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Abstract

The loop space of a string manifold supports an infinite-dimensional Fock space bundle, which is an analog of the spinor bundle on a spin manifold. This spinor bundle on loop space appears in the description of two-dimensional sigma models as the bundle of states over the configuration space of the superstring. We construct a product on this bundle that covers the fusion of loops, i.e. the merging of two loops along a common segment. For this purpose, we exhibit it as a bundle of bimodules over a certain von Neumann algebra bundle, and realize our product fibrewise using the Connes fusion of von Neumann bimodules. Our main technique is to establish novel relations between string structures, loop fusion, and the Connes fusion of Fock spaces. The fusion product on the spinor bundle on loop space was proposed by Stolz and Teichner as part of a programme to explore the relation between generalized cohomology theories, functorial field theories, and index theory. It is related to the pair of pants worldsheet of the superstring, to the extension of the corresponding smooth functorial field theory down to the point, and to a higher-categorical bundle on the underlying string manifold, the stringor bundle.

Contents

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1. Introduction

In this article we construct a Connes fusion product on the spinor bundle on the loop space of a string manifold and, thus, solve a problem formulated by Stolz and Teichner in 2005 [ST05]. We recall that a spin manifold M admits a string structure if and only if its first fractional Pontryagin class vanishes, i.e.

$$\tfrac{1}{2}p_1(M) = 0.$$

If M is a string manifold, then its free loop space $LM = C^\infty(S^1, M)$ is a spin manifold in the sense of Killingback [Kil87], i.e. it comes equipped with a certain principal bundle for the basic central extension

$$\mathrm{U}(1) \to \widetilde{L\,\mathrm{Spin}}(d) \to L\,\mathrm{Spin}(d)$$

of the loop group of $\mathrm{Spin}(d)$, where $d = \dim(M)$. In our previous work [KW20] we have constructed an infinite-dimensional Hilbert space bundle $F(LM)$ on LM by associating a certain unitary representation

$$\widetilde{L\,\mathrm{Spin}}(d) \to \mathrm{U}(F)$$

to this principal bundle, where F is the Fock space of 'free fermions on the circle'. We have proved [KW20] that the bundle $F(LM)$ realizes precisely what Stolz and Teichner called the *spinor bundle on loop space* [ST05].

In the present article, we construct a hyperfinite type III_1 von Neumann algebra bundle $\mathcal{N}$ over the space PM of smooth paths in M, and prove (Theorem 5.2.5) that the spinor bundle $F(LM)$ is a $p_1^*\mathcal{N}$–$p_2^*\mathcal{N}$-bimodule bundle, where $p_1, p_2 : LM \to PM$ are the maps that divide a loop into its two halves. From a fibrewise point of view, if $\beta_1, \beta_2 \in PM$ are paths with a common initial point and a common endpoint, and $\beta_1 \cup \beta_2$ denotes the corresponding loop, then $F(LM)_{\beta_2\cup\beta_1}$ is an $\mathcal{N}_{\beta_1}$–$\mathcal{N}_{\beta_2}$-bimodule. Our main result (Theorem 5.3.1) is the existence and unique characterization of unitary intertwiners

$$\chi_{\beta_1,\beta_2,\beta_3} : F(LM)_{\beta_2\cup\beta_3} \boxtimes F(LM)_{\beta_1\cup\beta_2} \to F(LM)_{\beta_1\cup\beta_3}$$

of $\mathcal{N}_{\beta_3}$–$\mathcal{N}_{\beta_1}$-bimodules, where $\boxtimes$ is the Connes fusion of bimodules over $\mathcal{N}_{\beta_2}$, and $\beta_1, \beta_2, \beta_3 \in PM$ is any triple of paths with a common initial point and a common endpoint. One may regard the loop $\beta_1 \cup \beta_3$ as the fusion of the loops $\beta_1 \cup \beta_2$ and $\beta_2 \cup \beta_3$ along the common segment β_2, and regard $\chi_{\beta_1,\beta_2,\beta_3}$ as lifting this loop-fusion product. We call the collection of intertwiners $\chi_{\beta_1,\beta_2,\beta_3}$, varying over all triples of paths with a common initial point and a common endpoint, the

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Connes fusion product on the spinor bundle on loop space; it is an associative product covering the fusion of loops (Proposition 5.3.3).

Our construction of the Connes fusion product on the spinor bundle on loop space became possible because we found a specific way to relate string structures, loop fusion, and Connes fusion.

(1) We use the recent discovery of a certain loop-fusion product on Killingback's spin structure on loop space [Wal16a, Wal15]. This fusion product belongs to a loop space equivalent formulation of a string structure on M, and thus provides a neat way to use a string structure while working on the loop space.
(2) A similar, Lie group-theoretical loop-fusion product exists on the basic central extension $\widetilde{L\,\mathrm{Spin}}(d)$, and it satisfies a certain compatibility condition with the fusion product of part (1). In our previous work [KW22] we have found an operator-theoretic description of this loop-fusion product, using Tomita–Takesaki theory of the free fermions F.
(3) We use that the free fermions F are a standard form for a certain Clifford–von Neumann algebra to construct a novel Connes fusion product between certain operators on F (Definition 3.4.5). We derive a new and crucial relation (Theorem 4.3.3) between the loop-fusion product of part (2) and this Connes fusion product.

Our construction of the Connes fusion product on the spinor bundle in Theorem 5.3.1 combines these three fusion products, and uses the above-mentioned correspondences between the differential geometric setting and the operator-algebraic setting. In particular, we discovered how the datum of a string structure on M is involved; this was probably the main issue that needed to be solved.

The question for the existence of an associative Connes fusion product on the spinor bundle $F(LM)$ was formulated as Theorem 1 in [ST05], but has not been carried out so far. In addition to proving its (fibrewise) existence, we address and solve the question in which way these structures can be upgraded from a purely topological setting into a *smooth* setting. In our previous work [KW20], we started this by generalizing the concept of a *rigged Hilbert space* to C*-algebras and bundle versions thereof. In particular, we proved there that the spinor bundle $F(LM)$ is a *rigged* Hilbert space bundle over LM. In the present article, we further extend this framework to von Neumann algebras, and exhibit our von Neumann algebra bundle $\mathcal{N}$ over PM as a *rigged* von Neumann algebra bundle, and the spinor bundle $F(LM)$ as a *rigged* von Neumann $p_1^*\mathcal{N}$–$p_2^*\mathcal{N}$-bimodule bundle (Theorem 5.2.5). Moreover, we show that our Connes fusion product on the spinor bundle is smooth with respect to these rigged structures (Proposition 5.3.5). Concerning smoothness aspects, we work as far as possible with Fréchet spaces and Fréchet manifolds. When it comes to fusion, we use the convenient setting of *diffeological spaces* in order to properly describe spaces of paths with common endpoints, on which operations like $(\beta_1, \beta_2) \mapsto \beta_1 \cup \beta_2$ are well-defined and smooth. The extension of, for instance, rigged von Neumann algebra bundles from Fréchet manifolds to diffeological spaces is performed in a concrete way that fits at the same time in a neat sheaf-theoretic perspective (Remark 2.3.3).

Our results here are part of a programme to explore the relation between functorial field theories, generalized cohomology theories, and index theory. Within this programme there are, among others, two big (and still open) problems, to which the results of this article make a contribution. The first problem is to realize the two-dimensional supersymmetric sigma model (the 'free superstring') rigorously as a smooth functorial field theory (FFT) in the sense of Atiyah [Ati88], Segal [Seg81], and Stolz and Teichner [ST04]. In this sigma model, the Hilbert space $F(LM)_\gamma$ is the 'state space' for superstrings with underlying 'world-sheet embedding'

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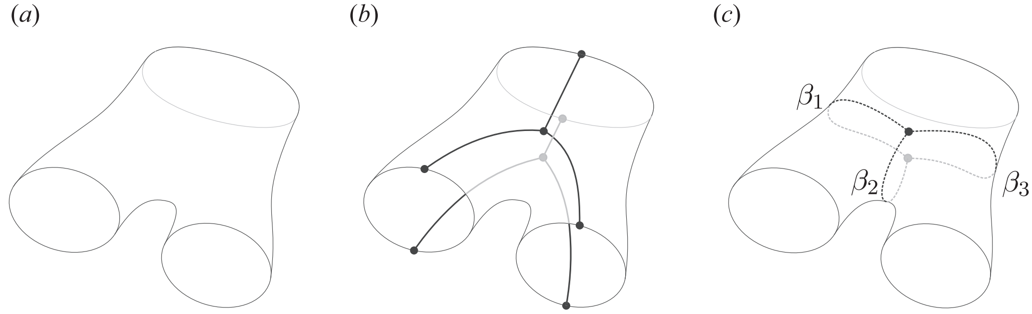

FIGURE 1. (a) A pair of pants. (b) A decoration where boundary loops are split into two halves. (c) The three paths with a common initial point and a common end point, at which we the halves of (b) meet. There, our Connes fusion product on the spinor bundle on loop space can be applied. Still missing is a notion of 'parallel transport' in the spinor bundle, which would take care of the passage from the boundary loops to the three loops $\beta_1 \cup \beta_2$, $\beta_2 \cup \beta_3$, and $\beta_1 \cup \beta_3$.

$\gamma : S^1 \to M$; in other words, it is the value of the FFT on the circle (equipped with map γ). Our result that the spinor bundle is a *rigged* Hilbert space bundle is at the basis of the statement that this FFT is *smooth*. The main contribution is, however, that our Connes fusion product on $F(LM)$ is the central ingredient to what the FFT assigns to a pair of pants, as sketched in Figure 1. A further contribution is that our rigged von Neumann algebra bundle $\mathcal{N}$ over PM is part of the answer to the question how the FFT can be extended to a point: we discuss in § 5.4 that $\mathcal{N}$ furnishes a certain *2-vector bundle* over M, the 'stringor bundle' of the string manifold M. The FFT then assigns to the point (equipped with a map $x : * \to M$) the fibre of this 2-vector bundle over x.

Yet, two major steps are still missing in the construction of the supersymmetric sigma model as a smooth extended FFT. First, a proper definition of an appropriate smooth bordism category over M (probably, it will be a sheaf of $(\infty, 2)$-categories). There have been very recent promising proposals [LS21, GP20] raising hope that this is accomplishable. Second, a construction and investigation of a *connection* on the spinor bundle $F(LM)$, which would provide assignments to cylinders (see Figure 1). Some ideas and results in this direction have already been reported by Stolz and Teichner under the name 'string connection' [ST04] or 'conformal connection' [ST05]. Further results have been obtained in [Wal13, Wal15], discussing string connections in terms of Killingback's spin structures on loop spaces. Altogether, we believe that a complete solution to the problem to cast the free superstring as an extended smooth FFT is now in reach.

The second problem that arises in the above-mentioned programme is the quest for a Dirac-type operator acting on spinors on loop space [Wit88, Tau89]. Admittedly, our results do not yet place this goal in easy reach. However, they might be relevant in order to specify the precise space of spinors on which such an operator may act. At this point, we would like to highlight the general philosophy behind fusion on loop space, following Stolz and Teichner [ST05]: *fusion characterizes geometric structure on LM that subtly encodes geometry on M*. This is supported by the following list of incarnations of this philosophy.

(a) A fusion product on a line bundle on LM encodes a bundle gerbe on M; similarly, a fusion product on a central extension of a loop group LG encodes a multiplicative bundle gerbe on G. The precise statements are the main results of [Wal12b, Wal17].
(b) A fusion product on a Killingback spin structure on LM encodes a string structure on M, as mentioned above. The precise statement is in [Wal15].

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(c) Our Connes fusion product on the spinor bundle on LM encodes the stringor bundle on M, as outlined above and described in § 5.4.
(d) Finally, the following lower-categorical analog of (b) has been proved by Stolz and Teichner [ST05]: the orientation bundle of LM carries a canonical fusion product, and the fusion-preserving sections encode precisely the spin structures on M.

In analogy to statement (d), we expect that the relevant space of spinors on loop space consists of *fusion-preserving sections* of the spinor bundle $F(LM)$. In this sense, we believe that our construction of the Connes fusion product on the spinor bundle will help to approach the mysterious Dirac operator on loop space.

For completeness, we remark that both problems described above are conjectured to be related by a yet unknown index theory with family indices taking values in a certain generalized cohomology theory, whose objects can be represented by certain smooth FFTs. It is expected that the index of the Dirac operator on loop space corresponds precisely to the free superstring FFT [ST04, ST05].

This article is organized as follows. In § 2 we develop the theory of rigged von Neumann algebra bundles and bimodule bundles, first over Fréchet manifolds and then over diffeological spaces. In § 3 we discuss the free fermions F, exhibit it as a rigged von Neumann bimodule, and construct the Connes fusion of certain operators on F. Section 4 is devoted to all aspects of loop fusion, including a discussion of the relation between Killingback spin structures and string structures. In § 5 we carry out our main constructions: the rigged von Neumann algebra bundle $\mathcal{N}$, the $\mathcal{N}$–$\mathcal{N}$-bimodule structure on the spinor bundle $F(LM)$, and, finally, the Connes fusion product on $F(LM)$. We include two appendices about bimodules of von Neumann algebras: Appendix A.1 collects results about the theory of standard forms that we mainly use in § 3.3; and Appendix A.2 contains definitions and properties of Connes fusion.

2. Rigged von Neumann algebras

One of the central objects of consideration in this article are certain bundles of von Neumann algebras over the space of smooth paths/loops in a manifold. It is highly desirable that these bundles are equipped with a smooth structure, because they are expected to host interesting differential operators. However, to the best of the authors' knowledge, no treatment of locally trivial smooth bundles of Hilbert spaces, C*-algebras, or von Neumann algebras is available. In our previous work [KW20] on spinor bundles on loop space, we found that these smoothness issues are best addressed in the setting of *rigged* Hilbert spaces, which we extended there to rigged C*-algebras and smooth bundles thereof.

In this section, we expand these notions further to include rigged von Neumann algebras and smooth bundles of rigged von Neumann algebras, as well as rigged bimodules over von Neumann algebras and bundles thereof. Then, in order to discuss fusion in loop spaces, we further extend these structures from Fréchet manifolds to diffeological spaces. This is necessary because the spaces of paths and tuples of paths we consider are not Fréchet manifolds anymore, but have nice and natural diffeologies.

2.1 Representations on rigged Hilbert spaces

In this paper, we work with rigged Hilbert spaces and rigged C*-algebras as introduced in § 2 of our paper [KW20]; we briefly review all required notions and results, and refer the reader to that paper for more motivation and context.

DEFINITION 2.1.1. A *rigged Hilbert space* is a Fréchet space equipped with a continuous (sesquilinear) inner product. A *morphism of rigged Hilbert spaces* is simply a continuous linear map. A morphism of rigged Hilbert spaces is called *bounded/isometric* if it is bounded/isometric with respect to the inner products, and it is called *unitary*, if it is an isometric isomorphism.

Given a rigged Hilbert space E, one obtains an honest Hilbert space, denoted $\hat{E}$, by completion with respect to the inner product. The prototypical example of a rigged Hilbert space is the Fréchet space of smooth functions on the circle, equipped with the L^2-inner product; its Hilbert completion is the space of square-integrable functions on the circle.

By a *smooth representation* of a Fréchet Lie group $\mathcal{G}$ on a rigged Hilbert space E we mean an action of $\mathcal{G}$ on E by unitary morphisms of rigged Hilbert spaces, such that the map $\mathcal{G} \times E \to E$ is smooth. The typical example in this paper is the rigged Hilbert space F_L^{s} of smooth vectors in a Fock space of a Lagrangian $L \subset V$, which carries a smooth representation of a Lie group $\mathrm{Imp}_L^{\theta}(V)$ of implementers, see Proposition 3.2.2.

DEFINITION 2.1.2. A *rigged C*-algebra* is a Fréchet algebra A, equipped with a continuous norm $\|\cdot\| : A \to \mathbb{R}_{\geqslant 0}$ and a continuous complex anti-linear involution $* : A \to A$, such that its completion with respect to the norm is a C*-algebra. A *morphism of rigged C*-algebras* is a morphism of Fréchet algebras that is bounded with respect to the norms and intertwines the involutions. A morphism of rigged C*-algebras is called *isometric*, if it is an isometry with respect to the norms.

By definition, an actual C*-algebra $\hat{A}$ can be obtained from a rigged C*-algebra A by norm completion. By a *smooth representation* of a Fréchet Lie group $\mathcal{G}$ on a rigged C*-algebra A we mean an action of $\mathcal{G}$ on A by isometric isomorphisms of rigged C*-algebras, such that the map $\mathcal{G} \times A \to A$ is smooth. The typical example in this paper is the Fréchet algebra $\mathrm{Cl}(V)^{\mathrm{s}}$ of smooth vectors in the Clifford algebra of a real Hilbert space V, which carries a smooth representation of the orthogonal group $\mathrm{O}(V)$ via Bogoliubov automorphisms, see Proposition 3.2.4.

Of major importance for this article is a certain type of representations of rigged C*-algebras on rigged Hilbert spaces.

DEFINITION 2.1.3. Let A be a rigged C*-algebra. A *rigged A-module* is a rigged Hilbert space E together with a representation ρ of (the underlying algebra of) A on (the underlying vector space of) E, such that the map $\rho : A \times E \to E$ is smooth and the following conditions hold for all $a \in A$ and all $v, w \in E$:

$$\langle \rho(a,v), \rho(a,v) \rangle \leqslant \|a\|^2 \langle v, v \rangle \quad \text{and} \quad \langle \rho(a,v), w \rangle = \langle v, \rho(a^*, w) \rangle. \tag{1}$$

A *(unitary) intertwiner* from a rigged A_1-module E_1 to a rigged A_2-module E_2 is a pair (ϕ, ψ) of a morphism $\phi : A_1 \to A_2$ of rigged C*-algebras and a bounded (unitary) morphism $\psi : E_1 \to E_2$ of rigged Hilbert spaces that intertwines the representations along ϕ.

We recall that if a map $\rho : A \times E \to E$ is bilinear, it is smooth if and only if it is continuous.

The typical example is the rigged Hilbert space F_L^{s}, which becomes under Clifford multiplication a rigged $\mathrm{Cl}(V)^{\mathrm{s}}$-module, see Proposition 3.2.5. A couple of remarks are in order.

Remark 2.1.4. (a) Conditions (1) in Definition 2.1.3 are chosen such that a rigged A-module E with underlying representation ρ induces a $*$-homomorphism $\hat{\rho} : \hat{A} \to \mathcal{B}(\hat{E})$, i.e. a representation of the C*-algebra $\hat{A}$ on the Hilbert space $\hat{E}$, see [KW20, Remark 2.2.11].

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(b) If $\phi : A \to B$ is a morphism of rigged C*-algebras, and E is a rigged B-module with representation ρ, then E becomes a rigged A-module under the induced representation $(a, v) \mapsto \rho(\phi(a), v)$. Moreover, the pair $(\phi, \mathbb{1})$ is a unitary intertwiner.

(c) Rigged C*-algebras and rigged modules are compatible with dualization. If A is a rigged C*-algebra, then its opposite algebra, A^{opp}, is a rigged C*-algebra in a natural way, and its completion $(A^{\mathrm{opp}})^{\|\cdot\|}$ is the usual opposite C*-algebra. Let E be a rigged A-module. The inner product
on E gives us a complex anti-linear injection $\iota : E \to E^*$ mapping E into its continuous linear dual E^*. Denote the image of ι by $E^\sharp$. We turn $E^\sharp$ into a Fréchet space using the identification with E. If ρ denotes the representation of A on E, then the map

$$\rho^\sharp : A^{\mathrm{opp}} \times E^\sharp \to E^\sharp, (a, \varphi) \mapsto \varphi \circ \rho(a, -),$$

is a representation of A^{opp} on $E^\sharp$, and it is straightforward to show that it turns $E^\sharp$ into a rigged A^{opp}-module.

Next, we extend our framework, which we have so far recalled from [KW20], by introducing new structures that ultimately lead to the notion of rigged bimodules over rigged von Neumann algebras.

Definition 2.1.5. Let A_1 and A_2 be rigged C*-algebras. A *rigged A_1–A_2-bimodule* is a rigged Hilbert space E with commuting representations of A_1 and A_2^{opp}, in such a way that E is a rigged A_1-module and a rigged A_2^{opp}-module. A *(unitary) intertwiner* from a rigged A_1–A_2-bimodule E to a rigged A_1'–A_2'-bimodule E' is a triple (ϕ^1, ϕ^2, ψ) consisting of morphisms $\phi^1 : A_1 \to A_1'$ and $\phi^2 : A_2 \to A_2'$ of rigged C*-algebras, and of a bounded (unitary) morphism $\psi : E \to E'$ of rigged Hilbert spaces that intertwines both representations along ϕ^1 and ϕ^2.

Under completion, a rigged A–B-bimodule E becomes a Hilbert space $\hat{E}$ with commuting representations of the C*-algebras $\hat{A}$ and $\hat{B}^{\mathrm{opp}} = \widehat{B^{\mathrm{opp}}}$. Our main example of a rigged bimodule is again the space F_L^{s} of smooth vectors in a Fock space, which we equip with a rigged bimodule structure over a certain subalgebra of $\mathrm{Cl}(V)^{\mathrm{s}}$, see § 3.3. Next, we develop the setting of rigged von Neumann algebras.

Definition 2.1.6. A *rigged von Neumann algebra* is a pair $N = (A, E)$ consisting of a rigged C*-algebra A and a rigged A-module E, with the property that the induced C*-representation $\hat{A} \to \mathcal{B}(\hat{E})$ is faithful. The representation underlying the rigged A-module E is called the *defining representation* of N.

Remark 2.1.7. If $N = (A, E)$ is a rigged von Neumann algebra, and ρ its defining representation, then the assumption that $\hat{\rho} : \hat{A} \to \mathcal{B}(\hat{E})$ is faithful implies that it is an isometry and, hence, a homeomorphism onto its image. We may thus identify $\hat{\rho}(\hat{A})$ with $\hat{A}$. Then, we define the von Neumann algebra

$$N'' \coloneqq (\hat{A})'' \subseteq \mathcal{B}(\hat{E}),$$

and conclude that it contains $\hat{A}$ as a weakly dense (and, thus, also σ-weakly dense) subspace. In particular, any *rigged* von Neumann algebra N gives rise to an *ordinary* von Neumann algebra N''.

An example of a rigged von Neumann algebra is the Clifford algebra $\mathrm{Cl}(V)^{\mathrm{s}}$ with its defining representation on Fock space F_L^{s}, see Proposition 3.2.5. The associated von Neumann algebra is $\mathcal{B}(F_L)$. More interesting examples will be constructed in Proposition 3.2.9; these give rise to type III_1-factors.

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Remark 2.1.8. If $N = (A, E)$ is a rigged von Neumann algebra, then $N^{\mathrm{opp}} := (A^{\mathrm{opp}}, E^{\sharp})$ is a rigged von Neumann algebra, see Remark 2.1.4.

Definition 2.1.9. Let $N_1 = (A_1, E_1)$ and $N_2 = (A_2, E_2)$ be rigged von Neumann algebras. A *morphism* of rigged von Neumann algebras from N_1 to N_2 is a morphism $\phi : A_1 \to A_2$ of rigged C*-algebras that extends to a normal $*$-homomorphism $N_1'' \to N_2''$.

We recall that 'normal' means to be continuous in the σ-weak topologies, and we recall that these topologies are generated by semi-norms $p_\rho(T) := |\mathrm{tr}(\rho T)|$, where ρ is a trace-class operator. Since $\hat{A} \subset N''$ is σ-weakly dense, it is clear that the extension $N_1'' \to N_2''$ is unique, if it exists. The following lemma gives us a useful sufficient criterion for the extendability.

Lemma 2.1.10. *Let $N_1 = (A_1, E_1)$ and $N_2 = (A_2, E_2)$ be rigged von Neumann algebras. Suppose that (ϕ, ψ) is a unitary intertwiner from E_1 to E_2 in the sense of Definition 2.1.3. Then, ϕ is a morphism of rigged von Neumann algebras.*

Proof. All that needs to be shown is that $\phi : A_1 \to A_2$ extends to a normal $*$-homomorphism $N_1'' \to N_2''$. According to Remark 2.1.7 we view A_1 as a subset of $\mathcal{B}(\hat{E}_1)$ and A_2 as a subset of $\mathcal{B}(\hat{E}_2)$. Now, we have, for all $a \in A_1$ and all $v \in \hat{E}_2$,

$$\phi(a)v = \phi(a)\psi\psi^* v = \psi a \psi^* v.$$

Here, $\psi\psi^* = \mathbb{1}$ since ψ is unitary and, thus, extends to a unitary operator $\psi : \hat{E}_1 \to \hat{E}_2$ on the completions. It follows that $\phi(a) = \psi a \psi^*$ as elements of $\mathcal{B}(\hat{E}_2)$, which implies that the map $C_\psi : \mathcal{B}(\hat{E}_1) \to \mathcal{B}(\hat{E}_2), a \mapsto \psi a \psi^*$ extends ϕ. Next, we prove that C_ψ is normal. If ρ is a trace-class operator on $\hat{E}_2$, then $\psi^*\rho\psi$ is a trace-class operator on $\hat{E}_1$, and we have

$$p_{\psi^*\rho\psi}(a) = p_\rho(C_\psi(a)),$$

for all $a \in \mathcal{B}(\hat{E}_1)$. This implies that C_ψ is σ-weakly continuous. Now, because A_1 is σ-weakly dense in N_1'', and A_2 is σ-weakly dense in N_2'' it follows that $C_\psi(N_1'') \subseteq N_2''$, and that, moreover, $C_\psi : N_1'' \to N_2''$ is a $*$-homomorphism. □

Remark 2.1.11. Lemma 2.1.10 motivates the following terminology: a *spatial morphism* from a rigged von Neumann algebra N_1 to another rigged von Neumann algebra N_2 is a unitary intertwiner (ϕ, ψ) between the underlying rigged modules. Lemma 2.1.10 implies then that ϕ is a morphism of rigged von Neumann algebras. In addition, a spatial morphism (ϕ, ψ) will be called *invertible*, or spatial *iso*morphism if it is invertible by another spatial morphism. We note that this is the case if and only if (ϕ, ψ) is an invertible unitary intertwiner, i.e. in addition to ψ being unitary, ϕ must be an isometric isomorphism. Spatial isomorphisms appear frequently in the context of *bundles* of rigged von Neumann algebras, whose local trivializations will be spatial isomorphisms in each fibre, see Definition 2.2.9 and Remark 2.2.12.

Next is the discussion of modules and bimodules for rigged von Neumann algebras.

Definition 2.1.12. Let $N = (A, E)$ be a rigged von Neumann algebra. A *rigged von Neumann N-module* is a rigged A-module F whose induced $*$-homomorphism $\hat{A} \to \mathcal{B}(\hat{F})$ extends to a normal $*$-homomorphism $N'' \to \mathcal{B}(\hat{F})$.

Remark 2.1.13. Again, the extension $N'' \to \mathcal{B}(\hat{F})$ in Definition 2.1.12 is unique, if it exists. It guarantees that, for N a rigged von Neumann algebra and F a rigged von Neumann N-module, the completion $\hat{F}$ is an N''-module in the usual von Neumann theoretical sense (see Appendix A.1). Observe, moreover, that the defining representation of any rigged von Neumann algebra is automatically a rigged von Neumann module.

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We will need the following lemma, which will be used later in the proof of Lemma 2.1.16.

Lemma 2.1.14. *Let $N = (A, E)$ be a rigged von Neumann algebra, and let F be a rigged von Neumann N-module with underlying representation $\rho : A \times F \to F$. Then, the normal $*$-homomorphism $N'' \to \mathcal{B}(\hat{F})$ factors through the von Neumann algebra $\hat{\rho}(\hat{A})'' \subset \mathcal{B}(\hat{F})$.*

Proof. Let $a \in N''$ be arbitrary. We wish to prove that its image in $\mathcal{B}(\hat{F})$ in fact lies in $\hat{\rho}(\hat{A})''$. Thus, let a_n be a sequence in $\hat{A}$ that σ-weakly converges to a. Then, by continuity of $N'' \to \mathcal{B}(\hat{F})$, the image of the sequence converges to some element in $\mathcal{B}(\hat{F})$, but because all elements a_n were in $\hat{A}$ to begin with, the limit must already lie in its completion $\hat{\rho}(\hat{A})''$. □

Finally, we come to bimodules for rigged von Neumann algebras. One of the fundamental results of this work exhibits the Fock space F^{s} as a rigged von Neumann bimodule, see Proposition 3.3.4.

Definition 2.1.15. If $N_1 = (A_1, E_1)$ and $N_2 = (A_2, E_2)$ are rigged von Neumann algebras, then a *rigged von Neumann N_1–N_2-bimodule* is a rigged A_1–A_2-bimodule F that is a rigged von Neumann N_1-module and a rigged von Neumann N_2^{opp}-module. A *(unitary) intertwiner* from a rigged von Neumann N_1–N_2-bimodule F to a rigged von Neumann $\tilde{N}_1$–$\tilde{N}_2$-bimodule $\tilde{F}$ is a (unitary) intertwiner (ϕ^1, ϕ^2, ψ) between the underlying rigged bimodules in the sense of Definition 2.1.5, such that $\phi^1 : N_1 \to \tilde{N}_1$ and $\phi^2 : N_2 \to \tilde{N}_2$ are morphisms of rigged von Neumann algebras. A *spatial intertwiner* from F to $\tilde{F}$ is a quintuple $(\phi^1, \psi^1, \phi^2, \psi^2, \psi)$ in which (ϕ^1, ϕ^2, ψ) is a unitary intertwiner from F to $\tilde{F}$, and $(\phi^1, \psi^1) : N_1 \to \tilde{N}_1$ and $(\phi^2, \psi^2) : N_2 \to \tilde{N}_2$ are spatial morphisms.

Here, Lemma 2.1.10 implies again that *spatial* intertwiners are in particular (unitary) intertwiners. We assure by the following result that completion brings us into the classical setting, see Appendix A.1.

Lemma 2.1.16. *If F is a rigged von Neumann N_1–N_2-bimodule, then its completion $\hat{F}$ is a N_1''–N_2''-bimodule in the ordinary von Neumann theoretical sense. Likewise, any (unitary) intertwiner between rigged von Neumann bimodules induces a bounded (unitary) intertwiner in the ordinary sense.*

Proof. For the first statement, we only have to argue that the actions of the von Neumann algebras N_1'' and N_2'' on $\hat{F}$ commute. Indeed, as the actions of the rigged C*-algebras commute, it is clear that the images $\hat{\rho}_1(\hat{A}_1)$ and $\hat{\rho}_2(\hat{A}_2)$ in $\mathcal{B}(\hat{F})$ commute, for instance, $\hat{\rho}_2(\hat{A}_2) \subset \hat{\rho}_1(\hat{A}_1)'$. Taking commutants, we obtain $\hat{\rho}_1(\hat{A}_1)'' \subset \hat{\rho}_2(\hat{A}_2)' = \hat{\rho}_2(\hat{A}_2)'''$. This shows that the von Neumann algebras $\hat{\rho}_1(\hat{A}_1)''$ and $\hat{\rho}_2(\hat{A}_2)''$ commute. Now, Lemma 2.1.14 proves the claim. The statement about intertwiners follows from the usual continuity arguments. □

We remark that ordinary von Neumann algebras, bimodules and intertwiners form a bicategory, in which the composition of morphisms is given by the Connes fusion of bimodules [Bro03, ST04]. We have, unfortunately, not yet been able to lift Connes fusion to the setting of *rigged* von Neumann bimodules and, thus, there is no corresponding bicategory of rigged von Neumann algebras. This is an important issue that we are going to address in future work.

