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PAPER

The patch topology in univalent foundations

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Abstract
Stone locales together with continuous maps form a coreflective subcategory of spectral locales and perfect maps. A proof in the internal language of an elementary topos was previously given by the second-named author. This proof can be easily translated to univalent type theory using *resizing axioms*. In this work, we show how to achieve such a translation *without* resizing axioms, by working with large, locally small, and small-complete frames with small bases. This requires predicative reformulations of several fundamental concepts of locale theory in predicative HoTT/UF, which we investigate systematically.

Keywords: Univalent foundations; constructive mathematics; pointfree topology; locale theory; patch topology; spectral locale

1. Introduction

The category **Stone** of Stone locales together with continuous maps forms a coreflective subcategory of the category **Spec** of spectral locales and *perfect* maps. A proof in the internal language of an elementary topos was previously constructed by Escardó (1999, 2001), defining the patch frame as the frame of Scott continuous nuclei on a given frame.

In the present work, we show that the same construction can be carried out in predicative and constructive univalent foundations. In the presence of resizing axioms, which were originally considered as rules by Voevodsky (2011), it is straightforward to translate this proof to univalent type theory. At the time of writing, however, there is no known computational interpretation of the resizing axioms. Therefore, the question of whether resizing can be avoided in the construction of the patch locale is of interest. In such a predicative situation, the usual approach to locale theory is to work with presentations of locales, known as *formal topologies* (Sambin 1987; Coquand et al. 2003; Coquand and Tosun 2021). We show in our work, however, that it is possible to work with locales directly. This requires a number of modifications to the proofs and constructions of Escardó (1999, 2001):

(1) The major modification is that we work with large, locally small, and small-complete frames with small bases. The reason for this is that, in the absence of propositional resizing, there are no nontrivial examples of frames (de Jong and Escardó 2021). See Section 3 for further discussion.
(2) The patch frame is defined as the frame of Scott continuous nuclei by Escardó (1999, 2001). In order to prove that this is indeed a frame, one starts with the frame of all nuclei and then exhibits the Scott continuous nuclei as a subframe. This procedure, however, does not seem

to be possible in the context of our work, as it is not clear whether all nuclei can be shown to form a frame predicatively; so we construct the frame of Scott continuous nuclei *directly*, which requires predicative reformulations of all proofs about it inherited from the frame of all nuclei.

(3) In an impredicative setting, any frame has all Heyting implications, which are needed to construct open nuclei. Again, this does not seem to be the case in our predicative setting. We show, however, that it is possible to construct Heyting implications in large, locally small, and small-complete frames with small bases, by an application of the posetal instance of the Adjoint Functor Theorem.

(4) Similar to (3), we use the posetal Adjoint Functor Theorem to define the right adjoint of a frame homomorphism, which we then use to define the notion of a *perfect map*, namely, a map whose defining frame homomorphism's right adjoint is Scott continuous. This notion is used by Escardó in his impredicative proof (1999; 2001).

For the purposes of our work, a *spectral locale* is a locale in which any open can be expressed as the join of a family of compact opens. A continuous map of spectral locales is *spectral* if its defining frame homomorphism preserves compact opens. A *Stone locale* is one that is compact and in which any open is expressible as the join of a family of clopens. Every Stone locale is spectral since the clopens coincide with the compact opens in Stone locales. The patch frame construction is the right adjoint to the inclusion **Stone** $\hookrightarrow$ **Spec**. The main contribution of our work is the construction of this right adjoint in the predicative context of univalent type theory. We have also formalized (Tosun 2023) the development of this paper in the Agda proof assistant (Norell 2009) as part of the TypeTopology library (Escardó and contributors 2018). Our presentation here, however, is self-contained and can be followed independently of the formalization.

The organization of this paper is as follows. In Section 2, we present the type-theoretical context in which we work. In Section 3, we introduce our notion of locale discussed above. In Section 4, we present our definitions of spectral and Stone locales, which provide a suitable basis for a predicative development. In Section 5, we present the posetal instance of the Adjoint Functor Theorem for the simplified context of locales, which is central to our development. In Section 6, we define the meet-semilattice of perfect nuclei as preparation for the complete lattice of perfect nuclei, which we then construct in Section 7. Finally in Section 8, we prove the desired universal property, namely, that the patch locale exhibits the category **Stone** as a coreflective subcategory of **Spec**, where we restrict ourselves to locales with small bases.

Finally, we note that a preliminary version of the work that we present here previously appeared in (Tosun and Escardó 2023). Our work here extends *loc. cit.* in several directions. The presentation that we provide through Theorem 36 and Theorem 54 here involves new results. Furthermore, we provide a complete proof of the universal property of Patch through a new approach, as given in Theorem 84 and Theorem 86.

2. Foundations

In this section, we introduce the type-theoretical setting in which we work and then present the type-theoretical formulations of some of the preliminary notions that form the basis of our work. Our type-theoretical conventions follow those of de Jong and Escardó (2021; 2023) and The Univalent Foundations Program (2013).

We work in Martin-Löf Type Theory with binary sums $(-) + (-)$, dependent products Π, dependent sums Σ, the identity type $(-) = (-)$, and inductive types including the empty type $\mathbf{0}$, the unit type $\mathbf{1}$, and the type $\mathsf{List}(A)$ of lists, or *words*, over any type A. We denote by π_1 and π_2 the first and the second projections of a Σ type. We use the abbreviation $\mathbf{2}$ for the sum type $\mathbf{1}+\mathbf{1}$, and refer to this as the type of Booleans. We adhere to the convention of the HoTT Book (2013) of using $(-)\equiv(-)$ for judgmental equality and $(-)=(-)$ for the identity type.

We work explicitly with universes, for which we adopt the convention of using the variables $\mathscr{U},\mathscr{V},\mathscr{W},\ldots$ The ground universe is denoted $\mathscr{U}_0$ and the successor of a given universe $\mathscr{U}$ is denoted $\mathscr{U}^+$. The least upper bound of two universes is given by the operator $(-)\sqcup(-)$ which is assumed to be associative, commutative, and idempotent. We do not assume that the universes are, or are not, cumulative. Furthermore, $(-)^+$ is assumed to distribute over $(-)\sqcup(-)$. Universes are computed for the given type formers as follows:

- Given types $X:\mathscr{U}$ and $Y:\mathscr{V}$, the type $X+Y$ inhabits universe $\mathscr{U}\sqcup\mathscr{V}$.
- Given a type $X:\mathscr{U}$ and an X-indexed family $Y:X\to\mathscr{V}$, both $\sum_{x:X}Y(x)$ and $\prod_{x:X}Y(x)$ inhabit the universe $\mathscr{U}\sqcup\mathscr{V}$.
- Given a type $X:\mathscr{U}$ and inhabitants $x,y:X$, the identity type $x=y$ inhabits universe $\mathscr{U}$.
- The type $\mathbb{N}$ of natural numbers inhabits $\mathscr{U}_0$.
- The empty type $\mathbf{0}$ and the unit type $\mathbf{1}$ have copies in every universe $\mathscr{U}$, which we occasionally make explicit using the notations $\mathbf{0}_{\mathscr{U}}$ and $\mathbf{1}_{\mathscr{U}}$.
- Given a type $A:\mathscr{U}$, the type $\mathsf{List}(A)$ inhabits $\mathscr{U}$.

We assume the univalence axiom and therefore function extensionality and propositional extensionality. We maintain a careful distinction between *structure* and *property*, and reserve logical connectives for *propositional types,* that is, types A satisfying $\mathsf{isProp}\,(A):\equiv\Pi_{(x,y\,:\,A)}x=y$. We denote by $\Omega_{\mathscr{U}}$ the type of propositional types in universe $\mathscr{U}$, that is, $\Omega_{\mathscr{U}}:\equiv\Sigma_{(A\,:\,\mathscr{U})}\mathsf{isProp}\,(A)$. A type A is called a *set* if its identity type is always a proposition, that is, $x=y$ is a propositional type, for every pair of inhabitants $x,y:A$.

We assume the existence of *propositional truncation*, given by a type former $\|-\|:\mathscr{U}\to\mathscr{U}$ and a unit operation $|-|:A\to\|A\|$. It is given by the recursion principle:

$$\Pi_{(P\,:\,\mathscr{V})}\mathsf{isProp}\,(P)\to(X\to P)\to(\|X\|\to P).$$

The existential quantification operator is defined using propositional truncation as

$$\exists_{(x\,:\,A)}\,B(x):\equiv\left\|\,\Sigma_{(x\,:\,A)}B(x)\right\|.$$

When presenting proofs informally, we adopt the following conventions for avoiding ambiguity between propositional and non-propositional types.

- For the anonymous inhabitation $\|A\|$ of a type A, we say that A is inhabited.
- For truncated Σ types, we use the terminologies *there is* and *there exists*.
- We say *specified inhabitant* of type A to contrast it with the anonymous inhabitation $\|A\|$. Similarly, we say there is a *specified* or *chosen* element to emphasize that we are using Σ instead of $\exists$.
- When we use the phrase *has a*, we take it to mean a *specified* inhabitant. In contrast, we say *has some* when talking about an unspecified inhabitant. If we want to completely avoid ambiguity, we prefer to use the more explicit terminology of *has specified* or *has unspecified*.

If a given type A is a proposition, it is clear that it is logically equivalent to its own propositional truncation. The converse, however, is not always true: there are types that are logically equivalent to their own propositional truncations, despite not being propositions themselves. Types that satisfy this more general condition of being logically equivalent to their own truncations are said to have *split support* and have previously been investigated by Kraus et al. (2017).

Definition 1 ($\mathcal{V}$-small type). *A type $A : \mathcal{U}$ is called $\mathcal{V}$-small if it has a copy in universe $\mathcal{V}$. That is to say, $\Sigma_{(B\,:\,\mathcal{V})} A \simeq B$.*

Lemma 2. *The following are equivalent.*

(1) *For every type $A : \mathcal{U}$ and universe $\mathcal{V}$, the property of being $\mathcal{V}$-small is a proposition.*
(2) *Univalence holds.*

We also have the notion of *local smallness* that refers to the equality type always being small with respect to some universe.

Definition 3 (Local $\mathcal{V}$-smallness). *A type $X : \mathcal{U}$ is called locally $\mathcal{V}$-small if the identity type $x = y$ is $\mathcal{V}$-small for every $x, y : X$.*

We have mentioned the *propositional resizing* axiom in Section 1. Having formally defined the notion of $\mathcal{V}$-smallness, we can now give the precise definition of this axiom.

Definition 4. *The propositional $(\mathcal{U}, \mathcal{V})$-resizing axiom says that any proposition $P : \Omega_{\mathcal{U}}$ is $\mathcal{V}$-small. The global propositional resizing axiom says that, for any two universes $\mathcal{U}$ and $\mathcal{V}$, the propositional $(\mathcal{U}, \mathcal{V})$-resizing axiom holds.*

By *impredicative mathematics* in the context of univalent foundations, we mean the use of the above axiom. Our work here is concerned with the development of locale theory *without* the use of the above axiom.

