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On the representations of Sp(p,q) and SO$^*(2n)$ unitarily induced from derived functor modules

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Abstract

We obtain a decomposition formula of a representation of $\mathrm{Sp}(p,q)$ or $\mathrm{SO}^*(2n)$ unitarily induced from a derived functor module, which enables us to reduce the problem of irreducible decompositions to the study of derived functor modules. In particular, we show that such an induced representation is decomposed into a direct sum of irreducible unitarily induced modules from derived functor modules under some regularity condition on the parameters. In particular, representations of $\mathrm{SO}^*(2n)$ and $\mathrm{Sp}(p,q)$ induced from one-dimensional unitary representations of their parabolic subgroups are irreducible.

Introduction

Our object of study is the decomposition of unitarily induced modules of a real reductive Lie group from derived functor modules. In [Mat96], the case of $\mathrm{U}(m,n)$ is treated. In this article, we study the case of $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$. Reducibilities of the representations of $\mathrm{U}(m,n)$ unitarily induced from derived functor modules come from the reducibility of particular degenerate principal series of $\mathrm{U}(n,n)$ found by Kashiwara and Vergne [KV79]. In the case of $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$ the situation is quite similar, at least for regular values of the parameters. The reducibilities are also reduced to the Kashiwara–Vergne decomposition.

Let us go into more detail. Put $G=\mathrm{Sp}(p,q)(p\geqslant q)$ or $G=\mathrm{SO}^*(2n)$. We fix a Cartan involution θ as usual. Let $\kappa=(k_1,\dots,k_s)$ be a finite sequence of positive integers such that

$$k_1+\cdots+k_s\leqslant\begin{cases} q & \text{if } G=\mathrm{Sp}(p,q),\\ n/2 & \text{if } G=\mathrm{SO}^*(2n).\end{cases}$$

If $G=\mathrm{Sp}(p,q)$, put $p'=p-k_1-\cdots-k_s$ and $q'=q-k_1-\cdots-k_s$. If $G=\mathrm{SO}^*(2n)$, put $r=n-2(k_1-\cdots-k_s)$. Then, there is a parabolic subgroup P_κ of G, whose Levi subgroup M_κ is written as

$$M_\kappa\cong\begin{cases}\mathrm{GL}(k_1,\mathbb{H})\times\cdots\times\mathrm{GL}(k_s,\mathbb{H})\times\mathrm{Sp}(p',q') & \text{if } G=\mathrm{Sp}(p,q),\\ \mathrm{GL}(k_1,\mathbb{H})\times\cdots\times\mathrm{GL}(k_s,\mathbb{H})\times\mathrm{SO}^*(2r) & \text{if } G=\mathrm{SO}^*(2n).\end{cases}$$

Here, formally, we denote by $\mathrm{Sp}(0,0)$ and $\mathrm{SO}^*(0)$ the trivial group $\{1\}$. Any parabolic subgroup of G is G-conjugate to some P_κ. $\mathrm{GL}(k,\mathbb{H})$ has some particular irreducible unitary representation, the so-called quaternionic Speh representation, defined as follows. We consider $\mathrm{GL}(k,\mathbb{C})$ as a subgroup of $\mathrm{GL}(k,\mathbb{H})$. For $\ell\in\mathbb{Z}$ and $t\in\sqrt{-1}\,\mathbb{R}$, we define a one-dimensional unitary representation $\xi_{\ell,t}$

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of $\mathrm{GL}(k,\mathbb{C})$ as follows:

$$\xi_{\ell,t}(g) = \left(\frac{\det(g)}{|\det(g)|}\right)^{\ell} |\det(g)|^{t}.$$

$\mathrm{GL}(k,\mathbb{C})$ is the centralizer in $\mathrm{GL}(k,\mathbb{H})$ of the group consisting of complex scalar matrices with eigenvalue of absolute value unity. So, there is a θ-stable parabolic subalgebra $\mathfrak{q}(k)$ with a Levi subgroup $\mathrm{GL}(k,\mathbb{C})$. We choose the nilradical $\mathfrak{n}(k)$ so that $\xi_{\ell,t}$ is good with respect to $\mathfrak{q}(k)$ for sufficiently large ℓ. Derived functor modules with respect to $\mathfrak{q}(k)$ are called quaternionic Speh representations. For $t \in \sqrt{-1}\,\mathbb{R}$, there is a one-dimensional unitary representation $\tilde{\xi}_t$ of $\mathrm{GL}(k,\mathbb{H})$ whose restriction to $\mathrm{GL}(k,\mathbb{C})$ is $\xi_{0,t}$. We put

$$A_k(\ell,t) = ({}^u\mathcal{R}^{\mathfrak{gl}(k,\mathbb{H})\otimes_{\mathbb{R}}\mathbb{C},\mathrm{Sp}(k)}_{\mathfrak{q}(k),\mathrm{O}(k)})^{k(k+1)}(\xi_{\ell+2k,t}) \quad (\ell \in \mathbb{Z}).$$

Here, ${}^u\mathcal{R}$ means the cohomological induction ([KV95], see also § 1.3). We also put

$$A_k(-\infty,t) = \tilde{\xi}_t.$$

For $\ell \in \mathbb{Z}$, $A_k(\ell,t)$ is a derived functor module in the good (respectively weakly fair) range in the sense of [Vog88] if and only if $\ell \geqslant 0$ (respectively $\ell \geqslant -k$). It is more or less known by [Vog86] that any derived functor module of $\mathrm{GL}(k,\mathbb{H})$ is a unitary parabolic induction from one-dimensional representations or quaternionic Speh representations. So, it suffices to consider the following induced representation:

$$\mathrm{Ind}^{G}_{P_\kappa}(A_{k_1}(\ell_1,t_1)\boxtimes\cdots\boxtimes A_{k_s}(\ell_s,t_s)\boxtimes Z). \tag{1}$$

Here, Z is a derived functor module of $\mathrm{Sp}(p',q')$ or $\mathrm{SO}^*(2r)$ in the weakly fair range. Moreover, $\ell_i \in \{\ell \in \mathbb{Z} \mid \ell \geqslant -k_i\} \cup \{-\infty\}$, and $t_i \in \sqrt{-1}\,\mathbb{R}$ for $1 \leqslant i \leqslant s$. If we apply the Harish-Chandra result that the equivalence class of a representation parabolically induced from a unitary representation (π_M, M) depends only on the conjugacy class of (π_M, M), we see that permuting the $A_{k_i}(\ell_i,t_i)$'s does not change the induced representation. We assume that $\ell_i + 1 \in 2\mathbb{Z}$ and $t_i = 0$ for some $1 \leqslant i \leqslant s$. Thus, we may assume $\ell_s + 1 \in 2\mathbb{Z}$ and $t_s = 0$. Let $\kappa' = (k_1,\dots,k_{s-1})$. Then from induction by stages, we have

$$\begin{aligned}
&\mathrm{Ind}^{G}_{P_\kappa}(A_{k_1}(\ell_1,t_1)\boxtimes\cdots\boxtimes A_{k_s}(\ell_s,t_s)\boxtimes Z)\\
&\quad\cong \mathrm{Ind}^{G}_{P'_\kappa}(A_{k_1}(\ell_1,t_1)\boxtimes\cdots\boxtimes A_{k_{s-1}}(\ell_{s-1},t_{s-1})\boxtimes \mathrm{Ind}^{M^\circ_{\kappa'}}_{P_{(k_s)}}(A_{k_s}(\ell_s,0)\boxtimes Z)).
\end{aligned}$$

Here, $M^\circ_{\kappa'}$ is $\mathrm{Sp}(p'+k_s, q'+k_s)$ or $\mathrm{SO}^*(2(r+2k_s))$.

Our reducibility result is thus the following (where the number in parentheses refers to the theorem number later in this paper).

Theorem A (Theorem 3.6.5). $\mathrm{Ind}^{M_{\kappa'}}_{P_{(k_s)}}(A_{k_s}(\ell_s,0)\boxtimes Z)$ *decomposes into a direct sum of derived functor modules of* $M^\circ_{\kappa'}$ *in the weakly fair range.*

We obtain an explicit decomposition formula.

Whenever there is $1 \leqslant i \leqslant s$ such that $\ell_i + 1 \in 2\mathbb{Z}$ and $t_i = 0$, we can apply the above procedure. Assuming that we understand the reducibility of derived functor modules, we can reduce the irreducible decomposition of the above induced module to the following:

$$\mathrm{Ind}^{G}_{P_\kappa}(A_{k_1}(\ell_1,0)\boxtimes\cdots\boxtimes A_{k_h}(\ell_h,0)\boxtimes A_{k_{h+1}}(\ell_{h+1},t_{h+1})\boxtimes\cdots\boxtimes A_{k_s}(\ell_s,t_s)\boxtimes Z). \tag{2}$$

Here, ℓ_i is not an odd integer if $1 \leqslant i \leqslant h$, $\sqrt{-1}\,t_i > 0$ if $h < i \leqslant s$, and Z is an irreducible representation of M°_κ whose infinitesimal character plus the half-sum of positive roots can be realized as a weight of a finite-dimensional representation of G. Put $\tau = (k_1,\dots,k_h)$ and $\tau' = (k_{h+1},\dots,k_s)$. Also put $a = k_1 + \cdots + k_h$ and $b = k_{h+1} + \cdots + k_s$. In this setting we have the following theorem.

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THEOREM B (Theorem 4.2.2). *The following are equivalent.*

i) *The induced representation (2) is irreducible.*

ii) *The induced module*

$$\mathrm{Ind}_{P_\tau}^{\mathrm{SO}^*(4a)}(A_{k_1}(\ell_1,0)\boxtimes\cdots\boxtimes A_{k_h}(\ell_h,0))$$

is irreducible.

