\Big) \in \mathcal{M}(3),\ k \in \mathbb{Z},\ k \geq 1 \Big\},$$

$$\mathcal{M}_4(n) = \Big\{ M\Big(-1, \frac{1}{2}, \frac{2}{5}, \frac{1}{10}, \frac{b_4}{10b_4+1}, \dots, \frac{b_n}{10b_n+1}\Big) \in \mathcal{M}(n),\ n > 3 \Big\}.$$

If $M \in \mathcal{M}_3 \cup \mathcal{M}_4(n)$, then $M \in \mathcal{Q}_n$ and is a non-integral Seifert fibered 3-manifold that admits the $\widetilde{SL}_2(\mathbb{R})$-geometry and a taut analytic foliation.

Proof First, considering Lemma 5.1, we can check that if $M \in \mathcal{M}_3 \cup \mathcal{M}_4(n)$, then M is a $\mathbb{Q}HS$ but not a $\mathbb{Z}HS$. Indeed, if $M \in \mathcal{M}_3$, then $e(M) > -1 + 1/2 + 2/5 + 1/8$, *i.e.*, $e(M) > 1/40$, so $e(M) \neq 0$ and M is a $\mathbb{Q}HS$. If $e(M) = \frac{\varepsilon}{a_1 a_2 a_3}$ (where $\varepsilon = \pm 1$) then $e(M) < 1/70$, which is not possible, so M is not a $\mathbb{Z}HS$. Similarly, if $M \in \mathcal{M}_4$, then $e(M) > -1 + 1/2 + 2/5 + 1/10 + 1/11$, *i.e.*, $e(M) > 1/11$; so $e(M) \neq 0$ and M is a $\mathbb{Q}HS$. If $e(M) = \frac{\varepsilon}{a_1 a_2 \dots a_n}$, then $e(M) < 1/100$, which is not possible, so M is not a $\mathbb{Z}HS$.

Now, we check that they all admit the $\widetilde{SL}_2(\mathbb{R})$-geometry. If $n \geq 4$, then it is a direct consequence of Proposition 5.4. If $n = 3$, that follows from $\sum_{i=1}^{n} \frac{1}{a_i} < 1$ and Proposition 5.3.

Finally, if we choose $\alpha = 3$ and $m = 7$, then (m, α) trivially satisfies Property $(*)$, which implies that they all admit a taut analytic foliation (by Lemma 3.1, Theorems 4.1 and 4.3). ∎

This also concludes the proof of Theorem 7.1. ∎

8 Proof of Theorem 1.1

This section is almost entirely devoted to the proof of Proposition 8.1, which implies Theorem 1.1, as will be shown below.

We may recall here (see Section 2) that if M is a Seifert fibered 3-manifold, then $M = M(-b_0, b_1/a_1, \dots, b_n/a_n)$, where b_0 is a positive integer and $0 < b_i < a_i$ for all $i \in \{1, \dots, n\}$. Note that n has to be greater than 2 (otherwise M cannot be a $\mathbb{Z}HS$ unless it is $\mathbb{S}^3$).

If M is also a $\mathbb{Z}HS$, then two rational coefficients cannot be the same (see Remark 5.2); therefore we may re-order them so that $b_1/a_1 > b_2/a_2 > \cdots > b_n/a_n$.

Thus, two positive integers m and α satisfy Property $(*)$ (for these rational coefficients) if and only if $\alpha < m$ and (i) to (iii) of Property $(*)$ are satisfied.

Proposition 8.1 *Let n be a positive integer and M be a $\mathbb{Z}HS$ Seifert fibered 3-manifold, which is neither homeomorphic to $\mathbb{S}^3$ nor to $\mathcal{P}$. We assume that*

$$M = M(-1, b_1/a_1, \dots, b_n/a_n),$$

where:

- $0 < b_i < a_i$ *for all* $i \in \{1, \dots, n\}$, *and*
- $b_1/a_1 > b_2/a_2 > \cdots > b_n/a_n$.

Then there exist two positive integers m and α that satisfy Property $(*)$.

Proof of Theorem 1.1 First of all, if M is either homeomorphic to $\mathbb{S}^3$ or to the Poincaré sphere $\mathcal{P}$, then we may recall that M cannot admit a taut foliation.

We assume that M is neither homeomorphic to $\mathbb{S}^3$ nor to the Poincaré sphere $\mathcal{P}$. We want to show that M always admits a taut analytic foliation. Let

$$M = M(-b_0, b_1/a_1, \dots, b_n/a_n),$$

where b_0 is a positive integer and $0 < b_i < a_i$ for all $i \in \{1, \dots, n\}$.

Then we note that Corollary 4.2 claims that if $b_0 \in \{2, \dots, n-2\}$, then M admits a horizontal analytic foliation that is a taut $\mathcal{C}^2$-foliation. Then we assume for the following that $b_0 \notin \{2, \dots, n-2\}$.

On the other hand, since M is a $\mathbb{Z}HS$, Lemma 5.1(i) implies that

$$b_0 = \sum_{i=1}^{n} \frac{b_i}{a_i} + \frac{\varepsilon}{a_1 a_2 \dots a_n}, \text{ where } \varepsilon \in \{-1, +1\}.$$

Then the property $0 < b_i/a_i < 1$ for all $i \in \{1, \dots, n\}$, implies that $0 < b_0 < n$. By the fiber-preserving homeomorphism Φ (see Section 2) we may assume that $b_0 = 1$. Hence, Proposition 8.1 implies that there exists a pair of positive integers (m, α) that satisfy Property $(*)$. This implies that M admits a horizontal foliation (Theorem 4.3) then a taut analytic foliation (Lemma 2.2 and Theorem 4.1), which concludes the proof of Theorem 1.1. ■

The rest of the paper is devoted to the proof of Proposition 8.1.

Schedule of the proof of Proposition 8.1 The proof of Proposition 8.1 is organized in four steps.

Step 1: If Proposition 8.1 is true for $n = 3$, then it is true for all $n \geq 3$.
Step 2: Considering $n = 3$ gives common notations and results for the following.
Step 3: We prove Proposition 8.1 for $n = 3$ and $\epsilon = -1$.
Step 4: We prove Proposition 8.1 for $n = 3$ and $\epsilon = 1$.

Before starting the proof, we fix some notation and conventions for the remainder of the paper.