We will use the following axiom of *set replacement*:

Definition 5 (Axiom: set replacement (cf. (Rijke, 2022, Axiom 18.1.18))). *Let $f : X \to Y$ be a function from a type $X : \mathcal{U}$ into a set $Y : \mathcal{V}$. The set replacement axiom says that if X is $\mathcal{U}$-small and the type Y is locally $\mathcal{W}$-small then the type $\mathsf{image}(f)$ is $(\mathcal{U} \sqcup \mathcal{W})$-small, where*

$$\mathsf{image}(f) :\equiv \Sigma_{(y\,:\,Y)} \exists_{(x\,:\,X)} f(x) = y.$$

Remark 6. *Notice that we obtain the following as a special case of the set replacement axiom: given a locally $\mathcal{U}$-small set $X' : \mathcal{U}^{+}$, the image of any function $f : X \to X'$ is $\mathcal{U}$-small whenever the type X is $\mathcal{U}$-small.*

An inhabitant of the set replacement principle can be constructed as a higher inductive type and is thus a theorem in a foundational setting where all higher inductive types are available (such as cubical type theory (Cohen et al. 2018)). We call this an *axiom* here since we would like to carefully keep track of which higher inductive types are used in our work. Moreover, de Jong (2023, Theorem 2.11.24) showed that the set replacement axiom is logically equivalent to the existence of small set quotients, which means that this principle does not require the use of an additional HIT if small set quotients are already available.

We now proceed to define, in the presented type-theoretical setting, some preliminary notions that are fundamental to our development of locale theory.

Definition 7 (Family). *A $\mathcal{U}$-family on a type A is a pair (I, α) where $I : \mathcal{U}$ and $\alpha : I \to A$. We denote the type of $\mathcal{U}$-families on type A by $\mathsf{Fam}_{\mathcal{U}}(A)$, that is, $\mathsf{Fam}_{\mathcal{U}}(A) :\equiv \Sigma_{(I\,:\,\mathcal{U})} I \to A$.*

Convention 8. *We often use the shorthand $(x)_{i:I}$ for families. In other words, instead of writing (I, x) for a family, we write $(x_i)_{i:I}$ where x_i denotes the application $x(i)$. Given a family (I, α), a subfamily* *of it is a family of the form $(J, \alpha \circ \beta)$ where (J, β) is a family on the index type I. When talking about a subfamily of some family $(x_i)_{i:I}$, we use the notation $(x_{i_j})_{j:J}$ to denote the subfamily given by a family $(i_j)_{j:J}$* □

We will also be talking about families (I, α) where the function α is an *embedding*. We refer to such families as *embedding families*.

Definition 9 (Embedding). *A function $f : X \to Y$ is called an embedding if, for every $y : Y$, the type $\Sigma_{(x:X)} f(x) = y$ is a proposition.*

Definition 10. *A $\mathscr{U}$-family (I, α) on a type A is said to be an embedding family if the function $\alpha : I \to A$ is an embedding. We denote the type of embedding families by $\mathsf{Fam}^{\hookrightarrow}_{\mathscr{U}}(A)$.*

3. Locales

A *locale* is a notion of space characterized solely by its lattice of opens. The lattice-theoretic notion abstracting the behavior of a lattice of open subsets is a *frame*: a lattice with finite meets and arbitrary joins in which the binary meets distribute over arbitrary joins.

Our type-theoretic definition of a frame is parameterized by three universes: (1) for the carrier set, (2) for the order, and (3) for the join-completeness, that is, the index types of families on which the join operation is defined. We adopt the convention of using the universe variables $\mathscr{U}$, $\mathscr{V}$, and $\mathscr{W}$ for these, respectively.

Definition 11 (Frame). *A $(\mathscr{U}, \mathscr{V}, \mathscr{W})$-frame L consists of*

- *a type $|L| : \mathscr{U}$,*
- *a partial order $(-) \leq (-) : |L| \to |L| \to \Omega_{\mathscr{V}}$,*
- *a top element $\mathbf{1} : |L|$,*
- *an operation $(-) \wedge (-) : |L| \to |L| \to |L|$ giving the greatest lower bound $U \wedge V$ of any two $U, V : |L|$,*
- *an operation $\bigvee(-) : \mathsf{Fam}_{\mathscr{W}}(|L|) \to |L|$ giving the least upper bound $\bigvee(I, \alpha)$ of any $\mathscr{W}$-family (I, α),*

such that binary meets distribute over arbitrary joins, that is, $U \wedge \bigvee_{i:I} x_i = \bigvee_{i:I} U \wedge x_i$ for every $x : |L|$ and $\mathscr{W}$-family $(x_i)_{i:I}$ in $|L|$.

As we have done in the definition above, we adopt the shorthand notation $\bigvee_{i:I} x_i$ for the join of a family $(x_i)_{i:I}$. We also adopt the usual abuse of notation and write L instead of $|L|$.

The reader might have noticed that we have not imposed a sethood condition on the underlying type of a frame in the above definition. The reason for this is that it follows automatically from the antisymmetry condition for partial orders that the underlying type of a frame is a set.

Lemma 12. *Let L be a $(\mathscr{U}^+, \mathscr{V}, \mathscr{U})$-frame and let $x, y : L$. We have that $x \leq y$ is $\mathscr{U}$-small if and only if the carrier of L is a locally $\mathscr{U}$-small type.*

Proof. Let $x, y : L$. It is a standard fact of lattice theory that $x \leq y \leftrightarrow x \wedge y = x$. ($\Rightarrow$) If the frame is locally small in the sense that $x \leq y$ is $\mathscr{U}$-small for every $x, y : L$, then the identity type $x = y$ must also be $\mathscr{U}$-small since $x = y$ is equivalent to the conjunction $(x \leq y) \wedge (y \leq x)$. ($\Leftarrow$) Conversely, if the carrier of L is a locally small type, then $x \leq y$ must be $\mathcal{U}$-small since it is equivalent to the identity type $x \wedge y = x$ which is $\mathcal{U}$-small by the local smallness of the carrier. □

We also note that, in the work of de Jong and Escardó (2021; 2021; 2023), the term $\mathcal{V}$-dcpo is used for a directed-complete partially ordered set whose directed joins are over $\mathcal{V}$-families. This terminology leaves the carrier and the order universes implicit, as the completeness universe is the only one relevant to the discussion in most cases.

We gave a highly general definition of the notion of frame in Definition 11: all three universes involved in the definition are permitted to live in separate universes. This generality, however, is never needed in our work since it is known that complete and small lattices cannot be constructed predicatively. This was first shown by Curi (2010a; 2010b), who proved that the existence of complete, small lattices is not provable in Aczel and Rathjen's Constructive[1] ZF set theory (2010), which is a predicative system.

In our setting, an analogous result was proved by de Jong and Escardó (2021), who showed that the existence of a nontrivial complete and small lattice is equivalent to a form of propositional resizing that is known to be independent of type theory—see (de Jong and Escardó 2021, Theorem 35) for details. Their result is more direct and is in the style of reverse constructive mathematics (Ishihara 2006): they show directly that the existence of a nontrivial complete and small lattice implies a form of propositional resizing. This formally shows that, when we adopt a predicative approach to locale theory, we are forced to work with large and small-complete lattices.

In light of this, we restrict attention to $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$ frames, which we refer to as *large, locally small, and small-complete* frames.

Convention 13. From now on, we fix a base universe $\mathcal{U}$ and refer to types that have isomorphic copies in $\mathcal{U}$ as small types. In contrast, we refer to types in $\mathcal{U}^+$ as large types. Accordingly, we hereafter take frame/locale to mean one that is large, locally small, and small complete with respect to the base universe $\mathcal{U}$. □

Definition 14 (Frame homomorphism). *Let K and L be two frames. A function $h : |K| \to |L|$ is called a frame homomorphism if it preserves the top element, binary meets, and joins of small families. We denote by* **Frm** *the category of frames and their homomorphisms (over our base universe $\mathcal{U}$).*

We adopt the notational conventions of Mac Lane and Moerdijk (1994). A *locale* is a frame considered in the opposite category, denoted **Loc** $:\equiv$ **Frm**$^{\mathrm{op}}$. To highlight this, we adopt the standard conventions of (1) using the letters $X, Y, Z, \ldots$ (or sometimes $A, B, C, \ldots$) for locales and (2) denoting the frame corresponding to a locale X by $\mathcal{O}(X)$. For variables that range over the frame of opens of a locale X, we use the letters $U, V, W, \ldots$ We use the letters f and g for continuous maps $X \to Y$ of locales. A continuous map $f : X \to Y$ is given by a frame homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$.

In point-set topology, a *basis* for a space X is a collection $\mathcal{B}$ of subsets of X such that every open $U \in \mathcal{O}(X)$ can be decomposed into a union of subsets contained in $\mathcal{B}$. These subsets are called the *basic opens*. In the context of point-free topology, the notion of a basis is captured by the lattice-theoretic notion of a *generating lattice*. In (Johnstone 2002, p. 548), for example, a basis for a locale $\mathcal{O}(X)$ is defined as a subset $\mathcal{B} \subseteq \mathcal{O}(X)$ such that every element of $\mathcal{O}(X)$ is expressible as a join of members of $\mathcal{B}$. We now give the formal definition of this notion in our type-theoretical setting.

Definition 15 (Directed family). *Let $(x_i)_{i:I}$ be a family on some type A that is equipped with a preorder $(-) \leq (-)$. This family is called directed if*

(1) *I is inhabited, and*
(2) *for every $i, j : I$, there exists some $k : I$ such that x_k is an upper bound of $\{x_i, x_j\}$.*

Definition 16 (Basis of a frame). *Let X be a locale. A basis for X is an embedding family $(B_i)_{i:I}$ such that for every $U : \mathscr{O}(X)$, there is a specified directed small family $(i_j)_{j:J}$ on the index type I, satisfying $U = \bigvee_{j:J} B_{i_j}$. We often drop the embedding requirement, in which case we speak of intensional bases. When we want to emphasize the distinction, we speak of extensional bases when we require the embedding condition.*

Notice that the embedding requirement says that the family in consideration does not have repetitions. However, it will often be convenient to work with families that might have repetitions. Such repetitions can be removed (constructively) if necessary, by factoring the family through its image. In the case of a spectral locale, for example, there can be different intensional bases but the extensional basis is unique.

Even though we do not require a basis to be given by a small family, we always work with small bases in our work.

Remark 17. *In the above definition, we required the covering families to be directed. This requirement is not essential since one can always construct an alternative basis in which the basic covering families are directified as follows:*

(1) *Given a base $(B_i)_{i:I}$, we take its closure under finite joins, that is, the map $B^{\uparrow} : \mathsf{List}(I) \to \mathscr{O}(X)$ defined by $B^{\uparrow}(i_0, \ldots, i_{n-1}) :\equiv B_{i_0} \vee \cdots \vee B_{i_{n-1}}$.*
(2) *Consider an open $U : \mathscr{O}(X)$ and denote by $(B_{i_j})_{j:J}$ the basic covering family that the basis $(B_i)_{i:I}$ gives for U. The basic covering family given by the new basis $B^{\uparrow}$ is defined to be the family $\alpha : \mathsf{List}(J) \to \mathscr{O}(X)$, defined by $\alpha(j_0, \ldots, j_{n-1}) :\equiv B_{i_{j_0}} \vee \cdots \vee B_{i_{j_{n-1}}}$.*

We thus obtain a basis $(B^{\uparrow}_s)_{s:\mathsf{List}(I)}$ in which the covering families are directed.