Under an appropriate regularity condition on $\ell_1,\ldots,\ell_h$, we may show the irreducibility of the induced module in part ii above.

On the induced representation in Theorem B, part ii, we have a partial answer.

LEMMA C (Lemma 5.1.1). *If $\ell_1,\ldots,\ell_h$ are all $-\infty$ (namely, if $A_{k_1}(\ell_1,0),\ldots,A_{k_h}(\ell_h,0)$ are trivial representations), then $\mathrm{Ind}_{P_\tau}^{\mathrm{SO}^*(4a)}(A_{k_1}(\ell_1,0)\boxtimes\cdots\boxtimes A_{k_h}(\ell_h,0))$ is irreducible.*

We have a corollary of this result.

COROLLARY D (Corollary 5.1.2). *Representations of $\mathrm{SO}^*(2n)$ and $\mathrm{Sp}(p,q)$ induced from one-dimensional unitary representations of their parabolic subgroups are irreducible.*

For some special parabolic subgroups, the irreducibility of the above kind of induced representation is known. If the parabolic subgroup is minimal, the irreducibility of the induced representation is a special case of a general result in [Kos69] (see also [Hel70]). The studies of Johnson, of Sahi, and of Howe and Tan [Joh90, Sah93, HT93] also include the irreducibility of the induced modules from a unitary one-dimensional representation of some maximal parabolic subgroups.

The remaining problems on the reducibility of the representations of $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$ unitarily induced from derived functor modules in the weakly fair region are:

i) vanishing and irreducibilities of derived functor modules of $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$ in the weakly fair range;

ii) irreducibilities of the induced representation of the form

$$\mathrm{Ind}_{P_\tau}^{\mathrm{SO}^*(4a)}(A_{k_1}(\ell_1,0)\boxtimes\cdots\boxtimes A_{k_h}(\ell_h,0)).$$

(Here, $\ell_i (1\leqslant i\leqslant h)$ are even integers or $-\infty$.)

Regrettably, I do not have a complete answer to the above problem. For a type A group $\mathrm{U}(m,n)$, general theories on translation principle are applicable to the above problem on irreducibilities. Together with Trapa's result [Tra01], we have a complete answer. Unfortunately, neither $\mathrm{Sp}(p,q)$ nor $\mathrm{SO}^*(2n)$ are of type A. So, the situation is more difficult than for the case of $\mathrm{U}(m,n)$. In fact, irreducibility of a derived functor module of $\mathrm{Sp}(p,q)$ fails in some singular parameter (Vogan). If the degeneration of the parameter is not so bad, Vogan [Vog88] found an idea to control irreducibilities. Using the idea, he proved irreducibility of discrete series of semisimple symmetric spaces. This idea works in this case. In fact, using Vogan's idea, Kobayashi studied irreducibilities of derived functor modules of $\mathrm{Sp}(p,q)$ in [Kob92]. In a subsequent article, I would like to take up this problem.

One of the main ingredients of this article is the change-of-polarization formula (Theorem 2.2.3). It means we may exchange, under some positivity condition, the order of cohomological induction and parabolic induction in the Grothendieck group of the category of Harish-Chandra modules. The change of polarization for a standard module was originated by Vogan [Vog83a] and completed by Hecht, Miličić, Schmid, and Wolf (cf. [Sch88]). See also [KV95, Theorem 11.87]. For the degenerate setting, some case is observed for $\mathrm{GL}(n)$ in [Vog86]. I applied this idea in [Mat96]. In Theorem 2.2.3, I give a formulation of the change of polarization in the general setting.

The other ingredient of this article is comparison of Hecke algebra module structures. In fact, the irreducible decomposition of the standard representation is determined only by the Hecke algebra module structure via the so-called Kazdhan–Lusztig algorithm [ABV92]. This deep result enables us to compare irreducibilities of induced representations of different groups with the same Hecke algebra module structures. Using this idea, we show Theorem B (Theorem 4.2.2).

1. Preliminaries

1.1 General notations

In this article, we use the following notations. As usual we denote the Hamilton quaternionic field, the complex number field, the real number field, the rational number field, the ring of integers, and the set of non-negative integers by $\mathbb{H}$, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively. We denote by $\emptyset$ the empty set and denote by $A-B$ the set theoretical difference of A from B. For each set A, we denote by $\operatorname{card} A$ the cardinality of A. For a complex number a (respectively a matrix X over $\mathbb{C}$), we denote by $\bar{a}$ (respectively $\overline{X}$) the complex conjugation. For the Hamilton quaternionic field, we also use a similar notation. If $p>q$, we put $\sum_{i=p}^{q}=0$.

Let R be a ring and let M be a left R-module. We denote by $\operatorname{Ann}_R(M)$ the annihilator of M in R. In this article, a character of a Lie group G means a (not necessarily unitary) continuous homomorphism of G to $\mathbb{C}^\times$. For a matrix $X=(a_{ij})$, we denote by ${}^{t}X$, $\operatorname{tr}X$, and $\det X$ the transpose (a_{ji}) of X, the trace of X, and the determinant of X respectively. For a positive integer k, we denote by I_k (respectively 0_k) the $k\times k$ identity (respectively zero) matrix.

Let $n, n_1, \ldots, n_\ell$ be positive integers such that $n = n_1+\cdots+n_\ell$. For $n_i\times n_i$ matrices X_i $(1\leqslant i\leqslant \ell)$, we put

$$\operatorname{diag}(X_1,\ldots,X_\ell)=\begin{pmatrix} X_1 & 0 & \cdots & \cdots & 0\\ 0 & X_2 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \cdots & 0 & X_\ell\end{pmatrix}$$

We denote by $\mathfrak{S}_\ell$ the ℓth symmetric group.

For a complex Lie algebra $\mathfrak{g}$, we denote by $U(\mathfrak{g})$ its universal enveloping algebra. We denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. For a Harish-Chandra module V, we denote by $[V]$ the corresponding distribution character. In this article, an irreducible Harish-Chandra module should be non-zero.

1.2 Notations for root systems

Let G be a connected real reductive linear group, and let $G_{\mathbb{C}}$ be its complexification. We fix a maximal compact subgroup K of G and denote by θ the corresponding Cartan involution. We denote by $\mathfrak{g}_0$ (respectively $\mathfrak{k}_0$) the Lie algebra of G (respectively K). Let H be a θ-stable Cartan subgroup of G and let $\mathfrak{h}_0$ be its Lie algebra. We denote by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{h}$ the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, and $\mathfrak{h}_0$, respectively. We denote by $\mathfrak{h}^*$ the complex dual of $\mathfrak{h}$. We denote the induced involution from θ on $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{h}^*$ by the same letter θ. We denote by σ the complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. We denote by $W(\mathfrak{g},\mathfrak{h})$ (respectively $\Delta(\mathfrak{g},\mathfrak{h})$) the Weyl group (respectively the root system) with respect to the pair $(\mathfrak{g},\mathfrak{h})$. Let $\langle\ .\ \rangle$ be a $W(\mathfrak{g},\mathfrak{h})$-invariant bilinear form on $\mathfrak{h}^*$ induced from an invariant non-degenerate bilinear form of $\mathfrak{g}$.

A root $\alpha\in\Delta(\mathfrak{g},\mathfrak{h})$ is called imaginary (respectively real) if $\theta(\alpha)=\alpha$ (respectively $\theta(\alpha)=-\alpha$). A root $\alpha\in\Delta(\mathfrak{g},\mathfrak{h})$ is called complex if α is neither real nor imaginary. An imaginary root $\alpha\in\Delta(\mathfrak{g},\mathfrak{h})$ is called compact (respectively non-compact) if the root space for α is contained (respectively not contained) in $\mathfrak{k}$.

We denote by $\mathcal{P}(\mathfrak{h})$ the integral weight lattice in $\mathfrak{h}^*$. Namely, we put

$$\mathcal{P}(\mathfrak{h}) = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; 2\frac{\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z} \quad (\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})) \right\}.$$

We also put

$$\mathcal{P}_G(\mathfrak{h}) = \{\lambda \in \mathfrak{h}^* \mid \lambda \text{ is a weight of some finite-dimensional representation of } G\}.$$

We denote by $\mathcal{Q}(\mathfrak{h})$ the root lattice, namely the set of integral linear combinations of elements of $\Delta(\mathfrak{g}, \mathfrak{h})$. We have $\mathcal{Q}(\mathfrak{h}) \subseteq \mathcal{P}_G(\mathfrak{h}) \subseteq \mathcal{P}(\mathfrak{h}) \subseteq \mathfrak{h}^*$.

For $\lambda \in \mathfrak{h}^*$, we denote by χ_λ the corresponding Harish-Chandra homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$.

We fix a θ-stable maximally split Cartan subgroup ${}^sH$ of G and denote by ${}^s\mathfrak{h}$ its complexified Lie algebra. For simplicity, we write Δ, W, $\mathcal{P}$, $\mathcal{P}_G$, $\mathcal{Q}$ for $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$, $W(\mathfrak{g}, {}^s\mathfrak{h})$, $\mathcal{P}({}^s\mathfrak{h})$, $\mathcal{P}_G({}^s\mathfrak{h})$, $\mathcal{Q}({}^s\mathfrak{h})$, respectively.