Notation and Conventions We keep the previous notation. Let

$$M = M(-1, b_1/a_1, \dots, b_n/a_n)$$

be a $\mathbb{Z}HS$ Seifert fibered 3-manifold, where $0 < b_i < a_i$ for all $i \in \{1, \dots, n\}$. By Lemma 5.1, M is a $\mathbb{Z}HS$ if and only if

$$\sum_{i=1}^{n} \frac{b_i}{a_i} = 1 + \frac{\epsilon}{a_1.a_2.\dots.a_n}, \text{ where } \epsilon \in \{-1, 1\}. \tag{8.1}$$

Let $\widehat{a}_i$ (for $i \in \{3, \dots, n\}$), $\alpha_1, \alpha_2, a_3', b_3'$ be the following positive rational numbers. Note that all are positive integers except α_1, α_2, which are rational numbers:

$$\alpha_1 = 1 - \frac{b_1}{a_1}, \qquad \alpha_2 = \frac{b_2}{a_2},$$

$$\widehat{a}_i = \frac{a_3 \dots a_n}{a_i} \qquad \forall i \in \{3, \dots, n\},$$

$$b_3' = \sum_{i=3}^{n} b_i \widehat{a}_i, \qquad a_3' = a_3 \dots a_n.$$

Thus,

$$\frac{b_3'}{a_3'} = \sum_{i=3}^{n} \frac{b_i}{a_i}.$$

Now, we fix the following inequalities by denoting them from (8.2) to (8.7). The first three are trivially always true. The last three are true when $n = 3$; see Claim 8.2. They concern Steps 2 to 4:

$$\frac{b_1}{a_1} > \frac{b_2}{a_2} > \dots > \frac{b_n}{a_n}, \tag{8.2}$$

$$\frac{b_1}{a_1} \geq \frac{1}{2}, \tag{8.3}$$

$$\alpha_1 \leq \frac{b_1}{a_1}. \tag{8.4}$$

When $n = 3$:

$$\frac{b_2}{a_2} < \frac{1}{2}, \tag{8.5}$$

$$\frac{b_3}{a_3} < \frac{1}{4}, \tag{8.6}$$

$$\alpha_2 > \alpha_1 - \alpha_2. \tag{8.7}$$

(Here, (8.2) is up to reordering; (8.3) follows by Corollary 4.4, which implies (8.4); (8.5), (8.6), and (8.7), follow by Claim 8.2.)

$$\frac{b_3}{a_3} = \alpha_1 - \alpha_2 + \frac{\epsilon}{a_1 a_2 a_3}, \text{ where } \epsilon \in \{-1, 1\}. \tag{8.8}$$

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Claim 8.2 *If $n = 3$, then $\frac{b_2}{a_2} < \frac{1}{2}$, $\frac{b_3}{a_3} < \frac{1}{4}$, and $\alpha_2 > \alpha_1 - \alpha_2$.*

Proof Since $b_1/a_1 \geq 1/2$, there exists a non-negative integer r_1 such that

$$2b_1 = a_1 + r_1.$$

If $b_2/a_2 + b_3/a_3 < 1/2$, then (8.2) implies (8.5) and (8.6). So, we may suppose that $b_2/a_2 + b_3/a_3 \geq 1/2$. Hence, there exists a non-negative integer r such that

$$2(b_2a_3 + a_2b_3) = a_2a_3 + r.$$

Then (8.8) implies that

$$\frac{r_1}{2a_1} + \frac{r}{2a_2a_3} = \frac{\epsilon}{a_1a_2a_3},$$

so $r_1a_2a_3 + ra_1 = 2\epsilon$.

Therefore, $r_1 = 0$, $r = 1$, $a_1 = 2$, and $\epsilon = +1$. Thus, $b_1/a_1 = 1/2$ and (8.2) implies (8.5) and $a_3b_2 > a_2b_3$. Then $2(b_2a_3 + a_2b_3) = a_2a_3 + r$ implies $1 + a_2a_3 > 4a_2b_3$, and so $a_2a_3 \geq 4a_2b_3$, which is equivalent to $1/4 \geq b_3/a_3$. Since $a_1 = 2$ and the a_i's are pairwise relatively prime, $1/4 > b_3/a_3$ which proves (8.6). By (8.8),

$$\alpha_1 - \alpha_2 = \frac{b_3}{a_3} - \frac{\epsilon}{a_1a_2a_3}.$$

On the other hand, (8.2) implies: $b_2a_3 \geq b_3a_2 + 1$ (since they are positive integers). Therefore,

$$\alpha_2 = \frac{b_2}{a_2} \geq \frac{b_3}{a_3} + \frac{1}{a_2a_3} > \frac{b_3}{a_3} - \frac{\epsilon}{a_1a_2a_3},$$

which implies (8.7). ∎

8.1 Step 1: From $n = 3$ to $n > 3$

We suppose that Proposition 8.1 is satisfied for $n = 3$. Now, we assume that $n \geq 4$ and $M = M(-1, b_1/a_1, \dots, b_n/a_n)$ is a $\mathbb{Z}HS$. We want to show that Property $(*)$ is satisfied for the rational coefficients of the Seifert invariant of M.

Let $M' = M(-1, b_1/a_1, b_2/a_2, b_3'/a_3')$. Note that (8.1) is satisfied because M is a $\mathbb{Z}HS$; therefore, M' is also a $\mathbb{Z}HS$, by the definition of b_3'/a_3'.

We separate the proof according to whether

$$\frac{b_3'}{a_3'} < \frac{b_2}{a_2} \quad \text{or} \quad \frac{b_2}{a_2} < \frac{b_3'}{a_3'}.$$

Note that $b_2/a_2 \neq b_3'/a_3'$ because the a_i's are pairwise relatively prime.

Case 1: $b_3'/a_3' < b_2/a_2$.

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First, we check that $M' \not\cong \mathcal{P}$. Indeed, otherwise $b_3'/a_3' = 1/5$, so we get $a_3' = a_3 = \cdots = a_n = 5$, with $n \geq 4$, a contradiction. Then there exist positive integers m and α such that $\alpha < m$ and:

(i) $$\frac{b_1}{a_1} < \frac{m-\alpha}{m},$$

(ii) $$\frac{b_2}{a_2} < \frac{\alpha}{m}, \text{ and}$$

(iii) $$\frac{b_3'}{a_3'} < \frac{1}{m}.$$

By definition, $\frac{b_i}{a_i} < \frac{b_3'}{a_3'}$ for $i \in \{3, 4, \ldots, n\}$, then the same positive integers m and α satisfy Property $(*)$ for the rational coefficients $\frac{b_i}{a_i}$ (for $i \in \{1, \ldots, n\}$).

Case 2: $b_2/a_2 < b_3'/a_3'$.

We repeat the same argument. Similarly, $M' \not\cong \mathcal{P}$; otherwise $\frac{b_3'}{a_3'} = \frac{1}{3}$, so $a_3 = \cdots = a_n = 3$, with $n \geq 4$, a contradiction. Then there exist positive integers m and α such that $\alpha < m$ and:

(i) $$\frac{b_1}{a_1} < \frac{m-\alpha}{m},$$

(ii) $$\frac{b_3'}{a_3'} < \frac{\alpha}{m}, \text{ and}$$

(iii) $$\frac{b_2}{a_2} < \frac{1}{m}.$$

Since $b_1/a_1 > b_2/a_2 > \cdots > b_n/a_n$, we obtain that $\frac{b_i}{a_i} < \frac{1}{m}$ for $i \in \{2, 3, \ldots, n\}$, which implies that m and α can be chosen so that they satisfy Property $(*)$ for the rational coefficients $\frac{b_i}{a_i}$ (for $i \in \{1, \ldots, n\}$).

8.2 Step 2: General results for $n = 3$

First, note that if m and α are positive integers such that $\alpha < m$, which satisfy Property $(*)$ then, by definition of α_1 and α_2, (i) and (ii) of Property $(*)$ are respectively equivalent to (I) and (II) below:

$$\begin{cases} \text{(I)} & \alpha < m\alpha_1 \\ \text{(II)} & m\alpha_2 < \alpha. \end{cases}$$

Let $a = a_1 a_2$ and $b = a - b_1 a_2 - b_2 a_1$; then $\frac{b}{a} = \alpha_1 - \alpha_2$.