Although the directedness requirement is not essential, as explained in the above remark, we prefer to include it in our definition of basis for technical convenience. Finally, we also note that the directification of a small basis gives a small basis.

Lemma 18. *For every small, intensional basis $(B_i)_{i:I}$ on a locale X, the image of $B : I \to \mathscr{O}(X)$ is small (assuming set replacement).*

Proof. By appealing to the set replacement axiom (Definition 5) as explained in Remark 6, it just remains to show that the carrier of $\mathscr{O}(X)$ is a locally small type, which is always the case in a locally small locale thanks to Lemma 12. □

4. Spectral and Stone Locales

The standard impredicative definition of a spectral locale is as one in which the compact opens form a basis closed under finite meets (see II.3.2 from (Johnstone 1982)). To talk about compactness, we define the *way-below* relation.

Definition 19 (Way-below relation). *We say that an open U of a locale X is way below an open V, written $U \ll V$, if for every directed family $(W_i)_{i:I}$ with $V \leq \bigvee_{i:I} W_i$ there is some $i : I$ with $U \leq W_i$.*

Lemma 20. *Given any two opens $U, V : \mathcal{O}(X)$, the type $U \ll V$ is a proposition.*

The statement $U \ll V$ is thought of as expressing that open U is compact *relative to* open V. An open is said to be compact if it is compact relative to itself:

Definition 21 (Compact open of a locale). *An open $U : \mathcal{O}(X)$ is called compact if $U \ll U$.*

Example 22. *The bottom open $\mathbf{0}_X$ of every locale X is compact. Let $(U_i)_{i:I}$ be a directed family of opens of X with $\mathbf{0} \leq \bigvee_{i:I} U_i$. The fact that the family $(U_i)_{i:I}$ is directed entails that it is inhabited by some U_i. We know that $\mathbf{0} \leq U_i$ as $\mathbf{0}$ is the bottom element.*

4.1 Spectral locales

Definition 23 (Compact locale). *A locale X is called compact if the top open $\mathbf{1}_X$ is compact.*

We denote by $\mathsf{K}(X)$ the type of compact opens of a locale X. In other words, we define

$$\mathsf{K}(X) \quad :\equiv \quad \Sigma_{(U\,:\,\mathcal{O}(X))}\, U \text{ is compact.}$$

Since the locales we consider are large, we have that the type $\mathsf{K}(X)$ lives in universe $\mathcal{U}^+$. A spectral locale, also sometimes called *coherent* in the literature (Johnstone 1982, II.3.2), is defined in impredicative locale theory as one in which the compact opens (1) are closed under finite meets and (2) form a basis. In our formulation of this notion, we capture the same idea but impose the additional requirement that the type $\mathsf{K}(X)$ be small.

Definition 24 (Spectral locale). *A locale X is called spectral if it satisfies the following conditions.*

- (SP1) *It is a compact locale (i.e., the empty meet $\mathbf{1}_X$ is compact).*
- (SP2) *Compact opens are closed under binary meets.*
- (SP3) *For any $U : \mathcal{O}(X)$, there exists some small directed family $(K_i)_{i:I}$, with each K_i compact, such that $U = \bigvee_{i:I} K_i$.*
- (SP4) *The type $\mathsf{K}(X)$ is small.*

Notice that in the impredicative notion of spectral locale, everything is small. One way of understanding the above definition is that we only make the *carrier* large, but everything else, including the basis, remains small. This is important for a variety of reasons. Without the smallness condition, it does not seem possible to (1) define Heyting implication, and hence open nuclei, (2) show that frame homomorphisms have right adjoints, and so define the notion of perfect frame homomorphism. Additionally, a natural Stone-type duality to consider is that between small distributive lattices and spectral locales, for which it becomes absolutely necessary that we work with small bases as in the above definition. Notice that the same kind of phenomenon occurs when we work with locally presentable categories, which are large, locally small categories with a small set of objects that suitably generate all objects by taking filtered colimits (Adámek and Rosicky (1994); Johnstone (2002)).

Lemma 25. *For any locale X, the statement that X is spectral is a proposition.*

Proof. Immediate from Lemma 20 and the fact that the Π type over a family of propositions is a proposition. □

The natural notion of morphism between spectral locales is that of a *spectral map*.

Definition 26 (Spectral map). *A continuous map $f : X \to Y$ of spectral locales X and Y is called spectral if it reflects compact opens, that is, the open $f^*(V) : \mathcal{O}(X)$ is compact whenever the open $V : \mathcal{O}(Y)$ is compact.*

We denote by **Spec** the category of spectral locales with spectral maps as the morphisms. We recall the following useful fact (see e.g. Escardó (1999)).

Lemma 27. *Let X be a spectral locale and let $U, V : \mathcal{O}(X)$ be two opens.*

$$U \leq V \quad \text{if and only if} \quad \Pi_{(K : \mathcal{O}(X))}\, K \text{ is compact} \to K \leq U \to K \leq V.$$

In the context of the definition of spectrality that we gave in Definition 24, where we have conditions ensuring that the compact opens form a small basis, it is natural to ask whether the basis in consideration is unique. This is indeed the case, but it is subtler than one might expect and is proved in Theorem 36 below. We first need some preparation.

Given that the conditions in Definition 24 capture the idea of the compact opens behaving like a small basis closed under finite meets, one might wonder if the same notion can be formulated by starting with a small basis closed under finite meets and requiring it to consist of compact opens.

Definition 28 (Intensional spectral basis). *An intensional spectral basis for a locale X is a small intensional basis $(B_i)_{i:I}$ for X satisfying the following three conditions.*

(1) *For every $i : I$, the open B_i is compact.*
(2) *There is some index $t : I$ such that $B_t = \mathbf{1}_X$.*
(3) *For any two $i, j : I$, there exists some $k : I$ such that $B_k = B_i \wedge B_j$.*

We say that the basis is extensional if the map $B : I \to \mathcal{O}(X)$ is an embedding. When we say spectral basis without any qualification, we mean an extensional spectral basis.

In Remark 17, we mentioned that a basis can always be directified by closing it under finite joins. The same applies to spectral bases as well since the join of a finite family of compact opens is compact.

Lemma 29. *Given any locale X with an intensional basis $(B_i)_{i:I}$, every compact open of X falls in the basis, that is, for every compact open K, there exists an index $i : I$ such that $B_i = K$.*

Proof. Let $K : \mathcal{O}(X)$ be a compact open. As $(B_i)_{i:I}$ is an intensional basis, there must be a specified directed family $(i_j)_{j:J}$ on I such that $K = \bigvee_{j:J} B_{i_j}$. By the compactness of K, there must be some $k : J$ such that $K \leq B_{i_k}$. Clearly, $B_{i_k} \leq K$ is also the case, since K is an upper bound of the family $(B_{i_j})_{j:J}$, which means we have that $K = B_{i_k}$. □

Corollary 30. *Given any locale X with an intensional basis $(B_i)_{i:I}$ consisting of compact opens, we have an equivalence of types* $\mathsf{image}(B) \simeq \mathsf{K}(X)$.

Lemma 31. *If a locale X has an unspecified, intensional, spectral, and small basis, then X is spectral.*

Proof. First, notice that the conclusion is a proposition since being spectral is a proposition (by Lemma 25). This means that we may appeal to the induction principle of propositional truncation and assume we have a specified intensional spectral basis $(B_i)_{i:I}$. We know that $\mathsf{image}(B) \simeq \mathsf{K}(X)$ by Corollary 30, which is to say that the type $\mathsf{K}(X)$ is small, by the smallness of $\mathsf{image}(B)$ (given by Lemma 18) meaning (SP4) is satisfied.

The top element falls in the basis and is thus compact so (SP1) holds. Given two compact opens K_1 and K_2, they must be basic meaning there exist k_1, k_2 such that $K_1 = B_{k_1}$ and $K_2 = B_{k_2}$. Because the basis is closed under binary meets there must be some k_3 with $B_{k_3} = K_1 \wedge K_2$ which means condition (SP2) also holds.

For (SP3), consider an open $U : \mathscr{O}(X)$. We know that there is a specified a small family $(i_j)_{j:J}$ of indices such that $U = \bigvee_{j:J} B_{i_j}$. The subfamily $(B_{i_j})_{j:J}$ is then clearly a small directed family with each B_{i_j} compact which is what we needed. □

Lemma 32. *If a locale X is spectral, then it has a specified, extensional, and spectral small basis.*

Proof. Let X be a spectral locale. We claim that the inclusion $\mathsf{K}(X) \hookrightarrow \mathscr{O}(X)$, which is formally given by the first projection, is an extensional and spectral small basis. The fact that it is small is given by (SP4). By (SP1) and (SP2), we know that this basis contains $\mathbf{1}_X$ and is closed under binary meets. It remains to show that it forms a basis. To define a covering family for an open $U : \mathscr{O}(X)$, we let the index type be the subtype of compact opens below U, which is again small by (SP4), and the family is again given by the first projection. It is clear that U is an upper bound of this family so it remains to show that it is its least upper bound. Consider some V that is an upper bound of this family. By Lemma 27, it suffices to show $K \leq U$ implies $K \leq V$, for every compact open K. Any such compact open $K \leq U$ is below U by construction, which implies $K \leq V$ since it is an upper bound. □

Definition 33 (Extensionalization of an intensional basis). *For an intensional basis $(B_i)_{i:I}$ on a locale X, its extensionalization is defined by taking the index set to be* $\mathsf{image}(B)$, *and the family to be the corestriction of B to its image, which is given by the first projection, and hence is an embedding.*

Lemma 34. *If $(B_i)_{i:I}$ is an intensional spectral basis, then so is its extensionalization.*

Lemma 35. *From an unspecified intensional spectral basis on a locale X, we can obtain a specified extensional spectral basis.*

Proof. Let X be a locale with an unspecified intensional spectral basis. By Lemma 31, we know that X is a spectral locale and therefore that it has a specified extensional basis by Lemma 32. □

Recall that a type X is said to have *split support* if $\|X\| \to X$ (Kraus et al. 2017). With this terminology, the above lemma says that the type of intensional spectral bases on a locale X has split support.

Theorem 36. *The following are logically equivalent for any locale X.*

(1) *X is spectral.*
(2) *X has an unspecified intensional small spectral basis.*
(3) *X has an unspecified extensional small spectral basis.*
(4) *The inclusion $\mathsf{K}(X) \hookrightarrow X$ is an extensional spectral basis, where to an open U we assign the directed family of compact opens below it as in the construction of Lemma.* 32

(5) *X has a specified intensional small spectral basis.*

(6) *X has a specified extensional small spectral basis.*

Notice that the first four conditions are propositions, but the last two are not in general.