We choose regular weights $\lambda \in \mathfrak{h}^*$ and ${}^s\lambda \in {}^s\mathfrak{h}^*$ such that $\chi_\lambda = \chi_{{}^s\lambda}$. Then, there is a unique isomorphism $\mathbf{i}_{{}^s\lambda,\lambda} : {}^s\mathfrak{h}^* \to \mathfrak{h}^*$ induced from an inner automorphism of G such that $\mathbf{i}_{{}^s\lambda,\lambda}({}^s\lambda) = \lambda$. We denote by the same letter $\mathbf{i}_{{}^s\lambda,\lambda}$ the corresponding isomorphism of W onto $W(\mathfrak{g}, \mathfrak{h})$.

1.3 Cohomological inductions

We fix the notations on the Vogan–Zuckerman cohomological inductions of Harish-Chandra modules. Here, we adapt the definition found in [KV95]. Let G be a real reductive linear Lie group which is contained in the complexification $G_{\mathbb{C}}$. We assume $G_{\mathbb{C}}$ is a connected complex reductive linear group.

Definition 1.3.1. Assume that a parabolic subalgebra $\mathfrak{q}$ has a Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ such that $\mathfrak{l}$ is stable under θ and σ. Such a Levi decomposition is called an orderly Levi decomposition.

A θ-stable or σ-stable parabolic subalgebra has a unique orderly Levi decomposition. In fact, if $\mathfrak{q}$ is θ-stable (respectively σ-stable), then $\mathfrak{l} = \mathfrak{q} \cap \sigma(\mathfrak{q})$ (respectively $\mathfrak{l} = \mathfrak{q} \cap \theta(\mathfrak{q})$).

Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ with an orderly Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. We fix a θ- and σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ and a Weyl group invariant non-degenerate bilinear form $\langle\ ,\ \rangle$. Let L be the corresponding Levi subgroup in G to $\mathfrak{l}$.

We denote by ${}^u\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K}$ the right adjoint functor of the forgetful functor of the category of $(\mathfrak{g}, K)$-modules to the category of $(\mathfrak{q}, L \cap K)$-modules. Introducing trivial $\mathfrak{u}$-action, we regard an $(\mathfrak{l}, L \cap K)$-module as a $(\mathfrak{q}, L \cap K)$-module. So, we also regard ${}^u\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K}$ as a functor of the category of $(\mathfrak{l}, L\cap K)$-modules to the category of $(\mathfrak{g}, K)$-modules. We denote by $({}^u\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^i$ the ith right derived functor (see [KV95, p. 671]).

We review a normalized version. We define a one-dimensional representation $\delta(\mathfrak{u})$ of $\mathfrak{l}$ by $\delta(\mathfrak{u})(X) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{\mathfrak{u}})$. Following [KV95, p. 720], we define a one-dimensional representation $\mathbb{C}_{2\delta(\mathfrak{u})'}$ of L as follows. (For later use, we introduce slightly more general setting.) Let V be a finite-dimensional semisimple $\mathfrak{l}$-module. We define a one-dimensional representation $\delta(V)$ of $\mathfrak{l}$ by $\delta(V)(X) = \frac{1}{2}\mathrm{tr}(X|_V)$. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ be the decomposition of V into irreducible $\mathfrak{l}$-modules. We distinguish between those V_i that are self-conjugate with respect to σ and those that are not. We define a one-dimensional representation $\xi_{2\delta(V)'}$ of L on a space $\mathbb{C}_{2\delta(V)'}$ by

$$\xi_{2\delta(V)'}(\ell) = \Bigg(\prod_{i \text{ with } V_i \text{ self-conjugate}} |\det(\ell|_{V=i})| \Bigg) \Bigg(\prod_{i \text{ with } V_i \text{ not self-conjugate}} \det(\ell|_{V=i}) \Bigg).$$

Let $L^\sim$ be the metaplectic double cover of L with respect to $\mathbb{C}_{2\delta(V)'}$, namely,

$$L^\sim = \{(\ell, z) \in L \times \mathbb{C}^\times \mid \xi_{2\delta(V)'}(\ell) = z^2\}.$$

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We define the one-dimensional $L^{\sim}$-module $\mathbb{C}_{\delta(V)'}$ by the projection to the second factor of $L^{\sim} \subseteq L \times \mathbb{C}^{\times}$. Of course, the definition of $L^{\sim}$ depends on V. Hereafter, we consider the case of $V = \mathfrak{u}$ (the adjoint action of L on $\mathfrak{u}$). Let $(K \cap L)^{\sim}$ be the maximal compact subgroup of L corresponding to $K \cap L$.

Let Z be a Harish-Chandra $(\mathfrak{l}, (K \cap L)^{\sim})$-module such that $Z \otimes \mathbb{C}_{\delta(\mathfrak{u})}$ is a Harish-Chandra $(\mathfrak{l}, K \cap L)$-module. We put

$$({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i}(Z) = ({}^{u}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i}(Z \otimes \mathbb{C}_{\delta(\mathfrak{u})'}).$$

Let λ be the infinitesimal character of Z with respect to $\mathfrak{h}$. (It is well defined up to the Weyl group action of $\mathfrak{l}$.) Then $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i}(Z)$ is a Harish-Chandra $(\mathfrak{g}, K)$-module of an infinitesimal character λ.

We consider three particular cases.

i) *Hyperbolic case.* If $\mathfrak{q}$ is σ-stable, then there is a parabolic subgroup $Q = LU$ whose complexified Lie algebra is $\mathfrak{q}$ and whose nilradical is U. In this case, we have $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i}(Z) = 0$ for all $i > 0$. In fact, $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{0}(Z)$ is nothing but the parabolic induction $\mathrm{Ind}_{Q}^{G}(Z)$.

We clarify the definition of the parabolic induction. First, we remark that $L^{\sim}$ is just a direct product $L \times \{\pm 1\}$ in this case and $\mathbb{C}_{\delta(\mathfrak{u})}$ can be reduced to a representation of L (say $(\xi_{\delta(\mathfrak{u})}, \mathbb{C}_{\delta(\mathfrak{u})})$).

$\mathrm{Ind}_{Q}^{G}(Z)$ (we also write $\mathrm{Ind}(Q \uparrow G; Z)$) is the K-finite part of

$$\{f \in C^{\infty}(G) \otimes H \mid f(g\ell n) = \pi(\ell^{-1}) f(g)\ (g \in G, \ell \in L, n \in U)\}.$$

Here, (π, H) is a Hilbert globalization of $Z \otimes \mathbb{C}_{\delta(\mathfrak{u})}$. If Z is unitarizable, so is $\mathrm{Ind}(Q \uparrow G; Z)$ (unitary induction). We also consider the unnormalized parabolic induction as follows:

$${}^{u}\mathrm{Ind}(Q \uparrow G; Z) = \mathrm{Ind}(Q \uparrow G; Z \otimes \mathbb{C}_{\delta(\bar{\mathfrak{u}})}).$$

ii) *Elliptic case.* Assume $\mathfrak{q}$ is θ-stable and put $S = \dim(\mathfrak{u} \cap \mathfrak{k})$. We call Z weakly good (or λ is in the weakly good range) if $\mathrm{Re}\langle \lambda, \alpha \rangle \geqslant 0$ holds for each root α of $\mathfrak{h}$ in $\mathfrak{u}$. We call Z integrally good (respectively weakly integrally good), if $\langle \lambda, \alpha \rangle > 0$ (respectively $\langle \lambda, \alpha \rangle \geqslant 0$) holds for each root α of $\mathfrak{h}$ in $\mathfrak{u}$ such that $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$.

Theorem 1.3.2 [Vog84, Theorem 2.6].

a) *If Z is weakly integrally good, then $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{i} = 0$ for $i \neq S$.*

b) *If Z is irreducible and weakly integrally good, $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{S}(Z)$ is irreducible or zero.*

c) *If Z is irreducible and integrally good, $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{S}(Z)$ is irreducible.*

d) *If Z is unitarizable and weakly good, $({}^{n}\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q},L\cap K})^{S}(Z)$ is unitarizable.*

iii) *Standard modules.* A regular character (H, Γ, λ) is a triple satisfying the following conditions R1–R6 (cf. [Vog82b]).

R1) H is a θ-stable Cartan subgroup of G.

R2) Γ is a (non-unitary) character of H.

R3) λ is in $\mathfrak{h}^{*}$. (Here, $\mathfrak{h}$ is the complexified Lie algebra of H.)

In order to write down the remaining conditions, we introduce some notations. Let $\mathfrak{t}$ (respectively $\mathfrak{a}$) be the $+1$ (respectively -1) eigenspace in $\mathfrak{h}$ with respect to θ. We denote by $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Then $\Delta(\mathfrak{m}, \mathfrak{h})$ is the set of imaginary roots in $\Delta(\mathfrak{g}, \mathfrak{h})$.

R4) λ is regular with respect to $\Delta(\mathfrak{m}, \mathfrak{h})$.

R5) $\langle \lambda, \alpha \rangle$ is real for any $\alpha \in \Delta(\mathfrak{m}, \mathfrak{h})$.

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Under the above conditions R4 and R5, there is a unique positive system $\Delta^+_\lambda(\mathfrak{m},\mathfrak{h})$ of $\Delta(\mathfrak{m},\mathfrak{h})$ such that $\langle\alpha,\lambda\rangle>0$ for all $\alpha\in\Delta^+_\lambda(\mathfrak{m},\mathfrak{h})$. We denote by $\rho_\lambda(\mathfrak{m},\mathfrak{h})$ (respectively $\rho^c_\lambda(\mathfrak{m},\mathfrak{h})$) the half-sum of positive imaginary roots (respectively positive compact imaginary roots) with respect to $\Delta^+_\lambda(\mathfrak{m},\mathfrak{h})$. We put $\mu_\lambda=\lambda+\rho_\lambda(\mathfrak{m},\mathfrak{h})-2\rho^c_\lambda(\mathfrak{m},\mathfrak{h})$.