2.2 Locally trivial rigged bundles

In this section we introduce locally trivial bundles of rigged von Neumann algebras and rigged von Neumann bimodules over Fréchet manifolds. The prerequisite notions of rigged Hilbert space bundles, rigged C*-algebra bundles, and rigged module bundles have been introduced in § 2 of

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[KW20], and are discussed there in more detail. Throughout this section, we let $\mathcal{M}$ be a Fréchet manifold.

Definition 2.2.1. Let E be a rigged Hilbert space. A *rigged Hilbert space bundle* over $\mathcal{M}$ with typical fibre E is a Fréchet vector bundle $\pi : \mathcal{E} \to \mathcal{M}$ with typical fibre E equipped with a map $g : \mathcal{E} \times_\pi \mathcal{E} \to \mathbb{C}$, such that g is fibrewise an inner product and local trivializations $\Phi : \mathcal{E}|_U \to E \times U$ of $\mathcal{E}$ can be chosen to be fibrewise isometric. A *morphism of rigged Hilbert space bundles* is a morphism of the underlying Fréchet vector bundles. A morphism is called *locally bounded/isometric* if it is locally bounded/isometric with respect to the inner products, and it is called *unitary* if it is isometric and an isomorphism of Fréchet vector bundles.

It is straightforward to see that the fibres of a rigged Hilbert space bundle $\mathcal{E}$ are rigged Hilbert spaces [KW20, Remark 2.1.10], and that the fibrewise completion of $\mathcal{E}$ is a locally trivial continuous Hilbert space bundle over $\mathcal{M}$ with typical fibre $\hat{E}$ (see [KW20, Lemma 2.1.13]). Here, a locally trivial continuous Hilbert space bundle with fibre H has continuous local transition functions $U \times H \to H$ or, equivalently, *strongly* continuous maps $U \to \mathrm{U}(H)$. Likewise, on the level of morphisms, any locally bounded morphism of rigged Hilbert space bundles extends uniquely to a continuous morphism of the corresponding continuous Hilbert space bundles. The following lemma [KW20, Proposition 2.1.15] shows that our notion of smooth representations (see § 2.1) fits well into the context of rigged Hilbert space bundles.

Lemma 2.2.2. *Let $\mathcal{G}$ be a Fréchet Lie group, $\mathcal{P}$ be a Fréchet principal $\mathcal{G}$-bundle over $\mathcal{M}$, and let $\rho : \mathcal{G} \times E \to E$ be a smooth representation of $\mathcal{G}$ on a rigged Hilbert space E. Then, the associated bundle $(\mathcal{P} \times E)/\mathcal{G}$ is a rigged Hilbert space bundle with typical fibre E in a unique way, such that every local trivialization $\Phi : \mathcal{P}|_U \to \mathcal{G} \times U$ of $\mathcal{P}$ induces a local trivialization $[p, v] \mapsto (\rho(g(p), v), \pi(p))$ of $(\mathcal{P} \times E)/\mathcal{G}$, where $p \in \mathcal{P}$, $v \in E$, and $\Phi(p) = (g(p), \pi(p))$.*

The spinor bundle on loop space is a rigged Hilbert space bundle over $\mathcal{M} = LM$, and will be defined via Lemma 2.2.2, see Definition 4.1.4. We proceed similarly for rigged C*-algebra bundles.

Definition 2.2.3. Let A be a rigged C*-algebra. A *rigged C*-algebra bundle* over $\mathcal{M}$ with typical fibre A is a Fréchet vector bundle $\pi : \mathcal{A} \to \mathcal{M}$, equipped with:

– a map $\|\cdot\| : \mathcal{A} \to \mathbb{R}_{\geqslant 0}$; and
– fibre-preserving maps $m : \mathcal{A} \times_\pi \mathcal{A} \to \mathcal{A}$ and $* : \mathcal{A} \to \mathcal{A}$;

such that the following conditions hold for each $x \in \mathcal{M}$.

– The map $\|\cdot\|_x : \mathcal{A}_x \to \mathbb{R}_{\geqslant 0}$ is a norm.
– The maps $m_x : \mathcal{A}_x \times \mathcal{A}_x \to \mathcal{A}_x$ and $*_x : \mathcal{A}_x \to \mathcal{A}_x$ turn $\mathcal{A}_x$ into a $*$-algebra.
– There exists a local trivialization around x that is fibrewise an isometric $*$-homomorphism.

A *morphism of rigged C*-algebra bundles* over $\mathcal{M}$ is a morphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ of Fréchet vector bundles that is fibrewise a morphism of $*$-algebras and locally bounded with respect to the norms. A morphism is called *isometric* if it is isometric with respect to the norms.

One can show that each fibre of a rigged C*-algebra bundle $\mathcal{A}$ is a rigged C*-algebra, and that the opposite multiplication produces a rigged C*-algebra bundle $\mathcal{A}^{\mathrm{opp}}$ with typical fibre A^{opp}. Further, the fibrewise norm completion gives a locally trivial continuous bundle of C*-algebras with typical fibre $\hat{A}$, and strongly continuous transition functions [KW20, Lemma 2.2.6]. Likewise, any morphism of rigged C*-algebra bundles extends uniquely to a continuous morphism of continuous bundles of C*-algebras. We remark that every locally trivial

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continuous bundle of C*-algebras yields a continuous field of C*-algebras by taking its continuous sections.

Rigged C*-algebra bundles can be obtained by associating a smooth representation to a principal bundle, as the following result [KW20, Proposition 2.2.8] shows.

Lemma 2.2.4. *Let $\mathcal{P}$ be a Fréchet principal $\mathcal{G}$-bundle over $\mathcal{M}$, and let $\rho : \mathcal{G} \times A \to A$ be a smooth representation on a rigged C*-algebra A. Then, the associated bundle $(\mathcal{P} \times A)/\mathcal{G}$ is a rigged C*-algebra bundle with typical fibre A in a unique way, such that every local trivialization $\Phi : \mathcal{P}|_U \to \mathcal{G} \times U$ of $\mathcal{P}$ induces a local trivialization $[p, a] \mapsto (\rho(g(p), a), \pi(p))$ of $(\mathcal{P} \times A)/\mathcal{G}$, where $p \in \mathcal{P}$, $a \in A$, and $\Phi(p) = (g(p), \pi(p))$.*

The Clifford bundle on loop space is a rigged C*-algebra bundle, and it will be defined via Lemma 2.2.4, see Definition 4.1.5. Next, we discuss module bundles and bimodule bundles for rigged C*-algebra bundles.

Definition 2.2.5. Let A be a rigged C*-algebra and E be a rigged A-module, with representation ρ_0, and let $\mathcal{A}$ be a rigged C*-algebra bundle over $\mathcal{M}$ with typical fibre A. A *rigged $\mathcal{A}$-module bundle with typical fibre E* is a rigged Hilbert space bundle $\mathcal{E}$ with typical fibre E, and a fibre-preserving map

$$\rho : \mathcal{A} \times_{\mathcal{M}} \mathcal{E} \to \mathcal{E}$$

with the property that around every point in $\mathcal{M}$ there exist local trivializations Φ of $\mathcal{A}$ and Ψ of $\mathcal{E}$ that fibrewise intertwine ρ with ρ_0, i.e. we have $\Psi_x(\rho(a, v)) = \rho_0(\Phi_x(a), \Psi_x(v))$ for all $x \in \mathcal{M}$ over which Φ and Ψ are defined, and all $a \in \mathcal{A}_x$ and $v \in \mathcal{E}_x$. A pair (Φ, Ψ) of local trivializations with this property is called *compatible*.

One can easily show that ρ is automatically a morphism of Fréchet vector bundles. Furthermore, for each $x \in \mathcal{M}$, the map ρ_x turns $\mathcal{E}_x$ into a rigged $\mathcal{A}_x$-module; and every pair of compatible local trivializations (Φ, Ψ) around x yields an invertible unitary intertwiner (Φ_x, Ψ_x) between the rigged $\mathcal{A}_x$-module $\mathcal{E}_x$ and the rigged A-module E.

Definition 2.2.6. A *(unitary) intertwiner* between a rigged $\mathcal{A}_1$-module bundle $\mathcal{E}_1$ and a rigged $\mathcal{A}_2$-module bundle $\mathcal{E}_2$ is a pair (Φ, Ψ) of a morphism $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ of rigged C*-algebra bundles, and a locally bounded (unitary) morphism $\Psi : \mathcal{E}_1 \to \mathcal{E}_2$ of rigged Hilbert space bundles, such that (Φ_x, Ψ_x) is a (unitary) intertwiner of rigged modules in the fibre over each point $x \in \mathcal{M}$.

The definition of a rigged *bimodule* bundle now follows naturally; it is, however, important enough that we give it in full detail.

Definition 2.2.7. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be rigged C*-algebra bundles over $\mathcal{M}$ with typical fibres A_1 and A_2, respectively, and let E be a rigged A_1–A_2-bimodule. A *rigged $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule bundle $\mathcal{E}$ with typical fibre E* is a rigged Hilbert space bundle $\mathcal{E}$ over $\mathcal{M}$ that is both a rigged $\mathcal{A}_1$-module bundle and a rigged $\mathcal{A}_2^{\mathrm{opp}}$-module bundle, such that around every point in $\mathcal{M}$ there exist local trivializations Φ^1 of $\mathcal{A}_1$, Φ^2 of $\mathcal{A}_2$, and Ψ of $\mathcal{E}$ with both (Φ^1, Ψ) and (Φ^2, Ψ) compatible. A triple (Φ^1, Φ^2, Ψ) of local trivializations with this property is again called *compatible*.

It is straightforward to see that the fibres $\mathcal{E}_x$ are rigged $(\mathcal{A}_1)_x$–$(\mathcal{A}_2)_x$-bimodules, for each $x \in \mathcal{M}$, and that compatible local trivializations around x establish an invertible unitary intertwiner between $\mathcal{E}_x$ and E. For completeness, we also note the bundle version of a bimodule intertwiner.

Definition 2.2.8. A *(unitary) intertwiner* from a rigged $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule bundle $\mathcal{E}$ to a rigged $\widetilde{\mathcal{A}}_1$–$\widetilde{\mathcal{A}}_2$-bimodule bundle $\widetilde{\mathcal{E}}$ is a triple (Φ^1, Φ^2, Ψ) consisting of morphisms $\Phi^1 : \mathcal{A}_1 \to \widetilde{\mathcal{A}}_1$ and $\Phi^2 : \mathcal{A}_2 \to \widetilde{\mathcal{A}}_2$ of rigged C*-algebra bundles and of a locally bounded (unitary) morphism $\Psi : \mathcal{E} \to \widetilde{\mathcal{E}}$

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of rigged Hilbert space bundles, such that over each point $x \in \mathcal{M}$ the triple $(\Phi^1_x, \Phi^2_x, \Psi_x)$ is a (unitary) intertwiner of rigged bimodules.

Later, we will exhibit the spinor bundle on loop space as a rigged module bundle for the Clifford bundle, see § 4. Next we come to the definition of rigged von Neumann algebra bundles and rigged von Neumann bimodule bundles. These definitions are a novelty introduced in this article; we are not aware of any other treatment of smooth locally trivial bundles of von Neumann algebras, or von Neumann bimodules.

DEFINITION 2.2.9. Let $N = (A, E)$ be a rigged von Neumann algebra. A *rigged von Neumann algebra bundle* $\mathcal{N} = (\mathcal{A}, \mathcal{E})$ with typical fibre N over $\mathcal{M}$ is a rigged C*-algebra bundle $\mathcal{A}$ with typical fibre A and a rigged $\mathcal{A}$-module bundle $\mathcal{E}$ with typical fibre E.

Thus, the difference between a rigged $\mathcal{A}$-module bundle $\mathcal{E}$ and a rigged von Neumann algebra bundle $\mathcal{N} = (\mathcal{A}, \mathcal{E})$ only lies in fact that the *typical fibre* is required to be a rigged von Neumann algebra. Via compatible local trivializations (Definition 2.2.5), this property extends to all fibres; thus, every fibre $\mathcal{N}_x = (\mathcal{A}_x, \mathcal{E}_x)$ is a rigged von Neumann algebra. Moreover, Lemma 2.1.10 guarantees that any choice of compatible local trivializations induces fibrewise spatial isomorphisms from $\mathcal{N}_x$ to N.

DEFINITION 2.2.10. A *morphism* $\Phi : \mathcal{N}_1 \to \mathcal{N}_2$ between rigged von Neumann algebra bundles $\mathcal{N}_1 = (\mathcal{A}_1, \mathcal{E}_1)$ and $\mathcal{N}_2 = (\mathcal{A}_2, \mathcal{E}_2)$ is a morphism $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ of rigged C*-algebra bundles that is fibrewise a morphism of rigged von Neumann algebras.

Remark 2.2.11. Again, it is useful to introduce the notion of a *spatial morphism* from $\mathcal{N}_1 = (\mathcal{A}_1, \mathcal{E}_1)$ to $\mathcal{N}_2 = (\mathcal{A}_2, \mathcal{E}_2)$ as a unitary intertwiner (Φ, Ψ) of rigged module bundles (Definition 2.2.6). In this situation, we have in the fibre over each point $x \in \mathcal{M}$ a unitary intertwiner (Φ_x, Ψ_x) from $(\mathcal{E}_1)_x$ to $(\mathcal{E}_2)_x$, i.e. a spatial morphism from $(\mathcal{N}_1)_x$ to $(\mathcal{N}_2)_x$. By Lemma 2.1.10, Φ_x is a morphism of rigged von Neumann algebras, and thus, by definition, Φ is a morphism of rigged von Neumann algebra bundles. This shows that spatial morphisms of rigged von Neumann algebra bundles are, in particular, morphisms in the sense of Definition 2.2.10.

Remark 2.2.12. Let $\mathcal{N} = (\mathcal{A}, \mathcal{E})$ be a rigged von Neumann algebra bundle with typical fibre $N = (A, E)$. Then, there exist compatible local trivializations (Φ, Ψ) over open subsets $U \subset \mathcal{M}$ as in Definition 2.2.5, which are invertible unitary intertwiners between the rigged module bundles $\mathcal{E}|_U$ and $E \times U$ and, thus, spatial isomorphisms of rigged von Neumann algebra bundles from $\mathcal{N}|_U$ to $N \times U$. In particular, rigged von Neumann algebra bundles are locally trivial in this strong sense of spatial isomorphisms.

The rigged von Neumann algebra bundles that appear in this paper are certain Clifford algebra bundles over loop spaces, and they appear in §§ 4.1 and 5.1. For now, it remains to define rigged von Neumann bimodule bundles.

DEFINITION 2.2.13. Let $N_1 = (A_1, E_1)$ and $N_2 = (A_2, E_2)$ be rigged von Neumann algebras, and let F be a rigged von Neumann N_1–N_2-bimodule. Further, let $\mathcal{N}_1 = (\mathcal{A}_1, \mathcal{E}_1)$ and $\mathcal{N}_2 = (\mathcal{A}_2, \mathcal{E}_2)$ be rigged von Neumann algebra bundles over $\mathcal{M}$ with typical fibres N_1 and N_2, respectively. A *rigged von Neumann $\mathcal{N}_1$–$\mathcal{N}_2$-bimodule bundle* with typical fibre F is a rigged $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule bundle $\mathcal{F}$ with typical fibre F, such that around each point $x \in \mathcal{M}$ there exist compatible local trivializations (Φ^1, Ψ^1) of $\mathcal{N}_1$ and (Φ^2, Ψ^2) of $\mathcal{N}_2$, and a local trivialization Ψ of $\mathcal{F}$ such that (Φ^1, Φ^2, Ψ) is compatible in the sense of Definition 2.2.7.

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The following lemma is to ensure that Definition 2.2.13 gives the correct structure in each fibre.

Lemma 2.2.14. *If $\mathcal{F}$ is a rigged von Neumann $\mathcal{N}_1$–$\mathcal{N}_2$-bimodule bundle with typical fibre F, then the fibre $\mathcal{F}_x$ is a rigged von Neumann $(\mathcal{N}_1)_x$–$(\mathcal{N}_2)_x$-bimodule, for each $x \in \mathcal{M}$.*

Proof. Because $\mathcal{F}$ is a rigged $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule bundle, we have that $\mathcal{F}_x$ is a rigged $(\mathcal{A}_1)_x$–$(\mathcal{A}_2)_x$-bimodule. Let $\widehat{(\rho_1)_x} : \widehat{(\mathcal{A}_1)_x} \to \mathcal{B}(\widehat{\mathcal{F}_x})$ be the corresponding $*$-homomorphism. The statement that $\mathcal{F}_x$ is a rigged $(\mathcal{N}_1)_x$-module is then equivalent to the statement that $\widehat{(\rho_1)_x}$ extends to $(\mathcal{N}_1)_x''$. But, since the typical fibre F is a rigged von Neumann bimodule, its corresponding maps $\hat{A}_1 \to \mathcal{B}(\hat{F})$ extend to N_1''. Next, consider local trivializations (Φ^1, Ψ^1) and Ψ around x as in Definition 2.2.13. Then, we have the extensions $\Phi_x^1 : (\mathcal{N}_1)_x'' \to N_1''$, and the conjugation by the unitary operator Ψ_x, which yields a normal $*$-homomorphism $\mathcal{B}(\hat{F}) \to \mathcal{B}(\widehat{\mathcal{F}_x})$. Using the compatibility of Φ^1 with Ψ it then follows that the map

$$(\mathcal{N}_1)_x'' \to N_1'' \to \mathcal{B}(\hat{F}) \to \mathcal{B}(\widehat{\mathcal{F}_x})$$

is an extension of $\widehat{(\rho_1)_x}$, which is moreover the composition of normal $*$-homomorphisms and, thus, a normal $*$-homomorphism itself. A similar argument for the $*$-homomorphism $\widehat{(\rho_2)_x}$ proves that $\mathcal{F}_x$ is a rigged von Neumann $(\mathcal{N}_2)_x^{\mathrm{opp}}$-module. $\square$

In the next section, we will glue rigged von Neumann bimodules bundles; therefore, we also need to introduce (spatial) intertwiners between them.

Definition 2.2.15. A *(unitary) intertwiner* from a rigged von Neumann $\mathcal{N}_1$–$\mathcal{N}_2$-bimodule bundle $\mathcal{F}$ to a rigged von Neumann $\widetilde{\mathcal{N}}_1$–$\widetilde{\mathcal{N}}_2$-bimodule bundle $\widetilde{\mathcal{F}}$ is a (unitary) intertwiner (Φ^1, Φ^2, Ψ) between the underlying rigged bimodule bundles in the sense of Definition 2.2.8, such that Φ^1 and Φ^2 are morphisms of rigged von Neumann algebra bundles. A *spatial intertwiner* from $\mathcal{F}$ to $\widetilde{\mathcal{F}}$ is a quintuple $(\Phi^1, \Psi^1, \Phi^2, \Psi^2, \Psi)$ in which (Φ^1, Φ^2, Ψ) is a unitary intertwiner of rigged von Neumann bimodule bundles, and $(\Phi^1, \Psi^1) : \mathcal{N}_1 \to \widetilde{\mathcal{N}}_1$ and $(\Phi^2, \Psi^2) : \mathcal{N}_2 \to \widetilde{\mathcal{N}}_2$ are spatial morphisms.

The definitions guarantees full compatibility with the fibrewise notions: any (spatial) intertwiner induces in the fibre over each point $x \in \mathcal{M}$ a (spatial) intertwiner between rigged von Neumann bimodules in the sense of Definition 2.1.15. Moreover, if $\mathcal{F}$ is a rigged von Neumann $\mathcal{N}_1$–$\mathcal{N}_2$-bimodule bundle with typical fibre F, any choice of local trivializations (Φ^1, Ψ^1), (Φ^2, Ψ^2), and Ψ over an open subset $U \subset \mathcal{M}$ as in Definition 2.2.13, assemble into an invertible spatial intertwiner $(\Phi^1, \Psi^1, \Phi^2, \Psi^2, \Psi)$ from $\mathcal{F}|_U$ to $F \times U$. In particular, rigged von Neumann bimodule bundles are locally trivial in this strong sense of invertible spatial intertwiners.

We would like to make clear and summarize the following important feature of our definitions. Rigged von Neumann algebra bundles and bimodule bundles over $\mathcal{M}$ have in the fibre over each point $x \in \mathcal{M}$ rigged von Neumann algebras and bimodules, as introduced in § 2.1 (see Lemma 2.2.14). Likewise, (spatial) morphisms between rigged von Neumann algebra bundles, and (spatial) intertwiners between rigged von Neumann bimodules restrict in each fibre to (spatial) morphisms of rigged von Neumann algebras and (spatial) intertwiners of rigged von Neumann bimodules, as introduced in § 2.1. In turn, we have shown in Remark 2.1.7 and Lemma 2.1.16 that these fibrewise structures induce, respectively, ordinary von Neumann algebras, ordinary (spatial) morphisms, ordinary bimodules between von Neumann algebras, and ordinary bounded

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intertwiners between these, in the classical sense. Thus, one can pass, at any time and over any point, to this classical setting.

2.3 Rigged bundles over diffeological spaces

For the discussion of fusion on loop space, we need to treat bundles over *diffeological spaces*. We recall briefly that a diffeology on a set X consists of a set of maps $c : U \to X$ called 'plots', where $U \subset \mathbb{R}^k$ is open and $k \in \mathbb{N}_0$ can be arbitrary, subject to three natural axioms; see [Igl13] for a comprehensive textbook. A map $f : X \to Y$ between diffeological spaces is called *smooth*, if its composition with any plot of X results in a plot of Y. We let $\mathcal{D}\!\mathit{iff}$ denote the category of diffeological spaces. Any smooth manifold M or Fréchet manifold $\mathcal{M}$ becomes a diffeological space by saying that every smooth map $c : U \to M$, for every open subset $U \subset \mathbb{R}^k$ and any k, is a plot. Maps between smooth manifolds or Fréchet manifolds are then smooth in the classical sense if and only if they are smooth in the diffeological sense. In other words, the category $\mathcal{F}\!\mathit{r\acute{e}ch}$ of Fréchet manifolds fully faithfully embeds into $\mathcal{D}\!\mathit{iff}$ (see [Los92]).

Next we describe a general procedure to extend a presheaf $\mathcal{F}$ of categories on $\mathcal{F}\!\mathit{r\acute{e}ch}$ to one on $\mathcal{D}\!\mathit{iff}$, and then apply this to various presheaves of bundles defined in the previous section. We briefly recall that a *presheaf $\mathcal{F}$ of categories on a category $\mathcal{C}$* is a weak functor $\mathcal{F} : \mathcal{C}^{\mathrm{opp}} \to \mathcal{C}\!\mathit{at}$ to the bicategory of categories, functors, and natural transformations. That is, $\mathcal{F}$ assigns to each object X of $\mathcal{C}$ a category $\mathcal{F}(X)$, to each morphism $f : X \to Y$ in $\mathcal{C}$ a functor $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$, and to each pair (f, g) of composable morphisms $f : X \to Y$ and $g : Y \to Z$ a natural equivalence $\eta_{f,g} : (g \circ f)^* \Rightarrow f^* \circ g^*$, such that the natural equivalences respect the associativity of the composition.

Remark 2.3.1. In § 2.2 we have encountered the following presheaves of categories over $\mathcal{F}\!\mathit{r\acute{e}ch}$.

- For a Fréchet Lie group $\mathcal{G}$, the presheaf of principal $\mathcal{G}$-bundles and bundle morphisms.
- For a rigged Hilbert space E, the presheaf of rigged Hilbert space bundles with typical fibre E, and with isometric morphisms (Definition 2.2.1).
- For a rigged C*-algebra A, the presheaf of rigged C*-algebra bundles with typical fibre A, and with isometric morphisms (Definition 2.2.3).
- For rigged C*-algebras A_1 and A_2, and a rigged A_1–A_2-bimodule E, the presheaf of triples $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{E})$ of rigged C*-algebra bundles $\mathcal{A}_1$ and $\mathcal{A}_2$ with typical fibres A_1 and A_2, respectively, and a rigged $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule bundle $\mathcal{E}$ with typical fibre E as defined in Definition 2.2.7, with unitary intertwiners as defined in Definition 2.2.8 as morphisms.
- For a rigged von Neumann algebra N, the presheaf of rigged von Neumann algebra bundles with typical fibre N, and with spatial morphisms (Definition 2.2.9 and Remark 2.2.11).
- For rigged von Neumann algebras N_1 and N_2, and a rigged von Neumann N_1–N_2-bimodule F, the presheaf of triples $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{E})$ of rigged von Neumann algebra bundles $\mathcal{N}_1$ and $\mathcal{N}_2$ with typical fibres N_1 and N_2, respectively, and a rigged von Neumann $\mathcal{N}_1$–$\mathcal{N}_2$-bimodule bundle $\mathcal{F}$ with typical fibre F as defined in Definition 2.2.13, with spatial intertwiners as defined in Definition 2.2.15 as morphisms.

Our aim is to assign to any presheaf $\mathcal{F}$ of categories over $\mathcal{F}\!\mathit{r\acute{e}ch}$ a presheaf $\mathcal{F}^{\mathcal{D}\!\mathit{iff}}$ of categories over $\mathcal{D}\!\mathit{iff}$. There is an abstract canonical procedure how to do this, which we explain later in Remark 2.3.3. In the following definition we spell out directly the result of this procedure, emphasising the concrete perspective.

DEFINITION 2.3.2. Let $\mathcal{F}$ be a presheaf of categories over $\mathcal{F}\!\mathit{r\acute{e}ch}$, and let X be a diffeological space. We define a category $\mathcal{F}^{\mathcal{D}\!\mathit{iff}}(X)$ in the following way.

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(a) An object of the category $\mathcal{F}^{\mathcal{D}\mathit{iff}}(X)$ is a pair $\mathcal{E} = ((\mathcal{E}_c), (\phi_{c_1,c_2,f}))$ consisting of:
  - a family $(\mathcal{E}_c)$, indexed by the plots $c : U \to X$ of X, with $\mathcal{E}_c$ an object of $\mathcal{F}(U)$;
  - a family $(\phi_{c_1,c_2,f})$, indexed the set of all triples (c_1, c_2, f) consisting of two plots $c_1 : U_1 \to X$ and $c_2 : U_2 \to X$ and of a smooth map $f : U_1 \to U_2$ such that $c_2 \circ f = c_1$, with a morphism $\phi_{c_1,c_2,f} : \mathcal{E}_{c_1} \to f^*\mathcal{E}_{c_2}$ in the category $\mathcal{F}(U_1)$.

  This structure is subject to the condition that whenever (c_1, c_2, f_{12}) and (c_2, c_3, f_{23}) are triples as above, the diagram

$$\begin{tikzcd} \mathcal{E}_{c_1} \arrow[r, "\phi_{f_{23}\circ f_{12}}"] \arrow[d, "\phi_{f_{12}}"'] & (f_{23}\circ f_{12})^*\mathcal{E}_{c_3} \arrow[d, "\eta_{f_{12},f_{23}}"] \\ f_{12}^*\mathcal{E}_{c_2} \arrow[r, "f_{12}^*\phi_{f_{23}}"'] & f_{12}^*f_{23}^*\mathcal{E}_{c_3} \end{tikzcd}$$

  of morphisms in $\mathcal{F}(U_1)$ is commutative.

(b) A morphism from an object $\mathcal{E} = ((\mathcal{E}_c), (\phi_{c_1,c_2,f}))$ to an object $\mathcal{E}' = ((\mathcal{E}'_c), (\phi'_{c_1,c_2,f}))$ in $\mathcal{F}^{\mathcal{D}\mathit{iff}}(X)$ is a family $\psi = (\psi_c)$, indexed by the plots $c : U \to X$ of X, with morphisms $\psi_c : \mathcal{E}_c \to \mathcal{E}'_c$ in $\mathcal{F}(U)$, such that the diagram

$$\begin{tikzcd} \mathcal{E}_{c_1} \arrow[r, "\psi_{c_1}"] \arrow[d, "\phi_f"'] & \mathcal{E}'_{c_1} \arrow[d, "\phi'_f"] \\ f^*\mathcal{E}_{c_2} \arrow[r, "f^*\psi_{c_2}"'] & f^*\mathcal{E}_{c_1} \end{tikzcd}$$

  is commutative for all triples (c_1, c_2, f). The composition of morphisms is plot-wise.

It is straightforward to complete the assignment $X \mapsto \mathcal{F}^{\mathcal{D}\mathit{iff}}(X)$ given in Definition 2.3.2 to a presheaf of categories on $\mathcal{D}\mathit{iff}$. This is how we extend presheaves of categories from $\mathcal{F}\mathit{r\acute{e}ch}$ to $\mathcal{D}\mathit{iff}$. If $\mathcal{M}$ is a Fréchet manifold, which we may consider as a diffeological space, then there is a faithful functor

$$\mathcal{F}(\mathcal{M}) \to \mathcal{F}^{\mathcal{D}\mathit{iff}}(\mathcal{M}), \tag{2}$$

which takes an object $\mathcal{E}$ of $\mathcal{F}(\mathcal{M})$ to the pair $((\mathcal{E}_c), (\phi_{c_1,c_2,f}))$, where $\mathcal{E}_c := c^*\mathcal{E}$ is just the pullback in $\mathcal{F}$ (recall that any plot c is a smooth map between Fréchet manifolds here) and $\phi_{c_1,c_2,f} : c_1^*\mathcal{E} \to f^*c_2^*\mathcal{E}$ is the isomorphism η_{f,c_2} provided by $\mathcal{F}$. A morphism $\psi : \mathcal{E} \to \mathcal{E}'$ is sent to the family with $\psi_c := c^*\psi$; it is obvious that this preserves composition and is faithful.