Let $[\,\cdot\,]$ denote the integral value, *i.e.*, $[x]$ is the integer k such that $k \leq x < k+1$, for all real x. Let $N = [a/b]$, hence $N = \left[\frac{1}{\alpha_1 - \alpha_2}\right]$.

Lemma 8.3 *Recall that α and m are integers. The following two properties are satisfied.*

(i) $N \geq 4$;

(ii) *If* $N\alpha_1 - 1 \leq \alpha \leq N\alpha_1$ *and* $N - 1 \leq m$, *then* $0 < \alpha < m$.

Proof

Proof of (i) By (8.8) and (8.6), $\alpha_1 - \alpha_2 < \frac{1}{4} - \frac{\varepsilon}{aa_3}$, *i.e.*, $4b < a - \frac{4\varepsilon}{a_3}$. Note that (8.6) implies that $a_3 \geq 5$ ($b_3 \geq 1$). Then (since a and b are positive integers) $4b \leq a$, so $N = [\frac{a}{b}] \geq 4$.

Proof of (ii) Let α and m such that $N\alpha_1 - 1 \leq \alpha \leq N\alpha_1$ and $N - 1 \leq m$. Now, we can check that $0 < \alpha < m$. The fact that $\alpha < m$ is trivial, because $\alpha_1 \leq 1/2$. Now we check that $\alpha \geq 1$. First, note that if $b = 1$, then

$$N\alpha_1 - 1 = a\frac{(a_1 - b_1)}{a_1} - 1 = a_2(a_1 - b_1) - 1 = b_2 a_1 > 1.$$

Then we assume that $b > 1$. We proceed by contradiction. Assume that $\alpha \leq 0$, then $N\alpha_1 \leq 1$. But $N\alpha_1 \leq 1 \Leftrightarrow \alpha_1 \leq \frac{1}{N}$, which is equal to $\frac{1}{[a/b]}$.

Hence, (8.8) implies

$$\frac{b_2}{a_2} + \frac{b_3}{a_3} \leq \frac{1}{[a/b]} + \frac{\epsilon}{a_1 a_2 a_3}.$$

Since $\frac{b_3}{a_3} < \frac{b_2}{a_2}$,

$$\frac{b_3}{a_3} \leq \frac{1}{2[a/b]} + \frac{\epsilon}{2a_1 a_2 a_3},$$

and so $2b_3[a/b] \leq a_3 + \frac{\epsilon[a/b]}{a}$. Now, $b > 1$ implies $\frac{[a/b]}{a} < 1$, hence $2b_3[a/b] \leq a_3$. Furthermore $[a/b] > a/b - 1 \Rightarrow \frac{a}{b} - 1 < \frac{a_3}{2b_3}$ and so: $ab_3 - bb_3 < \frac{a_3 b}{2}$. Then $ab_3 - \frac{a_3 b}{2} < bb_3$. Finally, note that $(E_2) \Leftrightarrow ab_3 - a_3 b = \epsilon$, *i.e.*, $ab_3 - \frac{a_3 b}{2} = \epsilon + \frac{a_3 b}{2}$. Hence

$$\epsilon + \frac{a_3 b}{2} < bb_3 \Leftrightarrow \frac{b_3}{a_3} > \frac{1}{2} + \frac{\epsilon}{ba_3}.$$

By (8.6) $\varepsilon = -1$ and $\frac{1}{4} > \frac{1}{2} + \frac{-1}{ba_3}$, *i.e.*, $\frac{1}{ba_3} > \frac{1}{4}$, so $ba_3 < 4$. This is a contradiction, because (8.6) implies that $a_3 \geq 5$ and $b \geq 2$. ■

Lemma 8.4 *Let* $r = N\alpha_1 - [N\alpha_1]$, $r' = a/b - [a/b]$ *and* $r'' = a\alpha_1/b - [a\alpha_1/b]$. *If* $N\alpha_1 \in \mathbb{N}$, *let* $(\alpha, m) = (N\alpha_1 - 1, N - 1)$. *If* $N\alpha_1 \notin \mathbb{N}$ *and* $r'\alpha_2 \leq r'' < \alpha_1 r'$, *let* $(\alpha, m) = ([N\alpha_1], N)$. *Otherwise, let* $(\alpha, m) = ([N\alpha_1], N - 1)$. *Then* m *and* α *are positive integers that satisfy* (I) *and* (II), *and* $\alpha < m$.

The proof of this lemma is the main part of Step 3, but does not depend on $\varepsilon = \pm 1$. The fact that $0 < \alpha < m$ is an immediate consequence of Lemma 8.3.

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8.3 Step 3: $n = 3$ and $\epsilon = -1$

Let us consider *Property* $(**)$:

$$(**) \begin{cases} \text{(I)} & \alpha < m\alpha_1 \\ \text{(II)} & m\alpha_2 < \alpha \\ \text{(III)} & \dfrac{b}{a} < \dfrac{1}{m} \end{cases}$$

By (8.8), $\epsilon = -1 \Rightarrow \frac{b_3}{a_3} < \frac{b}{a}$, so Property $(**)$ trivially implies Property $(*)$, *i.e.*, if there exist positive integers m and α such that $\alpha < m$ which satisfy Property $(**)$, then they satisfy Property $(*)$.

We will separate the cases where $N\alpha_1 \in \mathbb{N}$ or $N\alpha_1 \notin \mathbb{N}$. If $N\alpha_1 \notin \mathbb{N}$, let

$$r = N\alpha_1 - [N\alpha_1], \quad r' = a/b - [a/b], \quad r'' = a\alpha_1/b - [a\alpha_1/b].$$

Claim 8.5 $N\alpha_1 = \frac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r'$.

Proof By the definition of r', $N\alpha_1 = [a/b]\alpha_1 = (a/b - r')\alpha_1$. Then

$$N\alpha_1 = \frac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r'. \qquad \blacksquare$$

Claim 8.6 $N\alpha_1 = \left[\frac{\alpha_1}{\alpha_1 - \alpha_2}\right] + r'' - \alpha_1 r'$.

Proof By Claim 8.5 $\frac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r' = N\alpha_1$. Moreover,

$$\frac{\alpha_1}{\alpha_1 - \alpha_2} = \frac{a\alpha_1}{b} = \left[\frac{\alpha_1}{\alpha_1 - \alpha_2}\right] + r''$$

by the definition of r''. $\blacksquare$

Claim 8.7 *If* $[N\alpha_1] = \left[\frac{\alpha_1}{\alpha_1 - \alpha_2}\right] - 1$, *then* $r'' = r + \alpha_1 r' - 1$.