R6) μ_λ is the differential of Γ.

We fix a regular character $\gamma=(H,\Gamma,\lambda)$. We denote by M the centralizer of $\mathfrak{a}$ in G. The above conditions R1–R5 assure that there is a unique relative discrete series representation σ with infinitesimal character λ such that the Blattner parameter of σ is Γ. Here, a relative discrete series means a representation whose restriction to semisimple part is in discrete series. We do not require the unitarizability of σ itself. We fix a parabolic subgroup P of G such that M is a Levi part of P. We define the standard module $\pi_G(\gamma)$ (we simply write $\pi(\gamma)$, if there is no confusion) for a regular character $\gamma=(H,\Gamma,\lambda)$ by $\pi_G(\gamma)=\mathrm{Ind}^G_P(\sigma)$. The distribution character $[\pi_G(\gamma)]$ is independent of the choice of P.

We may describe $\pi_G(\gamma)$ in terms of the cohomological induction as follows. First, let $\mathfrak{b}_0$ be the Borel subalgebra of $\mathfrak{m}$ corresponding to $(\mathfrak{h},\Delta^+_\lambda(\mathfrak{m},\mathfrak{h}))$ and let $\mathfrak{u}_1$ be its nilradical. Then $\mathfrak{b}_1$ is θ-stable and $\sigma\cong({}^u\mathcal{R}^{\mathfrak{m},M\cap K}_{\mathfrak{b}_1,H})^{\dim\mathfrak{u}_1\cap\mathfrak{k}}(\Gamma\otimes\mathbb{C}_{2\delta(\mathfrak{u}_1\cap\mathfrak{k})'})$. Let $\mathfrak{n}$ be the nilradical of the complexified Lie algebra of P. We put $\mathfrak{b}=\mathfrak{b}_1+\mathfrak{n}$ and $\mathfrak{u}=\mathfrak{u}_1+\mathfrak{n}$. Then $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{u}$ is the nilradical of $\mathfrak{b}$. Using the induction-by-stage formula [KV95, Corollary 11.86], we have

$$\pi_G(\gamma)\cong({}^u\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{b},H})^{\dim\mathfrak{u}\cap\mathfrak{k}}(\Gamma\otimes\mathbb{C}_{2\delta(\mathfrak{u}_1\cap\mathfrak{k})'}\otimes\mathbb{C}_{\delta(n)}).$$

There are various presentations of the standard representation as a cohomological induction from a character on a Borel subalgebra (cf. [Sch88] and [KV95, XI]).

For a regular character $\gamma=(H,\Gamma,\lambda)$ and $k\in K$, we put $k\cdot\gamma=(\mathrm{Ad}(k)H,\Gamma\circ\mathrm{Ad}(k^{-1}),\lambda\circ\mathrm{Ad}(k^{-1}))$. Then, $k\cdot\gamma$ is also a regular character. For two regular characters γ_1 and γ_2, $[\pi_G(\gamma_1)]=[\pi_G(\gamma_2)]$ if and only if $k\cdot\gamma_1=\gamma_2$ for some $k\in K$.

Let $\gamma=(H,\Gamma,\lambda)$ be a regular character and assume λ is regular with respect to $\Delta(\mathfrak{g},\mathfrak{h})$. Then, a standard module $\pi_G(\gamma)$ has a unique irreducible subquotient (Langlands subquotient) $\bar\pi_G(\gamma)$ such that all the minimal K-types of $\pi_G(\gamma)$ are contained in $\bar\pi_G(\gamma)$. Subquotient $\bar\pi_G(\gamma)$ is independent of the choice of P. Each irreducible Harish-Chandra $(\mathfrak{g},K)$-module with a regular infinitesimal character is isomorphic to some $\bar\pi_G(\gamma)$, and for two regular characters γ_1 and γ_2, $\bar\pi_G(\gamma_1)\cong\bar\pi_G(\gamma_2)$ if and only if $k\cdot\gamma_1=\gamma_2$ for some $k\in K$ (Langlands classification).

For a θ-stable Cartan subgroup H of G and a regular weight $\eta\in{}^s\mathfrak{h}^*$, we denote by $R_G(H,\eta)$ the set of regular characters (H,Γ,λ) such that $\chi_\lambda=\chi_\eta$. For a regular weight $\eta\in{}^s\mathfrak{h}^*$, we denote by $R_G(\eta)$ the set of all the regular character γ such that $\pi(\gamma)$ has an infinitesimal character η. Set $R_G(\eta)$ is the union of $R_G(H,\eta)$'s. We call a θ-stable Cartan subgroup H of G η-integral if $R_G(H,\eta)\neq\emptyset$.

A root $\alpha\in\Delta$ is called real, complex, compact imaginary, non-compact imaginary with respect to $\gamma=(H,\Gamma,\lambda)\in R_G(\eta)$, if $\mathbf{i}_{\eta,\lambda}(\alpha)$ is real, complex, compact imaginary, non-compact imaginary, respectively. For $\gamma=(H,\Gamma,\lambda)\in R_G(\eta)$, we put $\theta_\gamma=\mathbf{i}^{-1}_{\eta,\lambda}\circ\theta\circ\mathbf{i}_{\eta,\lambda}$; and θ_γ acts on Δ. Obviously, θ_γ only depends on the K-conjugacy class of γ.

2. Change of polarization

2.1 The $\sigma\theta$-pair

We consider here the following setting. Let G be a real reductive linear Lie group which is contained in the complexification $G_{\mathbb{C}}$. We fix a maximal compact subgroup K of G and let θ be the corresponding Cartan involution. We denote by $\mathfrak{g}_0$ (respectively $\mathfrak{k}_0$) the Lie algebra of G

(respectively K) and denote by $\mathfrak{g}$ (respectively $\mathfrak{k}$) its complexification. We denote also by the same letter θ the complexified Cartan involution on $\mathfrak{g}$. We denote by σ the complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

Definition 2.1.1. A pair $(\mathfrak{p}, \mathfrak{q})$ is called a $\sigma\theta$-pair if it satisfies the following conditions S1 and S2.

S1) $\mathfrak{q}$ (respectively $\mathfrak{p}$) is a θ-stable (respectively σ-stable) parabolic subalgebra of $\mathfrak{g}$.

S2) There exists a θ- and σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{p} \cap \mathfrak{q}$.

Hereafter, we fix a $\sigma\theta$-pair $(\mathfrak{p}, \mathfrak{q})$. Let $\mathfrak{h}$ be any θ- and σ-stable Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p} \cap \mathfrak{q}$. For $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$, we denote by $\mathfrak{g}_\alpha$ (respectively s_α) the root space (respectively the reflection) corresponding to α. Since $\mathfrak{h}$ is θ-stable, θ and σ induce actions on $\Delta(\mathfrak{g}, \mathfrak{h})$. We easily see $\theta\alpha = -\sigma\alpha$ for any $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$.

For a subspace U in $\mathfrak{g}$, put $\Delta(U) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \mathfrak{g}_\alpha \subseteq U\}$ and $\rho(U) = \frac{1}{2}\sum_{\alpha \in \Delta(U)} \in \mathfrak{h}^*$.

We put

$$\mathfrak{m} = \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{p}) \cap (-\Delta(\mathfrak{p}))} \mathfrak{g}_\alpha, \quad \mathfrak{n} = \sum_{\alpha \in \Delta(\mathfrak{p}) - \Delta(\mathfrak{m})} \mathfrak{g}_\alpha, \quad \bar{\mathfrak{n}} = \sum_{\alpha \in \Delta(\mathfrak{n})} \mathfrak{g}_{-\alpha},$$
$$\mathfrak{l} = \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{q}) \cap (-\Delta(\mathfrak{q}))} \mathfrak{g}_\alpha, \quad \mathfrak{u} = \sum_{\alpha \in \Delta(\mathfrak{q}) - \Delta(\mathfrak{l})} \mathfrak{g}_\alpha, \quad \bar{\mathfrak{u}} = \sum_{\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}_{-\alpha}.$$

We immediately see that $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ (respectively $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$) is an orderly Levi decomposition of $\mathfrak{q}$ (respectively $\mathfrak{p}$) and the nilradical satisfies $\sigma(\mathfrak{u}) = \bar{\mathfrak{u}}$ (respectively $\theta(\mathfrak{n}) = \bar{\mathfrak{n}}$). Moreover, $\bar{\mathfrak{u}}$ (respectively $\bar{\mathfrak{n}}$) is the opposite nilradical to $\mathfrak{u}$ (respectively $\mathfrak{n}$).

We denote by $L_{\mathbb{C}}$, $P_{\mathbb{C}}$, and $M_{\mathbb{C}}$ the analytic subgroups of $G_{\mathbb{C}}$ with respect to $\mathfrak{l}$, $\mathfrak{p}$, and $\mathfrak{m}$, respectively. We put $L = L_{\mathbb{C}} \cap G$, $P = P_{\mathbb{C}} \cap G$, and $M = M_{\mathbb{C}} \cap G$.

In the setting above, we easily have the following proposition.

Proposition 2.1.2.

S3) $\mathfrak{l} \cap \mathfrak{p}$ *is a parabolic subalgebra of* $\mathfrak{l}$ *and* $L \cap P$ *is a parabolic subgroup of* L.