Remark 2.3.3. The extension of presheaves of categories from $\mathcal{F}\mathit{r\acute{e}ch}$ to $\mathcal{D}\mathit{iff}$ that we defined in Definition 2.3.2 fits into a more conceptual perspective, which we want to outline in this remark. To this end, we infer that any diffeological space X can be viewed as a presheaf $\underline{X}$ (of sets) on the category $\mathcal{O}\mathit{pen}$ of open subsets of cartesian spaces, see [BH11]. Namely, the presheaf $\underline{X}$ assigns to an object U in $\mathcal{O}\mathit{pen}$ the set $\underline{X}(U)$ of all plots $c : U \to X$ with domain U. In the following we regard $\underline{X}$ even as a presheaf of *categories* on $\mathcal{O}\mathit{pen}$, and we do this simply by regarding the sets $\underline{X}(U)$ as categories with only identity morphisms. Now, let $\mathcal{F}$ be a presheaf of categories on $\mathcal{F}\mathit{r\acute{e}ch}$, which we may restrict to $\mathcal{O}\mathit{pen} \subset \mathcal{F}\mathit{r\acute{e}ch}$. Presheaves of categories on $\mathcal{O}\mathit{pen}$ form a bicategory $\mathrm{PSh}(\mathcal{O}\mathit{pen})$, the bicategory of weak functors $\mathcal{O}\mathit{pen}^{\mathrm{opp}} \to \mathcal{C}\mathit{at}$. Now, we consider the

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functor represented by $\mathcal{F}|_{\mathcal{O}pen}$ and restricted along $\mathcal{D}iff \to \mathrm{PSh}(\mathcal{O}pen)$,

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{O}pen)}(\,\underline{\ }\,, \mathcal{F}|_{\mathcal{O}pen}) : \mathcal{D}iff^{\mathrm{opp}} \to \mathcal{C}at.$$

Explicitly, it assigns to a diffeological space X the category $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{O}pen)}(\underline{X}, \mathcal{F}|_{\mathcal{O}pen})$ of morphisms between the objects $\underline{X}$ and $\mathcal{F}|_{\mathcal{O}pen}$ of $\mathrm{PSh}(\mathcal{O}pen)$. It is a straightforward exercise to show that

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{O}pen)}(\underline{X}, \mathcal{F}|_{\mathcal{O}pen}) = \mathcal{F}^{\mathcal{D}iff}(X);$$

this embeds our Definition 2.3.2 into a proper topos-theoretical framework. The bicategorical version of the Yoneda lemma (see, e.g., [JY21]) implies now that $\mathcal{F}^{\mathcal{D}iff}(U) \cong \mathcal{F}(U)$ for every object U in $\mathcal{O}pen$. This exhibits $\mathcal{F}^{\mathcal{D}iff}$ as the (left) Kan extension of $\mathcal{F}|_{\mathcal{O}pen}$ along $\mathcal{O}pen^{\mathrm{opp}} \to \mathcal{D}iff^{\mathrm{opp}}$, which is precisely the standard way to extend presheaves from dense subsites to sites [Joh02]. This is our main justification for Definition 2.3.2.

We apply Definition 2.3.2 to the presheaves listed in Remark 2.3.1, obtaining neat definitions of rigged Hilbert space bundles, rigged C*-algebra bundles, rigged bimodule bundles, rigged von Neumann algebra bundles, and rigged von Neumann bimodule bundles over diffeological spaces. Most examples of such bundles that appear in this article are obtained via the functor (2), in the following way: consider a diffeological space X, a Fréchet manifold $\mathcal{M}$, and a smooth map $f : X \to \mathcal{M}$. Then, the composite

$$\mathcal{F}(\mathcal{M}) \xrightarrow{(2)} \mathcal{F}^{\mathcal{D}iff}(\mathcal{M}) \xrightarrow{f^*} \mathcal{F}^{\mathcal{D}iff}(X)$$

produces objects in $\mathcal{F}^{\mathcal{D}iff}(X)$ from objects in $\mathcal{F}(\mathcal{M})$.

Remark 2.3.4. It might be worth pointing out that bundles over a diffeological space X have fibres over points $x \in X$, just like ordinary bundles. Indeed, the axioms of diffeology imply that $c_x : \mathbb{R}^0 \to X : 0 \mapsto x$ is always a plot. Thus, if $\mathcal{N}$ is, say, a rigged von Neumann algebra bundle over X, then $\mathcal{N}_x := \mathcal{N}_{c_x}$ is a rigged von Neumann algebra bundle over the point $\mathbb{R}^0$, i.e. a rigged von Neumann algebra, the fibre of $\mathcal{N}$ at x. If $\mathcal{N} = f^*\mathcal{N}'$ is obtained by pullback of a rigged von Neumann algebra bundle $\mathcal{N}'$ over a Fréchet manifold $\mathcal{M}$ along a smooth map $f : X \to \mathcal{M}$, then $\mathcal{N}_x = \mathcal{N}'_{f(x)}$ is just the ordinary fibre of $\mathcal{N}'$ over $f(x) \in \mathcal{M}$.

3. The free fermions on the circle

In this section we consider a bimodule for certain von Neumann algebras, known as the free fermions on the circle. It plays the role of the typical fibre of the spinor bundle of the loop space. In the first three subsections we recall and then extend our earlier work [KW22, KW20] about the free fermions. Most importantly, we introduce a description of Clifford algebras and Fock spaces in the setting of rigged von Neumann algebras (Proposition 3.3.4), and throughout explore the effects of splitting circles into two halves, which is at the basis of loop fusion. In § 3.4 we then define a novel Connes fusion product of certain implementers on Fock space (Definition 3.4.5). Its relation to loop fusion proved later in Theorem 4.3.3 is a cornerstone of our construction of a fusion product on the spinor bundle.

3.1 Lagrangians and Fock spaces

We recall some aspects of infinite-dimensional Clifford algebras and their representations on Fock spaces, for which the book [PR94] is an excellent reference. The results we require on

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implementers are more spread out across the literature; see, for instance, [Ara87, Ott95, Nee10b, KW22]. In §§ 3.1 and 3.2 we consider in some generality a complex Hilbert space V equipped with a real structure α, i.e. an anti-unitary involution $\alpha : V \to V$. From § 3.3 on, we restrict to a specific example, described at the beginning of § 3.3.

Given a unital C*-algebra A, we say that a map $f : V \to A$ is a *Clifford map* if the following equations are satisfied, for all $v, w \in V$:

$$f(v)f(w) + f(w)f(v) = 2\langle v, \alpha(w)\rangle \mathbb{1}, \quad f(v)^* = f(\alpha(v)). \tag{3}$$

The Clifford C*-algebra $\mathrm{Cl}(V)$ is the unique (up to unique isomorphism) unital C*-algebra equipped with a Clifford map $\iota : V \to \mathrm{Cl}(V)$, such that for each unital C*-algebra A and each Clifford map $f : V \to A$, there exists a unique unital C*-algebra homomorphism $\mathrm{Cl}(f) : \mathrm{Cl}(V) \to A$ with the property that $\mathrm{Cl}(f) \circ \iota = f$. An explicit construction of $\mathrm{Cl}(V)$ is given in [PR94, § 1.2].

The orthogonal group of V, denoted by $\mathrm{O}(V)$, consists of those unitary transformations of V that commute with the real structure. If $g \in \mathrm{O}(V)$, then $\iota g : V \to \mathrm{Cl}(V)$ is a Clifford map. We write $\theta_g := \mathrm{Cl}(\iota g) : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ for its extension to $\mathrm{Cl}(V)$. The map θ_g is called the *Bogoliubov automorphism* associated to g. The map $\theta : g \mapsto \theta_g$ is a continuous homomorphism from $\mathrm{O}(V)$ to $\mathrm{Aut}(\mathrm{Cl}(V))$, where $\mathrm{O}(V)$ is equipped with the operator norm topology, and $\mathrm{Aut}(\mathrm{Cl}(V))$ is equipped with the strong operator topology; see, e.g., [Amb12, Proposition 4.35].

A *Lagrangian* in V is a subspace $L \subset V$ such that V splits as the orthogonal direct sum $V = L \oplus \alpha(L)$. If L is a Lagrangian, then the *Fock space* F_L is the Hilbert completion of the exterior algebra, ΛL, of L. We identify $\alpha(L)$ with the dual of L by identifying $w \in \alpha(L)$ with the linear map $L \ni v \mapsto \langle v, \alpha(w)\rangle$. If $v \in L$ and $w \in \alpha(L) \simeq L^*$, then we write $c(v) : F_L \to F_L$ for left multiplication with v, and $a(w) : F_L \to F_L$ for contraction with w. The maps $c(v)$ and $a(v)$ are bounded operators on F_L, and the map

$$\rho_L : V = L \oplus \alpha(L) \to \mathcal{B}(F_L), \quad (v, w) \mapsto \sqrt{2}(c(v) + a(w))$$

is a Clifford map. This means that $\mathrm{Cl}(\rho_L) : \mathrm{Cl}(V) \to \mathcal{B}(F_L)$ is a unital C*-algebra homomorphism, i.e. a representation of $\mathrm{Cl}(V)$ on F_L. This representation is irreducible [PR94, Theorem 2.4.2] and faithful; hence, we may identify $\mathrm{Cl}(V)$ with its image in $\mathcal{B}(F_L)$. Whenever convenient, we adopt the notation $a \triangleright v = \mathrm{Cl}(\rho_L)(a)(v)$ for $a \in \mathrm{Cl}(V)$ and $v \in F_L$.

If $g \in \mathrm{O}(V)$, then we say that g is *implementable*, if there exists $U \in \mathrm{U}(F_L)$ with the property that

$$\theta_g(a) = UaU^* \tag{4}$$

for all $a \in \mathrm{Cl}(V)$; the operator U is said to *implement* g. The problem to decide which $g \in \mathrm{O}(V)$ are implementable is called the 'implementability problem'. It is completely solved: an element $g \in \mathrm{O}(V)$ is implementable if and only if the operator $P_L g P_L^\perp$ is Hilbert–Schmidt, where P_L is the orthogonal projection to L, see [PR94, Theorem 3.3.5] or [Ara87, Theorem 6.3]. We write $\mathrm{O}_L(V)$ for the set consisting of those $g \in \mathrm{O}(V)$ which are implementable; the set $\mathrm{O}_L(V)$ is, in fact, a subgroup of $\mathrm{O}(V)$.

The group $\mathrm{O}(V)$ can be equipped with the structure of Banach Lie group in the standard way, with underlying topology the operator norm topology. The Lie algebra of $\mathrm{O}(V)$ is

$$\mathfrak{o}(V) = \{X \in \mathcal{B}(V) \mid [X, \alpha] = 0, X^* = -X\}.$$

The subgroup $\mathrm{O}_L(V)$ can also be equipped with the structure of a Banach Lie group, whose underlying topology is given by the norm $\|g\|_{\mathcal{J}} = \|g\| + \|P_L g P_L^\perp\|_2$, where $\|g\|$ is the operator

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norm of g, and where $\|\cdot\|_2$ is the Hilbert–Schmidt norm [KW22, § 3.4]. The inclusion $\mathrm{O}_L(V) \to \mathrm{O}(V)$ is smooth. The Lie algebra of $\mathrm{O}_L(V)$ is

$$\mathfrak{o}_L(V) = \{X \in \mathfrak{o}(V) \mid \|P_L X P_L^\perp\|_2 < \infty\}.$$

The *group of implementers*, $\mathrm{Imp}_L(V)$, is defined to be the subgroup of $\mathrm{U}(F_L)$ consisting of those operators $U \in \mathrm{U}(F_L)$, for which there exists a $g \in \mathrm{O}_L(V)$ such that (4) holds. If $U \in \mathrm{Imp}_L(V)$, then the element $g \in \mathrm{O}_L(V)$ that it implements is determined uniquely, and we obtain a group homomorphism $q : \mathrm{Imp}_L(V) \to \mathrm{O}_L(V)$. Using the irreducibility of the representation of $\mathrm{Cl}(V)$ on F_L together with Schur's lemma, we see that, for each $g \in \mathrm{O}_L(V)$, the fibre $q^{-1}\{g\}$ is a $\mathrm{U}(1)$-torsor. We equip the group $\mathrm{Imp}_L(V)$ with the structure of Banach Lie group as in [KW22, Theorem 3.15], see also [Wur01]. We then have that the exact sequence

$$\mathrm{U}(1) \to \mathrm{Imp}_L(V) \xrightarrow{q} \mathrm{O}_L(V),$$

is a central extension of Banach Lie groups. It is important to note that the topology underlying the Banach Lie group structure on $\mathrm{Imp}_L(V)$ is not the operator norm topology, and that the inclusion map $\mathrm{Imp}_L(V) \to \mathrm{U}(F_L)$ is not continuous, let alone smooth.

Let us assume that we are given a further orthogonal decomposition $V = V_- \oplus V_+$ of V into two Hilbert spaces $V_\pm$ which are preserved under α. In the sequel, such a splitting will implement the splitting of a loop into two paths. Given such a splitting, we are interested in the operators on V which preserve this splitting. We write $P_\pm$ for the projection onto $V_\pm$ and define

$$\mathfrak{o}^\theta(V) := \{X \in \mathfrak{o}(V) \mid P_\pm X P_\mp = 0\}, \quad \mathfrak{o}^\theta_L(V) := \{X \in \mathfrak{o}_L(V) \mid P_\pm X P_\mp = 0\}.$$

We observe that the maps $X \mapsto P_\pm X P_\mp$ are linear and continuous (with respect to the operator norm), which implies that $\mathfrak{o}^\theta(V)$ is a closed linear subspace of $\mathfrak{o}(V)$ and, hence, a Banach space. The subspace $\mathfrak{o}^\theta_L(V)$ is the pre-image of $\mathfrak{o}^\theta(V)$ under the continuous inclusion $\mathfrak{o}_L(V) \to \mathfrak{o}(V)$, and hence a Banach space as well. The spaces $\mathfrak{o}^\theta(V)$ and $\mathfrak{o}^\theta_L(V)$ are Lie subalgebras of $\mathfrak{o}(V)$ and $\mathfrak{o}_L(V)$, respectively. Next, we define the metric groups

$$\mathrm{O}^\theta(V) := \{g \in \mathrm{O}(V) \mid P_\pm g P_\mp = 0\}, \quad \mathrm{O}^\theta_L(V) := \{g \in \mathrm{O}_L(V) \mid P_\pm g P_\mp = 0\}.$$

Using standard techniques (see, e.g., [KW22, Lemma 3.13]), one may then show that the exponential maps $\mathfrak{o}^\theta(V) \to \mathrm{O}^\theta(V)$ and $\mathfrak{o}^\theta_L(V) \to \mathrm{O}^\theta_L(V)$ are local homeomorphisms, which allows us to equip $\mathrm{O}^\theta(V)$ and $\mathrm{O}^\theta_L(V)$ with the structure of Banach Lie groups, with Lie algebras $\mathfrak{o}^\theta(V)$ and $\mathfrak{o}^\theta_L(V)$, respectively. It is then clear from the construction that $\mathrm{O}^\theta(V)$ and $\mathrm{O}^\theta_L(V)$ are closed submanifolds of $\mathrm{O}(V)$ and $\mathrm{O}_L(V)$, respectively. If $g \in \mathrm{O}^\theta(V)$, then we set $g_\pm := P_\pm g P_\pm \in \mathrm{O}(V_\pm)$. We observe that the maps $g \mapsto g_\pm$ are smooth group homomorphisms. Finally, we restrict the group of implementers to the split setting, and define the Banach Lie group

$$\mathrm{Imp}^\theta_L(V) := \mathrm{Imp}_L(V)|_{\mathrm{O}^\theta_L(V)},$$

which then is a central extension of $\mathrm{O}^\theta_L(V)$ by $\mathrm{U}(1)$.

Next, we consider the Clifford algebras $\mathrm{Cl}(V_\pm)$. We observe that extending by zero gives isometries $V_\pm \to V$ which moreover intertwine the real structures, and thus induce isometric $*$-homomorphisms $\iota_\pm : \mathrm{Cl}(V_\pm) \to \mathrm{Cl}(V)$. The algebra product

$$\mathrm{Cl}(V_-) \times \mathrm{Cl}(V_+) \subset \mathrm{Cl}(V) \times \mathrm{Cl}(V) \to \mathrm{Cl}(V)$$

induces a unital isomorphism $\mathrm{Cl}(V_-) \otimes \mathrm{Cl}(V_+) \cong \mathrm{Cl}(V)$ of C*-algebras. (Here, $\otimes$ stands for any choice of tensor product of C*-algebras. The choice is immaterial, because the Clifford C*-algebra is uniformly hyperfinite and, hence, nuclear.) Under these identifications, we have $\iota_-(a_-) = a_- \otimes \mathbb{1}$ and $\iota_+(a_+) = \mathbb{1} \otimes a_+$, for $a_\pm \in \mathrm{Cl}(V_\pm)$. The following result expresses that the

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Bogoliubov automorphisms are compatible with the splitting; this will be used later in the proofs of Lemmas 3.4.2 and 5.2.1.

Lemma 3.1.1. *For all $g_- \oplus g_+ \in \mathrm{O}^\theta(V)$ and all $a_\pm \in \mathrm{Cl}(V_\pm)$ we have*

$$\theta_{g_-\oplus g_+}(a_- \otimes a_+) = \theta_{g_-}(a_-) \otimes \theta_{g_+}(a_+),$$

where $\theta_{g_\pm}$ are the Bogoliubov automorphisms of the Clifford algebras $\mathrm{Cl}(V_\pm)$.

Proof. This follows from the fact that $(g_- \oplus g_+) \circ \iota_\pm = \iota_\pm \circ g_\pm$ for all $g_- \oplus g_+ \in \mathrm{O}^\theta(V)$. □

3.2 Smooth Fock spaces

In §§ 4 and 5 we construct bundles of rigged Hilbert spaces and rigged C*-algebras over Fréchet manifolds. This requires a detailed study of smoothness properties of the representations obtained in § 3.1; this is the goal of this section. We continue working with a complex Hilbert space V with a real structure α, additionally equipped with an orthogonal decomposition into two Hilbert spaces $V_\pm$ which are preserved under α. Our first goal is to equip the Fock space F_L and the Clifford C*-algebra $\mathrm{Cl}(V)$ with the structure of a rigged Hilbert space and a rigged C*-algebra, respectively. We have done this already in our earlier paper [KW20]; however, there we have not taken the splitting $V = V_- \oplus V_+$ into account. For the purpose of this article, the splitting is essential, and it leads to a finer rigging (we compare them in Remark 3.2.6 below). Therefore, we describe the important steps again.

As a subgroup of $\mathrm{U}(F_L)$, the group $\mathrm{Imp}^\theta_L(V)$ comes equipped with a unitary representation on F_L. As is typical for infinite-dimensional representations, the action map $\mathrm{Imp}^\theta_L(V) \times F_L \to F_L$ is not smooth. The *subspace of smooth vectors* in F_L is defined as usual to be

$$F^{\mathrm{s}}_L := \{v \in F_L \mid \mathrm{Imp}^\theta_L(V) \to F_L, U \mapsto Uv \text{ is smooth}\}.$$

The following result follows directly from [KW22, Proposition 3.17], see also [Nee10b, § 10.1].

Lemma 3.2.1. *The set of smooth vectors F^{s}_L contains the exterior algebra ΛL and is, hence, a dense subspace of F_L.*

By definition of smooth vectors, the Lie algebra $\mathfrak{imp}^\theta(V)$ of $\mathrm{Imp}^\theta_L(V)$ acts infinitesimally on F^{s}_L, i.e. for $X \in \mathfrak{imp}^\theta(V)$ and $v \in F^{\mathrm{s}}_L$, we may define

$$Xv := \left.\frac{d}{dt}\right|_{t=0} \exp(tX)(v).$$

Let $\mathcal{P}(F_L)$ be the set of all continuous semi-norms on F_L. We define a topology on F^{s}_L by the following family of semi-norms [Nee10a, § 4]:

$$p_n(v) = \sup\{p(X_1 \dots X_n v) \mid X_i \in \mathfrak{imp}^\theta(V), \|X_i\| \leqslant 1\}, \quad p \in \mathcal{P}(F_L), n \in \mathbb{N}_0.$$

The following result is proved completely analogously to [KW20, Proposition 3.2.4].

Proposition 3.2.2. *The space of smooth vectors F^s_L is a rigged Hilbert space, and it carries a smooth representation of the Banach Lie group $\mathrm{Imp}^\theta_L(V)$.*

Just like the action map $\mathrm{Imp}^\theta_L(V) \times F_L \to F_L$ is not smooth, the map $\mathrm{O}^\theta(V) \times \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ for the action by Bogoliubov automorphisms is not smooth either, an issue that we handle in a similar way. We write $\mathrm{Cl}(V)^{\mathrm{s}}$ for the subspace of smooth vectors in $\mathrm{Cl}(V)$, i.e.

$$\mathrm{Cl}(V)^{\mathrm{s}} := \{a \in \mathrm{Cl}(V) \mid \mathrm{O}^\theta(V) \to \mathrm{Cl}(V), g \mapsto \theta_g(a) \text{ is smooth}\}.$$

Lemma 3.2.3. *The set of smooth vectors $\mathrm{Cl}(V)^{\mathrm{s}}$ contains the algebraic Clifford algebra of V and is, hence, dense in the C*-Clifford algebra $\mathrm{Cl}(V)$.*

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The Lie algebra $\mathfrak{o}^{\theta}_{res}(V)$ acts on $\mathrm{Cl}(V)^{\mathrm{s}}$, which allows us to proceed as follows. Let $\mathcal{R}(\mathrm{Cl}(V))$ be the set of continuous semi-norms on $\mathrm{Cl}(V)$. The topology on $\mathrm{Cl}(V)^{\mathrm{s}}$ is then defined by the family of semi-norms

$$r_n(a) = \sup\{r(Y_1 \dots Y_n a) \mid Y_i \in \mathfrak{o}^{\theta}(V), \|Y_i\| \leqslant 1\}, \quad r \in \mathcal{R}(\mathrm{Cl}(V)), n \in \mathbb{N}_0.$$

In analogy with [KW20, Proposition 3.2.7] we then have the following proposition.

Proposition 3.2.4. *The algebra of smooth vectors* $\mathrm{Cl}(V)^{\mathrm{s}}$ *is a rigged C*-algebra, and it carries a smooth representation of the Banach Lie group* $\mathrm{O}^{\theta}(V)$.

Finally, we adapt to this situation, and obtain the following result.

Proposition 3.2.5. *The Fock space representation* $\mathrm{Cl}(V) \times F_L \to F_L$ *restricts to a map* $\mathrm{Cl}(V)^{\mathrm{s}} \times F_L^{\mathrm{s}} \to F_L^{\mathrm{s}}$ *and exhibits* F_L^{s} *as a rigged* $\mathrm{Cl}(V)^{\mathrm{s}}$*-module. Moreover,* $\mathrm{Cl}_{\mathrm{vN}}(V)^{\mathrm{s}} := (\mathrm{Cl}(V)^{\mathrm{s}}, F_L^{\mathrm{s}})$ *is a rigged von Neumann algebra.*

Proof. Showing that the Fock space representation restricts and that the map $\rho : \mathrm{Cl}(V)^{\mathrm{s}} \times F_L^{\mathrm{s}} \to F_L^{\mathrm{s}}$ is continuous is completely analogous to the proof we gave in [KW20, Proposition 3.2.8] for the slightly coarser riggings. By [KW20, Remark 2.2.9] this implies that F_L^{s} as a rigged $\mathrm{Cl}(V)^{\mathrm{s}}$-module. In order to show that $\mathrm{Cl}_{\mathrm{vN}}(V)^{\mathrm{s}}$ is a rigged von Neumann algebra, we only have to note that the C*-representation induced by ρ is the Fock space representation, which is faithful. $\square$

We remark that the rigged von Neumann algebra $\mathrm{Cl}_{\mathrm{vN}}(V)^{\mathrm{s}}$ determines an ordinary von Neumann algebra $\mathrm{Cl}_{\mathrm{vN}}(V) := (\mathrm{Cl}_{\mathrm{vN}}(V)^{\mathrm{s}})''$ as the completion of $\mathrm{Cl}(V)$ acting on F_L (Remark 2.1.7). It is well-known that $\mathrm{Cl}_{\mathrm{vN}}(V) = \mathcal{B}(F_L)$.

Remark 3.2.6. In our earlier paper [KW20, § 3.2] we considered different Fréchet spaces as the riggings on F_L and $\mathrm{Cl}(V)$, obtained without assuming a splitting of V:

$$F_L^{\infty} := \{v \in F_L \mid \mathrm{Imp}_L(V) \to F_L, U \mapsto Uv \text{ is smooth}\},$$
$$\mathrm{Cl}(V)^{\infty} := \{a \in \mathrm{Cl}(V) \mid \mathrm{O}(V) \to \mathrm{Cl}(V), g \mapsto \theta_g(a) \text{ is smooth}\}.$$

Because of the fact that $\mathrm{Imp}_L^{\theta}(V)$ is contained in $\mathrm{Imp}_L(V)$ we have that $F_L^{\infty} \subset F_L^{\mathrm{s}}$. This means that we improve the rigging by passing to the subgroup $\mathrm{Imp}_L^{\theta}(V)$. The analogous statement is true for $\mathrm{Cl}(V)^{\infty}$ and $\mathrm{Cl}(V)^{\mathrm{s}}$. The riggings F_L^{∞} and $\mathrm{Cl}(V)^{\infty}$ have been appropriate for the construction of the spinor bundle on loop space and its Clifford action. For the construction of the fusion product we need the 'finer' riggings F_L^{s} and $\mathrm{Cl}(V)^{\mathrm{s}}$; in particular, we need these in Lemma 3.2.7, see Remark 3.2.8.

We now move beyond the analogy with [KW20, § 3.2] and lift the separate Clifford C*-algebras $\mathrm{Cl}(V_{\pm})$ to the setting of rigged C*-algebras, by defining

$$\mathrm{Cl}(V_{\pm})^{\mathrm{s}} := \{a \in \mathrm{Cl}(V_{\pm}) \mid \mathrm{O}(V_{\pm}) \to \mathrm{Cl}(V_{\pm}),\ g \mapsto \theta_g(a) \text{ is smooth}\}.$$

Just like $\mathrm{Cl}(V)^{\mathrm{s}}$ these are rigged C*-algebras.

Lemma 3.2.7. *The inclusion maps* $\iota_{\pm}$ *restrict to isometric morphisms* $\iota_{\pm} : \mathrm{Cl}(V_{\pm})^{\mathrm{s}} \to \mathrm{Cl}(V)^{\mathrm{s}}$ *of rigged C*-algebras.*

Proof. Let $a \in \mathrm{Cl}(V_-)$ be arbitrary. We then see from Lemma 3.1.1 that the map

$$\mathrm{O}^{\theta}(V) \to \mathrm{Cl}(V), \quad g \mapsto \theta_g(a \otimes \mathbb{1}) \tag{5}$$

is the composition of the Lie group homomorphism $\mathrm{O}^{\theta}(V) \to \mathrm{O}(V_-)$, $g \mapsto g_-$ with the action map $\mathrm{O}(V_-) \to \mathrm{Cl}(V_-)$, $g_- \mapsto \theta_{g_-}(a)$, followed by the smooth map $\iota_- : \mathrm{Cl}(V_-) \to \mathrm{Cl}(V)$. This proves

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that if $a \in \mathrm{Cl}(V_-)^{\mathrm{s}}$, then (5) is smooth, and we have $a \otimes \mathbb{1} \in \mathrm{Cl}(V)^{\mathrm{s}}$; and thus ι_- restricts to a map $\iota_- : \mathrm{Cl}(V_-)^{\mathrm{s}} \to \mathrm{Cl}(V)^{\mathrm{s}}$. What remains to be shown is that this restriction $\iota_- : \mathrm{Cl}(V_-)^{\mathrm{s}} \to \mathrm{Cl}(V)^{\mathrm{s}}$ is continuous with respect to the Fréchet space structures. Because ι_- is linear, it suffices to show continuity at 0. First, observe that if $r \in \mathcal{R}(\mathrm{Cl}(V))$, then $r \circ \iota_- \in \mathcal{R}(\mathrm{Cl}(V_-))$. Moreover, we claim that

$$r_n(a \otimes \mathbb{1}) = (r \circ \iota_-)_n(a), \tag{6}$$

indeed

$$\begin{aligned} r_n(a \otimes \mathbb{1}) &= \sup\{r(Y_1 \dots Y_n(a \otimes \mathbb{1})) \mid Y_i \in \mathfrak{o}^\theta(V), \|Y_i\| \leqslant 1\} \\ &= \sup\{r(Y_1|_{V_-} \dots Y_n|_{V_-}(a) \otimes \mathbb{1}) \mid Y_i \in \mathfrak{o}^\theta(V), \|Y_i\| \leqslant 1\} \\ &= \sup\{r \circ \iota_-(Y_{-,1} \dots Y_{-,n}(a)) \mid Y_{-,i} \in \mathfrak{o}(V_-), \|Y_i\| \leqslant 1\} \\ &= (r \circ \iota_-)_n(a). \end{aligned}$$

From (6) it follows that the preimage of the subbasis open neighbourhood of zero given by $\{x \in \mathrm{Cl}(V) \mid r_n(x) < \varepsilon\} \subset \mathrm{Cl}(V)$ is the subbasis open neighbourhood of zero given by $\{a \in \mathrm{Cl}(V_-) \mid (r \circ \iota_-)_n(a) < \varepsilon\}$. The discussion of $\mathrm{Cl}(V_+)$ is analogous. □

Remark 3.2.8. For Lemma 3.2.7 it is essential that we work with $\mathrm{O}^\theta(V)$ and not with $\mathrm{O}(V)$, because $\mathrm{O}(V)$ does not come equipped with a map to $\mathrm{O}(V_-)$ and, moreover, because (6) only holds because we use $\mathrm{O}^\theta(V)$. Further, the Fock space representation of $\mathrm{Cl}(V)$ on F_L does not restrict to a representation of $\mathrm{Cl}(V)^{\mathrm{s}}$ on F_L^∞; this is why we have to use the bigger rigging F_L^{s} on F_L a well.

We obtain the following result about the rigged C*-algebras $\mathrm{Cl}(V_\pm)^{\mathrm{s}}$.

Proposition 3.2.9. *The isometric morphisms $\iota_\pm$ induce on F_L^{s} the structure of a rigged $\mathrm{Cl}(V_\pm)^{\mathrm{s}}$-module. Moreover, $N_\pm(V)^{\mathrm{s}} := (\mathrm{Cl}(V_\pm)^{\mathrm{s}}, F_L^{\mathrm{s}})$ are rigged von Neumann algebras, and the pairs $(\iota_\pm, \mathbb{1})$ are spatial morphisms $N_\pm(V)^{\mathrm{s}} \to \mathrm{Cl}_{\mathrm{vN}}(V)^{\mathrm{s}}$.*

Proof. The first claim follows from Lemma 3.2.7 and Remark 2.1.4(b). The second claim (see Definition 2.1.6) follows since the representation of $\mathrm{Cl}(V_\pm)^{\mathrm{s}}$ on F_L^{s} extends to the representation of $\mathrm{Cl}(V_\pm)$ on F_L, which is faithful. □

The rigged von Neumann algebra $N_\pm(V)^{\mathrm{s}} = (\mathrm{Cl}(V_\pm)^{\mathrm{s}}, F_L^{\mathrm{s}})$ determines a von Neumann algebra $N_\pm(V) := (N_\pm(V)^{\mathrm{s}})''$ in the classical sense, as the von Neumann closure of the C*-algebra $\mathrm{Cl}(V_\pm)$ acting on F_L, see Remark 2.1.7. It is well known that these von Neumann algebras are III_1-factors (see [ST04, Example 4.3.2] and [Was98, § 16]).