Proof. We have already established one direction of the logical equivalence (2) $\leftrightarrow$ (5), since (6) $\to$ (5) and Lemma 35 gives (2) $\to$ (6). The converse is simply the unit of propositional truncation. Every extensional basis is intensional, and so (6) $\to$ (5) is clear. The implication (1) $\to$ (6) is Lemma 32. By Corollary 30, we know that $\mathsf{image}(B) \simeq \mathsf{K}(X)$ for any intensional spectral basis $(B_i)_{i:I}$. This implies that any extensional spectral basis is equivalent to the basis $\mathsf{K}(X) \hookrightarrow X$. Conversely, the inclusion $\mathsf{K}(X) \hookrightarrow X$ is always a small extensional basis, so that we get (6) $\leftrightarrow$ (4). We also clearly have (6) $\to$ (3) $\to$ (2), which concludes our proof. ☐

4.1.1 Examples of spectral locales

The *terminal locale* (denoted $\mathbf{1}_{\mathcal{U}}$) is the locale defined by the initial frame $\Omega_{\mathcal{U}}$.

Example 37 (The initial frame). *For every universe $\mathcal{U}$, the (large) type $\Omega_{\mathcal{U}}$ forms a poset under the order defined by implication. This poset is locally small since each proposition $P \Rightarrow Q$ is small. Note that the antisymmetry of this order is exactly propositional extensionality. Furthermore, this poset forms a frame: the top element is the true proposition $\top_{\mathcal{U}}$, the meet operation is given by the conjunction of two propositions, and the join of a family of propositions $(P_i)_{i:I}$ is defined as*

$$\bigvee_{i:I} P_i \quad :\equiv \quad \exists_{i:I} P_i,$$

which is a small proposition for every small family of propositions.

The fact that these define meet and join operations is easy to see. For the distributivity law, it follows directly from the recursion principle of propositional truncation that

$$P \wedge \exists_{i:I} Q_i \quad \leftrightarrow \quad \exists_{i:I} P \wedge Q_i,$$

for every $P : \Omega_{\mathcal{U}}$ and every small family of propositions $(Q_i)_{i:I}$.

Before proving the universal property of the initial frame, we first establish the following lemma:

Lemma 38. *Let $\beta : \mathbf{2}_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ be the function defined as $\beta(0) :\equiv \bot$, $\beta(1) :\equiv \top$. Every $P : \Omega_{\mathcal{U}}$ is equal to*

(1) *the join $\bigvee_{p:P} \top_{\mathcal{U}}$, and*

(2) *the directed join $\bigvee \left\{\beta(b) \mid b : \mathbf{2}_{\mathcal{U}}, \beta(b) \leq P\right\}$.*

Therefore, $\beta : \mathbf{2}_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ forms an intensional basis for the terminal locale.

Proof. (1) is easy to see since $P = \top$ if and only if P holds. For (2), we show that $P = \bigvee \left\{\beta(b) \mid b : \mathbf{2}, b \leq P\right\}$ by antisymmetry. If P holds, then $P = \top$ and hence $\top \leq P$. For the other direction, observe that P is an upper bound of the family since $\bot$ is trivially below P and $\top \leq P$ implies that P holds.

The family $\{\beta(b) \mid b : \mathbf{2}, \beta(b) \leq P\}$ is directed since it is always inhabited by $\bot = \beta(0)$, and for every pair of Booleans $b_1, b_2 : \mathbf{2}$ with $\beta(b_1) \leq P$ and $\beta(b_2) \leq P$, we have that

$$\beta(b_1 \vee b_2) = \beta(b_1) \vee \beta(b_2),$$

and also $\beta(b_1 \vee b_2) \leq P$, meaning the join $\beta(b_1) \vee \beta(b_2)$ falls in the family. □

Proposition 39. *The frame $\Omega_{\mathcal{U}}$ is the initial object in the category $\mathbf{Frm}_{\mathcal{U}}$.*

Proof. Let L be a frame. By Lemma 38, $P : \Omega_{\mathcal{U}}$ can be expressed as the small join $\bigvee_{p:P} \top$. This implies that every frame homomorphism $h : \Omega_{\mathcal{U}} \to L$ is necessarily unique as it must satisfy the equality:

$$h(P) = h\left(\bigvee_{p:P} \top\right) = \bigvee_{p:P} h(\top) = \bigvee_{p:P} \mathbf{1}_L.$$

We accordingly define the unique map $\Omega_{\mathcal{U}} \to L$ as above. It is easy to see that this defines a frame homomorphism. □

Lemma 40. *The terminal locale $\mathbf{1}_{\mathcal{U}}$ is compact.*

Proof. It follows directly from the definition of the join that, for every family $(P_i)_{i:I}$ of $\mathcal{U}$-propositions, we have that $\top \leq \bigvee_{i:I} P_i$ if and only if some P_i holds, that is, $\top \leq P_i$ for some $i : I$. □

Lemma 41. *The terminal locale $\mathbf{1}_{\mathcal{U}}$ is spectral.*

Proof. Using Theorem 36, it suffices to show that the basis constructed in Lemma 38 satisfies the conditions from Definition 28.

(1) We need to show that both $\beta(0)$ and $\beta(1)$ are compact. We have already established in Lemma 40 that $\beta(1) = \top$ is compact. The bottom open $\beta(0) = \bot$ is compact in any frame as shown in Example 22.
(2) The top proposition $\top_{\mathcal{U}}$ is obviously contained in the basis since we have $\beta(1) = \top_{\mathcal{U}}$.
(3) Let $b_1, b_2 : \mathbf{2}$ be a pair of Booleans. It is easy to see that the meet $\beta(b_1) \wedge \beta(b_2)$ falls in the basis, since we have $\beta(b_1 \wedge b_2) = \beta(b_1) \wedge \beta(b_2)$. □

4.2 Stone locales

Clopenness is central to the notion of Stone locale, similar to the fundamental role played by the notion of a compact open in the definition of a spectral locale. To define the clopens, we first define the well-inside relation.

Definition 42 (The well-inside relation). *We say that an open U of a locale X is well inside an open V, written $U \lessdot V$, if there is an open W with $U \wedge W = \mathbf{0}_X$ and $V \vee W = \mathbf{1}_X$.*

Definition 43 (Clopen). *An open U is called a clopen if it is well inside itself, which amounts to saying that it has a Boolean complement. We denote by $\mathsf{C}(X)$ the subtype of $\mathcal{O}(X)$ consisting of clopens.*

Before we proceed to the definition of a Stone locale, we record the following fact about the well-inside relation:

Lemma 44. *Given opens $U, V, W : \mathcal{O}(X)$ of a locale X,*

(1) *if $U \lessdot V$ and $V \leq W$ then $U \lessdot W$; and*
(2) *if $U \leq V$ and $V \lessdot W$ then $U \lessdot W$.*

Definition 45 (Stone locale). *A locale X is called Stone if it satisfies the following conditions:*

(ST1) *It is compact.*
(ST2) *For any $U : \mathcal{O}(X)$, there exists a small directed family $(C_i)_{i:I}$, with each C_i clopen, such that $U = \bigvee_{i:I} C_i$.*
(ST3) *The type $\mathsf{C}(X)$ is small.*

Lemma 46. *For any locale X, being Stone is a proposition.*

Proof. Being compact is a proposition (Lemma 20) and being small is a proposition by Lemma 2 (assuming univalence). The condition (ST2) is a proposition since the Π-type over a family of propositions is a proposition. □

We denote by **Stone** the category of Stone locales with continuous maps as the morphisms. Every continuous map reflects clopens automatically, so we do not need a special notion of continuous map in **Stone** like we do in the category **Spec**.

The following two lemmas are needed to prove that the compact opens and the clopens coincide in Stone locales, which we will need later. The proofs are standard (Johnstone, 1982, Lemma VII.3.5). We provide the proof of Lemma 48 for the sake of self-containment, since it uses our reformulation of the notion of a Stone locale.

Lemma 47. *In any compact locale, $U \lessdot V$ implies $U \ll V$ for any two opens U, V.*

Lemma 48. *In any Stone locale, $U \ll V$ implies $U \lessdot V$ for any two opens U, V.*

Proof. Let $U, V : \mathcal{O}(X)$ with $U \ll V$. We know that $V = \bigvee_{i:I} C_i$ for a family $(C_i)_{i:I}$ consisting of clopens. Since $V \leq \bigvee_{i:I} C_i$, it must be the case that there is some $k : I$ with $U \leq C_k$ as we know $U \ll V$. We then have $U \leq C_k \lessdot V$ which implies $U \lessdot V$ by Lemma 44. □

Corollary 49. *For every Stone locale X, we have a type equivalence $\mathsf{K}(X) \simeq \mathsf{C}(X)$.*

Corollary 50. *Every Stone locale X is spectral.*

A consequence of Corollary 49 is that we immediately get a characterization of Stone locales analogous to the notion of intensional spectral basis from Definition 28.

Definition 51 (Basis of clopens). *An intensional basis of clopens for a locale X is a small intensional basis $(B_i)_{i:I}$ for X that consists of clopens. We say that the basis is extensional if the map $B : I \to \mathcal{O}(X)$ is an embedding. When we say basis of clopens without any qualification, we mean an extensional basis of clopens.*

Notice that, because clopens are closed under finite joins, the directification of a basis of clopens is again a basis of clopens. For the next lemma, recall the notion of extensionalization given in Definition 33.

Lemma 52. *For every intensional basis $(B_i)_{i:I}$, if $(B_i)_{i:I}$ consists of clopens, then so does its extensionalization.*

Lemma 53. *If a locale X is compact and has an unspecified, intensional, and small basis of clopens, then it is Stone.*

Proof. Since being Stone is a proposition (by Lemma 46), we can work with a specified basis $(B_i)_{i:I}$ with each B_i clopen. Condition (ST2) is immediate since any covering family given by the basis is a subfamily of $\mathsf{C}(X)$. It remains to show that $\mathsf{C}(X)$ is a small type (ST3). Let $C : \mathsf{C}(X)$. Since X is compact, C must be a compact open by Lemma 47, and hence must fall in the basis by Lemma 29. The other direction is direct by construction meaning $\mathsf{C}(X) \simeq \mathsf{image}(B)$, which concludes that $\mathsf{C}(X)$ is small by Remark 6. □

Theorem 54. *The following are logically equivalent for any locale X.*

(1) *X is Stone.*
(2) *X is compact and has an unspecified intensional small basis of clopens.*
(3) *X is compact and has an unspecified extensional small basis of clopens.*
(4) *X is compact and the inclusion $\mathsf{C}(X) \hookrightarrow X$ is an extensional basis of clopens, where to an open U we assign the family of all clopens below it.*
(5) *X is compact has a specified intensional small basis of clopens.*
(6) *X is compact has a specified extensional small basis of clopens.*

Notice that the first four conditions are propositions, but the last two are not in general.