S4) $\mathfrak{m} \cap \mathfrak{q}$ *is a parabolic subalgebra of* $\mathfrak{m}$.

S5) $\mathfrak{l} \cap \mathfrak{m}$ *is a* θ*- and* σ*-stable Levi subalgebra of both* $\mathfrak{l} \cap \mathfrak{p}$ *and* $\mathfrak{m} \cap \mathfrak{q}$.

For a Borel subalgebra $\mathfrak{b}$, we also write $\Delta^+_{\mathfrak{b}}$ for $\Delta(\mathfrak{b})$. $\Delta^+_{\mathfrak{b}}$ is a positive system of $\Delta(\mathfrak{g}, \mathfrak{h})$.

Put $\tilde{\mathfrak{n}} = (\mathfrak{u} \cap \mathfrak{m}) + \mathfrak{n}$, $\tilde{\mathfrak{u}} = (\mathfrak{n} \cap \mathfrak{l}) + \mathfrak{u}$, $\tilde{\mathfrak{p}} = (\mathfrak{l} \cap \mathfrak{m}) + \tilde{\mathfrak{n}}$, and $\tilde{\mathfrak{q}} = (\mathfrak{l} \cap \mathfrak{m}) + \tilde{\mathfrak{u}}$. Then $\tilde{\mathfrak{p}}$ (respectively $\tilde{\mathfrak{q}}$) is a parabolic subalgebra of $\mathfrak{g}$ with a Levi part $\mathfrak{l} \cap \mathfrak{m}$ and the nilradical $\tilde{\mathfrak{n}}$ (respectively $\tilde{\mathfrak{u}}$).

We fix a Borel subalgebra $\mathfrak{b}^0$ of $\mathfrak{l} \cap \mathfrak{m}$ containing $\mathfrak{h}$. We put $\mathfrak{b}_1 = \mathfrak{b}^0 + \tilde{\mathfrak{n}}$ and $\mathfrak{b}_2 = \mathfrak{b}^0 + \tilde{\mathfrak{u}}$. Obviously, $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are Borel subalgebras of $\mathfrak{g}$. Let $\mathfrak{v}$, $\mathfrak{v}_1$, and $\mathfrak{v}_2$ be the nilradical of $\mathfrak{b}^0$, $\mathfrak{b}_1$, and $\mathfrak{b}_2$, respectively. Put $\mathfrak{d} = \mathfrak{v} + \mathfrak{n} \cap \mathfrak{l} + \mathfrak{u} \cap \mathfrak{m} + \mathfrak{u} \cap \mathfrak{n}$. Then, we easily see $\mathfrak{v}_1 = \mathfrak{d} \oplus (\mathfrak{n} \cap \bar{\mathfrak{u}})$ and $\mathfrak{v}_2 = \mathfrak{d} \oplus (\bar{\mathfrak{n}} \cap \mathfrak{u})$.

Lemma 2.1.3. *We have*

$$\dim \mathfrak{u} \cap \mathfrak{k} - \dim \mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{k} = \dim \mathfrak{u} \cap \bar{\mathfrak{n}}.$$

Proof. Since $\mathfrak{g} = \mathfrak{m} \oplus \bar{\mathfrak{n}} \oplus \mathfrak{n}$ and $\mathfrak{u}$ is θ-stable, we have $\dim \mathfrak{u} \cap \mathfrak{k} - \dim \mathfrak{u} \cap \mathfrak{m} \cap \mathfrak{k} = \dim((\mathfrak{u} \cap \bar{\mathfrak{n}}) \oplus (\mathfrak{u} \cap \mathfrak{n})) \cap \mathfrak{k}$. Let $p : (\mathfrak{u} \cap \bar{\mathfrak{n}}) \oplus (\mathfrak{u} \cap \mathfrak{n}) \to \mathfrak{u} \cap \bar{\mathfrak{n}}$ be the projection to the first factor. Since $\mathfrak{n} \cap \mathfrak{k} = 0$, the restriction of p to $((\mathfrak{u} \cap \bar{\mathfrak{n}}) \oplus (\mathfrak{u} \cap \mathfrak{n})) \cap \mathfrak{k}$ is an injection. On the other hand, for any $X \in \mathfrak{u} \cap \bar{\mathfrak{n}}$, we have $X \oplus \theta X \in ((\mathfrak{u} \cap \bar{\mathfrak{n}}) \oplus (\mathfrak{u} \cap \mathfrak{n})) \cap \mathfrak{k}$. So, the restriction of p to $((\mathfrak{u} \cap \bar{\mathfrak{n}}) \oplus (\mathfrak{u} \cap \mathfrak{n})) \cap \mathfrak{k}$ is onto. $\square$

Lemma 2.1.4. *Put* $d = \dim \mathfrak{u} \cap \bar{\mathfrak{n}}$. *There exists a sequence of complex roots* $\alpha_1, \ldots, \alpha_d \in \Delta(\mathfrak{g}, \mathfrak{h})$ *satisfying the following conditions i–vi. For* $1 \leqslant k \leqslant d$, *we put* $\Delta^+_k = s_{\alpha_k} \cdots s_{\alpha_1} \Delta^+_{\mathfrak{b}_1}$. *We also put* $\Delta^+_0 = \Delta^+_{\mathfrak{b}_1}$.

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i) *For* $1 \leqslant k \leqslant d$, $\alpha_k \in \Delta(\mathfrak{n} \cap \bar{\mathfrak{u}})$.

ii) *For* $1 \leqslant k \leqslant d$, $\Delta(\mathfrak{d}) \subseteq \Delta_k^+$.

iii) *For* $1 \leqslant k \leqslant d$, α_k *is simple with respect to* Δ_{k-1}^+.

iv) *For* $1 \leqslant k \leqslant d$, $\theta\alpha_k \notin \Delta_{k-1}^+$.

v) *For* $1 \leqslant k \leqslant d$, $\alpha_k \in \Delta_{k-1}^+$ *and* $-\theta\alpha_k \in \Delta_{k-1}^+$.

vi) $\Delta_d^+ = \Delta_{\mathfrak{b}_2}^+$.

Proof (cf. [KV95, Lemma 11.128]). For a positive system Δ^+ of $\Delta(\mathfrak{g},\mathfrak{h})$, we define $\mathrm{ht}(\Delta^+) = \mathrm{card}(\Delta^+ \cap \Delta(\bar{\mathfrak{n}} \cap \mathfrak{u}))$. We immediately see $\mathrm{ht}(\Delta_{\mathfrak{b}_1}^+) = 0$ and $\mathrm{ht}(\Delta_{\mathfrak{b}_2}^+) = d$.

We construct the sequence $\alpha_1, \ldots, \alpha_d$ inductively as follows. Let $1 \leqslant k \leqslant d$ and assume that $\alpha_1, \ldots, \alpha_{k-1}$ are already defined so that the conditions i–v hold. First, conditions i and iii imply $\mathrm{ht}(\Delta_{k-1}) = k-1$.

We have a disjoint union $\Delta(\mathfrak{g},\mathfrak{h}) = \Delta(\mathfrak{n} \cap \bar{\mathfrak{u}}) \sqcup \Delta(\mathfrak{v}_2) \sqcup -\Delta(\mathfrak{d})$. So, condition ii implies $\Delta_{k-1}^+ \subseteq \Delta(\mathfrak{n} \cap \bar{\mathfrak{u}}) \sqcup \Delta(\mathfrak{v}_2)$. If there is no simple root for Δ_{k-1}^+ contained in $\mathfrak{n} \cap \bar{\mathfrak{u}}$, we have any simple root for Δ_{k-1}^+ is contained in $\Delta(\mathfrak{v}_2) = \Delta_{\mathfrak{b}_2}^+$. Hence we have $\Delta_{k-1}^+ = \Delta_{\mathfrak{b}_2}^+$. However, this contradicts $\mathrm{ht}(\Delta_{k-1}^+) = k-1 < d = \Delta_{\mathfrak{b}_2}^+$. So, there exists some simple root α_k for Δ_{k-1}^+ such that $\alpha_k \in \Delta(\mathfrak{n} \cap \bar{\mathfrak{u}})$. Since $\theta(\mathfrak{n}) = \bar{\mathfrak{n}}$ and $\theta(\bar{\mathfrak{u}}) = \bar{\mathfrak{u}}$, we see α_k is complex and $\theta\alpha_k \in \Delta(\bar{\mathfrak{n}} \cap \bar{\mathfrak{u}}) \subseteq -\Delta(\mathfrak{d}) \subseteq -\Delta_{k-1}^+$. Hence, we see α_k satisfies conditions i–v. If $\Delta(\mathfrak{d}) \subseteq \Delta^+$ and $\mathrm{ht}(\Delta^+) = d$, then clearly $\Delta^+ = \Delta_{\mathfrak{b}_2}$. So, we have $\Delta_d^+ = \Delta_{\mathfrak{b}_2}$, since $\mathrm{ht}(\Delta_d^+) = d$. Thus, we have condition vi. $\square$

We immediately see the following.

Corollary 2.1.5. *The complex roots* $\alpha_1, \ldots, \alpha_d$ *in Lemma 2.1.4 are distinct from each other and we have* $\Delta(\mathfrak{n} \cap \bar{\mathfrak{u}}) = \{\alpha_1, \ldots, \alpha_d\}$.

Caution. The above numeration $\{\alpha_1, \ldots, \alpha_d\}$ of $\Delta(\mathfrak{n} \cap \bar{\mathfrak{u}})$ may depend on the choice of $\mathfrak{b}^0$.