3.3 Free fermions as a rigged bimodule

We now fix a concrete Hilbert space V, a real structure α, a Lagrangian L, and a splitting $V = V_- \oplus V_+$, which will remain the same for the rest of the paper. Let $\mathbb{S} \to S^1$ be the odd spinor bundle on the circle, i.e. that associated to the odd (i.e. the connected) spin structure on the circle. We set $V := L^2(S^1, \mathbb{S} \otimes \mathbb{C}^d)$, where d is a natural number (later it is the spacetime dimension). Pointwise complex conjugation gives a real structure $\alpha : V \to V$. In [KW22, § 2] it is explained how the space of smooth 2π-antiperiodic functions on the real line can be identified with the dense subspace $\Gamma(S^1, \mathbb{S} \otimes \mathbb{C}^d) \subset V$ of smooth sections. Under this identification, V has an orthonormal basis $\{\xi_{n,j}\}_{n \in \mathbb{N}, j=1,\dots,d}$ by setting

$$\xi_{n,j}(t) = e^{-i(n+1/2)t} e_j,$$

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where $\{e_j\}_{j=1,\dots,d}$ is the standard basis of $\mathbb{C}^d$. It is further shown that $\alpha(\xi_{n,j}) = \xi_{-n-1,j}$ for all n and all j. It follows that the closed linear span

$$L := \operatorname{span}\{\xi_{n,j} \mid n \geqslant 0,\ j = 1,\dots,d\}$$

is a Lagrangian in V. The corresponding Fock space $F := F_L$ is called the *free fermions*. Let us write $I_+ \subset S^1$ for the open upper semicircle and $I_- \subset S^1$ for the open lower semicircle. If $f \in V$, then we write $\operatorname{Supp}(f)$ for the support of f. We consider the subspaces

$$V_\pm := \{f \in V \mid \operatorname{Supp}(f) \subseteq I_\pm\};$$

they yield a decomposition $V = V_- \oplus V_+$, and α restricts to real structures on $V_\pm$. This puts us in the setting of § 3.2, and we thus obtain rigged von Neumann algebras $N^{\mathrm{s}}_\pm := N_\pm(V)^{\mathrm{s}} := (\mathrm{Cl}(V_\pm)^{\mathrm{s}}, F^{\mathrm{s}})$.

We denote by τ the complex-linear extension of the map $\xi_{n,j} \mapsto \xi_{-n-1,j}$. The map τ is orthogonal, exchanges L with $\alpha(L)$, and exchanges V_+ with V_-. We remark that while $\tau \in \mathrm{O}(V)$, one can show that it is not implementable. Since α interchanges L with $\alpha(L)$ as well, the anti-unitary isomorphism $\alpha\tau$ preserves both L and $\alpha(L)$. In particular, it induces an anti-unitary operator $\Lambda_{\alpha\tau} : F \to F$.

If $g \in \mathrm{O}(V)$, then we write $\tau(g) := \tau \circ g \circ \tau \in \mathrm{O}(V)$. With this notation we have, for all $f \in V$ and all $g \in \mathrm{O}(V)$, that $\tau(g(f)) = \tau(g)(\tau(f))$. Because τ interchanges L with $\alpha(L)$ we have that conjugation by τ yields an isometric (hence, smooth) group homomorphism from $\mathrm{O}_L(V)$ into $\mathrm{O}_L(V)$. Finally, because τ exchanges V_+ with V_- we have that τ preserves $\mathrm{O}^\theta_L(V)$, and moreover exchanges $\mathrm{O}(V_-)$ with $\mathrm{O}(V_+)$.

Remark 3.3.1. In the following discussion of the Clifford algebras $\mathrm{Cl}(V_\pm)$ we will focus on $\mathrm{Cl}(V_-)$ instead of $\mathrm{Cl}(V_+)$, and in the remainder of this work we will continue to do so. The rigged von Neumann algebra $N_- = (\mathrm{Cl}(V_-)^{\mathrm{s}}, F^{\mathrm{s}})$ will be denoted by just N in the following. Nothing is lost by this choice, the theory for $\mathrm{Cl}(V_+)$ is completely parallel.

We recall that the Fock space F is equipped with the left action of $\mathrm{Cl}(V_-)$ induced by the inclusion $\iota_- : \mathrm{Cl}(V_-) \to \mathrm{Cl}(V)$. Next, we equip it with a compatible right action of $\mathrm{Cl}(V_-)$. We note that F is naturally graded, as it is the completion of an exterior algebra. We let $\mathrm{k} : F \to F$ be the 'Klein transformation', that is, k acts on F as the identity on the even part, and as multiplication by i on the odd part. Note that k is unitary. Then we define the operator

$$J := \mathrm{k}^{-1}\,\Lambda_{\alpha\tau}, \tag{7}$$

which is an anti-unitary operator on F with $J^2 = 1$. Now, we define a right action as

$$F \times \mathrm{Cl}(V_-) \to F, \quad (v, a) \mapsto Ja^*Jv. \tag{8}$$

We shall then write $v \triangleleft a$ for the right action of a on v.

Lemma 3.3.2. *Right and left actions of* $\mathrm{Cl}(V_-)$ *on* F *commute.*

Proof. It is convenient to use Tomita–Takesaki theory to see this. The vector $\Omega := 1 \in \Lambda^0 L \subset F$ is cyclic and separating for the von Neumann algebra $\mathrm{Cl}(V_-)'' \subset \mathcal{B}(F)$, see [KW22, § 4.2]. In Tomita–Takesaki theory, one considers the triple $(\mathrm{Cl}(V_-)'', F, \Omega)$ and associates to it a so-called modular conjugation operator. In our case, this is precisely the operator J, see [KW22, § 4.2] and other references listed there. A main result of Tomita–Takesaki theory (recalled as Theorem A.1.3) is that $a \mapsto Ja^*J$ is an anti-isomorphism of von Neumann algebras from $\mathrm{Cl}(V_-)''$ onto its commutant. This shows that the action of $a \in \mathrm{Cl}(V_-)$ on F commutes with the one of Ja^*J. □

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Next, we pass to the rigged setting. In Propositions 3.2.5 and 3.2.9 we have seen that F^{s} is a rigged $\mathrm{Cl}(V_-)^{\mathrm{s}}$-module under the left action. The analog is true for the right action of (8), as the following result shows.

Lemma 3.3.3. *The right action $(v,a) \mapsto v \triangleleft a$ restricts to an action of $\mathrm{Cl}(V_-)^{\mathrm{s}}$ on F^{s}, and exhibits F^{s} as a rigged $(\mathrm{Cl}(V_-)^{\mathrm{s}})^{\mathrm{opp}}$-module. In particular, F^{s} is a rigged $\mathrm{Cl}(V_-)^{\mathrm{s}}$–$\mathrm{Cl}(V_-)^{\mathrm{s}}$-bimodule.*

Proof. The proof is mostly standard, and analogous to that of [KW20, Proposition 3.2.8]. To carry out the necessary computations one must make use of the formulas

$$J\theta_{\tau(g)}(a)J = \theta_g(JaJ) \quad (a \in \mathrm{Cl}(V), g \in \mathrm{O}(V)), \tag{9}$$

which is [KW22, Lemma 4.8], and

$$X(v \triangleleft a) = X(v) \triangleleft a + v \triangleleft \tau(X)(a) \quad (X \in \mathfrak{o}_L(V)),$$

where $\tau(X) = \tau \circ X \circ \tau \in \mathfrak{o}_L(V)$, which follows from (9). From this, one obtains a non-commutative binomial expansion, cf. [KW20, Equation (12)], at which point the proof is completely analogous to that of Proposition 3.2.5. □

We recall from Proposition 3.2.9 that $\mathrm{Cl}(V_-)^{\mathrm{s}}$ is not just a rigged C*-algebra but also forms a rigged von Neumann algebra $N^{\mathrm{s}} = N^{\mathrm{s}}_- = (\mathrm{Cl}(V_-)^{\mathrm{s}}, F^{\mathrm{s}})$, whose defining representation is the left action.

Proposition 3.3.4. *The rigged bimodule F^{s} is a rigged von Neumann N^{s}–N^{s}-bimodule.*

Proof. It remains to show that the right action, $(\mathrm{Cl}(V_-)^{\mathrm{s}})^{\mathrm{opp}} \times F^{\mathrm{s}} \to F^{\mathrm{s}}$, makes F^{s} a rigged von Neumann $(N^{\mathrm{s}})^{\mathrm{opp}}$-module, where $(N^{\mathrm{s}})^{\mathrm{opp}} = ((\mathrm{Cl}(V_-)^{\mathrm{s}})^{\mathrm{opp}}, (F^{\mathrm{s}})^{\sharp})$, see Definition 2.1.12. For this, all that needs to be shown is that the induced map

$$\hat{\rho} : \mathrm{Cl}(V_-)^{\mathrm{opp}} \to \mathcal{B}(F), a \mapsto Ja^*J$$

extends to a normal $*$-homomorphism

$$(\mathrm{Cl}(V_-)^{\mathrm{opp}})'' \to \mathcal{B}(F).$$

Now, let ρ_0 be the defining representation of $(N^{\mathrm{s}})^{\mathrm{opp}}$, let $c : F^{\sharp} \to F$ be the canonical anti-linear isomorphism, and let $\sigma : \mathcal{B}(F^{\sharp}) \to \mathcal{B}(F)$ be the $*$-homomorphism

$$\sigma : T \mapsto JcTc^{-1}J,$$

then one checks that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Cl}(V_-)^{\mathrm{opp}} \arrow[r, "\rho_0^\vee"] \arrow[dr, "\hat{\rho}"'] & \mathcal{B}(F^{\sharp}) \arrow[d, "\sigma"] \\ & \mathcal{B}(F) \end{tikzcd}$$

Thus, σ extends $\hat{\rho}$. Moreover, since $Jc : F^{\sharp} \to F$ is an isometric isomorphism, it follows that σ is a normal $*$-homomorphism. □

We recall that the rigged von Neumann algebra $N^{\mathrm{s}} = N^{\mathrm{s}}_- = (\mathrm{Cl}(V_-)^{\mathrm{s}}, F^{\mathrm{s}})$ determines a von Neumann algebra $N = (N^{\mathrm{s}})''$, see Remark 2.1.7. Moreover, the statement that F^{s} is a rigged von Neumann N^{s}–N^{s}-bimodule implies that its completion, the Fock space F, is an N–N-bimodule in the classical sense, see Lemma 2.1.16.

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As we recall in Appendix A.1, a cyclic and separating vector for a representation of a von Neumann algebra equips that representation with the structure of a so-called *standard form* of the von Neumann algebra. Using this in our case for $\Omega \in F$ (see [KW22, § 4.2]), we note the following result.

PROPOSITION 3.3.5. *The Fock space F is a standard form of N.*

Remark 3.3.6. (a) From the uniqueness of standard forms (see Theorem A.1.2) it follows that F is isomorphic to the canonical standard form $L^2_\Omega(N)$ constructed from the faithful and normal state $N \to \mathbb{C}, a \mapsto \langle a \triangleright \Omega, \Omega \rangle$. We recall that $L^2_\Omega(N)$ is defined as the completion of N with respect to the sesquilinear form $(a, b) \mapsto \langle a \triangleright \Omega, b \triangleright \Omega \rangle$. An explicit isomorphism $u : F \to L^2_\Omega(N)$ is given by the extension of the densely defined map $F \to N, a \triangleright \Omega \mapsto a$, see Lemma A.1.6.

(b) A standard form of a von Neumann algebra is in a natural way a bimodule for this von Neumann algebra, see Remark A.1.5. The left action is the given representation, whereas the right action is given by the formula $v \otimes a \mapsto Ja^*Jv$, where J is the modular conjugation for the triple $(\mathrm{Cl}(V_-)'', F, \Omega)$. In our case, this is precisely the right action we have defined in (8). In other words, the N–N-bimodule structure on F defined above coincides with the bimodule structure obtained from the theory of standard forms; moreover, the isomorphism $u : F \to L^2_\Omega(N)$ is an intertwiner.

A standard form of a von Neumann algebra, viewed as a bimodule, is neutral with respect to Connes fusion, as we recall in Proposition A.2.6 and Corollary A.2.7. More precisely, there are unitary intertwiners $\lambda_K : F \boxtimes K \to K$ and $\rho_K : K \boxtimes F \to K$ for any N–N-bimodule K, and for $K = F$ we have coincidence $\lambda_F = \rho_F$. We will denote this unitary intertwiner by

$$\chi : F \boxtimes F \to F.$$

We shall need a more explicit description of χ to prove one of our key results, Theorem 4.3.3. To make sense of this explicit description, we first recall the basic definition of the Connes fusion product, see Appendix A, and in particular Definition A.2.1, for more details. We start by writing $\mathcal{D}(F, \Omega) := \mathrm{Hom}_{-,N}(L^2_\Omega(N), F)$ for the space of bounded linear maps that intertwine the right N actions. We define a map

$$p_\Omega : \mathrm{Hom}_{-,N}(L^2_\Omega(N), L^2_\Omega(N)) \to N$$

by requiring $p_\Omega(x) \triangleright v = x(v)$ for all $v \in L^2_\Omega(N)$. Next, we consider the space $\mathcal{D}(F, \Omega) \otimes F$ equipped with the degenerate inner product

$$\langle x \otimes v, y \otimes w \rangle_\Omega := \langle v, p_\Omega(x^* y) \triangleright w \rangle.$$

The Connes fusion $F \boxtimes F$ is now the Hilbert completion of $\mathcal{D}(F, \Omega) \otimes F / \ker\langle \cdot, \cdot \rangle_\Omega$. On the space $\mathcal{D}(F, \Omega) \otimes F$ the map $\chi : F \boxtimes F \to F$ is given by (see Proposition A.2.6 and Corollary A.2.7)

$$x \otimes v \mapsto p_\Omega(u \circ x) \triangleright v, \tag{10}$$

where u is the invertible intertwiner $u : F \to L^2_\Omega(N)$ of Remark 3.3.6.

Remark 3.3.7. It is interesting to note that in our case the map $\chi : \mathcal{D}(F, \Omega) \otimes F \to F$ is surjective, which implies that the space $\mathcal{D}(F, \Omega) \otimes F / \ker\langle \cdot, \cdot \rangle_\Omega$, is already complete.

We remark that Connes fusion is coherently associative (Proposition A.2.5), which allows us to omit bracketing of multiple Connes fusions. The following lemma follows then directly from Corollary A.2.7.

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Lemma 3.3.8. *The isomorphism χ is associative in the sense that the following diagram commutes.*

$$\begin{tikzcd} F \boxtimes F \boxtimes F \arrow[r, "\mathbb{1}\boxtimes\chi"] \arrow[d, "\chi\boxtimes\mathbb{1}"'] & F \boxtimes F \arrow[d, "\chi"] \\ F \boxtimes F \arrow[r, "\chi"'] & F \end{tikzcd}$$

3.4 Connes fusion of implementers

The goal of this section is to use Connes fusion and the unitary intertwiner $\chi : F \boxtimes F \to F$ in order to define a product of certain implementers. Since Connes fusion is only available in an ungraded setting, we need to restrict the discussion to *even* implementers, i.e. to operators $U \in \mathrm{Imp}_L(V) \subset \mathcal{B}(F)$ that preserve the natural grading of F. The collection of all even implementers is, in fact, easy to find. Namely, the group $\mathrm{O}_L(V)$ has two connected components, thus $\mathrm{Imp}_L(V)$ has two connected components as well. All the elements of the identity component of $\mathrm{Imp}_L(V)$ are even, and all the remaining elements are odd [Ara87, Theorem 6.7]. Later, in § 4, we will pull back all this structure to a connected Lie group, and hence anyway only see the connected component of $\mathrm{O}_L(V)$.

We begin by specifying the domain of our Connes fusion product of implementers. It mimics the fusion condition in loop spaces and later turns out to imply that condition (see Theorem 4.3.3).

Definition 3.4.1. A pair (U, U') of even implementers $U, U' \in \mathrm{Imp}^\theta_L(V)$ is called *fusable*, if the elements $g_- \oplus g_+$ and $g'_- \oplus g'_+$ of $\mathrm{O}^\theta_L(V)$ implemented by U and U', respectively, satisfy $g'_- = \tau g_+ \tau$.

Next are some preparative steps.

Lemma 3.4.2. *Let $U \in \mathrm{Imp}^\theta_L(V)$ be even and implement an element $g = g_- \oplus g_+ \in \mathrm{O}^\theta_L(V)$. Then, the Bogoliubov automorphisms $\theta_{g_\pm} : \mathrm{Cl}(V_\pm) \to \mathrm{Cl}(V_\pm)$ extend uniquely to automorphisms of the von Neumann algebras $N_\pm$, induced by $a \mapsto UaU^*$.*

Proof. We discuss θ_{g_-}, the proof for θ_{g_+} is analogous. Because $\mathrm{Cl}(V_-)$ is dense in $N = N_-$, it is sufficient to prove the existence of an extension. We know that $\theta_g : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ restricts to θ_{g_-} (by Lemma 3.1.1) and at the same time extends to an automorphism of $\mathrm{Cl}(V)'' = \mathcal{B}(F)$, namely $a \mapsto UaU^*$. It is thus sufficient to prove that conjugation by U preserves $N \subset \mathrm{Cl}(V)''$. Let $c \in \mathrm{Cl}(V_-)$ and let $b \in \mathrm{Cl}(V_-)'$. Then, because $UcU^* = \theta_{g_-} c \in \mathrm{Cl}(V_-)$, we have

$$U^*bUc = U^*bUcU^*U = U^*UcU^*bU = cU^*bU,$$

and, hence, $U^*bU \in \mathrm{Cl}(V_-)'$. Now let $a \in N$, then we have

$$UaU^*b = UaU^*bUU^* = bUaU^*,$$

and hence $UaU^* \in N$. □

We emphasize that the resulting automorphism $\theta_{g_-} : N \to N$ depends on neither g_+ nor U, as it is the unique extension of a map only depending on g_-. Next we need to explore the relation between the automorphisms $\theta_{g_\pm}$ and the group homomorphism $\tau : \mathrm{O}^\theta_L(V) \to \mathrm{O}^\theta_L(V)$ that exchanges $\mathrm{O}(V_-)$ with $\mathrm{O}(V_+)$. To this end, we consider the orthogonal operator $\alpha\tau : V \to V$ from

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§ 3.3, which is not implementable in the classical sense but as it were implementable by an *anti-unitary* operator. The following lemma summarizes this and further results about $\alpha\tau$ from our earlier work [KW22, Lemmas 4.4 and 4.6, Propositions 4.9 and 4.11].

LEMMA 3.4.3. *The map $\alpha\tau : V \to V$ extends uniquely to a complex anti-linear $*$-automorphism $\kappa : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ of the Clifford algebra. Moreover, the anti-unitary operator $\Lambda_{\alpha\tau} : F \to F$ implements κ in the sense that for all $a \in \mathrm{Cl}(V)$ we have*

$$\kappa(a) = \Lambda_{\alpha\tau} a \Lambda_{\alpha\tau}$$

as elements of $\mathcal{B}(F)$. In particular, $\kappa : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ extends uniquely to an anti-unitary automorphism of $\mathcal{B}(F)$. From there, it restricts to a Lie group homomorphism $\kappa : \mathrm{Imp}_L(V) \to \mathrm{Imp}_L(V)$ covering $\tau : \mathrm{O}_L(V) \to \mathrm{O}_L(V)$.

It follows from Lemmas 3.4.2 and 3.4.3 that $\theta_{\tau(g_+)} : \mathrm{Cl}(V_-) \to \mathrm{Cl}(V_-)$ extends uniquely to an automorphism of N, induced by $a \mapsto \kappa(U)a\kappa(U)^*$. Now we are in position to prove a fundamental relation between implementability, the splitting $V = V_- \oplus V_+$, and the bimodule structure of the Fock space F.

PROPOSITION 3.4.4. *Let $U : F \to F$ be an even unitary map and let $g_\pm \in \mathrm{O}(V_\pm)$. Then, the following are equivalent:*

(a) *U implements $g_- \oplus g_+ \in \mathrm{O}^\theta(V)$;*
(b) *the triple $(\theta_{g_-}, \theta_{\tau(g_+)}, U)$ is a bimodule intertwiner.*

Proof. Suppose that part (a) holds. It then follows from Lemma 3.4.2 that U intertwines the left action along θ_{g_-}:

$$U(a \triangleright v) = \theta_{g_-}(a) \triangleright Uv \quad (a \in N, v \in F).$$

Because U is even, we have that U commutes with the Klein operator k and, thus, by Lemma 3.4.3 and the definition of the operator J in (7), we have $\kappa(U) = JUJ$. Using this, we compute, again for $a \in N$ and $v \in F$

$$(Uv) \triangleleft \theta_{\tau(g_+)}(a) = J\theta_{\tau(g_+)}(a^*)J \triangleright Uv = J\kappa(U)a^* \triangleright \kappa(U)^*JUv = UJa^* \triangleright Jv = U(v \triangleleft a).$$

This shows that the triple $(\theta_{g_-}, \theta_{\tau(g_+)}, U)$ is a bimodule intertwiner.

Now, assume part (b). Running the above arguments in reverse, we obtain for all $a_- \in \mathrm{Cl}(V_-)$ and all $a_+ \in \mathrm{Cl}(V_+)$ the equalities

$$Ua_-U^* = \theta_{g_-}(a_-), \quad Ua_+U^* = \theta_{g_+}(a_+).$$

Thus, we have that for all elements a of the algebraic tensor product of $\mathrm{Cl}(V_-)$ with $\mathrm{Cl}(V_+)$ the equation

$$UaU^* = \theta_{g_- \oplus g_+}(a)$$

holds. Because the algebraic tensor product is dense in the Clifford C*-algebra $\mathrm{Cl}(V_- \oplus V_+) = \mathrm{Cl}(V)$ this equation holds for all $a \in \mathrm{Cl}(V)$, and this completes the proof. □

Consider now two fusable implementers $U, U' \in \mathrm{Imp}_L^\theta(V)$, with U implementing $g_- \oplus g_+ \in \mathrm{O}_L^\theta(V)$ and U' implementing $g'_- \oplus g'_+ \in \mathrm{O}_L^\theta(V)$, where $g'_- = \tau g_+ \tau$. We have three $*$-automorphisms of N, namely θ_{g_-}, $\theta_{\tau(g_+)} = \theta_{g'_-}$, and $\theta_{\tau(g'_+)}$, and, additionally, we have the bimodule intertwiners $(\theta_{g_-}, \theta_{\tau(g_+)}, U)$ and $(\theta_{g'_-}, \theta_{\tau(g'_+)}, U')$ according to Proposition 3.4.4. Since Connes fusion is a functor (Proposition A.2.3) we obtain a bimodule intertwiner $(\theta_{g_-}, \theta_{\tau(g'_+)}, U \boxtimes U') : F \boxtimes F \to F \boxtimes F$.

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Definition 3.4.5. The *Connes fusion of fusable implementers U and U'* is the unitary $\hat{\mu}(U,U') \in \mathrm{U}(F)$ defined as the following composite.

$$F \xrightarrow{\chi^{-1}} F \boxtimes F \xrightarrow{U \boxtimes U'} F \boxtimes F \xrightarrow{\chi} F$$

By construction, the triple $(\theta_{g_-}, \theta_{\tau(g'_+)}, \hat{\mu}(U,U'))$ is an intertwiner. Moreover, we have the following.

Lemma 3.4.6. *The Connes fusion $\hat{\mu}(U,U') \in \mathrm{U}(F)$ of an even operator U implementing $g_- \oplus g_+$ and an even operator U' implementing $\tau g_+ \tau \oplus g'_+$ is an even operator and implements $g_- \oplus g'_+ \in \mathrm{O}^\theta_L(V)$. In particular, $\hat{\mu}(U,U') \in \mathrm{Imp}^\theta_L(V)$.*

Proof. It suffices to show that $\hat{\mu}(U,U')$ is even, the rest follows then from Proposition 3.4.4. In order to show that $\hat{\mu}(U,U')$ is even, we consider the grading involution $\mathrm{g} := \mathrm{k}^2 : F \to F$, which has the property that an operator on F is even if and only if it commutes with g. We also consider the grading involution $\nu : N \to N$ on the von Neumann algebra N, which implements the extension of the canonical grading of the Clifford algebra $\mathrm{Cl}(V_-)$. It is straightforward to check that the triple (ν, ν, g) is an intertwiner of N–N-bimodules. Therefore, functoriality of Connes fusion determines an operator $\mathrm{g} \boxtimes \mathrm{g} : F \boxtimes F \to F \boxtimes F$. Next we compute for $x \otimes v \in \mathcal{D}(F, \Omega) \otimes F$ using the formula of Proposition A.2.3 that

$$(\mathrm{g} \boxtimes \mathrm{g})(x \otimes v) = (\mathrm{g} \circ x \circ \overline{\nu}^*) \otimes \mathrm{g}(v).$$

Observe that we have used that the map u used in Proposition A.2.3 is the identity. Now, we apply χ, and calculate

$$\chi((\mathrm{g} \boxtimes \mathrm{g})(x \otimes v)) = \chi((\mathrm{g} \circ x \circ \overline{\nu}^*) \otimes \mathrm{g}(v)) = p_\Omega(u \circ \mathrm{g} \circ x \circ \overline{\nu}^*) \triangleright \mathrm{g}(v);$$

here, u is the invertible intertwiner of Remark 3.3.6, which is even. We recall that $p_\Omega(u \circ \mathrm{g} \circ x \circ \overline{\nu}^*)$ is the unique element of N that satisfies $p_\Omega(u \circ \mathrm{g} \circ x \circ \overline{\nu}^*) \triangleright w = u \circ \mathrm{g} \circ x \circ \overline{\nu}^*(w)$ for all $w \in L^2_\Omega(N)$. We then continue, for arbitrary $w \in L^2_\Omega(N)$,

$$u \circ \mathrm{g} \circ x \circ \overline{\nu}^*(w) = \overline{\nu} \circ u \circ x \circ \overline{\nu}^*(w) = \overline{\nu}(p_\Omega(u \circ x) \triangleright \overline{\nu}^*(w)) = \nu(p_\Omega(u \circ x)) \triangleright w,$$

whence $p_\Omega(u \circ \mathrm{g} \circ x \circ \overline{\nu}^*) = \nu(p_\Omega(u \circ x))$. We continue where we left off, and compute

$$p_\Omega(u \circ \mathrm{g} \circ x \circ \overline{\nu}^*) \triangleright \mathrm{g}(v) = \nu(p_\Omega(u \circ x)) \triangleright \mathrm{g}(v) = \mathrm{g}(p_\Omega(u \circ x) \triangleright v) = \mathrm{g}(\chi(x \otimes v)),$$

obtaining

$$\chi \circ (\mathrm{g} \boxtimes \mathrm{g}) = \mathrm{g} \circ \chi.$$

Using this identity twice together with the functoriality of Connes fusion (Proposition A.2.3) and the fact that U and U' are even and, hence, commute with g, we obtain the identity

$$\chi \circ (U \boxtimes U') \circ \chi^{-1} \circ \mathrm{g} = \mathrm{g} \circ \chi \circ (U \boxtimes U') \circ \chi^{-1};$$

this finishes the proof. □

The following result follows directly from the functoriality of Connes fusion (see Proposition A.2.3).

Proposition 3.4.7. *Connes fusion $\hat{\mu}$ of implementers is multiplicative in the following sense. Let $U, U', V, V' \in \mathrm{Imp}^\theta_L(V)$ be even implementers, of which U and U' are fusable, and V and V' are fusable. Then, UV and $U'V'$ are fusable, and we have*

$$\hat{\mu}(U,U')\hat{\mu}(V,V') = \hat{\mu}(UV, U'V').$$

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In § 4.3 we further study the Connes fusion of implementers. In particular, we relate it to the fusion product on the universal central extension of $L\,\mathrm{Spin}(d)$, see Theorem 4.3.3. This relation will be crucial for the proof of our main result, Theorem 5.3.1.

4. Fusive spin structures on loop space

In this section, we recall and relate the notions of spin structures on loop space, fusion products, and string structures. The relation between loop fusion and the Connes fusion of implementers is established in Theorem 4.3.3.

4.1 The spinor bundle on loop space

At the basis of our work is the notion of a spin structure on loop space according to Killingback [Kil87]. We suppose M is a spin manifold of dimension d, and consider its spin structure as a principal $\mathrm{Spin}(d)$-bundle $\mathrm{Spin}(M)$ over M that lifts the structure group of the oriented orthonormal frame bundle $\mathrm{SO}(M)$ along the central extension $\mathbb{Z}_2 \to \mathrm{Spin}(d) \to \mathrm{SO}(d)$.

By definition of the Fréchet manifold structure on LM, a tangent vector at a loop $\gamma \in LM$ is a smooth section of TM along γ. This is the motivation to consider $L\,\mathrm{Spin}(M)$, the Fréchet manifold of smooth free loops in the total space of $\mathrm{Spin}(M)$, as (a version of) the frame bundle of LM. We note that $L\,\mathrm{Spin}(M)$ is a Fréchet principal $L\,\mathrm{Spin}(d)$-bundle over LM, see [SW07, Proposition 1.8], where $L\,\mathrm{Spin}(d)$ is the loop group of $\mathrm{Spin}(d)$. The following definition is due to Killingback [Kil87].

DEFINITION 4.1.1. A *spin structure on* LM is a lift of the structure group of the frame bundle $L\,\mathrm{Spin}(M)$ of LM along the basic central extension

$$\mathrm{U}(1) \to \widetilde{L\,\mathrm{Spin}}(d) \to L\,\mathrm{Spin}(d).$$

Thus, a spin structure on LM is a principal $\widetilde{L\,\mathrm{Spin}}(d)$-bundle $\widetilde{L\,\mathrm{Spin}}(M)$ over LM together with a bundle map $\sigma : \widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M)$ that intertwines the group actions along the projection $\widetilde{L\,\mathrm{Spin}}(d) \to L\,\mathrm{Spin}(d)$. We remark that the map $\sigma : \widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M)$ has itself the structure of a principal $\mathrm{U}(1)$-bundle, where the $\mathrm{U}(1)$-action is the one of the central subgroup. Concerning the existence of spin structures on loop spaces, the following result was proved by McLaughlin [McL92].

PROPOSITION 4.1.2. *The loop space LM of a spin manifold M admits a spin structure if* $\frac{1}{2}p_1(M) = 0$.

The group $L\,\mathrm{SO}(d)$ acts in $V = L^2(S^1, \mathbb{S} \otimes \mathbb{C}^d)$ by pointwise multiplication; under the pointwise projection map $L\,\mathrm{Spin}(d) \to L\,\mathrm{SO}(d)$, this defines an action of $L\,\mathrm{Spin}(d)$ in V. It is clear that $L\,\mathrm{Spin}(d)$ acts on V through $\mathrm{O}^\theta(V)$, and a standard result that it acts through $\mathrm{O}_L(V)$ (see [PS03, Proposition 6.3.1]; also see [KW22, Lemma 3.21]). Moreover, the resulting map

$$\omega : L\,\mathrm{Spin}(d) \to \mathrm{O}_L^\theta(V)$$

is smooth, [KW22, Lemma 3.22]. The following result is [KW22, Theorem 3.26].

THEOREM 4.1.3. *If $1 < d \neq 4$, then the pullback of the central extension* $\mathrm{U}(1) \to \mathrm{Imp}_L^\theta(V) \xrightarrow{q} \mathrm{O}_L^\theta(V)$ *along* $\omega : L\,\mathrm{Spin}(d) \to \mathrm{O}_L^\theta(V)$ *is the basic central extension of* $L\,\mathrm{Spin}(d)$.