Proof First, we may note that $N\alpha_1 = [N\alpha_1] + r$, by definition of r. Assume that $[N\alpha_1] = \left[\frac{\alpha_1}{\alpha_1 - \alpha_2}\right] - 1$. By Claim 8.5,

$$\frac{\alpha_1}{\alpha_1 - \alpha_2} - \alpha_1 r' = N\alpha_1 = [N\alpha_1] + r = \left[\frac{\alpha_1}{\alpha_1 - \alpha_2}\right] - 1 + r.$$

Hence

$$\frac{\alpha_1}{\alpha_1 - \alpha_2} = \left[\frac{\alpha_1}{\alpha_1 - \alpha_2}\right] + \alpha_1 r' + r - 1.$$

So $r'' = r + \alpha_1 r' - 1$, by the definition of r''. $\blacksquare$

We want to find positive integers m and α such that $\alpha < m$, and that satisfy Property $(**)$. First, we consider separately the case $b = 1$.

Lemma 8.8 *If* $b = 1$, *then* $m = a - 1$ *and* $\alpha = a_1 b_2$ *satisfy property* $(*)$ *and* $0 < \alpha < m$.

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Proof Assume that $b = 1$ and let $m = a - 1$ and $\alpha = a_1 b_2$. First, we can check that $0 < \alpha < m$ because $a_1 b_2 \leq a_1(a_2 - 1) < a_1 a_2 - 1$. Now, we want to check (I)–(III).

For (I),

$$m\alpha_1 = (a_1 a_2 - 1)\frac{a_1 - b_1}{a_1} > a_2(a_1 - b_1) - 1.$$

Since $b = 1$, $a_2(a_1 - b_1) - 1 = a_1 b_2 < m\alpha_1$.

For (II),

$$\alpha_2 = (a_1 a_2 - 1)\frac{b_2}{a_2} = a_1 b_2 - \frac{b_2}{a_2} < \alpha.$$

Since $b = 1$ and $m = a - 1$, (III) is direct. ■

For the rest of this section, we assume that $b \neq 1$. We distinguish the three following cases.

Case A: $N\alpha_1 \in \mathbb{N}$. Then $(\alpha, m) = (N\alpha_1 - 1, N - 1)$.
Case B: $N\alpha_1 \notin \mathbb{N}$ and $r'\alpha_2 > r''$ or $r'' \geq \alpha_1 r'$. Then $(\alpha, m) = ([N\alpha_1], N)$.
Case C: $N\alpha_1 \notin \mathbb{N}$ and $r'\alpha_2 \leq r'' < \alpha_1 r'$. Then $(\alpha, m) = ([N\alpha_1], N - 1)$.

First, we prove (III) of Property $(**)$. Then Lemma 8.4 concludes this step. Note that, for $\varepsilon = 1$, Lemmata 8.8 and 8.4 imply that (I) to (III) are true, but (III) does not imply (iii).

Furthermore, we may note that $b \neq 1$ if and only if $\left[\frac{a}{b}\right] < \frac{a}{b}$, because a and b are positive coprime integers (since a_1 and a_2 are so).

Lemma 8.9 *We assume that $b \neq 1$. If the integers α and m are chosen as in Lemma 8.4 (according to Cases A, B, or C), then $\frac{b}{a} < \frac{1}{m}$.*

Proof Let α and m be integers as in Cases A, B, and C successively. Assume that Case A or Case C is satisfied. Then $m = N - 1$. Therefore (III) is trivial, because $m = N - 1 = [a/b] - 1 < a/b$, so $1/m > b/a$. Assume now that Case B is satisfied. Then (III) $\Leftrightarrow b/a < 1/N \Leftrightarrow N < a/b$, which is satisfied because $N = [a/b]$ and $b \neq 1$. ■

Proof of Lemma 8.4 We may recall that the proof does not depend on $\varepsilon = \pm 1$. We only have to show that the considered integers in Cases A, B, and C satisfy (I) and (II). We may recall that $0 < \alpha < m$ by Lemma 8.3.

Case A: $N\alpha_1 \in \mathbb{N}$, $(\alpha, m) = (N\alpha_1 - 1, N - 1)$. We have I $\Leftrightarrow \alpha < m\alpha_1$. So

$$\text{(I)} \Leftrightarrow N\alpha_1 - 1 < (N - 1)\alpha_1 \Leftrightarrow \alpha_1 < 1,$$

which is true because $0 < \frac{b_1}{a_1} < 1$.

Also, (II) $\Leftrightarrow m\alpha_2 < \alpha$. So

$$\text{(II)} \Leftrightarrow (N - 1)\alpha_2 < N\alpha_1 - 1 \Leftrightarrow 1 - \alpha_2 < N(\alpha_1 - \alpha_2).$$

Therefore,

$$\text{(II)} \Leftrightarrow \frac{1}{\alpha_1 - \alpha_2} - N < \frac{\alpha_2}{\alpha_1 - \alpha_2}.$$

But recall that $N = \left[\frac{1}{\alpha_1-\alpha_2}\right]$, hence $\frac{1}{\alpha_1-\alpha_2} - N < 1$. Thus, (II) follows from (8.7).

Case B: $N\alpha_1 \notin \mathbb{N}$, $r'' \geq \alpha_1 r'$ or $r'' < r'\alpha_2$, $(\alpha, m) = ([N\alpha_1], N)$

(I) is trivially satisfied, as it is equivalent to $[N\alpha_1] < N\alpha_1$.

Also, (II) $\Leftrightarrow$ $m\alpha_2 < \alpha \Leftrightarrow N\alpha_2 < [N\alpha_1] \Leftrightarrow N\alpha_2 < N\alpha_1 - r$. Then (II) $\Leftrightarrow r < N(\alpha_1 - \alpha_2) \Leftrightarrow r < (a/b - r')(\alpha_1 - \alpha_2)$, by the definition of r'. Recall that $b/a = \alpha_1 - \alpha_2$, so (II) $\Leftrightarrow r < 1 - r'(\alpha_1 - \alpha_2) \Leftrightarrow r + r'(\alpha_1 - \alpha_2) < 1$.

So we want to prove that $r + r'(\alpha_1 - \alpha_2) < 1$. Assume first, that $r'' \geq \alpha_1 r'$. Then Claim 8.6 implies that: $[N\alpha_1] = \left[\frac{\alpha_1}{\alpha_1-\alpha_2}\right]$. By Claim 8.5, $\frac{\alpha_1}{\alpha_1-\alpha_2} - \alpha_1 r' = [N\alpha_1] + r$, then $[N\alpha_1] = \frac{\alpha_1}{\alpha_1-\alpha_2} - \alpha_1 r' - r$. Thus,

$$\frac{\alpha_1}{\alpha_1 - \alpha_2} = \left[\frac{\alpha_1}{\alpha_1 - \alpha_2}\right] + \alpha_1 r' + r,$$

so $r'' = \alpha_1 r' + r < 1$. Now, we can see that $r + r'(\alpha_1 - \alpha_2) < r + \alpha_1 r' < 1$, which proves (II). Now, we may assume that $r'' < r'\alpha_2$. By the previous work, we may assume that $r'' < \alpha_1 r'$. Then Claim 8.6 implies that $[N\alpha_1] = \left[\frac{\alpha_1}{\alpha_1-\alpha_2}\right] - 1$.