2.2 Change of polarization

In this section, we fix a $\sigma\theta$-pair $(\mathfrak{p},\mathfrak{q})$. Let $\mathfrak{m}, \mathfrak{l}, \ldots$ be as in § 2.1.

Let $L^\sim$ (respectively $(L \cap M)^\sim$) be the metaplectic double covering of L (respectively $L \cap M$) with respect to $\delta(\mathfrak{u})$ (respectively $\delta(\mathfrak{u} \cap \mathfrak{m})$).

Lemma 2.2.1. *On* $\mathfrak{l} \cap \mathfrak{m}$, *we have*

$$\delta(\mathfrak{u}) - \delta(\mathfrak{u} \cap \mathfrak{m}) = \delta(\bar{\mathfrak{n}} \cap \mathfrak{l}) + \delta(\mathfrak{n}) + 2\delta(\mathfrak{u} \cap \bar{\mathfrak{n}}).$$

Proof. We remark first that $\delta(\bar{\mathfrak{n}} \cap \mathfrak{u}) = -\delta(\mathfrak{n} \cap \bar{\mathfrak{u}})$, $\delta(\bar{\mathfrak{n}} \cap \mathfrak{l}) = -\delta(\mathfrak{n} \cap \mathfrak{l})$, etc. So, we have the lemma from the computation below:

$$\begin{aligned}\delta(\mathfrak{u}) + \delta(\mathfrak{n} \cap \mathfrak{l}) &= \delta(\mathfrak{u} \cap \mathfrak{m}) + \delta(\mathfrak{u} \cap \bar{\mathfrak{n}}) + \delta(\mathfrak{u} \cap \mathfrak{n}) + \delta(\mathfrak{n} \cap \mathfrak{l}) \\ &= \delta(\mathfrak{u} \cap \mathfrak{m}) + 2\delta(\mathfrak{u} \cap \bar{\mathfrak{n}}) + \delta(\bar{\mathfrak{u}} \cap \mathfrak{n}) + \delta(\mathfrak{n} \cap \mathfrak{l}) + \delta(\mathfrak{u} \cap \mathfrak{n}) \\ &= \delta(\mathfrak{u} \cap \mathfrak{m}) + 2\delta(\mathfrak{u} \cap \bar{\mathfrak{n}}) + \delta(\mathfrak{n}).\end{aligned}$$ $\square$

We define a one-dimensional representation $\xi_{\mathfrak{p},\mathfrak{q}}$ of $L \cap M$ on a space $\mathbb{C}_{\mathfrak{p},\mathfrak{q}}$ by

$$\xi_{\mathfrak{p},\mathfrak{q}}(\ell) = \xi_{\delta(\bar{\mathfrak{n}} \cap \mathfrak{l})}(\ell)\xi_{\delta(\mathfrak{n})}(\ell)\xi_{2\delta(\bar{\mathfrak{n}} \cap \mathfrak{u})'}(\ell) \quad (\ell \in L \cap M).$$

From Lemma 2.2.1, we easily obtain the next lemma.

Lemma 2.2.2. *Assigning* $(\ell, z) \in (L \cap M)^\sim$ *to* $(\ell, z\xi_{\mathfrak{p},\mathfrak{q}}(\ell))$, *we have an embedding of the group* $(L \cap M)^\sim \hookrightarrow L^\sim$.

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Let $(P\cap L)^{\sim}$ be the parabolic subgroup of $L^{\sim}$ which is the pull-back of $P\cap L$ to $L^{\sim}$. Under the identification by the embedding in Lemma 2.2.2, we can regard $(L\cap M)^{\sim}$ as a Levi subgroup of $(P\cap L)^{\sim}$.

The following theorem is the main result of this section.

Theorem 2.2.3.

i) *Let Z be a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m}, L\cap M\cap K)$-module with an infinitesimal character $\lambda\in\mathfrak{h}^*$. Assume $\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m})-\delta(\mathfrak{n}),\alpha\rangle\geqslant 0$ for all $\alpha\in\Delta(\mathfrak{u})$ such that $2\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m})-\delta(\mathfrak{n}),\alpha\rangle/\langle\alpha,\alpha\rangle\in\mathbb{Z}$. Then, we have*
$$[{}^u\mathrm{Ind}_P^G(({}^u\mathcal{R}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K}^{\mathfrak{m},M\cap K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}(Z))]=[({}^u\mathcal{R}_{\mathfrak{q},L\cap K}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{k}}({}^u\mathrm{Ind}_{P\cap L}^L(Z\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})'}))].\tag{3}$$

ii) *Let Z be a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m}, L\cap M\cap K)$-module with an infinitesimal character $\lambda\in\mathfrak{h}^*$. We assume $\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m}),\alpha\rangle\geqslant 0$ for all $\alpha\in\Delta(\mathfrak{u})$ such that $2\langle\lambda-\delta(\mathfrak{u}\cap\mathfrak{m}),\alpha\rangle/\langle\alpha,\alpha\rangle\in\mathbb{Z}$. Then, we have*
$$[\mathrm{Ind}_P^G(({}^u\mathcal{R}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K}^{\mathfrak{m},M\cap K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}(Z))]=[({}^u\mathcal{R}_{\mathfrak{q},L\cap K}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{k}}(\mathrm{Ind}_{P\cap L}^L(Z\otimes\mathbb{C}_{\mathfrak{p},\mathfrak{q}}))].\tag{4}$$

iii) *Let $\tilde{Z}$ be a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m},(L\cap M\cap K)^{\sim})$-module with an infinitesimal character $\lambda\in\mathfrak{h}^*$ such that $Z=\tilde{Z}\otimes\mathbb{C}_{\delta(\mathfrak{u}\cap\mathfrak{m})'}$ is reduced to a Harish-Chandra $(\mathfrak{l}\cap\mathfrak{m}, L\cap M\cap K)$-module. We assume $\langle\lambda,\alpha\rangle\geqslant 0$ for all $\alpha\in\Delta(\mathfrak{u})$ such that $2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle\in\mathbb{Z}$. Then,*
$$[\mathrm{Ind}_P^G(({}^n\mathcal{R}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K}^{\mathfrak{m},M\cap K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}(\tilde{Z}))]=[({}^n\mathcal{R}_{\mathfrak{q},L\cap K}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{k}}(\mathrm{Ind}_{(P\cap L)^{\sim}}^{L^{\sim}}(\tilde{Z}))].\tag{5}$$

Proof. Parts ii and iii are rephrasings of part i. We remark that characters of standard modules form a basis of the Grothendieck group of the category of Harish-Chandra modules. Taking account of additivity of cohomological inductions, it suffices to show (3) in the case where Z is a standard module.

As in § 2.1, we fix a θ- and σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}\cap\mathfrak{m}$ and a Borel subalgebra $\mathfrak{b}^0$ of $\mathfrak{l}\cap\mathfrak{m}$ containing $\mathfrak{h}$. We denote by $\mathfrak{v}$ the nilradical of $\mathfrak{b}^0$. Let $H_{\mathbb{C}}$ be the analytic subgroup of $G_{\mathbb{C}}$ and put $H=H_{\mathbb{C}}\cap G$. Let Y be a one-dimensional H-representation whose differential is just λ. We consider the case of $Z=(\mathcal{R}_{\mathfrak{b},T}^{\mathfrak{l}\cap\mathfrak{m},L\cap M\cap K})^{\dim\mathfrak{v}\cap\mathfrak{k}}(Y)$. Put $\mathfrak{b}_1=\mathfrak{b}+\mathfrak{u}\cap\mathfrak{m}+\mathfrak{n}$ and $\mathfrak{b}_2=\mathfrak{b}+\mathfrak{n}\cap\mathfrak{l}+u$. Then, $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are Borel subalgebras of $\mathfrak{g}$. From [KV95, Corollary 11.86] (induction-by-stage formula), we have
$${}^u\mathrm{Ind}_P^G(({}^u\mathcal{R}_{\mathfrak{q}\cap\mathfrak{m},L\cap M\cap K}^{\mathfrak{m},M\cap K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}}((\mathcal{R}_{\mathfrak{b},T}^{\mathfrak{l}\cap\mathfrak{m},L\cap M\cap K})^{\dim\mathfrak{v}\cap\mathfrak{k}}(Y)))\cong({}^u\mathcal{R}_{\mathfrak{b}_1,T}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}+\dim\mathfrak{v}\cap\mathfrak{k}}(Y),$$
$$\begin{aligned}({}^u\mathcal{R}_{\mathfrak{q},L\cap K}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{k}}({}^u\mathrm{Ind}_{P\cap L}^L((\mathcal{R}_{\mathfrak{b},T}^{\mathfrak{l}\cap\mathfrak{m},L\cap M\cap K})^{\dim\mathfrak{v}\cap\mathfrak{k}}(Y)\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})'}))&\\ \cong({}^u\mathcal{R}_{\mathfrak{b}_2,T}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{k}+\dim\mathfrak{v}\cap\mathfrak{k}}(Y\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})'}).&\end{aligned}$$
So, we have only to show that
$$({}^u\mathcal{R}_{\mathfrak{b}_1,T}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{m}\cap\mathfrak{k}+\dim\mathfrak{v}\cap\mathfrak{k}}(Y)\cong({}^u\mathcal{R}_{\mathfrak{b}_2,T}^{\mathfrak{g},K})^{\dim\mathfrak{u}\cap\mathfrak{k}+\dim\mathfrak{v}\cap\mathfrak{k}}(Y\otimes\mathbb{C}_{2\delta(\mathfrak{u}\cap\bar{\mathfrak{n}})'}).\tag{6}$$
However, we have Lemmas 2.1.3 and 2.1.4. So, (6) can be obtained by the successive application of the transfer theorem [KV95, Theorem 11.87]. □

2.3 Derived functor modules; complex case

For complex connected reductive groups, irreducible unitary representation with regular integral infinitesimal character is a parabolic induction from a one-dimensional unitary representation [Enr79]. Moreover, Enright proved they have non-trivial $(\mathfrak{g},K)$-cohomologies. On the other hand, for general reductive Lie groups, Vogan and Zuckerman proved that any irreducible unitary representation with regular integral infinitesimal character and with non-trivial $(\mathfrak{g},K)$-cohomology is nothing but a

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derived functor module [VZ84]. Here we give an explanation of this phenomenon from the viewpoint of the change of polarization.