Proof. We know by Corollary 50 and Corollary 49 that every Stone locale X is spectral and has $\mathsf{K}(X) \simeq \mathsf{C}(X)$. Therefore, by Theorem 36, we know that $\mathsf{C}(X)$ is an extensional basis of clopens, which gives the implication (1) → (4). The implications (4) → (6) → (5) → (2) are direct, and the implication (2) → (1) is Lemma 53. We also clearly have (6) → (3) → (2), which concludes our proof. □

5. Predicative Form of the Posetal Adjoint Functor Theorem

We start with the definition of the notion of an adjunction in the simplified context of lattices.

Definition 55. *Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two preordered sets. An adjunction between X and Y consists of a pair of monotone maps $f : X \to Y$ and $g : Y \to X$ satisfying*

$$f \dashv g \quad :\equiv \quad f(x) \leq y \leftrightarrow x \leq g(y) \text{ for every } x : X, y : Y.$$

In locale theory, it is standard convention to denote by $f_* : \mathcal{O}(X) \to \mathcal{O}(Y)$ the right adjoint of a frame homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ corresponding to a continuous map of locales $f : X \to Y$. The right adjoint here is known to always exist thanks to a simple application of the Adjoint Functor Theorem which amounts to the definition:

$$f_*(U) \quad :\equiv \quad \bigvee \{V : \mathcal{O}(Y) \mid f^*(V) \leq U\}.$$

In the predicative setting of type theory, however, it is not clear how the right adjoint of a frame homomorphism would be defined as the family $\{V : \mathcal{O}(Y) \mid f^*(V) \leq U\}$ might be large in general. This gives rise to the problem that it is not *a priori* clear that its join in $\mathcal{O}(Y)$ exists. To resolve this problem, we use the assumption of a small basis. The use of a small basis for the posetal Adjoint Functor Theorem in a predicative setting was independently observed by Tom de Jong (personal communication).

Theorem 56 (Posetal Adjoint Functor Theorem). *Let X and Y be two locales and let $h : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a monotone map of frames. Assume that Y has some small basis. The map h has a right adjoint if and only if it preserves joins of small families.*

Proof. Let $h : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a monotone map of frames and assume that Y has some small basis.

The forward direction is easy: suppose h has a right adjoint $g : \mathcal{O}(X) \to \mathcal{O}(Y)$ and let $(U_i)_{i:I}$ be a family in $\mathcal{O}(Y)$. By the uniqueness of joins, it is sufficient to show that $h(\bigvee_i U_i)$ is the join of the family $(h(U_i))_{i:I}$. It is clearly an upper bound by the fact that h is monotone. Given any other upper bound V of $(h(U_i))_{i:I}$, we have that $h(\bigvee_i U_i) \leq V$ since $h(\bigvee_i U_i) \leq V \leftrightarrow \left(\bigvee_i U_i\right) \leq g(V)$ meaning it is sufficient to show $U_i \leq g(V)$ for each U_i. Since $U_i \leq g(V)$ if and only if $h(U_i) \leq V$, we are done as the latter can be seen to hold directly from the fact that V is an upper bound of the family in consideration.

For the converse, suppose $h(\bigvee_i U_i) = \bigvee_{i:I} h(U_i)$ for every small family of opens $(U_i)_{i:I}$. Since right adjoints are unique, we may appeal to the induction principle of propositional truncation and assume we have a specified basis $(B_i)_{i:I}$. We define the right adjoint of h as

$$g(U) \quad :\equiv \quad \bigvee \left\{B_i \mid i : I, h(B_i) \leq U\right\}.$$

We need to show that g is the right adjoint of h, hat is,

$$h(V) \leq U \leftrightarrow V \leq g(U)$$

for any two $V : \mathcal{O}(Y)$, $U : \mathcal{O}(X)$.

For the $(\to)$ direction, assume $h(V) \leq U$. It must be the case that $V = \bigvee_{j:J} B_{i_j}$ for some specified basic covering $(i_j)_{j:J}$. This means that we just have to show $B_{i_j} \leq g(U)$ for every $j : J$, which is the case since $h(B_{i_j}) \leq h(V) \leq U$.

For the $(\leftarrow)$ direction, assume $V \leq g(U)$. This means that we have

$$\begin{aligned} h(V) &\leq h(g(U)) \\ &= h\left(\bigvee \left\{B_i \mid i : I, h(B_i) \leq U\right\}\right) \\ &= \bigvee \left\{h(B_i) \mid i : I, h(B_i) \leq U\right\} \\ &\leq U \end{aligned}$$

so that $h(V) \leq U$, as required. □

Our primary use case for the Adjoint Functor Theorem is the construction of Heyting implications.

Definition 57 (Heyting implication). *Let X be a locale with some small basis. Given any open $U : \mathcal{O}(X)$, the map $(-) \wedge U : \mathcal{O}(X) \to \mathcal{O}(X)$ preserves joins by the frame distributivity law. This means, by Theorem 56, that it must have a right adjoint $U \Rightarrow (-) : \mathcal{O}(X) \to \mathcal{O}(X)$. The operation $(-) \Rightarrow (-)$ is known as Heyting implication.*

Definition 58. *By Theorem 56, for any continuous map $f : X \to Y$ of locales where Y has an unspecified small basis, the frame homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ has a right adjoint, denoted by*

$$f_* : \mathcal{O}(X) \to \mathcal{O}(Y).$$

Definition 59 (Perfect frame homomorphism). *Let X and Y be two locales and assume that Y has an unspecified small basis. A continuous map $f : X \to Y$ is said to be perfect if f_* is Scott continuous, that is, it preserves small directed suprema.*

Let us also record the following fact about perfect maps, which we will need later.

Lemma 60. *Let $f : X \to Y$ be a perfect map where Y is a locale with some small basis. The frame homomorphism f^* respects the way-below relation, that is, $U \ll V$ implies $f^*(U) \ll f^*(V)$, for any two $U, V : \mathcal{O}(Y)$.*

A proof of this fact can be found in (Escardó 1999), and it works in our predicative setting.

Corollary 61. *Perfect maps are spectral as they preserve the compact opens.*

In fact, the converse is also true in the case of spectral locales. The proof given in (Escardó 1999) works, once we know that the required adjoints are available, as established above. We include it in order to illustrate this point.

Lemma 62. *Let X and Y be spectral locales. A continuous map $f : X \to Y$ is perfect if and only if it is spectral.*

Proof. The forward direction is given by Corollary 61. For the backward direction, assume $f : X \to Y$ to be a spectral map. We have to show that the right adjoint $f_* : \mathcal{O}(X) \to \mathcal{O}(Y)$ of its defining frame homomorphism is Scott continuous. Letting $(U_i)_{i:I}$ be a directed family in $\mathcal{O}(X)$, we show $f_*(\bigvee_{i:I} U_i) = \bigvee_{i:I} f_*(U_i)$. The $\bigvee_{i:I} f_*(U_i) \le f_*(\bigvee_{i:I} U_i)$ direction is easy. For the $f_*(\bigvee_{i:I} U_i) \le \bigvee_{i:I} f_*(U_i)$ direction, we appeal to Lemma 27. Let K be a compact open with $K \le f_*(\bigvee_{i:I} U_i)$. By the adjunction $f^* \dashv f_*$, it must be the case that $f^*(K) \le \bigvee_{i:I} U_i$ and since $f^*(K)$ is compact, by the spectrality assumption of f^*, there must exist some $l : I$ such that $f^*(K) \le U_l$. Again by adjointness, $K \le f_*(U_l)$ which implies $K \le \bigvee_{i:I} f_*(U_i)$. □

6. Meet-Semilattice of Scott Continuous Nuclei

In this section, we construct the defining frame of the patch locale. It is an observation due to Escardó (1999, 2001) that the patch locale of a locale X can be defined by the frame of Scott continuous nuclei on X. We start by constructing the meet-semilattice of *all* nuclei on a frame.

Definition 63 (Nucleus). *A nucleus on a locale X is an endofunction $j : \mathcal{O}(X) \to \mathcal{O}(X)$ that is inflationary, idempotent, and preserves binary meets.*

In Section 7, we will work with inflationary and binary-meet-preserving functions that are not necessarily idempotent. Such functions are called *prenuclei*. We also note that, to show a prenucleus j to be idempotent, it suffices to show $j(j(U)) \le j(U)$ as the other direction follows from inflationarity. In fact, the notion of a nucleus could be defined as a prenucleus satisfying the inequality $j(j(U)) \le j(U)$, but we define it as in Definition 63 for the sake of simplicity and make implicit use of this fact in our proofs of idempotency.

Lemma 61. *The type of nuclei on a given frame $\mathcal{O}(X)$ forms a meet-semilattice under the pointwise order.*

Proof. The top nucleus is defined as the constant map with value $\mathbf{1}_X$ and the meet of two nuclei as $j \wedge k :\equiv U \mapsto j(U) \wedge k(U)$. It is easy to see that $j \wedge k$ is the greatest lower bound of j and k so it remains to show that $j \wedge k$ satisfies the nucleus laws.

Inflationarity can be seen to be satisfied from the inflationarity of j and k combined with the fact that $j(U) \wedge k(U)$ is the greatest lower bound of $j(U)$ and $k(U)$. To see that meet preservation holds, let $U, V : \mathcal{O}(X)$.

$$\begin{aligned}(j \wedge k)(U \wedge V) &= j(U \wedge V) \wedge k(U \wedge V)\\ &= j(U) \wedge j(V) \wedge k(U) \wedge k(V)\\ &= (j(U) \wedge k(U)) \wedge (j(V) \wedge k(V))\\ &= (j \wedge k)(U) \wedge (j \wedge k)(V)\end{aligned}$$

For idempotency, let $U : \mathcal{O}(X)$. We have

$$\begin{aligned}(j \wedge k)((j \wedge k)(U)) &= j(j(U) \wedge k(U)) \wedge k(j(U) \wedge k(U))\\ &= j(j(U)) \wedge j(k(U)) \wedge k(j(U)) \wedge k(k(U))\\ &\leq j(j(U)) \wedge k(k(U))\\ &= j(U) \wedge k(U)\\ &= (j \wedge k)(U)\end{aligned}$$

□

We now show that this meet-semilattice can be *refined* to the Scott continuous nuclei (i.e., the *perfect* nuclei).