From now on we will use this fixed model for the basic central extension. Most importantly, it comes with a smooth map $\tilde{\omega} : \widetilde{L\,\mathrm{Spin}}(d) \to \mathrm{Imp}_L^\theta(V)$. It pulls back the smooth representation

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of $\mathrm{Imp}^{\theta}_{L}(V)$ on Fock space F^{s} (Proposition 3.2.2) to a smooth representation of the basic central extension on F^{s}. The spinor bundle on loop space is obtained using the associated bundle construction of Lemma 2.2.2.

Definition 4.1.4. Let M be a spin manifold equipped with a spin structure $\widetilde{L\,\mathrm{Spin}}(M)$ on its loop space. The *spinor bundle on loop space* is the associated rigged Hilbert space bundle

$$F^{\mathrm{s}}(LM) \coloneqq \big(\widetilde{L\,\mathrm{Spin}}(M) \times F^{\mathrm{s}}\big)/\widetilde{L\,\mathrm{Spin}}(d).$$

The Clifford bundle is obtained in a similar way using Lemma 2.2.4 and the smooth representation of $L\,\mathrm{Spin}(d)$ on $\mathrm{Cl}(V)^{\mathrm{s}}$, induced via $\omega : L\,\mathrm{Spin}(d) \to \mathrm{O}(V)$ from the smooth representation of $\mathrm{O}(V)$ on $\mathrm{Cl}(V)^{\mathrm{s}}$ by Bogoliubov automorphisms (Proposition 3.2.4).

Definition 4.1.5. The *Clifford bundle on loop space* is the associated rigged C*-algebra bundle

$$\mathrm{Cl}^{\mathrm{s}}(LM) \coloneqq (L\,\mathrm{Spin}(M) \times \mathrm{Cl}(V)^{\mathrm{s}})/L\,\mathrm{Spin}(d).$$

Remark 4.1.6. The Clifford bundle on loop space can even be defined without a spin structure on M, since ω factors through $L\,\mathrm{SO}(d)$.

The Clifford bundle $\mathrm{Cl}^{\mathrm{s}}(LM)$ acts on the spinor bundle $F^{\mathrm{s}}(LM)$ by 'Clifford multiplication'. To make this precise, we consider the rigged $\mathrm{Cl}(V)^{\mathrm{s}}$-module F^{s} of Proposition 3.2.5, with representation $(a, v) \mapsto a \triangleright v$. We have proved in [KW20, Proposition 2.2.19] that this representation extends from the typical fibres to all fibres, resulting in the following statement.

Proposition 4.1.7. *There is a unique bundle map*

$$\mathrm{Cl}^{\mathrm{s}}(LM) \times_{LM} F^{\mathrm{s}}(LM) \to F^{\mathrm{s}}(LM)$$

such that

$$([\varphi, a], [\tilde{\varphi}, v]) \mapsto [\tilde{\varphi}, a \triangleright v],$$

for all $\tilde{\varphi} \in \widetilde{L\,\mathrm{Spin}}(M)$ lifting $\varphi \in L\,\mathrm{Spin}(M)$, and all $a \in \mathrm{Cl}(V)^{\mathrm{s}}$ and $v \in F^{\mathrm{s}}$. Moreover, this map equips the spinor bundle $F^{\mathrm{s}}(LM)$ with the structure of a rigged $\mathrm{Cl}^{\mathrm{s}}(LM)$-module bundle with typical fibre F^{s}.

As the typical fibre of the rigged $\mathrm{Cl}^{\mathrm{s}}(LM)$-module bundle $F^{\mathrm{s}}(LM)$ is, in fact, a rigged von Neumann algebra, $\mathrm{Cl}_{\mathrm{vN}}(V)^{\mathrm{s}} = (\mathrm{Cl}(V)^{\mathrm{s}}, F^{s})$ (Proposition 3.2.5), we note immediately the following consequence of Proposition 4.1.7.

Corollary 4.1.8. *The pair $\mathrm{Cl}^{\mathrm{s}}_{\mathrm{vN}}(LM) \coloneqq (\mathrm{Cl}^{\mathrm{s}}(LM), F^{\mathrm{s}}(LM))$ is a rigged von Neumann algebra bundle over LM with typical fibre $\mathrm{Cl}_{\mathrm{vN}}(V)^{\mathrm{s}}$.*

Remark 4.1.9. Let us comment on the relation between the spinor bundle on loop space defined in our earlier work using coarser riggings F^{∞} of Fock spaces and $\mathrm{Cl}(V)^{\infty}$ of Clifford algebras [KW20, § 4]. As remarked in Remark 3.2.6, we have $F^{\infty} \subset F^{\mathrm{s}}$, in other words, there is an isometric morphism of rigged Hilbert spaces $F^{\infty} \to F^{\mathrm{s}}$, which induces the identity on the completion F; and similarly for the rigged Clifford algebras. Comparing Definitions 4.1.4 and 4.1.5 with [KW20, Definitions 4.4 and 4.3], respectively, we observe that the above-mentioned isometric morphisms induce isometric morphisms on the level of the spinor bundle and the Clifford algebra bundle on loop space:

$$F^{\infty}(LM) \to F^{\mathrm{s}}(LM) \qquad \mathrm{Cl}^{\infty}(LM) \to \mathrm{Cl}^{\mathrm{s}}(LM).$$

Moreover, this pair of isometric morphisms, in fact, is an isometric intertwiner as in Definition 2.2.6. On the completions to continuous Hilbert space bundles and continuous C*-algebra-bundles (see § 2.2), these isometric intertwiners induce the identity maps.

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4.2 Fusion on loop space

We review first the general notion of a fusion product for principal U(1)-bundles over the loop space LM of a smooth manifold M, following [Wal16b]. In the subsequent subsections we apply this to two situations: central extensions of loop groups and spin structures on loop spaces. We write PM for the set of smooth paths in M with sitting instants, i.e.

$$PM := \{\beta : [0,\pi] \to M \mid \beta \text{ is smooth and constant around } 0 \text{ and } \pi\}. \tag{11}$$

We use sitting instants so that we are able to concatenate arbitrary paths with a common end point: the usual path concatenation $\beta_2 \star \beta_1$ is again a smooth path whenever $\beta_1(\pi) = \beta_2(0)$. Unfortunately, with sitting instants, PM is not any kind of manifold; however, we may regard it as a diffeological space, see § 2.3 for a quick review. The plots of PM are all maps $c : U \to PM$ whose adjoint map $c^\vee : U \times [0,\pi] \to M$, with $c^\vee(u,t) := c(u)(t)$, is smooth. We remark that path concatenation $\star$ and path reversal $\beta \mapsto \bar{\beta}$ are smooth maps. The evaluation map $\mathrm{ev} : PM \to M \times M, \beta \mapsto (\beta(0), \beta(\pi))$ is a smooth map, and since diffeological spaces admit arbitrary fibre products, the iterated fibre products $PM^{[k]} := PM \times_{M\times M} \cdots \times_{M \times M} PM$ are again diffeological spaces: their plots are simply tuples $(c_1, \dots, c_k)$ of plots of PM, such that $\mathrm{ev} \circ c_1 = \dots = \mathrm{ev} \circ c_k$. In the following, we will use the smooth map

$$PM^{[2]} \to LM, (\beta_1, \beta_2) \mapsto \beta_1 \cup \beta_2 := \bar{\beta}_2 \star \beta_1$$

that combines two paths with a common initial point and a common end point to a loop. If $(\beta_1, \beta_2, \beta_3) \in PM^{[3]}$, we will regard the loop $\beta_1 \cup \beta_3$ as the ‘fusion’ of the loops $\beta_1 \cup \beta_2$ and $\beta_2 \cup \beta_3$. A fusion product on a principal U(1)-bundle over LM is now a lift of this fusion operation to the total space.

DEFINITION 4.2.1. Let $\pi : \mathcal{L} \to LM$ be a Fréchet principal U(1)-bundle over LM. A *fusion product* on $\mathcal{L}$ assigns to each element $(\beta_1, \beta_2, \beta_3) \in PM^{[3]}$ a U(1)-bilinear map

$$\lambda_{\beta_1,\beta_2,\beta_3} : \mathcal{L}_{\beta_1\cup\beta_2} \times \mathcal{L}_{\beta_2\cup\beta_3} \to \mathcal{L}_{\beta_1\cup\beta_3},$$

such that the following two conditions are satisfied.

(i) Associativity: for all $(\beta_1, \beta_2, \beta_3, \beta_4) \in PM^{[4]}$ and all $q_{ij} \in \mathcal{L}_{\beta_i\cup\beta_j}$,

$$\lambda_{\beta_1,\beta_3,\beta_4}(\lambda_{\beta_1,\beta_2,\beta_3}(q_{12}, q_{23}), q_{34}) = \lambda_{\beta_1,\beta_2,\beta_4}(q_{12}, \lambda_{\beta_2,\beta_3,\beta_4}(q_{23}, q_{34})).$$

(ii) Smoothness: for every plot $(c_1, c_2, c_3) : U \to PM^{[3]}$ and all smooth maps $c_{12}, c_{23} : U \to \mathcal{L}$ such that $(\pi \circ c_{ij})(x) = c_i(x) \cup c_j(x)$ for all $x \in U$ and $ij \in \{12, 23\}$, the map

$$U \to \mathcal{L} : x \mapsto \lambda_{c_1(x),c_2(x),c_3(x)}(c_{12}(x), c_{23}(x))$$

is smooth.

Early versions of fusion products have been studied in [Bry93] and [ST05]. For a comprehensive treatment of this topic we refer to [Wal16b]. Fusion products are a characteristic feature of structure in the image of transgression. The main result of [Wal16b] is that principal U(1)-bundles over LM equipped with a fusion product (and a so-called thin homotopy equivariant structure) form a category that is equivalent to the category of bundle gerbes over M, with the equivalence established by transgression and regression functors. In particular, every principal U(1)-bundle over LM obtained by transgression of a bundle gerbe over M comes equipped with a fusion product.

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4.3 Fusion in the basic central extension

The first occurrence of a fusion product in this paper is when the smooth manifold M is a Lie group G, and the principal U(1)-bundle $\mathcal{L}$ is a central extension $\mathrm{U}(1) \to \mathcal{L} \to LG$ of the loop group. The diffeological space PG of paths with sitting instants becomes now a diffeological group, i.e. (pointwise) multiplication and (pointwise) inversion are smooth maps; further, path concatenation and path reversal are smooth group homomorphisms. We require compatibility between fusion products and these group structures in the following sense.

Definition 4.3.1. A fusion product μ on a central extension $\mathrm{U}(1) \to \mathcal{L} \to LG$ is called *multiplicative*, if it is a group homomorphism, i.e.

$$\mu_{\beta_1,\beta_2,\beta_3}(q_{12}, q_{23}) \cdot \mu_{\beta_1',\beta_2',\beta_3'}(q_{12}', q_{23}') = \mu_{\beta_1\beta_1',\beta_2\beta_2',\beta_3\beta_3'}(q_{12}q_{12}', q_{23}q_{23}')$$

for all $(\beta_1, \beta_2, \beta_3), (\beta_1', \beta_2', \beta_3') \in PG^{[3]}$, $q_{ij} \in \mathcal{L}_{\beta_i \cup \beta_j}$, and $q_{ij}' \in \mathcal{L}_{\beta_i' \cup \beta_j'}$.

The basic central extension of any compact simple Lie group can be obtained by transgression, and thus comes equipped with a fusion product, which is automatically multiplicative and unique up to isomorphism [Wal17]. In this paper, we need a precise formula for this fusion product μ on the basic central extension of $L\,\mathrm{Spin}(d)$. Within the operator-theoretic model for $\widetilde{L\,\mathrm{Spin}}(d)$ described above, such a formula has been obtained in [KW22], and we shall recall this.

A key ingredient to this work will be a relation between that fusion product μ and the Connes fusion of implementers defined in § 3.4.

We first note that the map $\omega : L\,\mathrm{Spin}(d) \to \mathrm{O}_L^\theta(V)$ that defines our operator-theoretic model via pullback, is compatible with all relations between loops and paths in $\mathrm{Spin}(d)$. To make this more precise, let $p_\pm : \mathrm{O}_L^\theta(V) \to \mathrm{O}(V_\pm)$ denote the projections. Further, let $\Delta : P\,\mathrm{Spin}(d) \to L\,\mathrm{Spin}(d), \beta \mapsto \beta \cup \beta$ be the doubling map. We define smooth maps $\omega_\pm : P\,\mathrm{Spin}(d) \to \mathrm{O}(V_\pm)$ by $\omega_\pm(\beta) := p_\pm(\omega(\Delta(\beta)))$. Since the action of $L\,\mathrm{Spin}(d)$ on V is pointwise, we have the following result.

Lemma 4.3.2. *The following diagram is commutative.*

$$\begin{tikzcd}
P\,\mathrm{Spin}(d) \arrow[dd, "\omega_+"'] & P\,\mathrm{Spin}(d)^{[2]} \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\cup"] & P\,\mathrm{Spin}(d) \arrow[dd, "\omega_-"] \\
 & L\,\mathrm{Spin}(d) \arrow[d, "\omega"] & \\
\mathrm{O}(V_+) & \mathrm{O}_L^\theta(V) \arrow[l, "p_+"] \arrow[r, "p_-"'] & \mathrm{O}(V_-)
\end{tikzcd}$$

Next, we recall from [KW22, Definition 5.5] that a *fusion factorization* for $\widetilde{L\,\mathrm{Spin}}(d)$ is a smooth group homomorphism $\rho : P\,\mathrm{Spin}(d) \to \widetilde{L\,\mathrm{Spin}}(d)$ such that the following diagram commutes.

$$\begin{tikzcd}
 & \widetilde{L\,\mathrm{Spin}}(d) \arrow[d] \\
P\,\mathrm{Spin}(d) \arrow[ur, "\rho"] \arrow[r, "\Delta"'] & L\,\mathrm{Spin}(d)
\end{tikzcd}$$

Such a fusion factorization was constructed in [KW22, § 5.3]; it is uniquely characterized by the property that $\rho(\beta)$ satisfies $\tilde{\omega}(\rho(\beta))J = J\tilde{\omega}(\rho(\beta))$ and $P_\Omega = P_{\tilde{\omega}(\rho(\beta))\Omega}$, where

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$\tilde{\omega} : \widetilde{L\,\mathrm{Spin}}(d) \to \mathrm{Imp}_L^\theta(V)$ is the map from § 4.1, J is the modular conjugation (see (7)) and P_v is the closed self-dual cone corresponding to a vector $v \in F$ (see Appendix A.1). Any fusion factorization induces a multiplicative fusion product [KW22, Theorem 5.6]. In the present case, ρ induces the fusion product μ on $\widetilde{L\,\mathrm{Spin}}(d)$:

$$\mu_{\beta_1,\beta_2,\beta_3}(g_1,g_2) := g_1\rho(\beta_2)^{-1}g_2,$$

where $(\beta_1,\beta_2,\beta_3) \in P\,\mathrm{Spin}(d)^{[3]}$, $g_1 \in \widetilde{L\,\mathrm{Spin}}(d)_{\beta_1\cup\beta_2}$, and $g_2 \in \widetilde{L\,\mathrm{Spin}}(d)_{\beta_2\cup\beta_3}$.

The following key result now tells us that the fusion product μ and the Connes fusion map $\hat{\mu}$ of § 3.4 coincide under the group homomorphism $\tilde{\omega} : \widetilde{L\,\mathrm{Spin}}(d) \to \mathrm{Imp}_L^\theta(V)$.

THEOREM 4.3.3. *Let $\beta = (\beta_1,\beta_2,\beta_3) \in P\,\mathrm{Spin}(d)^{[3]}$, $g_1 \in \widetilde{L\,\mathrm{Spin}}(d)_{\beta_1\cup\beta_2}$ and $g_2 \in \widetilde{L\,\mathrm{Spin}}(d)_{\beta_2\cup\beta_3}$. Then, the implementers $\tilde{\omega}(g_2)$ and $\tilde{\omega}(g_1)$ are fusable in the sense of Definition 3.4.1. Moreover, $\tilde{\omega}$ exchanges fusion on the basic central extension with the Connes fusion of implementers, i.e.*

$$\tilde{\omega}(\mu_{\beta_1,\beta_2,\beta_3}(g_1,g_2)) = \hat{\mu}(\tilde{\omega}(g_2),\tilde{\omega}(g_1)).$$

Proof. Since the map $\omega : L\,\mathrm{Spin}(d) \to \mathrm{O}_L(V)$ factors through $L\,\mathrm{SO}(d)$, its image is contained in the connected component of the identity of $\mathrm{O}_L^\theta(V)_0$. Thus, all implementers in the image of $\tilde{\omega}$ are even, which is a prerequisite for being fusable (Definition 3.4.1). For brevity, we set $U_i := \tilde{\omega}(g_i)$ for $i = 1,2$. According to Lemma 4.3.2, U_1 implements $\omega(\beta_1\cup\beta_2) = \omega_-(\beta_2)\oplus\omega_+(\beta_1)$, and U_2 implements $\omega(\beta_2\cup\beta_3) = \omega_-(\beta_3)\oplus\omega_+(\beta_2)$, and we have $\tau(\omega_+(\beta_2)) = \omega_-(\beta_2)$. This shows that the pair (U_2,U_1) is fusable in the sense of Definition 3.4.1.

Let $K := \tilde{\omega}(\rho(\beta_2))^{-1}$, so that

$$\tilde{\omega}(\mu_{\beta_1,\beta_2,\beta_3}(g_1,g_2)) = U_1KU_2. \tag{12}$$

We compute, using the multiplicativity of $\hat{\mu}$ (Proposition 3.4.7),

$$\hat{\mu}(U_2,U_1) = \hat{\mu}(U_2KK^{-1},U_1KK^{-1}) = \hat{\mu}(U_2K,U_1K)\hat{\mu}(K^{-1},K^{-1}).$$

We claim that

$$\hat{\mu}(U_2K,U_1K) = U_1KU_2K \quad \text{and} \quad \hat{\mu}(K^{-1},K^{-1}) = K^{-1};$$

this proves the theorem, in view of (12). The element K has the property that U_2K implements an operator of the form $g_-\oplus\mathbb{1} \in \mathrm{O}_L^\theta(V)$ (see [KW22, Equation (13)]), which implies that $U_2K \in (\mathrm{Cl}(V_+)'')'$, see Proposition 3.4.4. Moreover, because U_2K is even, this implies that $U_2K \in \mathrm{Cl}(V_-)''$. Similarly, one shows that $U_1K \in \mathrm{Cl}(V_+)''$ and, thus, $U_1KU_2K = U_2KU_1K$. Let $f : L^2_\Omega(\mathrm{Cl}(V_-)'') \to F$ be the inverse of the isomorphism u defined in Remark 3.3.6, given by $a \mapsto a \triangleright \Omega$. Let $v \in F$; then, to determine $\hat{\mu}(U_2K,U_1K)$ we compute

$$\begin{aligned}\chi((U_2K\boxtimes U_1K)\chi^{-1}v) &= \chi((U_2K\boxtimes U_1K)(f\otimes v)) = \chi(U_2Kf\otimes U_1K(v))\\ &= p_\Omega(uU_2Kf)\triangleright U_1K(v)\end{aligned}$$

using the definition of χ in (10) and the formula in Proposition A.2.3, with $\nu_2 = \mathbb{1}_N$ and $u = \mathbb{1}$. We then use the defining property of p_Ω, getting

$$p_\Omega(uU_2Kf)\triangleright w = uU_2Kfw = U_2K\triangleright w$$

for all $w \in L^2_\Omega(\mathrm{Cl}(V_-)'')$ and, thus,

$$\chi((U_2K\boxtimes U_1K)\chi^{-1}v) = U_2KU_1K(v) = U_1KU_2K(v).$$

This proves the first equation. Because of the fact that K^{-1} has the property that $JK^{-1} = K^{-1}J$ and $P_{K^{-1}\Omega} = P_\Omega$ (see [KW22, Lemma 5.16]), we see that the corresponding map

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$L^2_\Omega(\mathrm{Cl}(V_-)'') \to L^2_\Omega(\mathrm{Cl}(V_-)'')$ is $f^{-1}K^{-1}f$. We compute

$$(K^{-1} \boxtimes K^{-1})(f \otimes v) = K^{-1}ff^{-1}Kf \otimes K^{-1}v = f \otimes K^{-1}v;$$

this proves the second equation. □

Remark 4.3.4. Theorem 4.3.3 establishes a new relation between loop fusion and Connes fusion. It might fit into a larger framework, together with fusion of positive energy representations of loop groups established by Wassermann and Toledano-Laredo [Was98, Tol97], but we currently do not understand the full picture.

4.4 Fusive spin structures and string structures

We return to the discussion of spin structures on the loop space of a spin manifold M. We recall that a spin structure $\widetilde{L\,\mathrm{Spin}}(M)$ on LM can be viewed as a principal $\mathrm{U}(1)$-bundle over $L\,\mathrm{Spin}(M)$ and thus, in particular, over a loop space. As such, $\widetilde{L\,\mathrm{Spin}}(M)$ may host fusion products. The following definition is [Wal16a, Definition 3.6].

Definition 4.4.1. A *fusion product* on a spin structure $\widetilde{L\,\mathrm{Spin}}(M)$ on LM is a fusion product λ on the principal $\mathrm{U}(1)$-bundle $\widetilde{L\,\mathrm{Spin}}(M)$ over $L\,\mathrm{Spin}(M)$ that is equivariant with respect to the fusion product μ on $\widetilde{L\,\mathrm{Spin}}(d)$ under the principal action, i.e.

$$\lambda_{\beta_1\gamma_1,\beta_2\gamma_2,\beta_3\gamma_3}(\tilde{\varphi}_{12} \cdot g_{12} \otimes \tilde{\varphi}_{23} \cdot g_{23}) = \lambda_{\beta_1,\beta_2,\beta_3}(\tilde{\varphi}_{12} \otimes \tilde{\varphi}_{23}) \cdot \mu_{\gamma_1,\gamma_2,\gamma_3}(g_{12} \otimes g_{23}) \tag{13}$$

for all $(\beta_1,\beta_2,\beta_3) \in P\,\mathrm{Spin}(M)^{[3]}$, $(\gamma_1,\gamma_2,\gamma_3) \in P\,\mathrm{Spin}(d)^{[3]}$, $\tilde{\varphi}_{ij} \in \widetilde{L\,\mathrm{Spin}}(M)_{\beta_i \cup \beta_j}$, and $g_{ij} \in \widetilde{L\,\mathrm{Spin}}(d)_{\gamma_i \cup \gamma_j}$. A spin structure on LM with a fusion product is called a *fusive spin structure*.

Fusive spin structures bring us one step forward on the way from spin structures on LM to string structures on M. They already fix the missing 'only if' part of Proposition 4.1.2, as shown in [Wal16a, Theorem 1.4]: LM admits a fusive spin structure if and only if $\frac{1}{2}p_1(M) = 0$. A full string structure yet contains more information related to thin homotopy equivariance; however, this information is not needed for the construction of a fusion product on the spinor bundle on loop space, which we perform in § 5. There, we only need a fusive spin structure.

In the following we want to explain how a fusive spin structure on LM, the main ingredient of our construction in § 5, can be obtained from a (geometric) string structure on M. There are essentially four different, but equivalent, ways to say what a string structure on a spin manifold M is. All four versions have in common that their existence is obstructed by $\frac{1}{2}p_1(M) \in H^4(M,\mathbb{Z})$, and that, under appropriate notions of equivalence, they form a torsor over the group $H^3(M,\mathbb{Z})$.

(1) In purely topological terms, a string structure is a lift of the structure group of the spin frame bundle $\mathrm{Spin}(M)$ to the 3-connected covering group of $\mathrm{Spin}(d)$ (see [ST04]). That covering group, the *topological string group*, does not allow (finite-dimensional) Lie group structures (it has cohomology in infinitely many degrees).

(2) In terms of higher-categorical structures, a string structure is a lift of the structure group of the spin frame bundle $\mathrm{Spin}(M)$ along the central extension of Lie 2-groups,

$$B\,\mathrm{U}(1) \to \mathrm{String}(d) \to \mathrm{Spin}(d),$$

where $B\,\mathrm{U}(1)$ denotes the Lie 2-group with a single object, $\mathrm{Spin}(d)$ is considered as a Lie 2-group with only identity morphisms, and $\mathrm{String}(d)$ is, for instance, the *String Lie 2-group* constructed in [BSCS07] (strict, but infinite-dimensional) or in [Sch11] (finite-dimensional, but not strict), or in [Wal12a] (again strict, and with diffeological spaces). Geometric realization establishes the relation with part (1), see [BS09, NW13a].

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(3) In terms of bundle gerbes, a string structure is a trivialization of the *Chern–Simons 2-gerbe* over M (see [Wal13]). We explain below some more details about this approach. Its equivalence with part (2) was established in [NW13b, Theorem 7.9].
(4) In terms of loop space geometry, a string structure is a thin homotopy equivariant fusive spin structure on LM; this definition and its equivalence with part (3) is in [Wal15]; we will recall it below.

Now we want to describe some details about part (3) in condensed form. A trivialization of the Chern–Simons 2-gerbe (i.e. a string structure) is a triple $(\mathcal{S}, \mathcal{A}, \sigma)$ consisting of the following structure.

– A bundle gerbe $\mathcal{S}$ in the sense of Murray [Mur96] over $\mathrm{Spin}(M)$.
– A bundle gerbe isomorphism

$$\mathcal{A} : \delta^*\mathcal{G}_{bas} \otimes \mathrm{pr}_2^*\,\mathcal{S} \to \mathrm{pr}_1^*\,\mathcal{S}$$

over the double fibre product $\mathrm{Spin}(M)^{[2]}$ over M, where $\mathcal{G}_{bas}$ is the *basic gerbe* over $\mathrm{Spin}(d)$ constructed by Meinrenken [Mei03] and Gawedzki and Reis [Gaw88], and $\delta(\varphi, \varphi') \in \mathrm{Spin}(d)$ is defined by $\varphi\delta(p, p') = \varphi'$ for frames $\varphi, \varphi' \in L\,\mathrm{Spin}(M)$ at the same point.
– A certain 2-isomorphism σ over the triple fibre product, expressing a compatibility condition between $\mathcal{A}$ and the multiplicativity of $\mathcal{G}_{bas}$. For the details we refer to [Wal13].

A major advantage of this notion of a string structure is that it allows a differential refinement by *string connections*. A string connection consists of a connection on the bundle gerbe $\mathcal{S}$, such that the bundle gerbe morphism $\mathcal{A}$ is connection-preserving (the basic gerbe $\mathcal{G}_{bas}$ has a canonical connection). In the present context, string connections are useful for establishing the relation between parts (3) and (4), which we explain next. String connections always exist, and the space of string connections relative to a fixed string structure is affine [Wal13, Theorem 1.3.4].

The key technique is the earlier-mentioned transgression functor

$$\mathscr{T} : \mathrm{h}_1\mathcal{G}rb^{\nabla}(M) \to \mathcal{F}us\mathcal{B}un(LM)$$

from the homotopy 1-category of the bicategory of bundle gerbes with connection over M to the category of principal U(1)-bundles over LM equipped with fusion products, in the sense of Definition 4.2.1. Versions of this functor have been described in [Gaw88, Bry93], the complete construction is in [Wal16b]. As a prerequisite, we apply the transgression functor to the basic gerbe $\mathcal{G}_{bas}$ over $\mathrm{Spin}(d)$, and obtain a principal U(1)-bundle $\mathscr{T}(\mathcal{G}_{bas})$ with fusion product over the loop group $L\,\mathrm{Spin}(d)$. Functoriality allows to transgress multiplicativity; thus, what we really obtain is a central extension with a multiplicative fusion product. In fact, it is the basic central extension [Wal10], and we have proved in [KW22, Theorem 6.4] that there is even a canonical, fusion-preserving isomorphism

$$\mathscr{T}(\mathcal{G}_{bas}) \cong \widetilde{L\,\mathrm{Spin}}(d) \tag{14}$$

to our operator-theoretic model for the basic central extension of Theorem 4.1.3, equipped with the fusion product μ of § 4.3. Next, we apply transgression to a string structure $(\mathcal{S}, \mathcal{A}, \sigma)$ with a connection. The bundle gerbe $\mathcal{S}$ transgresses to a principal U(1)-bundle $S := \mathscr{T}(\mathcal{S})$ over $L\,\mathrm{Spin}(M)$ equipped with a fusion product. Taking the isomorphism (14) into account, the bundle gerbe morphism $\mathcal{A}$ transgresses to a fusion-preserving bundle morphism

$$\mathscr{T}(\mathcal{A}) : \delta^*\widetilde{L\,\mathrm{Spin}}(d) \otimes \mathrm{pr}_2^*\,S \to \mathrm{pr}_1^*\,S,$$

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from which one can extract an $\widetilde{L\,\mathrm{Spin}}(d)$-action on S turning S into a principal $\widetilde{L\,\mathrm{Spin}}(d)$-bundle over M. One can then show that S is a spin structure on LM, and further, that the fusion product on S turns it into a fusive spin structure. For the details, see [Wal16a].

Summarizing, a geometric string structure on M (i.e. a string structure with a string connection) induces in a canonical way a fusive spin structure on LM. Thus, our construction of the Connes fusion product on the spinor bundle on loop space, which we describe in the subsequent § 5, applies, in particular, to spin manifolds equipped with a geometric string structure.

5. Fusion in the spinor bundle on loop space

This section contains our main results: we first exhibit the spinor bundle $F^{\mathrm{s}}(LM)$ on loop space as a rigged von Neumann bimodule bundle, and then equip its completion with a Connes fusion product over the fusion of loops. Throughout this section, M is a spin manifold equipped with a spin structure $\widetilde{L\,\mathrm{Spin}}(M)$ on its loop space. In § 5.3 we will then require a *fusive* spin structure.

5.1 The von Neumann algebra bundle over path space

In this section we construct a rigged von Neumann algebra bundle $\mathcal{N}^{\mathrm{s}}$ on the path space PM of M, which, heuristically, has the property that the algebra $\mathcal{N}^{\mathrm{s}}_{\beta_2}$ sits in the Clifford algebra $\mathrm{Cl}^{\mathrm{s}}_{\mathrm{vN}}(LM)_{\beta_1\cup\beta_2}$ as $\mathrm{Cl}(V_-)^{\mathrm{s}}$ sits in $\mathrm{Cl}(V)^{\mathrm{s}}$, for paths $(\beta_1,\beta_2)\in PM^{[2]}$. In order to construct $\mathcal{N}^{\mathrm{s}}$, we start with the underlying rigged C*-algebra bundle $\mathcal{A}^{\mathrm{s}}$. We recall from § 4.3 that we have a Fréchet Lie group homomorphism $\widetilde{L\,\mathrm{Spin}}(d)\to \mathrm{O}^{\theta}_L(V)$, which is induced by the pointwise multiplication of $L\,\mathrm{SO}(d)$ on V. From § 3.1 we recall that we have a Fréchet Lie group homomorphism $\mathrm{O}^{\theta}_L(V)\to \mathrm{O}(V_-)$. Finally, $\mathrm{O}(V_-)$ acts smoothly on $\mathrm{Cl}(V_-)^{\mathrm{s}}$, and we thus obtain an induced smooth representation $\widetilde{L\,\mathrm{Spin}}(d)\times \mathrm{Cl}(V_-)^{\mathrm{s}}\to \mathrm{Cl}(V_-)^{\mathrm{s}}$. This allows us to define, via Lemma 2.2.4, an associated rigged C*-algebra bundle

$$\mathrm{Cl}^{\mathrm{s}}_-(LM) := \widetilde{L\,\mathrm{Spin}}(M)\times_{\widetilde{L\,\mathrm{Spin}}(d)} \mathrm{Cl}^{\mathrm{s}}(V_-)$$

over LM with typical fibre $\mathrm{Cl}(V_-)^{\mathrm{s}}$. Next we consider the diffeological space PM of paths in M, together with the doubling map $\Delta : PM\to LM, \beta\mapsto \beta\cup\beta$, which is well-defined and smooth (see § 4.2). Then, we define the rigged C*-algebra bundle

$$\mathcal{A}^{\mathrm{s}} := \Delta^* \,\mathrm{Cl}^{\mathrm{s}}_-(LM)$$

with typical fibre $\mathrm{Cl}^{\mathrm{s}}(V_-)$ over PM, as explained in § 2.3.