Let G be a complex connected reductive Lie group and we fix a Cartan involution θ. Here, we denote by $\mathfrak{g}_0$ the real Lie algebra of G. Then the complexification of $\mathfrak{g}_0$ can be identified with $\mathfrak{g}_0 \times \mathfrak{g}_0$. Let $\mathfrak{p}_0$ be any parabolic subalgebra of $\mathfrak{g}_0$ with a Levi decomposition $\mathfrak{p}_0 = \mathfrak{m}_0 + \mathfrak{n}_0$ such that m_0 is θ-stable. If we choose the identification appropriately, then the complexification $\mathfrak{p}$ of $\mathfrak{p}_0$ can be identified with $\mathfrak{p}_0 \times \mathfrak{p}_0 \subseteq \mathfrak{g}_0 \times \mathfrak{g}_0$. On the other hand, if we put $\mathfrak{q} = \mathfrak{p}_0 \times \bar{\mathfrak{p}}_0$, $\mathfrak{q}$ is a θ-stable parabolic algebra. Here, $\bar{\mathfrak{p}}_0$ means the opposite parabolic subalgebra to $\mathfrak{p}_0$. We immediately see $(\mathfrak{p}, \mathfrak{q})$ is a $\sigma\theta$-pair and $\mathfrak{p}$ and $\mathfrak{q}$ have a common Levi part $\mathfrak{m}_0 \times \mathfrak{m}_0$. Applying Theorem 2.2.3, we see that, for complex connected reductive groups, derived functor modules are actually certain irreducible degenerate principal series representations.

2.4 Derived functor modules; general case

For $G = \mathrm{GL}(n, \mathbb{R})$, derived functor modules are a parabolic induction from the external tensor product of some copies of distinguished derived functor modules, the so-called Speh representations and possibly a one-dimensional representation [Spe83].

We examine this phenomenon from the viewpoint of the change of polarization. Here, we use notations as in § 1.2, such as G, $G_{\mathbb{C}}$, K, $K_{\mathbb{C}}$, $\mathfrak{g}$, $\mathfrak{g}_0$, θ, σ, etc. Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra with an orderly Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. Let L be the Levi subgroup corresponding to $\mathfrak{l}$ defined as in § 1.1.

Let $\mathfrak{a}$ be the -1-eigenspace with respect to θ in the center of $\mathfrak{l}$. We call $\mathfrak{q}$ pure imaginary if $\mathfrak{a}$ is contained in the center of $\mathfrak{g}$.

Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Then $\mathfrak{m}$ is a Levi subalgebra of a σ-stable parabolic subgroup $\mathfrak{p}$. Obviously $(\mathfrak{p}, \mathfrak{q})$ is a $\sigma\theta$-pair and $\mathfrak{l} \subseteq \mathfrak{m}$. Then $\mathfrak{q}$ is pure imaginary if and only if $\mathfrak{m} = \mathfrak{g}$ holds.

Conversely, we assume that there is a σ-stable parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that $(\mathfrak{p}, \mathfrak{q})$ is a $\sigma\theta$-pair and there is an orderly Levi decomposition $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ such that $\mathfrak{l} \subseteq \mathfrak{m} \neq \mathfrak{g}$. Then, we have $\mathfrak{q}$ is not pure imaginary since the -1-eigenspace with respect to θ in the center of $\mathfrak{m}$ also centralizes $\mathfrak{l}$.

From Theorem 2.2.3, we have our next proposition.

PROPOSITION 2.4.1. *Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra with an orderly Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. Assume that $\mathfrak{q}$ is not pure imaginary. Then, there is a σ-stable parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with an orderly Levi decomposition $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ such that the derived functor modules of $\mathfrak{g}$ with respect to $\mathfrak{q}$ are isomorphic to the parabolic induction from a derived functor module of $\mathfrak{m}$.*

Obviously, if G has a compact Cartan subgroup, any θ-stable parabolic subalgebra is pure imaginary.

We interpret Speh's result as follows. So, for a while, we put $G = \mathrm{GL}(n, \mathbb{R})$. We fix a Cartan involution $\theta(g) = {}^t g^{-1}$ of G. So, we put $K = \mathrm{O}(n)$ here. For a positive integer k, we put

$$J_k = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}.$$

First, we assume n is even and write $n = 2k$. Put

$$\mathfrak{l}(k) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathfrak{gl}(2k, \mathbb{C}) \;\middle|\; A, B \in M_k(\mathbb{C}) \right\},$$

$$\mathfrak{u}(k) = \left\{ \begin{pmatrix} \sqrt{-1}\, S & S \\ S & -\sqrt{-1}\, S \end{pmatrix} \in \mathfrak{gl}(2k, \mathbb{C}) \;\middle|\; S \in M_k(\mathbb{C}) \right\},$$

$$\mathfrak{q}(k) = \mathfrak{l}(k) + \mathfrak{u}(k).$$

Then, $\mathfrak{q}(k)$ is a θ-stable parabolic subalgebra of $\mathfrak{gl}(2k,\mathbb{C})$ and $\mathfrak{q}(k)=\mathfrak{l}(k)+\mathfrak{u}(k)$ is a Levi decomposition such that $\mathfrak{l}(k)$ is a θ- and σ-stable Levi part. The derived functor module with respect to $\mathfrak{q}(k)$ is a Speh representation of $\mathrm{GL}(2k,\mathbb{C})$. Actually, we have the following proposition.

Proposition 2.4.2. *If n is odd, there is no proper pure imaginary θ-stable parabolic subalgebra.*

If n is even, any proper pure imaginary θ-stable parabolic subalgebra is $\mathrm{SO}(n)$-conjugate to $\mathfrak{q}(n/2)$.

Next, we consider general θ-stable parabolic subalgebras. For a sequence of positive integers $\vec{n}=(n_1,\dots,n_\ell)$ such that $0\leqslant n-2n_1+\cdots+2n_\ell$, we put $q=n-2n_1+\cdots+2n_\ell$ and

$$\mathfrak{t}(\vec{n})=\{\mathrm{diag}(t_1J_{n_1},\dots,t_\ell J_{n_\ell},0_q)\in\mathfrak{gl}(n,\mathbb{C})\mid t_1,\dots,t_\ell\in\mathbb{C}\}.$$

We denote by $\mathfrak{l}(\vec{n})$ the centralizer of $\mathfrak{t}(\vec{n})$ in $\mathfrak{gl}(n,\mathbb{C})$. Then we have

$$\mathfrak{l}(\vec{n})=\{\mathrm{diag}(A_1,\dots,A_\ell,D)\in\mathfrak{gl}(n,\mathbb{C})\mid A_i\in\mathfrak{l}(n_i)\ (1\leqslant i\leqslant\ell),D\in\mathfrak{gl}(q,\mathbb{C})\},$$

$\mathfrak{l}_0(\vec{n})=\mathfrak{l}(\vec{n})\cap\mathfrak{gl}(n,\mathbb{R})$ is a real form of $\mathfrak{l}(\vec{n})$ and

$$\mathfrak{l}_0(\vec{n})\cong\mathfrak{gl}(n_1,\mathbb{C})\times\cdots\times\mathfrak{gl}(n_\ell,\mathbb{C})\times\mathfrak{gl}(q,\mathbb{R}).$$

Put

$$\mathfrak{m}(\vec{n})=\{\mathrm{diag}(A_1,\dots,A_\ell,D)\in\mathfrak{gl}(n,\mathbb{C})\mid A_i\in\mathrm{GL}(2n_i,\mathbb{C})\ (1\leqslant i\leqslant\ell),D\in\mathfrak{gl}(q,\mathbb{C})\}.$$

There is a θ-stable parabolic subalgebra $\mathfrak{q}(\vec{n})$ such that

$$\mathfrak{m}(\vec{n})\cap\mathfrak{q}(\vec{n})=\{\mathrm{diag}(A_1,\dots,A_\ell,D)\in\mathfrak{gl}(n,\mathbb{C})\mid A_i\in\mathfrak{q}(n_i)\ (1\leqslant i\leqslant\ell),D\in\mathfrak{gl}(q,\mathbb{C})\}.$$

Any θ-stable parabolic subalgebra in $\mathfrak{gl}(n,\mathbb{C})$ is $\mathrm{O}(n,\mathbb{C})$-conjugate to some $\mathfrak{q}(\vec{n})$. Let $\mathfrak{n}$ be the Lie algebra of the upper triangular matrices in $\mathfrak{gl}(n,\mathbb{C})$ and put $\mathfrak{p}(\vec{n})=\mathfrak{m}(\vec{n})+\mathfrak{n}$. We denote by $\mathfrak{n}(\vec{n})$ the nilradical of $\mathfrak{p}(\vec{n})$. Then, $(\mathfrak{p}(\vec{n}),\mathfrak{q}(\vec{n}))$ is a $\sigma\theta$-pair. Applying Theorem 2.2.3 to the $\sigma\theta$-pair, we get Speh's result [Spe83, Theorem 4.2.2].