Lemma 65. *The Scott continuous nuclei on any locale form a meet-semilattice.*

Proof. Let X be a locale. The construction is the same as the one from Lemma 61; the top element is the constant map with value $\mathbf{1}_X$, which is trivially Scott continuous so it remains to show that the meet of two Scott continuous nuclei is Scott continuous. Consider two Scott continuous nuclei j and k on $\mathcal{O}(X)$, and let $(U_n)_{n:N}$ be a directed, small family of opens.

$$\begin{aligned}(j \wedge k)\left(\bigvee_{n:N} U_n\right) &\equiv j\left(\bigvee_{m:N} U_m\right) \wedge k\left(\bigvee_{n:N} U_n\right)\\ &= \left(\bigvee_{m:N} j(U_m)\right) \wedge \left(\bigvee_{n:N} k(U_n)\right) && \text{[Scott continuity of } j \text{ and } k\text{]}\\ &= \bigvee_{(m,n):N\times N} j(U_m) \wedge k(U_n) && \text{[distributivity]}\\ &= \bigvee_{n:N} j(U_n) \wedge k(U_n) && [\,\dagger\,]\\ &\equiv \bigvee_{n:N} (j \wedge k)(U_n) && \text{[meet preservation]}\end{aligned}$$

where, for the (†) step, we use antisymmetry. The ($\geq$) direction is immediate. For the ($\leq$) direction, we need to show that $\bigvee_{(m,n):N\times N} j(U_m) \wedge k(U_n) \leq \bigvee_{n:N} j(U_n) \wedge k(U_n)$, for which it suffices to show that $\bigvee_{n:N} j(U_n) \wedge k(U_n)$ is an upper bound of $\{j(U_m) \wedge k(U_n)\}_{(m,n):N\times N}$. Let $m, n : N$ be two indices. As $(U_n)_{n:N}$ is directed, there must exist some o such that U_o is an upper bound of $\{U_m, U_n\}$. Using the monotonicity of j and k, we get $j(U_m) \wedge k(U_n) \leq j(U_o) \wedge k(U_o) \leq \bigvee_{n:N} j(U_n) \wedge k(U_n)$ as desired. □

7. Joins in the Frame of Scott Continuous Nuclei

The nontrivial component of the patch frame construction is the join of a family $(k_i)_{i:I}$ of Scott continuous nuclei, as the pointwise join fails to be idempotent in general, and not inflationary when the family in consideration is empty. When the family in consideration is directed, however, the pointwise join is idempotent, and so it is the join in the poset of Scott continuous nuclei.

Lemma 66. *Given a directed family $(k_i)_{i:I}$ of Scott continuous nuclei, their join is computed pointwise, that is, $\left(\bigvee_{i:I} k_i\right)(U) = \bigvee_{i:I} k_i(U)$.*

Proof. The argument given in the paragraph preceding (Escardó 1998, Lemma 3.1.8) works in our setting. □

Regarding arbitrary joins, the situation is more complicated. A construction of the join, due to Escardó (1998), is based on the idea that, if we start with a family $(k_i)_{i:I}$ of nuclei, we can consider the family with index type $\mathsf{List}(I)$ of words over I, defined by

$$(i_0 i_1 \cdots i_{n-1}) \mapsto k_{i_{n-1}} \circ \cdots \circ k_{i_1} \circ k_{i_0}.$$

This family is easily seen to be directed.

To talk about such families of finite compositions over a given family of (pre)nuclei, we adopt the following notation. Recall that a word over type I is either empty or consists of the insertion of some $i : I$ onto some other word s. We denote the latter by is. Similarly, we write st for the concatenation of two words $s, t : \mathsf{List}(I)$.

To define the join operation, we will use the iterated composition operator $(-)^*$ which we define below.

Definition 67 (Family of finite compositions). *Given a small family $(k_i)_{i:I}$ of nuclei on a given locale X, its family of finite compositions is the family defined by*

$$k^*(i_0 \cdots i_{n-1}) \quad :\equiv \quad k_{i_{n-1}} \circ \cdots \circ k_{i_0}.$$

By an abuse of notation, we omit the superscript "$*$" when there is no possibility of confusion.

The order of composition is actually not important. A finite composition k_s is, in general, not a nucleus. It is, however, always a prenucleus which we prove below.

Lemma 68. *Given a family $(k_i)_{i:I}$ of nuclei on a locale, k_s is a prenucleus for every $s : \mathsf{List}(I)$.*

Proof. If s is the empty list, we are done as it is immediate that the identity function id is a prenucleus. If s is of the form is', with $i : I$, we need to show that $k_{s'} \circ k_i$ is a prenucleus. For meet preservation, let $U, V : \mathcal{O}(X)$.

$$\begin{aligned}
(k_{s'} \circ k_i)(U \wedge V) &= k_{s'}(k_i(U \wedge V)) \\
&= k_{s'}(k_i(U) \wedge k_i(V)) && [k_i \text{ is a nucleus}] \\
&= k_{s'}(k_i(U)) \wedge k_{s'}(k_i(V)) && [\text{inductive hypothesis}] \\
&= (k_{s'} \circ k_i)(U) \wedge (k_{s'} \circ k_i)(V)
\end{aligned}$$

For inflationarity, consider some $U : \mathcal{O}(X)$. We have that $U \leq k_i(U) \leq k_{s'}(k_i(U))$, by the inflationarity properties of k_i and $k_{s'}$ (the latter following from the inductive hypothesis). □

Lemma 69. *Given a nucleus j and a family $(k_i)_{i:I}$ of nuclei on a locale, if j is an upper bound then it is also an upper bound of the family of finite compositions.*

Proof. Let j and $(k_i)_{i:I}$ be, respectively, a nucleus and a family of nuclei on a locale and assume that j is an upper bound of the family $(k_i)_{i:I}$. We proceed by list induction. Base case: the empty composition is the identity function, and it is easy to see that we have $\mathsf{id}(U) \equiv U \leq j(U)$. Inductive step: consider a list of the form is, with $i : I$, and assume inductively that $k_s \leq j$.

$$\begin{aligned} k_s(k_i(U)) &\leq k_s(j(U)) && \text{[monotonicity of } k_s \text{ (Lemma 68 and monotonicity of prenuclei)]} \\ &\leq j(j(U)) && \text{[inductive hypothesis]} \\ &\leq j(U) && \text{[idempotency of } j\text{]} \end{aligned}$$

□

Lemma 70. *Given a family $(k_i)_{i:I}$ of Scott continuous nuclei on a locale, the prenucleus k_s is Scott continuous, for every $s : \mathsf{List}(I)$*

Proof. Any composition of finitely many Scott continuous functions is Scott continuous. □

Lemma 71. *Given a family $(k_i)_{i:I}$ of nuclei on a locale, the family of finite compositions is directed.*

Proof. The family $(k_s)_{s:\mathsf{List}(I)}$ is indeed always inhabited by the empty composition, which is defined to be the identity nucleus. The upper bound of nuclei k_s and k_t is given by k_{st}, which is equal to $k_t \circ k_s$. The fact that this is an upper bound of $\{k_s, k_t\}$ follows from monotonicity and inflationarity. □

Lemma 72. *Let j and $(k_i)_{i:I}$ be, respectively, a nucleus and a family of nuclei on a locale. Consider the family of finite compositions over the family $(j \wedge k_i)_{i:I}$. Each finite composition $(j \wedge k)^*_s$ is a lower bound of k_s and j, for every $s : \mathsf{List}(I)$.*

We are now ready to construct the join operation in the meet-semilattice of Scott continuous nuclei, hence constructing the defining frame of the patch locale of a locale X.

Theorem 73 (Join of Scott continuous nuclei). *Let $(k_i)_{i:I}$ be a family of Scott continuous nuclei. The join of K can be calculated as*

$$\left(\bigvee_{i:I} k_i\right)(U) :\equiv \bigvee_{s:\mathsf{List}(I)} k_s(U).$$

Proof. It must be checked that this is (1) indeed the join, (2) is a Scott continuous nucleus, that is, it is inflationary, binary-meet-preserving, idempotent, and Scott continuous. The inflationarity property is direct. For meet preservation, consider some $U, V : \mathcal{O}(X)$. We have:

$$\begin{aligned} \left(\bigvee_{i:I} k_i\right)(U \wedge V) &= \bigvee_{s:\mathsf{List}(I)} k_s(U \wedge V) \\ &= \bigvee_{s:\mathsf{List}(I)} k_s(U) \wedge k_s(V) && \text{[Lemma 70]} \\ &= \bigvee_{s,t:\mathsf{List}(I)} k_s(U) \wedge k_t(V) && [\,\dagger\,] \end{aligned}$$

$$= \left(\bigvee_{s:\mathsf{List}(I)} k_s(U) \right) \wedge \left(\bigvee_{t:\mathsf{List}(I)} k_t(V) \right) \qquad \text{[distributivity]}$$

$$= \left(\bigvee_{i:I} k_i \right)(U) \wedge \left(\bigvee_{i:I} k_i \right)(V)$$

where step (†) uses antisymmetry. The (≤) direction is direct whereas for the (≥) direction, we show that $\bigvee_{s:\mathsf{List}(I)} k_s(U) \wedge k_s(V)$ is an upper bound of the family $(k_s(U) \wedge k_t(V))_{s,t:\mathsf{List}(I)}$. Consider arbitrary $s, t : \mathsf{List}(I)$. By the directedness of the family of finite compositions, we know that there exists some $u : \mathsf{List}(I)$ such that k_u is an upper bound of $\{k_s, k_t\}$. We then have

$$k_s(U) \wedge k_t(V) \leq k_u(U) \wedge k_u(V) \leq \bigvee_{s:\mathsf{List}(I)} k_s(U) \wedge k_s(V).$$

For idempotency, let $U : \mathscr{O}(X)$.

$$\begin{aligned}
\left(\bigvee_i k_i \right)\left(\left(\bigvee_i k_i \right)(U) \right) &\equiv \bigvee_{s:\mathsf{List}(I)} k_s \left(\bigvee_{t:\mathsf{List}(I)} k_t(U) \right) \\
&= \bigvee_{s:\mathsf{List}(I)} \bigvee_{t:\mathsf{List}(I)} k_s(k_t(U)) && \text{[Lemma 70]} \\
&\leq \bigvee_{s,t:\mathsf{List}(I)} k_s(k_t(U)) \\
&\leq \bigvee_{s:\mathsf{List}(I)} k_s(U) && [\,\dagger\,] \\
&\equiv \left(\bigvee_i k_i \right)(U),
\end{aligned}$$

where for the step (†) it suffices to show that $\bigvee_{s:\mathsf{List}(I)} k_s(U)$ is an upper bound of the family $(k_s(k_t(U)))_{s,t:\mathsf{List}(I)}$. The prenucleus k_{st} is an upper bound of k_s and k_t (as in Lemma 71). We have that

$$k_t(k_s(U)) = k_{st}(U) \leq \bigvee_{u:\mathsf{List}(I)} k_u(U).$$

For Scott continuity, let $(U_j)_{j:J}$ be a directed family over $\mathscr{O}(X)$. We then have:

$$\begin{aligned}
\left(\bigvee_{i:I} k_i \right)\left(\bigvee_{j:J} U_j \right) &\equiv \bigvee_{s:\mathsf{List}(I)} k_s \left(\bigvee_{j:J} U_j \right) \\
&= \bigvee_{s:\mathsf{List}(I)} \bigvee_{j:J} k_s(U_j) && \text{[Lemma 70]} \\
&= \bigvee_{j:J} \bigvee_{s:\mathsf{List}(I)} k_s(U_j) && \text{[joins commute]} \\
&\equiv \bigvee_{j:J} \left(\bigvee_{i:I} k_i \right)(U_j)
\end{aligned}$$

as required.