Remark 5.1.1. Analogous to Remark 4.1.6, our construction of the rigged C*-algebra bundle $\mathcal{A}^{\mathrm{s}}$ holds if M is merely oriented, it neither needs a spin structure on M nor on LM. The following discussion requires both, however.

We have proved in Lemma 3.2.7 that the inclusion $\iota_- : \mathrm{Cl}(V_-)^{\mathrm{s}}\to \mathrm{Cl}(V)^{\mathrm{s}}, a\mapsto a\otimes \mathbb{1}$ is smooth, and since ι_- obviously intertwines the $\widetilde{L\,\mathrm{Spin}}(d)$-actions, we obtain an induced isometric morphism

$$\mathrm{Cl}^{\mathrm{s}}_-(LM)\to \mathrm{Cl}^{\mathrm{s}}(LM), [\varphi,a]\mapsto[\varphi,a\otimes\mathbb{1}] \tag{15}$$

of rigged C*-algebra bundles over LM. Further, we have considered the induced representation of $\mathrm{Cl}(V_-)^{\mathrm{s}}$ on Fock space F^{s}, turning F^{s} into a rigged $\mathrm{Cl}(V_-)^{\mathrm{s}}$-module. On the level of rigged module *bundles*, no general induction procedure exists; however, the following result shows that it works in the present case.

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PROPOSITION 5.1.2. *The restriction of Clifford multiplication* $\mathrm{Cl}^{\mathrm{s}}(LM) \times F^{\mathrm{s}}(LM) \to F^{\mathrm{s}}(LM)$ *along* (15) *equips the spinor bundle* $F^{\mathrm{s}}(LM)$ *with the structure of a rigged* $\mathrm{Cl}^{\mathrm{s}}_{-}(LM)$*-module bundle with typical fibre the rigged* $\mathrm{Cl}(V_{-})^{\mathrm{s}}$*-module* F^{s}*.*

Proof. In order to meet the assumptions of Definition 2.2.5, we have to find *compatible* local trivializations Ψ of $F^{\mathrm{s}}(LM)$ and Φ of $\mathrm{Cl}^{\mathrm{s}}_{-}(LM)$. As associated bundles, local trivializations can be induced from local trivializations of the underlying principal bundles (Lemmas 2.2.2 and 2.2.4); here, $\widetilde{L\,\mathrm{Spin}}(M)$ and $L\,\mathrm{Spin}(M)$. Let $\tilde{\varphi}: U \to \widetilde{L\,\mathrm{Spin}}(M)$ be any local section, and let $\varphi: U \to L\,\mathrm{Spin}(M)$ be the local section obtained by composing $\tilde{\varphi}$ with the projection $\widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M)$. The corresponding local trivializations of principal bundles induce the following local trivializations of associated bundles:

$$\begin{aligned} &\Psi: F^{\mathrm{s}}(LM)|_U \to F^{\mathrm{s}} \times U, && \Psi([\tilde{\varphi}(x), v]) = (v, x), \\ &\Phi: \mathrm{Cl}^{\mathrm{s}}_{-}(LM)|_U \to \mathrm{Cl}(V_{-})^{\mathrm{s}} \times U, && \Phi([\varphi(x), a]) = (a, x). \end{aligned}$$

It is obvious that these exchange the rigged module structure on the fibres with that on the typical fibre and, hence, are compatible in the sense of Definition 2.2.5. $\square$

Again via pullback along the doubling map Δ, we obtain the rigged $\mathcal{A}^{\mathrm{s}}$-module bundle $\Delta^* F^{\mathrm{s}}(LM)$, with typical fibre the rigged $\mathrm{Cl}(V_{-})^{\mathrm{s}}$-module F^{s}. Since the latter is a rigged von Neumann algebra, $N^{\mathrm{s}} = (\mathrm{Cl}(V_{-})^{\mathrm{s}}, F^{\mathrm{s}})$, we have that

$$\mathcal{N}^{\mathrm{s}} := (\mathcal{A}^{\mathrm{s}}, \Delta^* F^{\mathrm{s}}(LM))$$

is a *rigged von Neumann algebra bundle* over PM with typical fibre N^{s} in the sense of Definition 2.2.9. This implies that for each $\beta \in PM$ the fibre $\mathcal{N}^{\mathrm{s}}_{\beta} = (\mathcal{A}^{\mathrm{s}}_{\beta}, F^{\mathrm{s}}(LM)_{\Delta(\beta)})$, is a rigged von Neumann algebra, and thus induces an ordinary von Neumann algebra $\mathcal{N}_{\beta} := (\mathcal{N}^{\mathrm{s}}_{\beta})''$, see Remark 2.1.7. Any choice of compatible local trivializations as in Proposition 5.1.2 establishes a normal $*$-isomorphism $u: \mathcal{N}_{\beta} \to N$ of von Neumann algebras, see Remark 2.2.12. In particular, the von Neumann algebras $\mathcal{N}_{\beta}$ are type III_1-factors.

5.2 The spinor bundle as a bundle of bimodules

The goal of this section is to exhibit the spinor bundle $F^{\mathrm{s}}(LM)$ as a rigged von Neumann $\mathcal{N}^{\mathrm{s}}$–$\mathcal{N}^{\mathrm{s}}$-bimodule bundle. We start fibrewise, and shall, for each loop of the form $\gamma = \beta_1 \cup \beta_2$, where $(\beta_1, \beta_2) \in PM^{[2]}$, equip the Fock spaces $F^{\mathrm{s}}(LM)_{\gamma}$ with representations of the rigged C*-algebras $(\mathcal{A}^{\mathrm{s}}_{\beta_1})^{\mathrm{opp}}$ and $\mathcal{A}^{\mathrm{s}}_{\beta_2}$. We recall that F^{s} is a rigged $\mathrm{Cl}(V_{-})^{\mathrm{s}}$–$\mathrm{Cl}(V_{-})^{\mathrm{s}}$-bimodule, with representations denoted by $a_1 \triangleright v \triangleleft a_2$ (Lemma 3.3.3).

LEMMA 5.2.1. *For each* $(\beta_1, \beta_2) \in PM^{[2]}$ *and* $\gamma := \beta_1 \cup \beta_2$*, there exist unique maps*

$$(\rho_1)_{\beta_1,\beta_2} : (\mathcal{A}^{\mathrm{s}}_{\beta_1})^{\mathrm{opp}} \times F^{\mathrm{s}}(LM)_{\gamma} \to F^{\mathrm{s}}(LM)_{\gamma} \quad \text{and} \quad (\rho_2)_{\beta_1,\beta_2} : \mathcal{A}^{\mathrm{s}}_{\beta_2} \times F^{\mathrm{s}}(LM)_{\gamma} \to F^{\mathrm{s}}(LM)_{\gamma}$$

with the property that the equations

$$(\rho_1)_{\beta_1,\beta_2}([\Delta(\varphi_1), a], [\tilde{\varphi}, v]) = [\tilde{\varphi}, v \triangleleft a] \quad \text{and} \quad (\rho_2)_{\beta_1,\beta_2}([\Delta(\varphi_2), a], [\tilde{\varphi}, v]) = [\tilde{\varphi}, a \triangleright v]$$

hold for all $(\varphi_1, \varphi_2) \in P\,\mathrm{Spin}(M)^{[2]}$ *lifting* (β_1, β_2)*, all* $\tilde{\varphi} \in \widetilde{L\,\mathrm{Spin}}(M)$ *lifting* $\varphi_1 \cup \varphi_2$*, all* $a \in \mathrm{Cl}(V_{-})^{\mathrm{s}}$ *and all* $v \in F^{\mathrm{s}}_L$*.*

Proof. It is clear that the maps are determined uniquely by the given equations, provided that choices of (φ_1, φ_2) and $\tilde{\varphi}$ exist for arbitrary (β_1, β_2). To see this, we choose a lift $\tilde{y} \in \mathrm{Spin}(M)$ of the initial point $y := \beta_1(0) = \beta_2(0)$, and define $\varphi_1', \varphi_2 : [0, \pi] \to \mathrm{Spin}(M)$ as the horizontal lifts of β_1, β_2, with respect to some fixed connection and with initial point $\varphi_1'(0) = \varphi_2(0) = \tilde{y}$. Since horizontal lifts of constant paths are again constant paths, the lifts φ_1' and φ_2 have the same

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sitting instants as β_1 and β_2, respectively. Now we let $g \in \mathrm{Spin}(d)$ be the unique element such that $\varphi_2(\pi) = \varphi_1'(\pi)g$. Since $\mathrm{Spin}(d)$ is connected, there exists a path $\tilde{g} : [0,\pi] \to \mathrm{Spin}(d)$ connecting e with g, and we may choose it with sitting instants. Then, $\varphi_1 := \varphi_1'\tilde{g}$ and φ_2 form a pair $(\varphi_1, \varphi_2) \in P\,\mathrm{Spin}(M)^{[2]}$ as required. Moreover, $\varphi_1 \cup \varphi_2 \in L\,\mathrm{Spin}(M)$, and since $\widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M)$ is surjective, a lift $\tilde{\varphi}$ exists.

For existence, we have to check that the given equations can be used as definitions, i.e. that they are independent of the involved choices. To see this, we suppose $(\varphi_1', \varphi_2') \in P\,\mathrm{Spin}(M)^{[2]}$ lifts the same pair (β_1, β_2), and $\tilde{\varphi}'$ lifts $\varphi_1' \cup \varphi_2'$. Then, using the pointwise principal actions, we have $\varphi_i' = \varphi_i g_i$ for $g_i \in P\,\mathrm{Spin}(d)$ and $\tilde{\varphi}' = \tilde{\varphi}\tilde{g}$ for $\tilde{g} \in \widetilde{L\,\mathrm{Spin}}(d)$. By definition of $\mathcal{A}^{\mathrm{s}}$ and $F^{\mathrm{s}}(LM)$ as associated bundles, we have $[\Delta(\varphi_2'), a] = [\Delta(\varphi_2), \theta_{\omega_-(g_2)}(a)]$ and $[\tilde{\varphi}', v] = [\tilde{\varphi}, Uv]$, with $\omega_- : P\,\mathrm{Spin}(d) \to \mathrm{O}(V_-)$ defined in § 4.3, and $U \coloneqq \tilde{\omega}(\tilde{g}) \in \mathrm{Imp}_L^{\theta}(V)$. Thus, we get

$$
\begin{aligned}
(\rho_2)_{\beta_1,\beta_2}([\Delta(\varphi_2'), a], [\tilde{\varphi}', v]) &= [\tilde{\varphi}, (\theta_{\omega_-(g_2)}(a) \otimes \mathbb{1}) \triangleright Uv] \\
&= [\tilde{\varphi}, (\theta_{\omega(g_1 \cup g_2)_-}(a) \otimes \theta_{\omega(g_1 \cup g_2)_+}(\mathbb{1})) \triangleright Uv] \\
&= [\tilde{\varphi}, \theta_{\omega(g_1 \cup g_2)}(a \otimes \mathbb{1}) \triangleright Uv] \\
&= [\tilde{\varphi}, U((a \otimes \mathbb{1}) \triangleright v)] \\
&= [\tilde{\varphi}', a \triangleright v],
\end{aligned}
$$

as intended. In the second step we have used the commutativity of diagram Lemma 4.3.2, together with the fact that Bogoliubov automorphisms are unital. The third step uses Lemma 3.1.1, and the fourth step uses the fact that U implements $\omega(g_1 \cup g_2)$. For ρ_2, we have $[\Delta(\varphi_1'), a] = [\Delta(\varphi_1), \theta_{\omega_-(g_1)}(a)]$, and compute

$$
\begin{aligned}
(\rho_1)_{\beta_1,\beta_2}([\Delta(\varphi_1'), a], [\tilde{\varphi}', v]) &= [\tilde{\varphi}, J(\theta_{\omega_-(g_1)}(a) \otimes \mathbb{1})^* J \triangleright Uv] \\
&= [\tilde{\varphi}, J\theta_{\omega(g_2 \cup g_1)}(a \otimes \mathbb{1})^* J \triangleright Uv] \\
&= [\tilde{\varphi}, \theta_{\omega(g_1 \cup g_2)}(J(a \otimes \mathbb{1})^* J) \triangleright Uv] \\
&= [\tilde{\varphi}, U(J(a \otimes \mathbb{1})^* J \triangleright v)] \\
&= [\tilde{\varphi}', v \triangleleft a],
\end{aligned}
$$

where, in the third step, we have used (9) from [KW22, Lemma 4.8] together with the obvious identity $\tau(\omega(g_1 \cup g_2)) = \omega(g_2 \cup g_1)$. ☐

It is clear from the formulas in Lemma 5.2.1 that the maps $(\rho_1)_{\beta_1,\beta_2}$ and $(\rho_2)_{\beta_1,\beta_2}$ are indeed left actions of $(\mathcal{A}^{\mathrm{s}}_{\beta_1})^{\mathrm{opp}}$ and $\mathcal{A}^{\mathrm{s}}_{\beta_2}$, respectively, and that they commute. We remark that it is further true (this follows later from Proposition 5.2.4) that $F^{\mathrm{s}}(LM)_{\beta_1 \cup \beta_2}$ is a rigged $\mathcal{A}^{\mathrm{s}}_{\beta_2}$–$\mathcal{A}^{\mathrm{s}}_{\beta_1}$-bimodule in the sense of Definition 2.1.5.

Trying to assemble the maps $(\rho_1)_{\beta_1,\beta_2}$ and $(\rho_2)_{\beta_1,\beta_2}$ into bundle morphisms, we face the problem that the bundle $F^{\mathrm{s}}(LM)$ lives over LM, whereas the algebra bundle $\mathcal{A}^{\mathrm{s}}$ lives over PM. Therefore, we work over the space $PM^{[2]}$ of pairs of paths with common end points, and work with the pullbacks of bundles to this space, along the following maps.

$$
\begin{tikzcd}
LM & PM^{[2]} \arrow[l, "\cup"'] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & PM
\end{tikzcd}
$$

We recall from § 2.3 that over the diffeological space $PM^{[2]}$, bundles consist of plot-wise defined bundles, and bundle isomorphisms for the transition of plots. Let $c : U \to PM^{[2]}$ be a plot.

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We write $c_i := \Delta \circ p_i \circ c$ for $i = 1, 2$ and $\tilde{c} := \cup \circ c$. By definition of the involved diffeologies, the maps $c_1, c_2, \tilde{c} : U \to LM$ are smooth maps between (Fréchet) manifolds. The rigged C*-algebra bundles $p_1^* \mathcal{A}^{\mathrm{s}}$ and $p_2^* \mathcal{A}^{\mathrm{s}}$, consist of the plot-wise defined rigged C*-algebra bundles $(p_1^* \mathcal{A}^{\mathrm{s}})_c = c_1^* \, \mathrm{Cl}_-^{\mathrm{s}}(LM)$ and $(p_2^* \mathcal{A}^{\mathrm{s}})_c = c_2^* \, \mathrm{Cl}_-^{\mathrm{s}}(LM)$ over U, respectively, and the rigged Hilbert space bundle $\cup^* F^{\mathrm{s}}(LM)$ consists of the plot-wise defined rigged Hilbert space bundles $(\cup^* F^{\mathrm{s}}(LM))_c = \tilde{c}^* F^{\mathrm{s}}(LM)$ over U. Moreover, if $c' : U' \to PM^{[2]}$ is another plot, and $f : U \to U'$ is a smooth map such that $c' \circ f = c$, then the bundle isomorphisms $(p_i^* \mathcal{A}^{\mathrm{s}})_c \to f^*(p_i^* \mathcal{A}^{\mathrm{s}})_{c'}$ are the canonical ones obtained from the equality $c_i = c_i' \circ f$. Similarly, the bundle isomorphism $(\cup^* F^{\mathrm{s}}(LM))_c \to f^*(\cup^* F^{\mathrm{s}}(LM))_{c'}$ is the canonical one obtained from the equality $\tilde{c} = \tilde{c}' \circ f$.

Our goal is to show that the maps of Lemma 5.2.1 equip the rigged Hilbert space bundle $\cup^* F^{\mathrm{s}}(LM))$ over $PM^{[2]}$ with the structure of a rigged $p_2^* \mathcal{A}^{\mathrm{s}}$–$p_1^* \mathcal{A}^{\mathrm{s}}$-bimodule bundle. We will first discuss the situation on a fixed plot $c : U \to PM^{[2]}$, and to this end, consider $x \in U$ and $(\beta_1, \beta_2) := c(x)$. Then, for the fibres over x we find

$$((p_1^* \mathcal{A}^{\mathrm{s}})_c)_x = c_1^* \, \mathrm{Cl}_-^{\mathrm{s}}(LM)_x = \mathrm{Cl}_-^{\mathrm{s}}(LM)_{\Delta(\beta_1)} = \mathcal{A}^{\mathrm{s}}_{\beta_1}$$

and, similarly,

$$((p_2^* \mathcal{A}^{\mathrm{s}})_c)_x = \mathcal{A}^{\mathrm{s}}_{\beta_2} \quad \text{and} \quad (\cup^* F^{\mathrm{s}}(LM)_c)_x = F^{\mathrm{s}}(LM)_{\beta_1 \cup \beta_2}.$$

Now we see that the maps $(\rho_1)_{\beta_1, \beta_2}$ and $(\rho_2)_{\beta_1, \beta_2}$ of Lemma 5.2.1 assemble into fibre-preserving maps

$$\rho_{1,c} : (p_1^* (\mathcal{A}^{\mathrm{s}})^{\mathrm{opp}})_c \times_U \cup^* F^{\mathrm{s}}(LM)_c \to \cup^* F^{\mathrm{s}}(LM)_c,$$
$$\rho_{2,c} : (p_2^* \mathcal{A}^{\mathrm{s}})_c \times_U \cup^* F^{\mathrm{s}}(LM)_c \to \cup^* F^{\mathrm{s}}(LM)_c.$$

LEMMA 5.2.2. *Let $c : U \to PM^{[2]}$ be a plot. The maps $\rho_{1,c}$ and $\rho_{2,c}$ equip the rigged Hilbert space bundle $(\cup^* F^{\mathrm{s}}(LM))_c$ over U with the structure of a rigged $(p_2^* \mathcal{A}^{\mathrm{s}})_c$–$(p_1^* \mathcal{A}^{\mathrm{s}})_c$-bimodule bundle with typical fibre the rigged $\mathrm{Cl}(V_-)^{\mathrm{s}}$–$\mathrm{Cl}(V_-)^{\mathrm{s}}$-bimodule F^{s}.*

For the proof of Lemma 5.2.2 we require the following lemma, which delivers us appropriate local trivializations that at each point meet the conditions of Lemma 5.2.1.

LEMMA 5.2.3. *Each $x \in U$ has an open neighbourhood $W \subset U$ admitting smooth maps $\varphi_1, \varphi_2 : W \to P\,\mathrm{Spin}(M)$ and a smooth map $\tilde{\varphi} : W \to \widetilde{L\,\mathrm{Spin}}(M)$, such that $(\varphi_1(u), \varphi_2(u)) \in P\,\mathrm{Spin}(M)^{[2]}$ for all $u \in W$, and the diagram*

$$\begin{tikzcd}
 & & \widetilde{L\,\mathrm{Spin}}(M) \arrow[d] \\
 & P\,\mathrm{Spin}(M)^{[2]} \arrow[r, "\cup"'] \arrow[d] & L\,\mathrm{Spin}(M) \arrow[d] \\
W \arrow[r, "c"'] \arrow[ru, bend left, "(\varphi_1,\varphi_2)"'] \arrow[rruu, bend left, "\tilde{\varphi}"] & PM^{[2]} \arrow[r, "\cup"'] & LM
\end{tikzcd}$$

is commutative.

Proof. We consider the map $y : U \to M$ sending $u \in U$ to the common initial point of the pair of paths $c(u)$. We choose W such that $\mathrm{Spin}(M) \to M$ has a section $\tilde{y} : W \to \mathrm{Spin}(M)$ along $y|_W$. As in Lemma 5.2.1, we define $\varphi_1'(u), \varphi_2(u) \in P\,\mathrm{Spin}(M)$ as the horizontal lifts of $(\beta_1, \beta_2) := c(u)$ starting at $\tilde{y}(u)$. Since horizontal lifts depend smoothly on the initial condition and on the base

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path, this yields smooth maps $\varphi_1', \varphi_2 : W \to P\,\mathrm{Spin}(M)$. The endpoints differ by a smooth map $g : W \to \mathrm{Spin}(d)$. After shrinking W further to a contractible set, and since $\mathrm{Spin}(d)$ is connected, g is homotopic to the constant map with value $e \in \mathrm{Spin}(d)$. Such a homotopy $\tilde{g} : W \times [0,\pi] \to \mathrm{Spin}(d)$ can be chosen to have sitting instants for each $u \in W$; then, $\varphi_1(u) := \varphi_1'(u)\tilde{g}(u,-)$ yields a smooth map $\varphi_1 : W \to \mathrm{Spin}(M)$ such that φ_1, φ_2 have all required properties. Now, the map $W \to L\,\mathrm{Spin}(M) : u \mapsto \varphi_1(u) \cup \varphi_2(u)$ is smooth, and since $\widetilde{L\,\mathrm{Spin}(M)} \to L\,\mathrm{Spin}(M)$ is a locally trivial bundle, a section $\tilde{\varphi}$ exists, possibly after a further shrinking of W. $\square$

An immediate consequence of the definitions of the bundles $(p_i^*\mathcal{A}^{\mathrm{s}})_c$ and $(\cup^* F^{\mathrm{s}}(LM))_c$ as (pullbacks of) associated bundles, is that the sections into the corresponding principal bundles obtained from Lemma 5.2.3 induce local trivializations

$$\begin{aligned}
&\Phi^1 : (p_1^*\mathcal{A}^{\mathrm{s}})_c|_W \to \mathrm{Cl}(V_-)^{\mathrm{s}} \times W && \Phi^1([\Delta(\varphi_1(x)), a]) = (a, x),\\
&\Phi^2 : (p_2^*\mathcal{A}^{\mathrm{s}})_c|_W \to \mathrm{Cl}(V_-)^{\mathrm{s}} \times W && \Phi^2([\Delta(\varphi_2(x)), a]) = (a, x),\\
&\Psi : (\cup^* F^{\mathrm{s}}(LM))_c|_W \to F^{\mathrm{s}} \times W && \Psi([\tilde{\varphi}(x), v]) = (v, x);
\end{aligned} \tag{16}$$

see Lemmas 2.2.2 and 2.2.4.

Proof of Lemma 5.2.2. According to Definition 2.2.7, the statement can be proved locally in a neighbourhood of any point $x \in U$. We may assume an open neighbourhood $V \subset U$ that admits smooth maps $\varphi_1, \varphi_2 : W \to P\,\mathrm{Spin}(M)$ and $\tilde{\varphi} : W \to \widetilde{L\,\mathrm{Spin}(M)}$ satisfying the conditions in Lemma 5.2.3. Then, we let Φ_1, Φ_2, and Ψ be the local trivializations of (16). Inspecting the definition of $\rho_{1,c}$ and $\rho_{2,c}$ in Lemma 5.2.1, we find that

$$\Psi(\rho_{1,c}(a_1, v)) = \Psi(v) \triangleleft \Phi^1(a_1) \quad \text{and} \quad \Psi(\rho_{2,c}(a_2, v)) = \Phi^2(a_2) \triangleright \Psi(v)$$

for all appropriate $a_1 \in (p_1^*\mathcal{A}^{\mathrm{s}})_c$, $a_2 \in (p_2^*\mathcal{A}^{\mathrm{s}})_c$ and $v \in (\cup^* F^{\mathrm{s}}(LM))_c$. This proves that (Φ^1, Ψ) and (Φ^2, Ψ) are compatible local trivializations. $\square$

Lemma 5.2.2 establishes the plot-wise definition of the rigged $p_2^*\mathcal{A}^{\mathrm{s}}$–$p_1^*\mathcal{A}^{\mathrm{s}}$-bimodule bundle $\cup^* F^{\mathrm{s}}(LM)$; now it remains to assure the correct gluing behaviour for the transition between plots. Thus, suppose $c : U \to PM^{[2]}$ and $c' : U' \to PM^{[2]}$ are plots, and $f : U \to U'$ is a smooth map with $c' \circ f = c$. But the canonical bundle isomorphisms $(p_i^*\mathcal{A}^{\mathrm{s}})_c \to f^*(p_i^*\mathcal{A}^{\mathrm{s}})_{c'}$ and $(\cup^* F^{\mathrm{s}}(LM))_c \to f^*(\cup^* F^{\mathrm{s}}(LM))_{c'}$ are under the fibre-wise identifications used in the definition of $\rho_{1,c}$ and $\rho_{2,c}$ over each point $x \in U$ the identity maps of $\mathrm{Cl}^{\mathrm{s}}_-(LM)_{\beta_i \cup \beta_i}$ and $F^{\mathrm{s}}(LM)_{\beta_1 \cup \beta_2}$, respectively, where $(\beta_1, \beta_2) := c(x)$. In particular, they form a unitary intertwiner. This shows the following.

Proposition 5.2.4. *The plot-wise bimodule structure of Lemma 5.2.2 exhibits the spinor bundle $\cup^* F^{\mathrm{s}}(LM)$ as a rigged $p_2^*\mathcal{A}^{\mathrm{s}}$–$p_1^*\mathcal{A}^{\mathrm{s}}$-bimodule bundle, with typical fibre the rigged $\mathrm{Cl}(V_-)^{\mathrm{s}}$–$\mathrm{Cl}(V_-)^{\mathrm{s}}$-bimodule F^{s}.*

The bundle $\mathcal{A}^{\mathrm{s}}$ is a rigged C*-algebra bundle; however, in § 5.1 we discussed how to upgrade it to a rigged von Neumann algebra bundle $\mathcal{N}^{\mathrm{s}} = (\mathcal{A}^{\mathrm{s}}, \Delta^* F^{\mathrm{s}}(LM))$ with typical fibre N^{s}. Moreover, the typical fibre F^{s} of the rigged bimodule bundle $\cup^* F^{\mathrm{s}}(LM)$ is a rigged von Neumann N^{s}–N^{s}-bimodule. Our main result in this section is that these rigged von Neumann structures carry over to the spinor bundle.

Theorem 5.2.5. *The spinor bundle on loop space $\cup^* F^{\mathrm{s}}(LM)$ is a rigged von Neumann $p_2^*\mathcal{N}^{\mathrm{s}}$–$p_1^*\mathcal{N}^{\mathrm{s}}$-bimodule bundle with typical fibre the rigged von Neumann N^{s}–N^{s}-bimodule F^{s}.*

For the proof, we note the following improvement of Lemma 5.2.3.

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LEMMA 5.2.6. *Let $c : U \to PM^{[2]}$ be a plot. Then, each point $x \in U$ has an open neighbourhood $W \subset U$ admitting smooth maps $\varphi_1, \varphi_2 : W \to P\,\mathrm{Spin}(M)$ and $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi} : W \to \widetilde{L\,\mathrm{Spin}}(M)$ such that φ_1, φ_2, and $\tilde{\varphi}$ satisfy the conditions of Lemma 5.2.3, and $\tilde{\varphi}_i$ lifts $\Delta \circ \varphi_i : W \to L\,\mathrm{Spin}(M)$, for $i = 1, 2$.*

Proof. The additional maps $\tilde{\varphi}_i$ exist, possibly after a further shrinking of W, since $\widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M)$ is a locally trivial principal bundle. □

Proof of Theorem 5.2.5. Using Lemma 5.2.6 We obtain the local trivializations Φ^1 of $(p_1^*\mathcal{A}^{\mathrm{s}})_c$, Φ^2 of $(p_2^*\mathcal{A}^{\mathrm{s}})_c$, and Ψ of $(\cup^* F^{\mathrm{s}}(LM))_c$ of (16), and additionally from $\tilde{\varphi}_i$, since $\Delta \circ \varphi_i$ is a section along

$$U \xrightarrow{c} PM^{[2]} \xrightarrow{p_i} PM \xrightarrow{\Delta} LM$$

local trivializations

$$\Psi^i : (p_i^*\Delta^* F^{\mathrm{s}}(LM))_c|_W \to F^{\mathrm{s}} \times W, \quad \Psi^i([\tilde{\varphi}_i(x), v]) = (v, x).$$

By inspection of the involved formulas, we see that (Φ^i, Ψ^i) is the pullback of the compatible local trivializations of Proposition 5.1.2 and, hence, is a compatible local trivialization of $(p_i^*\mathcal{N}^{\mathrm{s}})_c$. Consulting Definition 2.2.13, this proves that $(\cup^* F^{\mathrm{s}}(LM))_c$ is a rigged von Neumann $(p_2^*\mathcal{N}^{\mathrm{s}})_c$–$(p_1^*\mathcal{N}^{\mathrm{s}})_c$-bimodule bundle. Concerning the transition between plots, there is nothing to add to the argument given in the proof of Proposition 5.2.4. □

We recall from § 2.2 that the fibrewise completion of the spinor bundle $F^{\mathrm{s}}(LM)$ results in a *continuous* Hilbert space bundle over LM, which we denote by $F(LM)$. The following result follows from Theorem 5.2.5 via the theory of rigged von Neumann algebra bundles developed in § 2.2. First, for each $(\beta_1, \beta_2) \in PM^{[2]}$ we have by Lemma 2.2.14 that the fibre $F^{\mathrm{s}}(LM)_{\beta_1 \cup \beta_2}$ is a rigged $\mathcal{N}^{\mathrm{s}}_{\beta_2}$–$\mathcal{N}^{\mathrm{s}}_{\beta_1}$-bimodule. Lemma 2.1.16 implies then the following result.

COROLLARY 5.2.7. *The completion $F(LM)_{\beta_1 \cup \beta_2}$ of each fibre of the spinor bundle on loop space is an $\mathcal{N}_{\beta_2}$–$\mathcal{N}_{\beta_1}$-bimodule in the classical von Neumann theoretical sense.*

In the next section it will be important to identify the bimodule $F(LM)_{\beta_1 \cup \beta_2}$ with the typical fibre bimodule F in a precise way, using the methods developed in the results above, reduced to a single point. For later reference, we summarize this in the following remark.