Therefore, Claim 8.7 implies that $r'' = r + \alpha_1 r' - 1$. Recall that we want to show that $r + r'(\alpha_1 - \alpha_2) < 1$. Since $r'' < r'\alpha_2$, we obtain

$$r + r'(\alpha_1 - \alpha_2) = r + \alpha_1 r' - r'\alpha_2 < r + \alpha_1 r' - r''.$$

Here, $r + \alpha_1 r' - r'' = 1$, which gives the required inequality.

Case C: $N\alpha_1 \notin \mathbb{N}$, $r'' < \alpha_1 r'$ and $r'' \geq r'\alpha_2$, $(\alpha, m) = ([N\alpha_1], N - 1)$

(I) $\Leftrightarrow \alpha < m\alpha_1$. (I) $\Leftrightarrow [N\alpha_1] < (N-1)\alpha_1 \Leftrightarrow \alpha_1 < r$. Since $r'' < \alpha_1 r'$, by Claim 8.6 $[N\alpha_1] = \left[\frac{\alpha_1}{\alpha_1-\alpha_2}\right] - 1$. Then by Claim 8.7, $r'' = r + \alpha_1 r' - 1$. Thus (I) $\Leftrightarrow \alpha_1 < r'' - \alpha_1 r' + 1 \Leftrightarrow \alpha_1 r' - r'' < 1 - \alpha_1$. Hence, (I) $\Leftrightarrow \alpha_1 r' - r'' < \frac{b_1}{a_1}$, because $1 - \alpha_1 = \frac{b_1}{a_1}$.

On the other hand, $\alpha_1 r' - r'' < \alpha_1 - r''$ and $\alpha_1 - r'' \leq \alpha_1 \leq \frac{b_1}{a_1}$ by (8.4). Therefore (I) is satisfied.

(II) $\Leftrightarrow m\alpha_2 < \alpha \Leftrightarrow (N-1)\alpha_2 < [N\alpha_1]$. By Claim 8.6 and the definition of r'', and since $r'' < \alpha_1 r'$: $[N\alpha_1] = [\frac{\alpha_1}{\alpha_1-\alpha_2}] - 1 = \frac{\alpha_1}{\alpha_1-\alpha_2} - r'' - 1$. Moreover, by the definition of r': $N\alpha_2 = \frac{\alpha_2}{\alpha_1-\alpha_2} - \alpha_2 r'$. Hence,

$$\text{(II)} \Leftrightarrow N\alpha_2 < [N\alpha_1] + \alpha_2 \Leftrightarrow \frac{\alpha_2}{\alpha_1 - \alpha_2} - \alpha_2 r' < \frac{\alpha_1}{\alpha_1 - \alpha_2} - r'' - 1 + \alpha_2.$$

Therefore, (II) $\Leftrightarrow r'' - r'\alpha_2 < \alpha_2$. On the other hand, $\alpha_2 > \alpha_1 - \alpha_2$ by (8.7) and $r'\alpha_2 \leq r'' < \alpha_1 r'$. Then $r'' - r'\alpha_2 < r'(\alpha_1 - \alpha_2) < r'\alpha_2 < \alpha_2$, which proves that (II) is satisfied. ■

In conclusion, Lemma 8.8 solves the case $b = 1$. If $b \neq 1$, then for the α and m chosen as in Lemma 8.3, we get that $0 < \alpha < m$ and Lemmata 8.4 and 8.9 show that they satisfy (I), (II), and (III). Therefore, Property $(*)$ is satisfied for $n = 3$ and $\varepsilon = -1$.

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8.4 Step 4: $n = 3$ and $\epsilon = 1$

Recall that $a = a_1 a_2$ and $b = a - b_1 a_2 - b_2 a_1$. We assume that $\epsilon = 1$ then (8.8) gives:

$$\frac{b_3}{a_3} = \frac{b}{a} + \frac{1}{a_1 a_2 a_3},$$

so

$$ab_3 - ba_3 = 1. \tag{8.9}$$

Then (Bezout relation) there exists a unique pair of positive coprime integers (u, v) such that:

$$au - bv = 1, \quad 0 < u \leq b \text{ and } 0 < v \leq a. \tag{8.10}$$

Now, (8.9) implies that there exists $p \in \mathbb{N}$ such that

$$b_3 = u + bp, \quad a_3 = v + ap.$$

Moreover, for all $p \in \mathbb{N}$, we have:

$$\frac{u}{v} \geq \frac{u + bp}{v + ap} > \frac{u + b(p+1)}{v + a(p+1)} > \frac{b}{a}. \tag{8.11}$$

We want to find positive integers α and m such that $\alpha < m$ and satisfy Property $(*)$. We consider separately the three following cases.

Case I: $u \neq 1$.
Case II: $u = 1$ and $b = 1$.
Case III: $u = 1$ and $b \neq 1$.

Case I: $u \neq 1$.

We will choose the integers α and m as in Lemma 8.4, so $m \in \{N-1, N\}$. By (8.11), if $\frac{u}{v} < \frac{1}{m}$, then (iii) of Property $(*)$ is satisfied. Therefore, Lemma 8.4 and the following lemma conclude Case I.

Lemma 8.10 *If $N - 1 \leq m \leq N$ and $u \neq 1$, then $\frac{u}{v} < \frac{1}{m}$.*

Proof Assume that $N - 1 \leq m \leq N$ and $u \neq 1$. First, note that $b \neq 1$, because $0 < u \leq b$ implies that if $b = 1$, then $u = 1$.

We suppose that $\frac{u}{v} \geq \frac{1}{m}$, and we look for a contradiction. Note that $v = um$ cannot happen, because u, v are coprime integers, and u and m are at least 2 by Lemma 8.3. Thus $\frac{u}{v} > \frac{1}{m}$.

Moreover, by Lemma 8.9: $\frac{b}{a} < \frac{1}{m}$. Then

$$\frac{b}{a} < \frac{1}{m} < \frac{u}{v}$$

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By (8.10)

$$\frac{a}{b} - \frac{v}{u} = \frac{1}{ub}.$$

We obtain

$$0 < m - \frac{v}{u} < \frac{a}{b} - \frac{v}{u} = \frac{1}{ub} < 1$$

which implies that $\left[\frac{v}{u}\right] = m - 1$. Now, let

$$r' = \frac{a}{b} - \left[\frac{a}{b}\right] < 1 \quad \text{and} \quad \rho = \frac{v}{u} - \left[\frac{v}{u}\right] < 1$$

We consider separately the cases $m = N$ and $m = N - 1$.

First, assume that $m = N = \left[\frac{a}{b}\right]$. Then

$$\left[\frac{v}{u}\right] = \left[\frac{a}{b}\right] - 1 \Leftrightarrow \frac{a}{b} - r' - 1 = \frac{v}{u} - \rho,$$

hence $\frac{1}{ub} = 1 + r' - \rho \Rightarrow 1 + r' - \rho < \frac{1}{b}$ because $u \neq 1$. Thus $r' < \frac{1}{b}$ because $\rho < 1$. Nevertheless, $r' = \frac{a}{b} - \left[\frac{a}{b}\right]$, a and b are coprime, and $a > b$. Hence $a = bk + l$, where $k \in \mathbb{N}^*$, and $1 \leq l \leq b - 1$; so r' can be written

$$r' = k + \frac{l}{b} - \left[k + \frac{l}{b}\right] = \frac{l}{b} \Rightarrow r' \geq \frac{1}{b};$$

which is a contradiction.