The fact that $\bigvee_i k_i$ is an upper bound of $(k_i)_{i:I}$ is easy to verify. To see that it is *the least* upper bound, consider a Scott continuous nucleus j that is an upper bound of $(k_i)_{i:I}$. Let $U : \mathscr{O}(X)$. We need to show that $(\bigvee_i k_i)(U) \leq j(U)$. We know by Lemma 69 that j is an upper bound of the family of finite compositions, since it is an upper bound of $(k_i)_{i:I}$, which is to say $k_s(U) \leq j(U)$ for every $s : \mathsf{List}(I)$, that is, $j(U)$ is an upper bound of the family $(k_s(U))_{s:\mathsf{List}(I)}$. Since $\left(\bigvee_i k_i\right)(U)$ is the least upper bound of this family by definition, it follows that it is below $j(U)$. □

We use Lemma 72 to prove the distributivity law.

Lemma 74 (Distributivity). *For any Scott continuous nucleus j and any family $(k_i)_{i:I}$ of Scott continuous nuclei, we have the equality:*

$$j \wedge \left(\bigvee_{i:I} k_i \right) = \bigvee_{i:I} j \wedge k_i.$$

It follows that the Scott continuous nuclei form a frame.

Definition 75 (Patch locale of a spectral locale). *The patch locale of a spectral locale X, written $\mathsf{Patch}(X)$, is the locale defined by the frame of Scott continuous nuclei on X.*

Notice that the truth value of the relation $j \leq k$ between two Scott continuous nuclei lives by default in universe $\mathcal{U}^+$. However, it always has an equivalent copy in the universe $\mathcal{U}$ if the locale in consideration is spectral.

Lemma 76. *For any spectral locale X and any two Scott continuous nuclei $j, k : \mathcal{O}(X) \to \mathcal{O}(X)$, we have that*

$$j \leq k \quad \text{if and only if} \quad j(K) \leq k(K) \;\text{ for all }\; K : \mathsf{K}(X),$$

and hence $\mathsf{Patch}(X)$ is locally small.

Proof. The usual pointwise ordering obviously implies the basic ordering so we address the other direction. Let j and k be two Scott continuous nuclei on a spectral locale X and assume $j(K) \leq k(K)$, for all $K : \mathsf{K}(X)$. We need to show that $j(U) \leq k(U)$ for every open U so let $U : \mathcal{O}(X)$. By (SP3), U can be decomposed as $U = \bigvee_{i:I} K_i$ for some directed covering family $(K_i)_{i:I}$ consisting of compact opens. We then have $j(\bigvee_{i:I} K_i) = \bigvee_{i:I} j(K_i)$ by Scott continuity and

$$\bigvee_{i:I} j(K_i) \leq \bigvee_{i:I} k(K_i)$$

since $j(K_i) \leq k(K_i)$ for every $i : I$. Finally, because the quantification is over the small type $\mathsf{K}(X)$, the proposition "$j(K) \leq k(K)$ for all $K : \mathsf{K}(X)$" is small. □

8. The Coreflection Property of Patch

We prove in this section that our construction of Patch has the desired universal property: it exhibits **Stone** as a coreflective subcategory of **Spec**. The notions of *closed* and *open* nuclei are crucial for proving this universal property. We first give the definitions of these. Let U be an open of a locale X.

(1) The *closed nucleus* induced by U is the map $V \mapsto U \vee V$.
(2) The *open nucleus* induced by U is the map $V \mapsto U \Rightarrow V$.

We denote the closed nucleus induced by open U by $\mathbf{c}(U)$ and the open nucleus induced by U by $\mathbf{o}(U)$.

Lemma 77. *For every locale Y and spectral locale X, a monotone map $h : \mathcal{O}(Y) \to \mathcal{O}(X)$ is Scott continuous if it satisfies the condition that, for every $U : \mathcal{O}(Y)$ and compact $K : \mathcal{O}(X)$ with $K \leq h(U)$, there is some compact $K' \leq U$ such that $K \leq h(K')$.*

Proof. Let $h : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a monotone map satisfying the above condition. Consider a directed family of opens $(U_i)_{i:I}$. We need to show that the relation $h(\bigvee_{i:I} U_i) \leq \bigvee_{i:I} h(U_i)$ holds. Since X is spectral, let $(K_j)_{j:J}$ be a small family of compact opens such that $\bigvee_{j:J} K_j = h(\bigvee_{i:I} U_i)$. For any $j : J$, we have $K_j \leq h(\bigvee_{i:I} U_i)$, so by assumption there is a compact open $K \leq \bigvee_{i:I} U_i$ such that $K_j \leq h(K)$. By compactness of K there is an $i : I$ such that $K \leq U_i$, and so $K_j \leq h(K) \leq h(U_i) \leq \bigvee_{i:I} h(U_i)$ and so we can take $K' :\equiv K_j$. □

Lemma 78. *For every open $U : \mathcal{O}(X)$ of spectral locale X,*

- *the closed nucleus $\mathbf{c}(U)$ is Scott continuous, and*
- *the open nucleus $\mathbf{o}(U)$ is Scott continuous if the open U is compact.*

Proof. The Scott continuity of the closed nucleus is easy to see. For the open nucleus, let D be a compact open of a locale. By Lemma 77, it is sufficient to show that for any open V and any compact open C_1 with $C_1 \leq D \Rightarrow V$, there exists some compact $C_2 \leq V$ such that $C_1 \leq D \Rightarrow C_2$. Let V and C_1 be two opens with C_1 compact and satisfying $C_1 \leq D \Rightarrow V$. Pick $C_2 :\equiv D \wedge C_1$. We know that this is compact by (SP2). It remains to check (1) $C_2 \leq V$ and (2) $C_1 \leq D \Rightarrow C_2$, both of which are direct. □

Consider the closed nucleus formation operation $U \mapsto \mathbf{c}(U)$. This defines a frame homomorphism $\mathcal{O}(X) \to \mathcal{O}(\mathsf{Patch}(X))$, whose corresponding continuous map is denoted

$$\varepsilon : \mathsf{Patch}(X) \to X.$$

In Lemma 80, we will show that this map is perfect. Before Lemma 80, however, we record an auxiliary result:

Lemma 79. *Let X be a spectral locale. The right adjoint $\varepsilon_* : \mathcal{O}(\mathsf{Patch}(X)) \to \mathcal{O}(X)$ to the closed-nucleus formation operation $\mathbf{c}(-)$ is given by $\varepsilon_*(j) = j(\mathbf{0})$ for every Scott continuous nucleus j on X.*

The proof of Lemma 79 can be found in Escardó (1999). It is omitted here as it is unchanged in our predicative setting.

Lemma 80. *The function $\mathbf{c}(-)$ is a perfect frame homomorphism $\mathcal{O}(X) \to \mathcal{O}(\mathsf{Patch}(X))$.*

Proof. We have to show that the right adjoint ε_* of $\mathbf{c}(-)$ is Scott continuous. Let $(k_i)_{i:I}$ be a directed family of Scott continuous nuclei. Thanks to Lemma 79, it suffices to show

$$\left(\bigvee_{i:I} k_i\right)(\mathbf{0}) = \bigvee_{i:I} \varepsilon_*(k_i).$$

By Lemma 66, we have that $\left(\bigvee_{i:I} k_i\right)(\mathbf{0}) = \bigvee_{i:I} k_i(\mathbf{0})$. □

We will later show that the perfect map $\varepsilon : \mathsf{Patch}(X) \to X$ is the counit of the coreflection of interest.

8.1 Patch *is Stone*

Before we proceed to showing that the patch locale has the desired universal property, we first need to show that $\mathsf{Patch}(X)$ is Stone for any spectral locale X. For this purpose, we need to (1) show that it is compact and (2) construct a basis for it consisting of clopens. We start with compactness.

Lemma 81. *For every spectral locale X, the locale $\mathsf{Patch}(X)$ is compact.*

Proof. Recall that the top element $\mathbf{1}$ of $\mathsf{Patch}(X)$ is defined as $\mathbf{1}_{\mathsf{Patch}} :\equiv U \mapsto \mathbf{1}_X$. Because ε^* is a frame homomorphism, it must be the case that $\mathbf{1}_{\mathsf{Patch}} = \varepsilon^*(\mathbf{1}_X)$, meaning it suffices to show $\varepsilon^*(\mathbf{1}_X) \ll \varepsilon^*(\mathbf{1}_X)$. By Lemma 60, it suffices to show $\mathbf{1}_X \ll \mathbf{1}_X$ which is immediate by (SP1). □

To construct a basis consisting of clopens, we will use the following fact, which was already mentioned above:

Lemma 82. *The open nucleus $\mathbf{o}(U)$ is the Boolean complement of the closed nucleus $\mathbf{c}(U)$.*

Lemma 83. *Let X be a spectral locale. Given any Scott continuous nucleus $j : \mathscr{O}(\mathsf{Patch}(X))$, we have that*

$$j \;=\; \bigvee_{K:\mathsf{K}(X)} \mathbf{c}(j(K)) \wedge \mathbf{o}(K) \;=\; \bigvee_{\substack{K_1,K_2:\mathsf{K}(X)\\ K_1\leq j(K_2)}} \mathbf{c}(K_1) \wedge \mathbf{o}(K_2).$$

Proof. The second equality in the statement is clear, so let us show the first one. We use the fact, proved in (Johnstone 1982, Lemma II.2.7), that for any nucleus j on any locale,

$$j = \bigvee_{U:\mathscr{O}(X)} \mathbf{c}(j(U)) \wedge \mathbf{o}(U).$$

Suppose now additionally that X is spectral and the nucleus j is Scott continuous. The inequality $\bigvee_{K:\mathsf{K}(X)} \mathbf{c}(j(K)) \wedge \mathbf{o}(K) \leq \bigvee_{U:\mathscr{O}(X)} \mathbf{c}(j(U)) \wedge \mathbf{o}(U) = j$ is trivial, so let us show the reverse one. Let $K : \mathsf{K}(X)$ and notice that $\big(\mathbf{c}(j(K)) \wedge \mathbf{o}(K)\big)\,(K) = (j(K) \vee K) \wedge (K \Rightarrow K) = j(K)$. Therefore

$$j(K) = \big(\mathbf{c}(j(K)) \wedge \mathbf{o}(K)\big)\,(K) \leq \left(\bigvee_{K':\mathsf{K}(X)} \mathbf{c}(j(K')) \wedge \mathbf{o}(K')\right)(K),$$

so the required relation follows from Lemma 76. □

Theorem 84. $\mathsf{Patch}(X)$ *is a Stone locale, for every spectral locale X.*

Proof. Compactness (ST1) was given in Lemma 81. For (ST2), let $j : \mathscr{O}(\mathsf{Patch}(X))$. We take the intensional basis given by the family $(\mathbf{c}(K_1) \wedge \mathbf{o}(K_2))_{(K_1,K_2):\mathsf{K}(X)\times\mathsf{K}(X)}$. For each Scott continuous nucleus j, we define its basic covering family to be $\gamma : I_j \to \mathscr{O}(\mathsf{Patch}(X))$, defined by

$$\begin{aligned} I_j \quad &:\equiv \quad \Sigma_{(K_1,K_2\,:\,\mathsf{K}(X))}(K_1 \leq j(K_2)), \text{ and}\\ \gamma(K_1, K_2) \quad &:\equiv \quad \mathbf{c}(K_1) \wedge \mathbf{o}(K_2). \end{aligned}$$

Even though this basic covering family is not a priori directed, we know that it can be directified as explained in Remark 17. We therefore obtain a specified, small, intensional basis of clopens by Lemma 83. We denote the directed form of the basis by $B^{\uparrow}$ which is small by Remark 17 since $\mathsf{K}(X)$ is small by (SP4). Since clopens are compact in compact locales by Lemma 47, the clopens fall in the basis by Lemma 29. Therefore, we get an equivalence of types

$$\mathsf{C}(X) \simeq \mathsf{image}(B^{\uparrow}),$$

where $B^{\uparrow}$ denotes the closure of the above basis under finite joins (as explained in Remark 17). This concludes (ST3) by Lemma 18. □

8.2 The universal property of Patch

We now show that Patch is the right adjoint to the inclusion **Stone** $\hookrightarrow$ **Spec**.