Remark 5.2.8. For a pair $(\beta_1, \beta_2) \in PM^{[2]}$ of paths with common endpoints, consider a lift $(\varphi_1, \varphi_2) \in P\,\mathrm{Spin}(M)^{[2]}$ and a lift $\tilde{\varphi} \in \widetilde{L\,\mathrm{Spin}}(M)$ of $\varphi_1 \cup \varphi_2$. Then, there exist unique isomorphisms $\phi^i : \mathcal{N}^{\mathrm{s}}_{\beta_i} \to N^{\mathrm{s}}$ of rigged von Neumann algebras and a unique isometric isomorphism $\psi : F^{\mathrm{s}}(LM)_{\beta_1 \cup \beta_2} \to F^{\mathrm{s}}$ of rigged Hilbert spaces with $\phi^i([\Delta\varphi_i, a]) = a$ and $\psi([\tilde{\varphi}, v]) = v$. Moreover, (ϕ^2, ϕ^1, ψ) is an invertible unitary intertwiner from the rigged von Neumann $p_2^*\mathcal{N}^{\mathrm{s}}$–$p_1^*\mathcal{N}^{\mathrm{s}}$-bimodule $F^{\mathrm{s}}(LM)_{\beta_1 \cup \beta_2}$ to the rigged von Neumann N^{s}–N^{s}-bimodule F^{s} in the sense of Definition 2.1.15. The isomorphisms ϕ^i induce normal $*$-isomorphisms $u_i : N_{\beta_i} \to N$ of von Neumann algebras, and by Lemma 2.1.16, the isometric isomorphism ψ induces a unitary map $\nu : F^{\mathrm{s}}(LM)_{\beta_1 \cup \beta_2} \to F$, such that the triple (u_2, u_1, ν) is an intertwiner between the $\mathcal{N}_{\beta_2}$–$\mathcal{N}_{\beta_1}$-bimodule $F(LM)_{\beta_1 \cup \beta_2}$ and the N–N-bimodule F.

5.3 Fusion of spinors

The goal of this section is to construct the Connes fusion of spinors on loop space. Now we require $\widetilde{L\,\mathrm{Spin}}(M)$ to be a fusive spin structure on LM (see Definition 4.4.1). Its fusion product

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is the essential ingredient to our construction, and will be denoted by λ. The main result is the following.

Theorem 5.3.1. *Let $(\beta_1, \beta_2, \beta_3) \in PM^{[3]}$ be a triple of paths with common endpoints, let $\mathcal{N}_{\beta_i}$ be the von Neumann algebras over the paths β_i, and let $F(LM)_{\beta_i \cup \beta_j}$ be the completions of the fibres of the spinor bundle on loop space, viewed as von Neumann $\mathcal{N}_{\beta_j}$–$\mathcal{N}_{\beta_i}$-bimodules over the loops $\beta_i \cup \beta_j$. Then, there exists a unique unitary intertwiner*

$$\chi_{\beta_1,\beta_2,\beta_3} : F(LM)_{\beta_2 \cup \beta_3} \boxtimes_{\mathcal{N}_{\beta_2}} F(LM)_{\beta_1 \cup \beta_2} \to F(LM)_{\beta_1 \cup \beta_3}$$

of $\mathcal{N}_{\beta_3}$–$\mathcal{N}_{\beta_1}$-bimodules, where $\boxtimes$ is Connes fusion, such that the following condition is satisfied.

For all $(\varphi_1, \varphi_2, \varphi_3) \in P\,\mathrm{Spin}(M)^{[3]}$ such that φ_i lifts β_i, and all $\tilde{\varphi}_{12}, \tilde{\varphi}_{23}, \tilde{\varphi}_{13} \in \widetilde{L\,\mathrm{Spin}}(M)$ such that $\tilde{\varphi}_{ij}$ lifts $\varphi_i \cup \varphi_j$ and such that $\lambda(\tilde{\varphi}_{12}, \tilde{\varphi}_{23}) = \tilde{\varphi}_{13}$, the diagram

$$\begin{tikzcd} F(LM)_{\beta_2 \cup \beta_3} \boxtimes_{\mathcal{N}_{\beta_2}} F(LM)_{\beta_1 \cup \beta_2} \arrow[r, "\chi_{\beta_1,\beta_2,\beta_3}"] \arrow[d, "(u_3,u_1,\nu_{23}\boxtimes\nu_{12})"'] & F(LM)_{\beta_1 \cup \beta_3} \arrow[d, "(u_3,u_1,\nu_{13})"] \\ F \boxtimes_N F \arrow[r, "\chi"'] & F \end{tikzcd} \tag{17}$$

is commutative, where (u_j, u_i, ν_{ij}) are the unitary intertwiners determined by φ_i, φ_j and $\tilde{\varphi}_{ij}$ as in Remark 5.2.8, and χ is the Connes fusion of the free fermions of § 3.3.

Proof. For fixed φ_i and $\tilde{\varphi}_{ij}$ as in the theorem statement, there is exactly one map $\chi_{\beta_1,\beta_2,\beta_3}$ such that diagram (17) commutes, namely $\nu_{13}^{-1} \circ \chi \circ \nu_{23} \boxtimes \nu_{12}$. Thus, to prove the theorem, we must prove that: (1) such choices of φ_i and $\tilde{\varphi}_{ij}$ exist; and (2) the resulting map $\chi_{\beta_1,\beta_2,\beta_3}$ does not depend on any of these choices.

We start with part (1). We may use Lemma 5.2.1 separately for (β_1, β_2) and then (β_2, β_3) to obtain choices of $(\varphi_1, \varphi_2) \in P\,\mathrm{Spin}(M)^{[2]}$ lifting (β_1, β_2), and $(\varphi_2', \varphi_3') \in P\,\mathrm{Spin}(M)^{[2]}$ lifting (β_2, β_3). Since $\mathrm{Spin}(M)$ is a principal $\mathrm{Spin}(d)$-bundle, there exists a unique $g \in P\,\mathrm{Spin}(d)$ such that $\varphi_2 = \varphi_2' g$. Then, $(\varphi_2' g, \varphi_3' g) \in P\,\mathrm{Spin}(M)^{[2]}$ also lifts (β_2, β_3), and with $\varphi_3 := \varphi_3' g$ we have $(\varphi_1, \varphi_2, \varphi_3) \in P\,\mathrm{Spin}(M)^{[3]}$. Now, choices of $\tilde{\varphi}_{12}$ and $\tilde{\varphi}_{23}$ obviously exist, since $\widetilde{L\,\mathrm{Spin}}(M) \to L\,\mathrm{Spin}(M)$ is surjective, and we set $\tilde{\varphi}_{13} := \lambda(\tilde{\varphi}_{12}, \tilde{\varphi}_{23})$. This finishes the proof of existence of choices.

We now prove part (2), i.e. $\chi_{\beta_1,\beta_2,\beta_3}$ does not depend on any of the choices. Indeed, suppose $(\varphi_1', \varphi_2', \varphi_3') \in P\,\mathrm{Spin}(M)^{[3]}$ such that φ_i lifts β_i, suppose $\tilde{\varphi}_{ij}' \in \widetilde{L\,\mathrm{Spin}}(M)$ lift $\varphi_i' \cup \varphi_j'$ and $\tilde{\varphi}_{13}' = \lambda(\tilde{\varphi}_{12}', \tilde{\varphi}_{23}')$. Then, there are unique elements $\tilde{g}_{12}, \tilde{g}_{23} \in \widetilde{L\,\mathrm{Spin}}(d)$ such that $\tilde{\varphi}_{12}' = \tilde{\varphi}_{12} \cdot \tilde{g}_{12}$ and $\tilde{\varphi}_{23}' = \tilde{\varphi}_{23} \cdot \tilde{g}_{23}$. The compatibility between the fusion products μ on $\widetilde{L\,\mathrm{Spin}}(d)$ and λ on $\widetilde{L\,\mathrm{Spin}}(M)$ in Definition 4.4.1 then implies that $\tilde{\varphi}_{13}' = \tilde{\varphi}_{13} \cdot \mu(\tilde{g}_{12} \otimes \tilde{g}_{23})$. We write (u_j', u_i', ν_{ij}') for the unitary intertwiners corresponding to $\tilde{\varphi}_{ij}'$ and φ_i' according to Remark 5.2.8. Using the definition of ν_{ij} and ν_{ij}' via $\tilde{\varphi}_{ij}$ and $\tilde{\varphi}_{ij}'$, respectively, we obtain $\nu_{12}' = U_{12} \circ \nu_{12}$ and $\nu_{23}' = U_{23} \circ \nu_{23}$, for the implementers $U_{12} := \tilde{\omega}(\tilde{g}_{12})$ and $U_{23} := \tilde{\omega}(\tilde{g}_{23})$ in $\mathrm{Imp}_L^{\theta}(V)$. By Theorem 4.3.3, we have then $\nu_{13}' = \hat{\mu}(U_{23}, U_{12}) \circ \nu_{13}$, where $\hat{\mu}$ denotes the Connes fusion of implementers defined in § 3.4. Using properties and the definition of $\hat{\mu}$, and the functoriality

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of Connes fusion, we compute

$$\begin{aligned}(\nu'_{13})^* \circ \chi \circ (\nu'_{23} \boxtimes \nu'_{12}) &= \nu^*_{13} \circ \hat{\mu}(U^*_{23}, U^*_{212}) \circ \chi \circ (U_{23} \boxtimes U_{12}) \circ (\nu_{23} \boxtimes \nu_{12}) \\ &= \nu^*_{13} \circ \chi \circ (U^*_{23} \boxtimes U^*_{12}) \circ (U_{23} \boxtimes U_{12}) \circ (\nu_{23} \boxtimes \nu_{12}) \\ &= \nu^*_{13} \circ \chi \circ (\nu_{23} \boxtimes \nu_{12}).\end{aligned}$$

This proves that one can define $\chi_{\beta_1,\beta_2,\beta_3} := \nu^*_{13} \circ \chi \circ (\nu_{23} \boxtimes \nu_{12})$ for arbitrary choices of φ_i and $\tilde{\varphi}_{ij}$. It remains to note that this definition indeed yields a unitary intertwiner of $\mathcal{N}_{\beta_3}$–$\mathcal{N}_{\beta_1}$-bimodules, as all three components are unitary intertwiners. □

The collection $\chi = (\chi_{\beta_1,\beta_2,\beta_3})$ with $(\beta_1, \beta_2, \beta_3)$ ranging over $PM^{[3]}$ will be called the *Connes fusion product* on the spinor bundle the loop space. Its construction proves the conjectured Theorem 1 of [ST05]. In the remainder of this section we derive three fundamental properties.

Proposition 5.3.2. *For a path $\beta \in PM$, the $\mathcal{N}_\beta$–$\mathcal{N}_\beta$-bimodule $F(LM)_{\beta\cup\beta}$ is neutral with respect to Connes fusion. Moreover, if $(\beta_1, \beta_2) \in PM^{[2]}$, the bimodules $F(LM)_{\beta_1\cup\beta_2}$ and $F(LM)_{\beta_2\cup\beta_1}$ are inverses of each other with respect to Connes fusion. In particular, $F(LM)_{\beta_1\cup\beta_2}$ is a Morita equivalence between the von Neumann algebras $\mathcal{N}_{\beta_2}$ and $\mathcal{N}_{\beta_1}$.*

Proof. The $\mathcal{N}_\beta$–$\mathcal{N}_\beta$-bimodule $F(LM)_{\beta\cup\beta}$ is a standard form of $\mathcal{N}_\beta$, since it is isomorphic via a unitary intertwiner to the N–N-bimodule F, which is a standard form of N (Remark 5.2.8 and Proposition 3.3.5). In particular, $F(LM)_{\beta\cup\beta}$ is neutral with respect to Connes fusion. If $(\beta_1, \beta_2) \in PM^{[2]}$, then evaluating the Connes fusion product over the triples $(\beta_1, \beta_2, \beta_1)$ and $(\beta_2, \beta_1, \beta_2)$ exhibits the bimodules $F(LM)_{\beta_1\cup\beta_2}$ and $F(LM)_{\beta_2\cup\beta_1}$ as inverses of each other. □

For the second result, we recall that Connes fusion is coherently associative (Proposition A.2.5), which allows us to omit bracketing of multiple Connes fusions.

Proposition 5.3.3. *The Connes fusion product on the spinor bundle is associative in the sense that the diagram*

$$\begin{tikzcd} F(LM)_{\beta_3\cup\beta_4} \boxtimes_{\mathcal{N}_{\beta_3}} F(LM)_{\beta_2\cup\beta_3} \boxtimes_{\mathcal{N}_{\beta_2}} F(LM)_{\beta_1\cup\beta_2} \arrow[r, "\mathbb{1} \boxtimes \chi_{\beta_1,\beta_2,\beta_3}"] \arrow[d, "\chi_{\beta_2,\beta_3,\beta_4} \boxtimes \mathbb{1}"'] & F(LM)_{\beta_3\cup\beta_4} \boxtimes_{\mathcal{N}_{\beta_3}} F(LM)_{\beta_1\cup\beta_3} \arrow[d, "\chi_{\beta_1,\beta_3,\beta_4}"] \\ F(LM)_{\beta_2\cup\beta_4} \boxtimes_{\mathcal{N}_{\beta_2}} F(LM)_{\beta_1\cup\beta_2} \arrow[r, "\chi_{\beta_1,\beta_3,\beta_4}"'] & F(LM)_{\beta_1\cup\beta_4} \end{tikzcd}$$

is commutative for all $(\beta_1, \beta_2, \beta_3, \beta_4) \in PM^{[4]}$.

Proof. Let $(\beta_1, \beta_2, \beta_3, \beta_4) \in PM^{[4]}$. Iterating the procedure described at the beginning of the proof of Theorem 5.3.1, one can see that there exists $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in P\,\mathrm{Spin}(M)^{[4]}$ such that φ_i lifts β_i. Choose $\tilde{\varphi}_{ij} \in \widetilde{L\,\mathrm{Spin}}(M)$ lifting $\varphi_i \cup \varphi_j$, for $ij \in \{12, 23, 34\}$. Then, we set $\tilde{\varphi}_{13} := \lambda(\tilde{\varphi}_{12} \otimes \tilde{\varphi}_{23})$, $\tilde{\varphi}_{24} := \lambda(\tilde{\varphi}_{23} \otimes \tilde{\varphi}_{34})$, and $\tilde{\varphi}_{14} := \lambda(\tilde{\varphi}_{13} \otimes \tilde{\varphi}_{34})$. Note that by the associativity of the fusion product λ (Definition 4.2.1), we have that $\tilde{\varphi}_{14} = \lambda(\tilde{\varphi}_{12} \otimes \tilde{\varphi}_{24})$. We let (u_j, u_i, ν_{ij}) be the unitary intertwiners induced by φ_i and $\tilde{\varphi}_{ij}$ according to Remark 5.2.8. We consider the

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following diagram of unitary isomorphisms.

$$\begin{tikzcd}
F(LM)_{\beta_3\cup\beta_4} \boxtimes_{\mathcal{N}_{\beta_3}} F(LM)_{\beta_2\cup\beta_3} \boxtimes_{\mathcal{N}_{\beta_2}} F(LM)_{\beta_1\cup\beta_2} \arrow[rrr, "\mathbb{1} \boxtimes \chi_{\beta_1,\beta_2,\beta_3}"] \arrow[dr, "\nu_{34} \boxtimes \nu_{23} \boxtimes \nu_{12}"] \arrow[ddd, "\chi_{\beta_2,\beta_3,\beta_4} \boxtimes \mathbb{1}"'] & & & F(LM)_{\beta_3\cup\beta_4} \boxtimes_{\mathcal{N}_{\beta_3}} F(LM)_{\beta_1\cup\beta_3} \arrow[dl, "\nu_{34} \boxtimes \nu_{13}"'] \arrow[ddd, "\chi_{\beta_1,\beta_3,\beta_4}"] \\
& F \boxtimes_N F \boxtimes_N F \arrow[r, "\mathbb{1} \boxtimes \chi"] \arrow[d, "\chi \boxtimes \mathbb{1}"'] & F \boxtimes_N F \arrow[d, "\chi"] & \\
& F \boxtimes_N F \arrow[r, "\chi"'] & F & \\
F(LM)_{\beta_2\cup\beta_4} \boxtimes_{\mathcal{N}_{\beta_2}} F(LM)_{\beta_1\cup\beta_2} \arrow[rrr, "\chi_{\beta_1,\beta_3,\beta_4}"'] \arrow[ur, "\nu_{24} \boxtimes \nu_{12}"] & & & F(LM)_{\beta_1\cup\beta_4} \arrow[ul, "\nu_{14}"']
\end{tikzcd}$$

The four outer diagrams are commutative by Theorem 5.3.1, using above relations between fusion products of the various $\tilde{\varphi}_{ij}$. The ones on top and on the left additionally use the functoriality of Connes fusion (Proposition A.2.3). The diagram in the middle is commutative by Lemma 3.3.8; this completes the proof. $\square$

The third result concerns the smoothness of the Connes fusion product. Since we have not yet been able to lift Connes fusion to the setting of rigged Hilbert spaces, we cannot claim that the fibre-wise maps $\chi_{\beta_1,\beta_2,\beta_3}$ assemble into a smooth (or only continuous) bundle homomorphism. Instead we will describe smoothness by showing that certain smooth sections are mapped to smooth sections.

We consider the (set-theoretical) bundle $p_{23}^*F(LM) \boxtimes p_{12}^*F(LM)$ over $PM^{[3]}$, whose fibre over $(\beta_1, \beta_2, \beta_3) \in PM^{[3]}$ is the Hilbert space $F(LM)_{\beta_2\cup\beta_3} \boxtimes_{\mathcal{N}_{\beta_2}} F(LM)_{\beta_1\cup\beta_2}$. The Connes fusion product on the spinor bundle assembles then into a (set-theoretical) bundle morphism

$$\chi : p_{23}^*F(LM) \boxtimes p_{12}^*F(LM) \to p_{13}^*F(LM)$$

over $PM^{[3]}$. We will now specify a natural choice of smooth sections into $p_{23}^*F(LM) \boxtimes p_{12}^*F(LM)$. To start with, we recall that a smooth section $\tilde{\varphi} : U \to \widetilde{L\,\mathrm{Spin}}(M)$ and a smooth map $v : U \to F^{\mathrm{s}}$ induce a smooth section $\sigma_{\tilde{\varphi},v} : U \to F^{\mathrm{s}}(LM)$ into the spinor bundle, with $\sigma_{\tilde{\varphi},v}(x) = [\tilde{\varphi}(x), v(x)]$. Since $F^{\mathrm{s}} \subset F$ is dense, the image of all such sections is dense in each fibre. The following lemma guarantees that similar local sections exist in a situation appropriate for fusion.

Lemma 5.3.4. *Let $(\beta_1, \beta_2, \beta_3) : U \to PM^{[3]}$ be a plot. Then, for each $x \in U$ there exists an open neighbourhood $W \subset U$ of x, smooth maps $\varphi_1, \varphi_2, \varphi_3 : W \to P\,\mathrm{Spin}(M)$, and smooth maps $\tilde{\varphi}_{12}, \tilde{\varphi}_{23}, \tilde{\varphi}_{13} : W \to \widetilde{L\,\mathrm{Spin}}(M)$ such that $(\varphi_1(x), \varphi_2(x), \varphi_3(x)) \in P\,\mathrm{Spin}(M)^{[3]}$ for all $x \in W$, φ_i lifts β_i for all $i \in \{1, 2, 3\}$, $\tilde{\varphi}_{ij}$ lifts $\varphi_i \cup \varphi_j$ for all $ij \in \{12, 23, 13\}$, and $\tilde{\varphi}_{13}(x) = \lambda(\tilde{\varphi}_{12}(x), \tilde{\varphi}_{23}(x))$ for all $x \in W$.*

Proof. As in the proof of Theorem 5.3.1, we apply Lemma 5.2.3 separately to the plots (β_1, β_2) and (β_2, β_3), obtaining smooth maps (φ_1, φ_2) and (φ_2', φ_3'). The difference between φ_2 and φ_2' is now a smooth map $g : W \to P\,\mathrm{Spin}(d)$ with $\varphi_2(x) = \varphi_2'(x)g(x)$ for all $x \in W$. With $\varphi_3(x) := \varphi_3'(x)g(x)$ we obtain the desired triple $(\varphi_1, \varphi_2, \varphi_3) : W \to P\,\mathrm{Spin}(M)^{[3]}$. As explained in the proof of Lemma 5.2.3, there exist $\tilde{\varphi}_{12}$ and $\tilde{\varphi}_{13}$ as claimed. Finally, since the fusion product λ on $\widetilde{L\,\mathrm{Spin}}(M)$ is a smooth in the sense of Definition 4.2.1, the pointwise definition $\tilde{\varphi}_{13}(x) := \lambda(\tilde{\varphi}_{12}(x), \tilde{\varphi}_{23}(x))$ yields a smooth map. $\square$

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In the situation of Lemma 5.3.4, at each point $x \in W$, the elements $\tilde{\varphi}_{ij}(x)$ and $\varphi_i(x)$ induce unitary intertwiners $(u_x^2, u_x^1, \nu_x^{12})$ and $(u_x^3, u_x^2, \nu_x^{23})$ as in Remark 5.2.8. For any smooth map $v : W \to F^{\mathrm{s}}$, we then obtain a section σ into the bundle $p_{23}^*F(LM) \boxtimes p_{12}^*F(LM)$ by setting

$$\sigma(x) := (\nu_x^{23} \boxtimes \nu_x^{12})^* \chi^*(v(x)) \in (p_{23}^*F(LM) \boxtimes p_{12}^*F(LM))_{c(x)} \tag{18}$$

for all $x \in W$. We will call any section that is locally of this form *smooth*. Again, since each fibre of the bundle $p_{23}^*F(LM) \boxtimes p_{12}^*F(LM)$ is isomorphic to F, the image of all smooth sections is dense. We have the following result.

Proposition 5.3.5. *The Connes fusion product χ on $F(LM)$ is smooth in the sense that it sends smooth sections to smooth sections.*

Proof. Let σ be a smooth section. Smoothness of $\chi \circ \sigma$ can be checked locally on an open set W, on which sections as in Lemma 5.3.4 and a smooth map $v : W \to F^{\mathrm{s}}$ exist such that (18) holds. The conditions in Lemma 5.3.4 imply that at each $x \in W$, diagram (17) in Theorem 5.3.1 is commutative, saying that

$$\sigma(x) = (\nu_x^{13})^{-1}(v(x)) \in (p_{13}^*F(LM))_{c(x)}.$$

Thus, $\sigma|_W$ is (the completion of) the smooth section $\sigma_{\tilde{\varphi}_{13},v} : W \to p_{13}^*F^{\mathrm{s}}(LM)$ and, hence, smooth. □

5.4 The stringor bundle

Given the spinor bundle $F^{\mathrm{s}}(LM)$ on the loop space LM together with its Connes fusion product, it is possible to assemble a structure that deserves to be called the *stringor bundle* of the string manifold M (this terminology is due to Stolz and Teichner [ST05]). While Stolz and Teichner understood the stringor bundle as the pair of the spinor bundle on loop space and its Connes fusion product (both not yet constructed at that time), our aim is to exhibit the stringor bundle as a higher structure on the manifold M, rather than on its loop space.

Suitable for our stringor bundle is the framework of 2-vector bundles. Roughly speaking, a *2-vector bundle* is meant to be a bundle version of a *2-vector space*, where the bicategory of 2-vector spaces is by definition the bicategory of algebras, bimodules, and intertwiners [ST04, Sch09]. A comprehensive study of 2-vector bundles can be found in [KLW21]. Let us first take all algebras and modules to be finite-dimensional and over $\mathbb{C}$. In this setting, a 2-vector bundle over a smooth manifold M consists of a surjective submersion $\pi : Y \to M$, for instance, the disjoint union of open sets of a cover, and:

(a) an algebra bundle $\mathcal{A}$ over Y;
(b) an invertible $p_2^*\mathcal{A}$–$p_1^*\mathcal{A}$-bimodule bundle $\mathcal{M}$ over the double fibre product $Y^{[2]}$;
(c) an invertible intertwiner $\mu : p_{23}^*\mathcal{M} \otimes_{p_2^*\mathcal{A}} p_{12}^*\mathcal{M} \to p_{13}^*\mathcal{M}$ of $p_3^*\mathcal{A}$–$p_1^*\mathcal{A}$-bimodule bundles over $Y^{[3]}$;

such that μ satisfies an associativity condition over $Y^{[4]}$.

An example of a 2-vector bundle is a line bundle gerbe; there, the algebra bundle $\mathcal{A}$ is trivial with fibre $\mathbb{C}$, and the invertible bimodule bundle $\mathcal{M}$ is just a complex line bundle. A result in bundle gerbe theory is that every bundle gerbe can be obtained, via a procedure called *regression*, from a line bundle $\mathcal{L}$ on loop space and a fusion product λ in the sense of Definition 4.2.1, see [Wal16b, Wal12b], at least if one allows Y to be a diffeological space. Namely, one chooses a base point $x \in M$, and considers $Y = P_xM$, the diffeological space of smooth paths with sitting instants starting at x, with $\pi := ev_1$ the end-point evaluation. Then, we have the smooth map $\cup : P_xM^{[2]} \to LM$ and obtain the line bundle $\mathcal{M} := \cup^*\mathcal{L}$. The intertwiner μ is the restriction of the fusion product λ to $P_xM^{[3]} \subset PM^{[3]}$; this defines a (diffeological) bundle gerbe over M.

We extend regression to the spinor bundle and its Connes fusion product. Let M be a string manifold with a fixed string structure and a fixed base point $x \in M$. Then, the *stringor 2-vector bundle* on M consists of the following structure.

(a) The diffeological space P_xM together with the end-point evaluation $ev_1 : P_xM \to M$.
(b) The rigged von Neumann algebra bundle $\mathcal{N}^{\mathrm{s}}$ over P_xM, obtained as the restriction of the rigged von Neumann algebra bundle constructed in § 5.1 to $P_xM \subset PM$.
(c) The rigged von Neumann $p_2^*\mathcal{N}^{\mathrm{s}}$–$p_1^*\mathcal{N}^{\mathrm{s}}$-bimodule bundle $\cup^*F^{\mathrm{s}}(LM)$ over $P_xM^{[2]}$, constructed in Theorem 5.2.5 and restricted to $P_xM^{[2]} \subset PM^{[2]}$.
(d) The Connes fusion product

$$\chi : p_{23}^* \cup^* F(LM) \boxtimes_{p_2^*\mathcal{N}} p_{12}^* \cup^* F(LM) \to p_{13}^* \cup^* F(LM)$$

over $P_xM^{[3]}$, constructed fibrewise in Theorem 5.3.1, which is associative (Proposition 5.3.3).

Though this stringor bundle is a perfectly well-defined structure, we do not yet have a well-defined von Neumann theoretical version of 2-vector bundles, and thus at the moment cannot claim that the stringor bundle is such a 2-vector bundle. To expand on this problem, we remark that there exists a bicategory of von Neumann algebras, bimodules, and intertwiners, with the composition by Connes fusion [Bro03, ST04]. However, as explained in § 2, a proper discussion of a bundle version requires *rigged* von Neumann algebras and bimodules. So far we have not been able to lift Connes fusion to the setting of rigged von Neumann bimodules. Thus, we currently do not have a properly defined bicategory of rigged von Neumann algebra bundles, bimodule bundles, and intertwiners, and hence cannot establish the stringor bundle as an object in a corresponding bicategory of 'rigged von Neumann 2-vector bundles'.

Yet, our stringor bundle realizes another claim of Stolz and Teichner [ST04, Corollary 5.0.4], namely that, after picking a basepoint $x \in M$, a string structure on M gives rise to a family of von Neumann algebras parameterized by the points of M. For $y \in M$, we obtain from our stringor bundle a family $(\mathcal{N}_\beta)$ of von Neumann algebras, indexed by the set $P_{x,y}M$ of paths connecting the base point x with y. Moreover, these von Neumann algebras are pairwise canonically Morita equivalent, via the invertible $\mathcal{N}_{\beta_1}$–$\mathcal{N}_{\beta_2}$-bimodule $F(LM)_{\beta_1 \cup \beta_2}$, in a way compatible with triples and quadruples of paths. Such a structure is as good as a single von Neumann algebra; thus, our stringor bundle may be seen as a family of von Neumann algebras parameterized by the points of M, associated to a string structure.

On the other hand, it is clear that the described definition of the stringor bundle is not quite optimal. For example, it depends on the choice of a base point x, which is a typical disadvantage of regression. Then, its construction depends rather indirectly on the string structure, and the *string 2-group* makes no direct appearance. We expect that there will be a more direct construction in the future that will avoid these problems. But even if such a construction is found, we believe that the results of the present article will remain useful, for instance to construct differential operators acting on spinors on loop spaces, whereas no theory of operators acting on sections of 2-vector bundles is available.

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Appendix A. Bimodules of von Neumann algebras

A.1 Standard forms of a von Neumann algebra

In this appendix, we recall some facts about standard forms of von Neumann algebras. We first recall the notion of modules of von Neumann algebras. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be von Neumann algebras. A *left* $\mathcal{A}_1$*-module* is a Hilbert space H equipped with a normal (i.e. σ-weakly continuous) $*$-homomorphism $\mathcal{A}_1 \to \mathcal{B}(H)$. We adopt the notation $a \triangleright v$ for the element $a \in \mathcal{A}_1$ acting on the vector $v \in H$. A *right* $\mathcal{A}_2$*-module* is a Hilbert space H equipped with a normal $*$-homomorphism $\mathcal{A}_2^{\mathrm{opp}} \to \mathcal{B}(H)$ (where $\mathcal{A}_2^{\mathrm{opp}}$ is the opposite algebra of $\mathcal{A}_2$). We adopt the notation $v \triangleleft a$ for the right action of $a \in \mathcal{A}_2$ on $v \in H$. An $\mathcal{A}_1$–$\mathcal{A}_2$*-bimodule* is a Hilbert space H which is a left $\mathcal{A}_1$-module and at the same time a right $\mathcal{A}_2$-module, such that the left and right actions commute. If H is an $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule, and $\tilde{H}$ is an $\widetilde{\mathcal{A}}_1$–$\widetilde{\mathcal{A}}_2$-bimodule, then an *intertwiner* from H to $\tilde{H}$ is a triple (f_1, f_2, T) consisting of normal $*$-homomorphisms $f_1 : \mathcal{A}_1 \to \widetilde{\mathcal{A}}_1$ and $f_2 : \mathcal{A}_2 \to \widetilde{\mathcal{A}}_2$, and of a bounded linear operator $T : H \to \tilde{H}$ that intertwines both actions along f_1 and f_2, i.e.

$$T(a_1 \triangleright v \triangleleft a_2) = f_1(a_1) \triangleright T(v) \triangleleft f_2(a_2)$$

for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$, and $v \in H$. If f_1 and f_2 are identities, we just say that T is an intertwiner of $\mathcal{A}_1$–$\mathcal{A}_2$-bimodules.

Next, we recall the notion of a standard form, see [Tak10, Chapter IX, Definition 1.13] or [Haa75].