Now, assume that $m = N - 1 = \left[\frac{a}{b}\right] - 1$. Then

$$\left[\frac{v}{u}\right] = \left[\frac{a}{b}\right] - 2 \Leftrightarrow \frac{a}{b} - r' - 2 = \frac{v}{u} - \rho,$$

hence $\frac{1}{ub} = 2 + r' - \rho$. This implies that $\frac{1}{ub} > 1$ because r' and ρ lie in $[0, 1[$. On the other hand, $\frac{1}{ub} \leq \frac{1}{4}$, because $b \geq 2$ and $u \geq 2$. These are in contradiction. ∎

Case II: $u = 1$ and $b = 1$. We assume that $u = b = 1$. Then $au - bv = 1$ gives $v = a - 1$.

We consider separately the cases where $\frac{b_1}{a_1} > \frac{1}{2}$ or $\frac{b_1}{a_1} = \frac{1}{2}$. First, assume that $\frac{b_1}{a_1} > \frac{1}{2}$. Let $m = a - 2$, and $\alpha = a_2(a_1 - b_1) - 1 = a - a_2 b_1 - 1 = b_2 a_1$ (because $b = a - a_2 b_1 - a_1 b_2 = 1$. Then $0 < \alpha < m$. We want to check (I), (II), and (iii).

$$\text{(I)} \Leftrightarrow \alpha < m\alpha_1 \Leftrightarrow a_2(a_1 - b_1) - 1 < (a_1 - b_1)a_2 - 2\alpha_1 \Leftrightarrow \frac{1}{2} < \frac{b_1}{a_1},$$

which is satisfied here.

$$\text{(II)} \Leftrightarrow m\alpha_2 < \alpha \Leftrightarrow (a - 2)\alpha_2 < a_2(a_1 - b_1) - 1 \Leftrightarrow 1 - 2\alpha_2 < a_2(a_1 - b_1) - a\alpha_2,$$

which is satisfied because

$$a_2(a_1 - b_1) - a\alpha_2 = a_2(a_1 - b_1) - a_1 b_2 = b = 1.$$

By (8.11), (iii) is satisfied if $\frac{u}{v} < \frac{1}{m}$, which is true because $\frac{u}{v} = \frac{1}{a-1}$ and $\frac{1}{m} = \frac{1}{a-2}$.

Now, assume that $\frac{b_1}{a_1} = \frac{1}{2}$.

Then $a_1 = 2$ and $b_1 = 1$. Since $1 = b = a_2(a_1 - b_1) - a_1 b_2$, $a_2 = 1 + 2b_2$. So

$$\frac{b_2}{a_2} = \frac{b_2}{1 + 2b_2},$$

and by (8.8), $\frac{b_3}{a_3} = \frac{1}{2} - \frac{b_2}{1+2b_2} + \frac{1}{2(2b_2+1)a_3}$.

Thus,

$$\frac{b_3}{a_3} = \frac{(2b_2 + 1)a_3 - 2b_2 a_3 + 1}{2(2b_2 + 1)a_3}, \quad \textit{i.e.}, \quad \frac{b_3}{a_3} = \frac{a_3 + 1}{2(2b_2 + 1)a_3}.$$

We consider separately the cases $b_2 = 1$ and $b_2 > 1$. Assume first $b_2 = 1$. Then $a_2 = 3$ so $\frac{b_3}{a_3} = \frac{a_3+1}{6a_3}$. Therefore, we can check easily that $\alpha = 2$ and $m = 5$ satisfy Property $(*)$.

(i) $$\frac{b_1}{a_1} = \frac{1}{2} < \frac{m - \alpha}{m}, \text{ which is } \frac{3}{5},$$

(ii) $$\frac{b_2}{a_2} = \frac{1}{3} < \frac{\alpha}{m}, \text{ which is } \frac{2}{5},$$

(iii) $$\frac{b_3}{a_3} = \frac{a_3 + 1}{6a_3} < \frac{1}{m} \text{ which is } \frac{1}{5} \text{ if and only if } a_3 > 5.$$

By (8.6) $a_3 \geq 5$, but if $a_3 = 5$, then $M \cong \mathcal{P}$, so $a_3 > 5$. Now we assume that $b_2 \geq 2$. Let $\alpha = 2b_2 - 1$ and $m = 4b_2 - 1$. Since $b_2 \geq 2$: $0 < \alpha < m$. We want to check (i), (ii), and (iii):

(i) $\frac{b_1}{a_1} = \frac{1}{2} < \frac{m-\alpha}{m}$, which is $\frac{2b_2}{4b_2-1}$, so (i) is satisfied.

(ii) $\frac{b_2}{a_2} = \frac{b_2}{2b_2+1} < \frac{\alpha}{m}$, which is $\frac{2b_2-1}{4b_2-1}$ and $\frac{b_2}{2b_2+1} < \frac{2b_2-1}{4b_2-1}$ if and only if $4b_2^2 - b_2 < 4b_2^2 - 1$, *i.e.*,$b_2 > 1$, so (ii) is satisfied.

(iii) $\frac{b_3}{a_3} = \frac{a_3+1}{2(2b_2+1)a_3} < \frac{1}{m}$, which is $\frac{1}{4b_2-1}$.

Then (iii) is satisfied if and only if: $(a_3+1)(4b_2-1) < (4b_2+2)a_3$, *i.e.*, $4b_2 < 3a_3+1$. Since $\frac{b_3}{a_3} = \frac{a_3+1}{2(2b_2+1)a_3}$, $b_3 = \frac{a_3+1}{2(2b_2+1)} \geq 1$ (because b_3 is a positive integer). So $a_3 + 1 \geq 4b_2 + 2$, thus (iii) is satisfied.

Case III: $u = 1$ and $b \neq 1$. We assume $u = 1$ and $b \geq 2$. Then $a - bv = 1$ by (8.10).

Claim 8.11 *If $\frac{b_2}{a_2} < \frac{1}{v}$, then $m = v$ and $\alpha = 1$ satisfy Property $(*)$.*

Proof Assume that $\frac{b_2}{a_2} < \frac{1}{v}$. To prove that $m = v$ and $\alpha = 1$ satisfy Property $(*)$, it remains to prove that $\frac{b_1}{a_1} < \frac{v-1}{v}$. Indeed, (II) and (iii) are trivially satisfied, because $\frac{b_3}{a_3} < \frac{b_2}{a_2} < \frac{1}{v}$. By (8.10),

$$1 + \frac{1}{av} = \frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{1}{v}.$$

But $\frac{b_2}{a_2} > \frac{1}{av}$, otherwise $b_2 < \frac{1}{a_1 v}$, which is impossible. Therefore,

$$\frac{b_1}{a_1} = 1 + \frac{1}{av} - \frac{b_2}{a_2} - \frac{1}{v} < 1 - \frac{1}{v},$$

so $\frac{b_1}{a_1} < \frac{v-1}{v}$. ∎

Hence, in the following, we assume that $\frac{b_2}{a_2} > \frac{1}{v}$ (note that equality is impossible because the integers are coprime). Let α be the integer such that

$$(v-1)\alpha_1 - 1 \leq \alpha < (v-1)\alpha_1 \quad \text{and} \quad m = \min(v-1, M),$$

where M is the positive integer such that $\frac{\alpha}{\alpha_2} - 1 \leq M < \frac{\alpha}{\alpha_2}$. Then $\alpha = (v-1)\alpha_1 - r$ where $0 < r \leq 1$; $M = \frac{\alpha}{\alpha_2} - r'$ where $0 < r' \leq 1$; and $m = \min(M, v-1)$.