Lemma 85. *Given any spectral map $f : X \to A$ from a Stone locale into a spectral locale, define a map $\bar{f}^* : \mathscr{O}(\mathsf{Patch}(A)) \to \mathscr{O}(X)$ by*

$$\bar{f}^*(j) \quad :\equiv \quad \bigvee_{K:\mathsf{K}(A)} f^*(j(K)) \wedge \neg f^*(K).$$

Then, the map $\bar{f}_* : \mathscr{O}(X) \to \mathscr{O}(\mathsf{Patch}(A))$ defined by

$$\bar{f}_*(V) \quad :\equiv \quad f_* \circ \mathbf{c}(V) \circ f^*$$

is the right adjoint of $\bar{f}^*$.

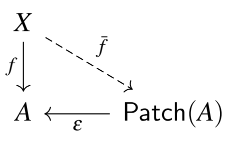
Proof. First, note that $f_* \circ \mathbf{c}(V) \circ f^*$ is indeed a Scott continuous nucleus, and both $\bar{f}^*$ and $\bar{f}_*$ are clearly monotone. Let us first prove the forward implication. Assume that for $j : \mathscr{O}(\mathsf{Patch}(A))$ and $V : \mathscr{O}(X)$, the relation $\bar{f}^*(j) \leq V$ holds. By Lemma 76, in order to show that $j \leq \bar{f}_*(V)$, it suffices to show that for any compact open $K : \mathsf{K}(A)$, the inequality $j(K) \leq \bar{f}_*(V)(K)$ holds. Hence, let $K : \mathsf{K}(A)$, and note that

$$f^*(j(K)) \wedge \neg f^*(K) \leq \bar{f}^*(j) \leq V.$$

Notice that $f^*(K)$ is clopen, by the fact that f^* is a spectral map and Lemma 48. It is therefore complemented in the lattice $\mathscr{O}(X)$, and so we have $f^*(j(K)) \leq V \vee f^*(K) = \mathbf{c}(V)(f^*(K))$, which by adjunction yields $j(K) \leq f_*(\mathbf{c}(V)(f^*(K))) = \bar{f}_*(K)$, as required.

Let us now show the reverse implication. Let $j : \mathscr{O}(\mathsf{Patch}(A))$ and $V : \mathscr{O}(X)$ and assume that $j \leq \bar{f}_*(V)$. Once again, by the definition of the ordering on $\mathscr{O}(\mathsf{Patch}(A))$, for all $K : \mathsf{K}(A)$ we have $j(K) \leq f_*(V \vee f^*(K))$, which by adjunction equivalently yields $f^*(j(K)) \leq V \vee f^*(K)$. Since $f^*(K)$ is clopen, and hence complemented in the lattice $\mathscr{O}(X)$ it follows that $f^*(j(K)) \wedge \neg f^*(K) \leq V$. Hence, $\bar{f}^*(j) \leq V$. □

Theorem 86. *For every spectral map $f : X \to A$ from a Stone locale into a spectral locale, there exists a unique continuous map $\bar{f} : X \to \mathsf{Patch}(A)$ satisfying $\varepsilon \circ \bar{f} = f$, as illustrated in the following diagram in* **Spec**:



Proof. Assume that a locale map $\bar{f} : X \to \mathsf{Patch}(A)$ satisfies the condition in the theorem. Then, for any $j : \mathscr{O}(\mathsf{Patch}(A))$, one has

$$\begin{aligned}
\bar{f}^*(j) &= \bar{f}^*\left(\bigvee_{K:\mathsf{K}(A)} \mathbf{c}(j(K)) \wedge \mathbf{o}(K)\right) && \text{[Lemma 83]}\\
&= \bigvee_{K:\mathsf{K}(A)} \bar{f}^*\left(\mathbf{c}(j(K)) \wedge \mathbf{o}(K)\right) && [\bar{f}^* \text{ preserves small joins}]\\
&= \bigvee_{K:\mathsf{K}(A)} \bar{f}^*\left(\mathbf{c}(j(K))\right) \wedge \neg\bar{f}^*\left(\mathbf{c}(K)\right) && [\bar{f}^* \text{ preserves binary meets and complements}]\\
&= \bigvee_{K:\mathsf{K}(A)} f^*\left(j(K)\right) \wedge \neg f^*\left(K\right) && \text{[commutativity of the diagram]}
\end{aligned}$$

and hence $\bar{f}$ is uniquely determined. If we now define a monotone map $\bar{f}^* : \mathscr{O}(\mathsf{Patch}(A)) \to \mathscr{O}(X)$ by

$$\bar{f}^*(j) \quad :\equiv \quad \bigvee_{K:\mathsf{K}(A)} f^*(j(K)) \wedge \neg f^*(K),$$

it is easy to show it preserves the top element (namely the top nucleus with constant value $\mathbf{1}_A$) because $\mathbf{0}_A$ is compact. It also preserves binary (pointwise) meets as

$$\begin{aligned}
&\bar{f}^*(j_1) \wedge \bar{f}^*(j_2)\\
&= \bigvee_{K_1,K_2:\mathsf{K}(A)} f^*\left(j_1(K_1)\right) \wedge \neg f^*\left(K_1\right) \wedge f^*\left(j_2(K_2)\right) \wedge \neg f^*\left(K_2\right) && \text{[distributivity]}\\
&= \bigvee_{K_1,K_2:\mathsf{K}(A)} f^*\left(j_1(K_1) \wedge j_2(K_2)\right) \wedge \neg f^*\left(K_1\right) \wedge \neg f^*\left(K_2\right) && [f^* \text{ preserves binary meets}]\\
&\leq \bigvee_{K_1,K_2:\mathsf{K}(A)} f^*\big(j_1(K_1 \vee K_2) \wedge j_2(K_1 \vee K_2)\big) \wedge \neg f^*(K_1) \wedge \neg f^*(K_2) && \text{[monotonicity]}\\
&= \bigvee_{K_1,K_2:\mathsf{K}(A)} f^*\left((j_1 \wedge j_2)(K_1 \vee K_2)\right) \wedge \neg f^*\left(K_1\right) \wedge \neg f^*\left(K_2\right)\\
&= \bigvee_{K_1,K_2:\mathsf{K}(A)} f^*\left((j_1 \wedge j_2)(K_1 \vee K_2)\right) \wedge \neg f^*\left(K_1 \vee K_2\right) && \text{[De Morgan law]}\\
&= \bar{f}^*(j_1 \wedge j_2). && [\mathsf{K}(X) \text{ closed under } (-) \vee (-)]
\end{aligned}$$

Moreover, Lemma 85 and Theorem 56 ensure that $\bar{f}^*$ preserves small joins and so it is a frame homomorphism.

Let us finally show that $\bar{f}$ makes the diagram commute. Since compact opens from a small basis for A, it suffices to show that $\bar{f}^*(\mathbf{c}(K)) = f^*(K)$ for any $K : \mathsf{K}(A)$. Let $K : \mathsf{K}(A)$ and note that

$$\begin{aligned}
\bar{f}^*(\mathbf{c}(K)) &\equiv \bigvee_{K':\mathsf{K}(A)} f^*\left(K \vee K'\right) \wedge \neg f^*\left(K'\right)\\
&= \bigvee_{K':\mathsf{K}(A)} f^*\left(K\right) \wedge \neg f^*\left(K'\right) && [f^* \text{ preserves binary joins}]\\
&= f^*(K) \wedge \bigvee_{K':\mathsf{K}(A)} \neg f^*\left(K'\right) && \text{[distributivity]}
\end{aligned}$$

$$= f^*(K) \wedge \mathbf{1}_X \qquad [\mathbf{0}_X \text{ is compact}]$$
$$= f^*(K),$$

as required. □

9. Summary and Discussion

We have constructed the patch locale of a spectral locale in the predicative and constructive setting of univalent type theory. Furthermore, we have shown that the patch functor

$$\mathsf{Patch} : \mathbf{Spec} \to \mathbf{Stone}$$

is the right adjoint to the inclusion **Stone** $\hookrightarrow$ **Spec**, which is to say that Patch exhibits the category **Stone** as a coreflective subcategory of **Spec**. As we have elaborated on in Section 4, answering this question in a predicative setting involves several new ingredients, compared to Escardó (1999, 2001):

(1) We have reformulated the notions of spectrality and Stone-ness in our predicative type-theoretic setting and have shown that crucial topological facts about these notions remain valid under these reformulations.
(2) We have developed several predicative formulations of the notion of spectral (resp. Stone) locale and have shown that they are equivalent in Theorem 36 (resp. Theorem 54).
(3) In Escardó's (1999) construction of the patch locale, the proof of the universal property relies on the existence of the frame of *all* nuclei. As it is not clear that the poset of all nuclei can be shown to form a frame predicatively, we developed a new proof of the universal property using Lemma 85, which is completely independent of the existence of the frame of all nuclei.

We have formalized all of our development, most importantly Theorem 84 and Theorem 86. The formalization has been carried out by the third-named author (Tosun 2023) as part of the TypeTopology library (Escardó and contributors, 2018).

In previous work (Escardó 1999, 2001), which forms the basis of the present work, the patch construction was used to

(1) exhibit **Stone** as a coreflective subcategory of **Spec**, and
(2) exhibit the category of compact regular locales and continuous maps as a coreflective subcategory of stably compact locales and perfect maps.

In our work, we have focused on item (1). The question of taking a predicative approach to item (2) was previously tackled by Coquand and Zhang (2003) using formal topology. We conjecture that it is possible to instead use the approach we have presented here, namely, working with locales with small bases and constructing the patch as the frame of Scott continuous nuclei. It should also be possible to show predicatively that the category of small distributive lattices is dually equivalent to the category of spectral locales as defined in this paper.

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Competing interests:. The authors declare none.

Note

1 Note that in the context of set theory, the term *constructive* corresponds to what we call *constructive and predicative*. The term *intuitionistic* is used for what we call constructive, hat is, mathematics without the use of classical principles.

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