Definition A.1.1. A *standard form* of a von Neumann algebra $\mathcal{A}$ is a quadruple $(\mathcal{A}, H, J, P)$, where H is a left $\mathcal{A}$-module, J is an anti-linear isometry with $J^2 = 1$, and P is a closed self-dual cone in H, subject to the following conditions:

(i) $J\mathcal{A}J = \mathcal{A}'$;
(ii) $JaJ = a^*$ for all $a \in \mathcal{A} \cap \mathcal{A}'$;
(iii) $Jv = v$ for all $v \in P$;
(iv) $aJaJP \subseteq P$ for all $a \in \mathcal{A}$.

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The following result, proved in [Haa75, Theorem 2.3], tells us that standard forms are unique up to unique isomorphism.

Theorem A.1.2. *Suppose that* $(\mathcal{A}_1, H_1, J_1, P_1)$ *and* $(\mathcal{A}_2, H_2, J_2, P_2)$ *are standard forms, and that* π *is an isomorphism of* $\mathcal{A}_1$ *onto* $\mathcal{A}_2$*. Then, there exists a unique unitary operator* u *from* H_1 *onto* H_2 *such that:*

(i) $\pi(a) = uau^*$ *for all* $a \in \mathcal{A}_1$*;*
(ii) $J_2 = uJ_1u^*$*;*
(iii) $P_2 = uP_1$.

Let H be a left $\mathcal{A}$-module. A vector $\xi \in H$ is called *cyclic* if $\mathcal{A} \triangleright \xi$ is dense in H, and it is called *separating* if the map $\mathcal{A} \to H, a \mapsto a \triangleright \xi$ is injective. If a cyclic and separating vector $\xi \in H$ is given, then one can equip H with the structure of a standard form of $\mathcal{A}$, a fact that we make use of in § 3.3. We give the main points of the construction here.

First, we consider the densely defined Tomita operator $S : H \to H$. It is defined to be the closure of the operator

$$\mathcal{A} \triangleright \xi \to \mathcal{A} \triangleright \xi, \quad a \triangleright \xi \mapsto a^* \triangleright \xi.$$

We then write $S = J\Delta^{1/2}$ for the polar decomposition of S, where J is an anti-unitary map called the *modular conjugation* and $\Delta^{1/2}$ is a positive unbounded map called the *modular operator*. A fundamental result of Tomita–Takesaki theory is the following, proven e.g. in [Tak10, Chapter IX].

Theorem A.1.3. *The assignment* $a \mapsto Ja^*J$ *is an anti-isomorphism of von Neumann algebras from* $\mathcal{A}$ *onto its commutant* $\mathcal{A}'$*.*

We define $P \subset H$ to be the closure of $\{JaJa \triangleright \xi \in H \mid a \in \mathcal{A}\}$. Then, P is a closed self-dual cone in H. It is proved in [Ara74, Theorem 4] that $Jv = v$ for all $v \in P$ and that $aJaJP \subseteq P$ for all $a \in \mathcal{A}$. In conclusion, we have the following result.

Proposition A.1.4. *The quadruple* $(\mathcal{A}, H, J, P)$ *is a standard form of* $\mathcal{A}$*.*

Remark A.1.5. We remark that any standard form $(\mathcal{A}, H, J, P)$ of a von Neumann algebra $\mathcal{A}$ can be equipped with the structure of an $\mathcal{A}$–$\mathcal{A}$-bimodule by defining the right action by $v \triangleleft a := Ja^*J \triangleright v$. Theorem A.1.3 readily shows that left and right action commute. Further, in Theorem A.1.2, the triple (π, π, u) is automatically a unitary intertwiner of bimodules.

It is well-known that, through the Gelfand–Naimark–Segal (GNS) construction, any normal state on a von Neumann algebra $\mathcal{A}$ produces a representation of $\mathcal{A}$ which has a cyclic vector. If the state is faithful, then it has a cyclic and separating vector. This construction will be useful later, so we review the main steps now. Let $\phi : \mathcal{A} \to \mathbb{C}$ be a faithful and normal state on a von Neumann algebra $\mathcal{A}$. Then the assignment $\mathcal{A} \times \mathcal{A} \to \mathbb{C}, (a, b) \mapsto \phi(b^*a)$ is a non-degenerate sesquilinear form on $\mathcal{A}$. We write $L^2_\phi(\mathcal{A})$ for the completion of $\mathcal{A}$ with respect to this sesquilinear form. The algebra $\mathcal{A}$ acts from the left on $L^2_\phi(\mathcal{A})$ by (the extension of) left multiplication. Clearly the identity $\mathbb{1} \in \mathcal{A} \subseteq L^2_\phi(\mathcal{A})$ is a cyclic and separating vector for this left action. It follows that the quadruple $(\mathcal{A}, L^2_\phi(\mathcal{A}), J, P)$ is a standard form of $\mathcal{A}$.

Lemma A.1.6. *Let* H *be a left* $\mathcal{A}$*-module, and let* $\xi \in H$ *be a cyclic and separating vector. Let* $\phi : \mathcal{A} \to \mathbb{C}$ *be the faithful and normal state* $\phi(a) = \langle a \triangleright \xi, \xi \rangle$*. The unitary map* $u : H \to L^2_\phi(\mathcal{A})$ *from Theorem A.1.2 with respect to* $\pi = 1$ *is given by the closure of the map*

$$u : a \triangleright \xi \mapsto a.$$

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Proof. A straightforward verification shows that u satisfies properties (i)–(iii) from Theorem A.1.2. $\square$

A.2 Connes fusion of bimodules

In this appendix we review Connes fusion, and provide results about functoriality and associativity specifically adapted to our situation. Let $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$ be von Neumann algebras. Let H be an $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule and let K be an $\mathcal{A}_2$–$\mathcal{A}_3$-bimodule. In short, the Connes fusion of H with K is an $\mathcal{A}_1$–$\mathcal{A}_3$-bimodule $H \boxtimes_\phi K$, which is defined with respect to some faithful and normal state $\phi : \mathcal{A}_2 \to \mathbb{C}$. The result will be, up to unique isomorphism, independent of the state ϕ. Our main references are [Tho11] and [Tak10, Chapter IX]. The articles [Bro03] and [BDH14] provide results similar to ours concerning functoriality and associativity of Connes fusion.

We consider the standard form $L^2_\phi(\mathcal{A}_2)$ of $\mathcal{A}_2$ as an $\mathcal{A}_2$–$\mathcal{A}_2$-bimodule, as explained in Appendix A.1. We write $\mathcal{D}(H,\phi) := \mathrm{Hom}_{-,\mathcal{A}_2}(L^2_\phi(\mathcal{A}_2), H)$ for the space of bounded right module maps from $L^2_\phi(\mathcal{A}_2)$ into H. This space is canonically an $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule; explicitly, if $x \in \mathcal{D}(H,\phi)$, $v \in L^2_\phi(\mathcal{A}_2)$, $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$, then we have

$$(a_1 \triangleright x)(v) := a_1 \triangleright x(v) \quad \text{and} \quad (x \triangleleft a_2)(v) := x(a_2 \triangleright v).$$

We note that $\mathcal{D}(H,\phi)$ includes into H as a dense subspace through the map $x \mapsto x(\mathbb{1})$. This map is, however, not an intertwiner between bimodules on the nose, in fact, one may verify that the right action is twisted by conjugation by the modular operator $\Delta^{1/2}$.

There is a canonical $\mathcal{A}_2$-valued inner product on $\mathcal{D}(H,\phi)$, defined as follows. If $x \in \mathcal{D}(H,\phi)$, then its adjoint, written x^*, is an element of $\mathrm{Hom}_{-,\mathcal{A}_2}(H, L^2_\phi(\mathcal{A}_2))$. Hence, if $x, y \in \mathcal{D}(H,\phi)$, then

$$y^* x \in \mathrm{Hom}_{-,\mathcal{A}_2}(L^2_\phi(\mathcal{A}_2), L^2_\phi(\mathcal{A}_2)).$$

There is a canonical isomorphism $p_\phi : \mathrm{Hom}_{-,\mathcal{A}_2}(L^2_\phi(\mathcal{A}_2), L^2_\phi(\mathcal{A}_2)) \to \mathcal{A}_2$, which is determined by the relation $p_\phi(x) \triangleright v = x(v)$, for $x \in \mathrm{Hom}_{-,\mathcal{A}_2}(L^2_\phi(\mathcal{A}_2), L^2_\phi(\mathcal{A}_2))$ and $v \in L^2_\phi(\mathcal{A}_2)$. The aforementioned $\mathcal{A}_2$-valued inner product on $\mathcal{D}(H,\phi)$ is given by $(x,y) = p_\phi(y^*x)$. On the algebraic tensor product $\mathcal{D}(H,\phi) \otimes K$ we define a (degenerate) sesquilinear form by

$$\langle (x \otimes v), (y \otimes w) \rangle_\phi = \langle p_\phi(y^* x) \triangleright v, w \rangle_K.$$

Definition A.2.1. The *Connes fusion product* $H \boxtimes_\phi K$ *of* H *with* K *relative to the faithful and normal state* ϕ of $\mathcal{A}_2$ is the completion of

$$\mathcal{D}(H,\phi) \otimes K / \ker\langle \cdot, \cdot \rangle_\phi$$

with respect to the inner product $\langle \cdot, \cdot \rangle_\phi$, with the left $\mathcal{A}_1$ action obtained from the $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule structure of $\mathcal{D}(H,\phi)$, and the right $\mathcal{A}_3$ action obtained from the $\mathcal{A}_2$–$\mathcal{A}_3$ bimodule structure on K.

The definition above does not treat H and K on equal footing, the following observation tells us that this is just an artifact of our description. We define $\mathcal{D}'(K,\phi) = \mathrm{Hom}_{\mathcal{A}_2 -}(L^2_\phi(\mathcal{A}_2), K)$ to be the space of bounded left module maps from $L^2_\phi(\mathcal{A}_2)$ into K. We then have that $\mathcal{D}'(K,\phi)$ includes into K as a dense subspace, through the map $x \mapsto x(\mathbb{1})$. Using the canonical isomorphism $p'_\phi : \mathrm{Hom}_{\mathcal{A}_2 -}(L^2_\phi(\mathcal{A}_2), L^2_\phi(\mathcal{A}_2)) \to \mathcal{A}_2$ we define sesquilinear forms on $H \otimes \mathcal{D}'(K,\phi)$ and on $\mathcal{D}(H,\phi) \otimes \mathcal{D}'(K,\phi)$, respectively,

$$\begin{aligned} &\langle (v \otimes x'), (w \otimes y') \rangle'_\phi = \langle v \triangleleft p'_\phi((y')^* x'), w \rangle_H, && v, w \in H, \quad x', y' \in \mathcal{D}'(K,\phi), \\ &\langle (x \otimes x'), (y \otimes y') \rangle''_\phi = \langle p_\phi(y^* x) \triangleright \mathbb{1} \triangleleft p'_\phi((y')^* x'), \mathbb{1} \rangle_{L^2_\phi(\mathcal{A}_2)} && x, y \in \mathcal{D}(H,\phi) \quad x', y' \in \mathcal{D}'(K,\phi). \end{aligned}$$

In [Tho11, § 5.2] or [Tak10, Proposition 3.15 and Definition 3.16] the following is proved.

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Lemma A.2.2. *The spaces $H \otimes \mathcal{D}'(K,\phi)/\ker\langle\cdot,\cdot\rangle'_\phi$ and $\mathcal{D}(H,\phi)\otimes\mathcal{D}'(K,\phi)/\ker\langle\cdot,\cdot\rangle''_\phi$ are dense in $H \boxtimes_\phi K$.*

Lemma A.2.2 allows us to identify $H \boxtimes_\phi K$ with the completion of $H \otimes \mathcal{D}'(K,\phi)/\ker\langle\cdot,\cdot\rangle'_\phi$, which will be used in Proposition A.2.5 in order to write down the associator for the Connes fusion product in a nice way.

Next is the functoriality of the Connes fusion product. Since we have not seen this written up in the way we need it, we will discuss this in detail. Let $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$ be further von Neumann algebras. Let H' be a $\mathcal{B}_1$–$\mathcal{B}_2$-bimodule and let K' be a $\mathcal{B}_2$–$\mathcal{B}_3$-bimodule, let $\nu_i : \mathcal{A}_i \to \mathcal{B}_i$, for $i = 1,2,3$, be isomorphisms of von Neumann algebras, and let $\nu_H : H \to H'$ and $\nu_K : K \to K'$ be unitary maps, such that (ν_1,ν_2,ν_H) and (ν_2,ν_3,ν_K) are intertwiners. Let $\phi' : \mathcal{B}_2 \to \mathbb{C}$ be a faithful and normal state. Finally, denote by $\psi : \mathcal{A}_2 \to \mathbb{C}$ the faithful and normal state $\phi' \circ \nu_2$. Let $u : L^2_\phi(\mathcal{A}_2) \to L^2_\psi(\mathcal{A}_2)$ be the unitary given by Theorem A.1.2, with $\pi = \mathbb{1}$.

Proposition A.2.3. *Connes fusion is a functor. Explicitly, the homomorphism $\nu_2 : \mathcal{A}_2 \to \mathcal{B}_2$ extends to a unitary map $\overline{\nu}_2 : L^2_\psi(\mathcal{A}_2) \to L^2_{\phi'}(\mathcal{B}_2)$ such that $(\nu_2,\nu_2,\overline{\nu}_2)$ is an intertwiner. Then, the map*

$$\mathcal{D}(H,\phi)\otimes K \to \mathcal{D}(H',\phi')\otimes K', \quad x \otimes v \mapsto \nu_H x u^* \overline{\nu}_2^* \otimes \nu_K(v), \tag{A.1}$$

induces a unitary map

$$\nu_H \boxtimes \nu_K : H \boxtimes_\phi K \to H' \boxtimes_{\phi'} K',$$

such that $(\nu_1,\nu_3,\nu_H\boxtimes\nu_K)$ is an intertwiner. Moreover, the construction of this intertwiner is compatible with composition.

Here, compatibility with composition means the following. Suppose we are given the following further data: von Neumann algebras $\mathcal{C}_i$ and von Neumann algebra isomorphisms $\nu'_i : \mathcal{B}_i \to \mathcal{C}_i$ for $i = 1,2,3$, a $\mathcal{C}_1$–$\mathcal{C}_2$-bimodule H'' and a $\mathcal{C}_2$–$\mathcal{C}_3$-bimodule K'', a faithful and normal state $\phi'' : \mathcal{C}_2 \to \mathbb{C}$, and unitary maps $\nu_{H'} : H' \to H''$ and $\nu_{K'} : K' \to K''$ such that $(\nu'_1,\nu'_2,\nu_{H'})$ and $(\nu'_2,\nu'_3,\nu_{K'})$ are intertwiners. Then the following diagram commutes.

$$\begin{tikzcd}
 & H' \boxtimes_{\phi'} K' \arrow[dr, "\nu_{H'} \boxtimes \nu_{K'}"] & \\
H \boxtimes_\phi K \arrow[ur, "\nu_H \boxtimes \nu_K"] \arrow[rr, "\nu_{H'}\nu_H \boxtimes \nu_{K'}\nu_K"'] & & H'' \boxtimes_{\phi''} K''
\end{tikzcd} \tag{A.2}$$

Proof of Proposition A.2.3. That ν_2 extends to a unitary map follows from the fact that it intertwines the inner product $\mathcal{A}_2 \times \mathcal{A}_2 \to \mathbb{C}, (a_1,a_2) \mapsto \psi(a_2^* a_1)$ with the inner product $\mathcal{B}_2 \times \mathcal{B}_2 \to \mathbb{C}, (a_1,a_2) \mapsto \phi'(a_2^* a_1)$. That $(\nu_2,\nu_2,\overline{\nu}_2)$ is an intertwiner follows from the fact that ν_2 is a isomorphism, which implies that $J_{\phi'} = \overline{\nu}_2 J_\phi \overline{\nu}_2^*$.

To prove that the map $x \otimes v \mapsto \nu_H x u^* \overline{\nu}_2^* \otimes \nu_K(v)$ induces an isomorphism it suffices to show that it intertwines the inner products. Explicitly, we need to prove that for all $x \otimes v$ and $y \otimes w$ in $\mathcal{D}(H,\phi)\otimes K$ we have

$$\langle \nu_H x u^* \overline{\nu}_2^* \otimes \nu_K(v), \nu_H y u^* \overline{\nu}_2^* \otimes \nu_K(w)\rangle = \langle x \otimes v, y \otimes w\rangle.$$

We start from the left-hand side

$$\begin{aligned}\langle \nu_H x u^* \overline{\nu}_2^* \otimes \nu_K(v), \nu_H y u^* \overline{\nu}_2^* \otimes \nu_K(w)\rangle &= \langle p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*)\nu_K(v), \nu_K(w)\rangle \\ &= \langle \nu_K^* p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*)\nu_K(v), w\rangle.\end{aligned}$$

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Hence, it suffices to show that

$$\nu_K^* p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*) \nu_K(v) = p_\phi(y^* x) v. \tag{A.3}$$

Using the fact that ν_K is an intertwiner along ν_2 we obtain

$$\nu_K^* p_{\phi'}\big(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*\big) \nu_K = \nu_2^*\big(p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*)\big). \tag{A.4}$$

Now, we compute the action of the right-hand side of (A.4) on an element $a \in L^2_\psi(\mathcal{B}_2)$, using the definitions of $p_{\phi'}$ and p_ϕ:

$$\nu_2^*(p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*))a = \overline{\nu}_2^*(p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*)\overline{\nu}_2(a)) = u y^* x u^* a = u p_\phi(y^* x) u^*(a) = p_\phi(y^* x) a,$$

since this holds for all a in $L^2_\psi(\mathcal{B}_2)$ we obtain $\nu_2^*(p_{\phi'}(\overline{\nu}_2 u y^* x u^* \overline{\nu}_2^*)) = p_\phi(y^* x)$. Together with (A.4), this implies (A.3).

Finally, to prove that the diagram (A.2) commutes we shall prove that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{D}(H,\phi) \otimes K \arrow[r] \arrow[rr, bend right] & \mathcal{D}(H',\phi') \otimes K' \arrow[r] & \mathcal{D}(H'',\phi'') \otimes K''
\end{tikzcd}$$

The composition of the arrows on top is the map

$$x \otimes v \mapsto \nu_{H'} \nu_H x u^* \overline{\nu}_2^* (u')^* (\overline{\nu}_2')^* \otimes \nu_{K'} \nu_K(v), \tag{A.5}$$

where $u' : L^2_{\phi'}(\mathcal{B}_2) \to L^2_{\phi''\nu_2'}(\mathcal{B}_2)$ is the unitary given by Theorem A.1.2, with $\pi = \mathbb{1}$. On the other hand, the bottom map is given by

$$x \otimes v \mapsto \nu_{H'} \nu_H x (u'')^* \overline{\nu}_2^* (\overline{\nu}_2')^* \otimes \nu_{K'} \nu_K(v), \tag{A.6}$$

where $u'' : L^2_\phi(\mathcal{A}_2) \to L^2_{\phi''\nu_2'\nu_2}(\mathcal{A}_2)$ is the unitary given by Theorem A.1.2, with $\pi = \mathbb{1}$. Note that in this expression, $\overline{\nu}_2$ is viewed as an isomorphism from $L^2_{\phi''\nu_2'\nu_2}(\mathcal{A}_2)$ into $L^2_{\phi''\nu_2'}(\mathcal{B}_2)$ (instead of from $L^2_{\phi'\nu_2}(\mathcal{B}_2)$ into $L^2_{\phi'}(\mathcal{B}_2)$ as before). Comparing (A.5) with (A.6), we see that it is sufficient to prove that

$$u^* \overline{\nu}_2^* (u')^* (\overline{\nu}_2')^* = (u'')^* \overline{\nu}_2^* (\overline{\nu}_2')^*.$$

Taking the adjoint on both sides of the equation and cancelling $\overline{\nu}_2'$, we see that this is equivalent to

$$u' \overline{\nu}_2 u = \overline{\nu}_2 u''.$$

One checks that both $u'\overline{\nu}_2 u$ and $\overline{\nu}_2 u''$ are isomorphisms from $L^2_\phi(\mathcal{A}_2)$ into $L^2_{\phi''\nu_2'}(\mathcal{B}_2)$. We claim that they both satisfy properties (i)–(iii) from Theorem A.1.2, hence that by the uniqueness statement in that theorem we conclude they must be equal. Let us check that they do, in fact, satisfy properties (i)–(iii).

(i) Both maps intertwine the left action along ν_2.
(ii) By construction we have $\overline{\nu_2} u'' J_\phi (u'')^* \overline{\nu_2}^* = J_{\phi''\nu_2'} = u' \overline{\nu}_2 u J_\phi u^* \overline{\nu}_2^* (u')^*$.
(iii) From Theorem A.1.2 we have that $u P_\phi = P_{\phi'\nu_2}$. Now we argue that $\overline{\nu}_2 P_{\phi'\nu_2} = P_{\phi'}$. Let $J_{\phi'\nu_2} a J_{\phi'\nu_2} a \in P_{\phi'\nu_2}$ be arbitrary. Then we compute

$$\overline{\nu}_2 J_{\phi'\nu_2} a J_{\phi'\nu_2} a = \overline{\nu}_2 J_{\phi'\nu_2} \overline{\nu}_2^* \overline{\nu}_2 a \overline{\nu}_2^* \overline{\nu}_2 J_{\phi'\nu_2} \overline{\nu}_2^* \overline{\nu}_2 a = J_{\phi'} \nu_2(a) J_{\phi'} \overline{\nu}_2 a,$$

from which the claim follows. Then, another application of Theorem A.1.2 yields $u' P_{\phi'} = P_{\phi''\nu_2'}$, hence $u' \overline{\nu_2} u P_\phi = P_{\phi''\nu_2'}$. A similar argument then shows that $\overline{\nu}_2 u'' P_\phi = P_{\phi''\nu_2'}$.

This completes the proof. □

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Remark A.2.4. In [BDH14, Theorem 6.23] a statement more general than Proposition A.2.3 is given, where none of the maps ν_i, ν_H, or ν_K is assumed to be an isomorphism. However, in this more general setting (A.1) does not make sense, and it is this explicit form that we make use of in the main text.

We may specialize Proposition A.2.3 to the case that all isomorphisms ν are identities, to conclude that for any two faithful and normal states ϕ, ϕ' on $\mathcal{B}_2$, there exists a natural isomorphism

$$H \boxtimes_\phi K \to H \boxtimes_{\phi'} K.$$

The fact that the diagram (A.2) commutes then tells us that these isomorphisms are coherent, which allows to define the Connes fusion product $H \boxtimes_{\mathcal{A}_2} K$ as the colimit of $H \boxtimes_\phi K$, where ϕ ranges over all faithful and normal states of $\mathcal{A}_2$. If no state is preferred, then we always refer to this limit, and if later a state is picked, then we have a unique isomorphism. In [Tak10, Exercise IX.3.8 (p. 210)] another approach to defining a tensor product of bimodules without reference to a state is given.

Proposition A.2.5. *Connes fusion is associative. Explicitly, if H is an $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule, K is an $\mathcal{A}_2$–$\mathcal{A}_3$-bimodule, and L is an $\mathcal{A}_3$–$\mathcal{A}_4$-bimodule, ϕ is a faithful and normal state on $\mathcal{A}_2$, and ψ is a faithful and normal state on $\mathcal{A}_3$, then the map*

$$(\mathcal{D}(H,\phi) \otimes K) \otimes \mathcal{D}'(L,\psi) \to \mathcal{D}(H,\phi) \otimes (K \otimes \mathcal{D}'(L,\psi))$$
$$((x \otimes v) \otimes y) \mapsto (x \otimes (v \otimes y)),$$

induces a unitary intertwiner

$$\alpha_{H,K,L} : (H \boxtimes_\phi K) \boxtimes_\psi L \to H \boxtimes_\phi (K \boxtimes_\psi L)$$

of $\mathcal{A}_1$–$\mathcal{A}_4$-bimodules. Moreover, these intertwiners are natural and satisfy the pentagon identity.

Here, naturality means the following. Let H, K, L be $\mathcal{A}_1$–$\mathcal{A}_2$-, $\mathcal{A}_2$–$\mathcal{A}_3$-, and $\mathcal{A}_3$–$\mathcal{A}_4$-bimodules respectively, and we let H', K', L' be $\mathcal{B}_1$–$\mathcal{B}_2$-, $\mathcal{B}_2$–$\mathcal{B}_3$-, and $\mathcal{B}_3$–$\mathcal{B}_4$-bimodules, respectively. Moreover, we consider isomorphisms $\nu_i : \mathcal{A}_i \to \mathcal{B}_i$ for $i = 1,2,3,4$, and unitary maps $\nu_H : H \to H'$, $\nu_K : K \to K'$ and $\nu_L : L \to L'$ such that (ν_1, ν_2, ν_H), (ν_2, ν_3, ν_K), and (ν_3, ν_4, ν_L) are intertwiners. Finally, let ϕ, ψ, ϕ', ψ' be faithful and normal states on $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{B}_2$, and $\mathcal{B}_3$, respectively. Then, naturality means the commutativity of the following diagram.

$$\begin{tikzcd}
(H \boxtimes_\phi K) \boxtimes_\psi L \arrow[r, "\alpha_{H,K,L}"] \arrow[d, "(\nu_H \boxtimes \nu_K) \boxtimes \nu_L"'] & H \boxtimes_\phi (K \boxtimes_\psi L) \arrow[d, "\nu_H \boxtimes (\nu_K \boxtimes \nu_L)"] \\
(H' \boxtimes_{\phi'} K') \boxtimes_{\psi'} L' \arrow[r, "\alpha_{H',K',L'}"'] & H' \boxtimes_{\phi'} (K' \boxtimes_{\psi'} L')
\end{tikzcd}$$

Proof of Proposition A.2.5. That the above map indeed induces an isomorphism of bimodules is [Tak10, Theorem 3.20]. A proof of the pentagon identity can be found in [Bro03]. Naturality can be proved by a straightforward computation using the explicit forms of the isomorphisms on the appropriate dense subspaces from Propositions A.2.3 and A.2.5 □

Finally, we discuss the fact that the standard form of a von Neumann algebra is neutral with respect to Connes fusion.

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PROPOSITION A.2.6. *The $\mathcal{A}_2$–$\mathcal{A}_2$-bimodule $L^2_\phi(\mathcal{A}_2)$ is neutral with respect to Connes fusion. Explicitly, for every $\mathcal{A}_2$–$\mathcal{A}_3$-module K the map*

$$\mathcal{D}(L^2_\phi(\mathcal{A}_2), \phi) \otimes K \to K, \quad x \otimes v \mapsto p_\phi(x) \triangleright v,$$

induces a unitary intertwiner

$$\lambda_K : L^2_\phi(\mathcal{A}_2) \boxtimes_\phi K \to K$$

of $\mathcal{A}_2$–$\mathcal{A}_3$-bimodules. Likewise, for every $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule H, the map

$$H \otimes \mathcal{D}'(L^2_\phi(\mathcal{A}_2), \phi) \to H, \quad w \otimes y \mapsto w \triangleleft p'_\phi(y),$$

induces a unitary intertwiner

$$\rho_H : H \boxtimes_\phi L^2_\phi(\mathcal{A}_2) \to H$$

of $\mathcal{A}_1$–$\mathcal{A}_2$-bimodules. Moreover, these maps are natural, and compatible with the associator in the sense that

$$\rho_H \boxtimes \mathbb{1}_K = (\mathbb{1}_H \boxtimes \lambda_K) \circ \alpha_{H, L^2_\phi(\mathcal{A}_2), K}.$$

Finally, we have $\rho_{L^2_\phi(\mathcal{A}_2)} = \lambda_{L^2_\phi(\mathcal{A}_2)}$.

Proof. It is straightforward to see that the given maps induce the claimed intertwiners, and that naturality and the compatibility condition is satisfied. For more detail, we refer to [Bro03], see in particular Proposition 3.5.3 therein. To prove that $\rho_{L^2_\phi(\mathcal{A}_2)} = \lambda_{L^2_\phi(\mathcal{A}_2)}$ it suffices to prove that the diagram

$$\begin{tikzcd}
\mathcal{D}(L^2_\phi(\mathcal{A}_2), \phi) \otimes \mathcal{D}'(L^2_\phi(\mathcal{A}_2), \phi) \arrow[r] \arrow[d] & \mathcal{D}(L^2_\phi(\mathcal{A}_2), \phi) \otimes L^2_\phi(\mathcal{A}_2) \arrow[d] \\
L^2_\phi(\mathcal{A}_2) \otimes \mathcal{D}'(L^2_\phi(\mathcal{A}_2), \phi) \arrow[r] & L^2_\phi(\mathcal{A}_2)
\end{tikzcd}$$

commutes, which follows from the computation

$$x(\mathbb{1}) \triangleleft p'_\phi(y) = p_\phi(x) \triangleright \mathbb{1} \triangleleft p'_\phi(y) = p_\phi(x) \triangleright y(\mathbb{1}),$$

where $x \in \mathcal{D}(L^2_\phi(\mathcal{A}_2), \phi)$, $y \in \mathcal{D}'(L^2_\phi(\mathcal{A}_2), \phi)$ and $\mathbb{1}$ is the unit of $\mathcal{A}_2$. $\square$

At the end, we want to transfer the results of Proposition A.2.6 from the canonical standard form to other standard forms. Let I be a left $\mathcal{A}_2$-module, and let $\xi \in I$ be a cyclic and separating vector, so that I becomes a standard form of $\mathcal{A}_2$, see Proposition A.1.4. Let $\phi : \mathcal{A}_2 \to \mathbb{C}$ be the faithful and normal state $\phi(a) = \langle a \triangleright \xi, \xi \rangle$, and consider the corresponding standard form $L^2_\phi(\mathcal{A}_2)$. By Theorem A.1.2, both standard forms are isomorphic under a unique isomorphism $u : I \to L^2_\phi(\mathcal{A}_2)$, which is by Lemma A.1.6 given by the extension of the map $a \triangleright \xi \mapsto a$. Further, we recall from Remark A.1.5 that standard forms are $\mathcal{A}_2$–$\mathcal{A}_2$-bimodules, and that u is an intertwiner of $\mathcal{A}_2$–$\mathcal{A}_2$-bimodules.

COROLLARY A.2.7. *Let I be a left $\mathcal{A}_2$-module, and let $\xi \in I$ be a cyclic and separating vector. Then, I is neutral with respect to Connes fusion. More explicitly, for every $\mathcal{A}_2$–$\mathcal{A}_3$-bimodule K and every $\mathcal{A}_1$–$\mathcal{A}_2$-bimodule H the unitary intertwiners*

$$\lambda^I_K : I \boxtimes_\phi K \to K, \quad \lambda^I_K := \lambda_K \circ (u \boxtimes \mathbb{1}_K),$$

$$\rho^I_H : H \boxtimes_\phi I \to H, \quad \rho^I_H := \rho_H \circ (\mathbb{1}_H \boxtimes u)$$

are natural and compatible with the associator. Moreover, we have $\lambda^I_I = \rho^I_I$.

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Proof. Compatibility with the associator follows from the definition of λ_K^I and ρ_H^I and the naturality of the associator proved in Proposition A.2.5. Naturality and the coincidence $\lambda_I^I = \rho_I^I$ follow from the naturality of λ_K and ρ_H. □

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