First, we will check that $m > \alpha > 0$, then we will show that the integers m and α satisfy Property $(*)$.

Claim 8.12 *The integers m and α satisfy* $1 \leq \alpha < m$

Proof First, we check that $\alpha \geq 1$, where $\alpha = (v-1)\alpha_1 - r$, $0 < r \leq 1$. We show that $(v-1)\alpha_1 > 1$, then $\alpha > 0$. Since $\alpha \in \mathbb{N}$, $\alpha \geq 1$. By (8.10):

$$\alpha_1 = \frac{b_2}{a_2} + \frac{1}{v} - \frac{1}{a_1 a_2 v}.$$

Since $\frac{b_2}{a_2} > \frac{1}{v}$,

$$\alpha_1 > \frac{2}{v} - \frac{1}{a_1 a_2 v}, \quad \textit{i.e.}, \quad \alpha_1 > \frac{2a_1 a_2 - 1}{a_1 a_2 v}.$$

Therefore,

$$v > \frac{2a_1 a_2 - 1}{a_1 a_2 \alpha_1},$$

so

$$(v-1)\alpha_1 > \frac{a_1 a_2 (2 - \alpha_1) - 1}{a_1 a_2}.$$

Finally, recall that $1 - \alpha_1 = \frac{b_1}{a_1}$. Thus,

$$(v-1)\alpha_1 > \frac{a_1 a_2 (1 + \frac{b_1}{a_1}) - 1}{a_1 a_2}, \quad \textit{i.e.}, \quad (v-1)\alpha_1 > \frac{a_1 a_2 + a_2 b_1 - 1}{a_1 a_2}.$$

Since $a_2 \geq 3$, $(v-1)\alpha_1 > \frac{a_1 a_2 + 2}{a_1 a_2} > 1$.

Now, we check that $m > \alpha$. If $m = v - 1$, this is trivial. So, we may assume that $m = \frac{\alpha}{\alpha_2} - r'$, where $0 < r' \leq 1$. Therefore, $m = \alpha(\frac{1}{\alpha_2} - \frac{r'}{\alpha})$. Since $\alpha \geq 1$, $\frac{r'}{\alpha} \leq r' \leq 1$, so $m \geq \alpha(\frac{1}{\alpha_2} - 1)$. Finally, (8.5) implies that $\alpha_2 < \frac{1}{2}$ and so that $m > \alpha$. ∎

To show that α and m satisfy Property $(*)$, we need the following claim.

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Claim 8.13 $\frac{\alpha_1-\alpha_2}{\alpha_1}+\alpha_1<1-\frac{1}{a}$.

Proof We first assume $\frac{b_1}{a_1}=\frac{1}{2}$. Then $\alpha_1=\frac{1}{2}$ and $a=2a_2$, so

$$1-\frac{1}{a}=\frac{2a_2-1}{2a_2}.$$

On the other hand,

$$\frac{\alpha_1-\alpha_2}{\alpha_1}+\alpha_1=\frac{3}{2}-2\alpha_2=\frac{3a_2-4b_2}{2a_2}.$$

Then

$$\frac{\alpha_1-\alpha_2}{\alpha_1}+\alpha_1<1-\frac{1}{a}$$

if and only if $a_2-4b_2<-1$. Now, (8.7) implies

$$\alpha_2>\frac{\alpha_1}{2},\quad \textit{i.e.},\quad 4b_2>a_2,$$

so $a_2-4b_2\leq -1$.

We are going to show that $a_2\neq 4b_2-1$ by contradiction. First, note that since $b_1/a_1=1/2$, $a=2a_2$ and $b=a_2-2b_2\neq 1$. On the other hand, since $a-bv=1$,

$$v=\frac{a-1}{b}=\frac{2a_2-1}{a_2-2b_2}.$$

If $a_2=4b_2-1$, then $v=\frac{8b_2-3}{2b_2-1}$. Now, $v=4+\frac{1}{2b_2-1}\in\mathbb{N}$ implies that $b_2=1$, $v=5$, and $a_2=3$. Then $b=3-2=1$, which is a contradiction. Therefore, $a_2<4b_2-1$, which is the required inequality.

Now, we assume that $\frac{b_1}{a_1}>\frac{1}{2}$, so $2b_1-a_1>0$. Then $a_1-b_1<a_1b_2(2b_1-a_1)$, so

$$(a_1-b_1)+a_1^2b_2<2a_1b_1b_2,\quad\text{and}\quad (a_1-b_1)-2a_1b_1b_2+2a_1^2b_2<a_1^2b_2.$$

Therefore,

$$(a_1-b_1)(1+2a_1b_2)<a_1^2b_2. \tag{8.12}$$

On the other hand, (8.7) implies that $2\alpha_2>\alpha_1$, *i.e.*, $2a_1b_2>a_2(a_1-b_1)$. Hence, $2a_1b_2(a_1-b_1)+(a_1-b_1)>a_2(a_1-b_1)^2+(a_1-b_1)$, *i.e.*,

$$(2a_1b_2+1)(a_1-b_1)>a_2(a_1-b_1)^2+(a_1-b_1).$$

Therefore, by the inequality (8.12):

$$a_2(a_1-b_1)^2+(a_1-b_1)<a_1^2b_2.$$

So

$$\frac{a_1-b_1}{a_1^2a_2}<\frac{a_1^2b_2-a_2(a_1-b_1)^2}{a_1^2a_2};\quad \textit{i.e.},\quad \frac{\alpha_1}{a}<\alpha_2-\alpha_1^2.$$

Thus, we obtain $\frac{1}{a}<\frac{\alpha_2}{\alpha_1}-\alpha_1$, which gives the required inequality. ■

Now, we will show successively that α and m satisfy (iii), (II), and (I) of Property $(*)$.

- *α and m satisfy* (iii)*:* $\frac{b_3}{a_3} < \frac{1}{m}$.

This is trivially satisfied because by (8.11), $\frac{b_3}{a_3} \leq \frac{1}{\nu}$, and $m \leq \nu - 1$.

- *α and m satisfy* (II)*:* $m\alpha_2 < \alpha$.

Since $m \leq M$, $m\alpha_2 \leq \alpha - r'\alpha_2 < \alpha$ (because $r' > 0$) then (α, m) trivially satisfies (II).

- *α and m satisfy* (I)*:* $\alpha < m\alpha_1$.

Since $r > 0$, $(\nu - 1)\alpha_1 - r < (\nu - 1)\alpha_1$. Hence, $\alpha < m\alpha_1$ if $m = \nu - 1$. Thus, we may assume that $m = M \leq \nu - 2$.

So, we want to show that $(\nu - 1)\alpha_1 - r < (\frac{\alpha}{\alpha_2} - r')\alpha_1$. Now,

$$
\begin{aligned}
(\nu - 1)\alpha_1 - r &< \left(\frac{\alpha}{\alpha_2} - r'\right)\alpha_1 \\
&\Leftrightarrow \nu - \frac{r}{\alpha_1} < \frac{(\nu - 1)\alpha_1 - r}{\alpha_2} - r' + 1 \\
&\Leftrightarrow \nu\alpha_1\alpha_2 - r\alpha_2 < \nu\alpha_1^2 - \alpha_1^2 - r\alpha_1 - r'\alpha_1\alpha_2 + \alpha_1\alpha_2 \\
&\Leftrightarrow r(\alpha_1 - \alpha_2) + r'\alpha_1\alpha_2 + \alpha_1(\alpha_1 - \alpha_2) < \nu\alpha_1(\alpha_1 - \alpha_2) \\
&\Leftrightarrow \nu(\alpha_1 - \alpha_2) > \alpha_1 - \alpha_2 + r\frac{\alpha_1 - \alpha_2}{\alpha_1} + r'\alpha_2
\end{aligned}
$$

Recall that $\frac{b}{a} = \alpha_1 - \alpha_2$ and $a - b\nu = 1$, so

$$\nu(\alpha_1 - \alpha_2) = \frac{\nu b}{a} = \frac{a - 1}{a} = 1 - \frac{1}{a}.$$

Therefore, α and m satisfy (I) if and only if

$$1 - \frac{1}{a} > \alpha_1 - \alpha_2 + r\frac{\alpha_1 - \alpha_2}{\alpha_1} + r'\alpha_2.$$

Since r and r' both lie in $]0, 1]$:

$$\alpha_1 - \alpha_2 + r\frac{\alpha_1 - \alpha_2}{\alpha_1} + r'\alpha_2 < \alpha_1 - \alpha_2 + \frac{\alpha_1 - \alpha_2}{\alpha_1} + \alpha_2,$$

i.e.,

$$\alpha_1 - \alpha_2 + r\frac{\alpha_1 - \alpha_2}{\alpha_1} + r'\alpha_2 < \alpha_1 + \frac{\alpha_1 - \alpha_2}{\alpha_1} < 1 - \frac{1}{a},$$

by Claim 8.13.

Hence, α and m satisfy (I), which ends the proof of Proposition 8.1. ■

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References

[1] M. Brittenham, *Essential laminations in Seifert-fibered spaces*. Topology **32**(1993), no. 1, 61–85. http://dx.doi.org/10.1016/0040-9383(93)90038-W

[2] M. Brittenham, R. Naimi, and R. Roberts, *Graph manifolds and taut foliations.* J. Differential Geom. **45**(1997), no. 3, 446–470.

[3] S. Caillat-Gibert and D. Matignon, *Compact leaves in Reebless or taut foliations.* arxiv:1105.3103v2

[4] D. Calegari, *Promoting essential laminations*. Invent. Math. **166**(2006), no. 3, 583–643. http://dx.doi.org/10.1007/s00222-006-0004-3

[5] E. Clauss, *Essential laminations in closed Seifert-fibered spaces*. Thesis, University of Texas at Austin, 1991.

[6] D. Eisenbud, U. Hirsch, and W. Neumann, *Transverse foliations on Seifert bundles and self-homeomorphisms of the circle.* Comment. Math. Helvetici **56**(1981), no. 4, 638–660. http://dx.doi.org/10.1007/BF02566232

[7] Y. Eliashberg and W. P. Thurston, *Confoliations.* University Lecture Series, 13, American Mathematical Society, Providence, RI, 1998.

[8] D. B. A. Epstein, *Periodic flows on three-manifolds.* Ann. of Math. **95**(1972), 66–82. http://dx.doi.org/10.2307/1970854

[9] D. Gabai, *Foliations and the topology of 3-manifolds.* J. Differential Geometry. **18**(1983), no. 3, 445–503.

[10] M. Jankins and W. Neumann, *Rotation numbers of products of circle homeomorphisms.* Math. Ann. **271**(1985), no. 3, 381–400. http://dx.doi.org/10.1007/BF01456075

[11] G. Levitt, *Feuilletages des variétés de dimension* 3 *qui sont des fibres en cercles.* Comment. Math. Helv. **53**(1978), no. 4, 572–594. http://dx.doi.org/10.1007/BF02566099

[12] P. Lisca and G. Matić, *Transverse contact structures on Seifert* 3*-manifolds.* Algebr. Geom. Topol. **4**(2004), 1125–1144. http://dx.doi.org/10.2140/agt.2004.4.1125

[13] P. Lisca and A. I. Stipsicz, *Ozsváth-Szabó invariants and tight contact* 3*-manifolds, III.* J. Symplectic Geom. **5**(2007), no. 4, 357–384.

[14] S. Matsumoto, *Foliations of Seifert-fibered spaces over* $\mathbb{S}^2$. In: Foliations (Tokyo, 1983), Adv. Stud. Pure Math., 5, North-Holland, Amsterdam, 1985, pp. 325–339.

[15] R. Naimi, *Foliations transverse to fibers of Seifert manifolds.* Comment. Math. Helv. **69**(1994), no. 1, 155–162. http://dx.doi.org/10.1007/BF02564479

[16] S. P. Novikov, *Foliations of codimension* 1 *on manifolds*. (Russian) Dokl. Akad. Nauk SSSR **155**(1964), 1010–1013.

[17] C. Palmeira, *Open manifolds foliated by planes*. Ann. Math. **107**(1978), no. 1, 109–131. http://dx.doi.org/10.2307/1971256

[18] R. Rustamov, *On Heegard Floer homology of three-manifolds.* Ph.D. Dissertation, Princeton University, 2005. arxiv:math/0505349

[19] R. Roberts, J. Shareshian, and M. Stein, *Infinitely many hyperboloic* 3*-manifolds which contain no Reebless foliation.* J. Amer. Math. Soc. **16**(2003), no. 3, 639–679. http://dx.doi.org/10.1090/S0894-0347-03-00426-0

[20] N. Saveliev, *Invariants for homology 3-spheres.* In: Encyclopaedia of Mathematical Sciences, 140, Low-Dimensional Topology, I, Springer-Verlag, Berlin, 2002.

[21] P. Scott, *The geometry of* 3*-manifolds*. Bull. London Math. Soc. **15**(1983), no. 5, 401–487. http://dx.doi.org/10.1112/blms/15.5.401

[22] H. Seifert, *Topologie dreidimensionaler gefaserter Raume.* Acta Math. **60**(1933), no. 1, 147–238. http://dx.doi.org/10.1007/BF02398271

[23] H. Seifert and W. Threlfall, *Topology of* 3*-dimensional fibered spaces.* Pure and Applied Mathematics, 89, Academic Press, 1980, pp. 360–418.

[24] W. P. Thurston, *Foliations of* 3*-manifolds which are circle bundles*. Thesis, University of California at Berkeley, 1972.

[25] F. Waldhausen, *Eine classe von* 3*-dimensionalen mannigfaltigkeiten I*. Invent. Math. **3**(1967), 308–333.

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