Next, we consider the case of $G=\mathrm{GL}(k,\mathbb{H})$. Write $\mathbb{H}=\mathbb{C}+j\mathbb{C}$. In this case we put $K=\mathrm{Sp}(n)=\{g\in\mathrm{GL}(k,\mathbb{H})\mid {}^{\mathrm{t}}\bar{g}g=I_k\}$. Then we regard $\mathfrak{gl}(k,\mathbb{C})$ as a real Lie subalgebra of $\mathfrak{gl}(k,\mathbb{H})$. For $\ell\in\mathbb{Z}$ and $t\in\sqrt{-1}\,\mathbb{R}$, we define a one-dimensional unitary representation $\xi_{\ell,t}$ of $\mathrm{GL}(k,\mathbb{C})$ as follows:

$$\xi_{\ell,t}(g)=\left(\frac{\det(g)}{|\det(g)|}\right)^{\ell}|\det(g)|^{t}.$$

Let $\mathfrak{q}(k)$ be a θ-stable parabolic subalgebra with an orderly Levi decomposition $\mathfrak{q}(k)=\mathfrak{l}(k)+\mathfrak{u}(k)$. We choose the nilradical $\mathfrak{n}(k)$ so that $\xi_{\ell,t}$ is good with respect to $\mathfrak{q}(k)$ for sufficiently large ℓ. Derived functor modules with respect to $\mathfrak{q}(k)$ are called quaternionic Speh representations.

For $t\in\sqrt{-1}\,\mathbb{R}$, there is a one-dimensional unitary representation $\tilde{\xi}_t$ of $\mathrm{GL}(k,\mathbb{H})$ whose restriction to $\mathrm{GL}(k,\mathbb{C})$ is $\xi(0,t)$.

Definition 2.4.3. We put

$$A_k(\ell,t)=({}^{u}\mathcal{R}^{\mathfrak{gl}(k,\mathbb{H})\otimes_{\mathbb{R}}\mathbb{C},\mathrm{Sp}(k)}_{\mathfrak{q}(k),\mathrm{O}(k)})^{k(k+1)}(\xi_{\ell+2k,t})\quad(\ell\in\mathbb{Z}).\tag{7}$$

We also put

$$A_k(-\infty,t)=\tilde{\xi}_t.$$

For $\ell\in\mathbb{Z}$, $A_k(\ell,t)$ is a derived functor module in the good (respectively weakly fair) range in the sense of [Vog88] if and only if $\ell\geqslant 0$ (respectively $\ell\geqslant -k$).

We immediately see that

$$A_k(\ell,t)\cong A_k(\ell,0)\otimes\tilde{\xi}_t.$$

We easily obtain the next proposition.

PROPOSITION 2.4.4. *Any proper pure imaginary θ-stable parabolic subalgebra is* $\mathrm{Sp}(k)$*-conjugate to* $\mathfrak{q}(k)$.

As in the case of $\mathrm{GL}(k,\mathbb{R})$, any derived functor module of $\mathrm{GL}(k,\mathbb{H})$ is a parabolic induction from the external tensor product of some copies of quaternionic Speh representations and possibly a one-dimensional representation (cf. [Vog86]).

Next, we consider the case of $G = \mathrm{SO}_0(2p+1, 2q+1)$. In this case, a Levi part of a non-pure imaginary θ-stable parabolic subalgebra $\mathfrak{q}$ is isomorphic to $\mathfrak{so}(1,1) \oplus \mathfrak{u}(p_1,q_1) \oplus \cdots \mathfrak{u}(p_k,q_k)$. Here, $p_1 + \cdots + p_k = p$ and $q_1 + \cdots + q_k = q$.

Let $\mathfrak{p}$ be a maximal cuspidal parabolic subalgebra whose Levi part is isomorphic to $\mathfrak{so}(2p,2q) \oplus \mathfrak{so}(1,1)$. The derived functor module with respect to the above $\mathfrak{q}$ is a parabolic induction with respect to $\mathfrak{p}$ from a derived functor module of $\mathfrak{so}(2p,2q)$ with respect to a θ-stable parabolic subalgebra whose Levi part is isomorphic to $\mathfrak{u}(p_1,q_1) \oplus \cdots \mathfrak{u}(p_k,q_k)$.

Among the exceptional real simple Lie algebras, only E I and E IV have non-pure imaginary θ-stable parabolic subalgebras.

3. Application of the change of polarization to $\mathbf{SO^*(2n)}$ and $\mathbf{Sp(p,q)}$

Throughout this section, we assume G is either $\mathrm{SO}^*(2n)$ or $\mathrm{Sp}(n-q,q)$ with $2q \leqslant n$. For $G = \mathrm{Sp}(n-q,q)$, we put $p = n-q$. For $G = \mathrm{SO}^*(2n)$, we put $q = [n/2]$. In both cases $G = \mathrm{Sp}(p,q)$ and $G = \mathrm{SO}^*(2n)$, q is the real rank of G.

3.1 Root systems

We fix a maximal compact subgroup K of $\mathrm{SO}^*(2n)$ (respectively $\mathrm{Sp}(p,q)$), which is isomorphic to $U(n)$ (respectively $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$). We denote by $G_{\mathbb{C}}$ the complexification of G as in § 1.2. So, $G_{\mathbb{C}}$ is isomorphic to $\mathrm{SO}(2n,\mathbb{C})$ or $\mathrm{Sp}(n,\mathbb{C})$. We denote by θ the Cartan involution corresponding to K as in § 1.2. We fix a θ-stable maximally split Cartan subgroup ${}^sH$ of G. We remark that all the Cartan subgroups of G are connected. We stress that we use notations introduced in § 1.

First, we consider the root system $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$ for $G = \mathrm{SO}^*(2n)$. Then we can choose an orthonormal basis $e_1, \ldots, e_n$ of ${}^s\mathfrak{h}^*$ such that

$$\Delta(\mathfrak{g}, {}^s\mathfrak{h}) = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\}.$$

If n is even, we write $n = 2q$. In this case, we choose the above $e_1, \ldots, e_n$ so that $\theta(e_{2i-1}) = -e_{2i}$ and $\theta(e_{2i}) = -e_{2i-1}$ for all $1 \leqslant i \leqslant q$. If n is odd, we write $n = 2q+1$. In this case, we choose the above $e_1, \ldots, e_n$ so that $\theta(e_{2i-1}) = -e_{2i}$ and $\theta(e_{2i}) = -e_{2i-1}$ for all $1 \leqslant i \leqslant q$ and $\theta(e_{2q+1}) = e_{2q+1}$. We immediately see that $\pm(e_{2i-1} - e_{2i})$ (respectively $\pm(e_{2i-1} + e_{2i})$) ($1 \leqslant i \leqslant q$) are compact imaginary (respectively real) and the other roots are complex.

If $G = \mathrm{Sp}(n-q,q)$, put $p = n-q$ and choose $e_1, \ldots, e_n$ such that

$$\Delta(\mathfrak{g}, {}^s\mathfrak{h}) = \{\pm e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{\pm 2e_i \mid 1 \leqslant i \leqslant n\},$$

$\theta(e_{2i-1}) = -e_{2i}, \theta(e_{2i}) = -e_{2i-1}$ for $1 \leqslant i \leqslant q$, and $\theta(e_i) = e_i$ for $2q < i \leqslant n$.

We fix a simple system for $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$ as follows. If $G = \mathrm{SO}^*(2n)$, then put $\Pi = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}$. If $G = \mathrm{Sp}(p,q)$, then put $\Pi = \{e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n\}$.

We denote by Δ^+ the corresponding positive system of $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$. Let $E_1, \ldots, E_n$ be the dual basis of ${}^s\mathfrak{h}$ to $e_1, \ldots, e_n$.

3.2 Square quadruplets

One famous realization of $\mathrm{Sp}(p,q)$ is as the automorphism group of an indefinite Hermitian form on an $\mathbb{H}$-vector space, namely,

$$\mathrm{Sp}(p,q)=\{g\in\mathrm{GL}(p+q,\mathbb{H})\mid {}^{\mathrm{t}}\bar{g}I_{p,q}g=I_{p,q}\}. \tag{8}$$

Here,

$$I_{p,q}=\begin{pmatrix} I_p & 0\\ 0 & -I_q\end{pmatrix}.$$

Similarly, we consider the complex indefinite unitary group

$$\mathrm{U}(p,q)=\{g\in\mathrm{GL}(p+q,\mathbb{C})\mid {}^{\mathrm{t}}\bar{g}I_{p,q}g=I_{p,q}\}.$$

$\mathrm{U}(p,q)$ is regarded as a subgroup of $\mathrm{Sp}(p,q)$ in the obvious way. We fix a maximal compact subgroup of $\mathrm{Sp}(p,q)$ as follows:

$$K=\mathrm{Sp}(p)\times\mathrm{Sp}(q)=\left\{\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}\;\middle|\; A\in\mathrm{Sp}(p), B\in\mathrm{Sp}(q)\right\}.$$

We denote by θ the corresponding Cartan involution.

For the case of $p=q$, we also consider another realization:

$$\mathrm{Sp}(q,q)=\{g\in\mathrm{GL}(2q,\mathbb{H})\mid {}^{\mathrm{t}}\bar{g}J_qg=J_q\}.$$

We put $n=2q$. Here,

$$J_q=\begin{pmatrix} 0 & I_q\\ I_q & 0\end{pmatrix}.$$

Then, identifying $\mathrm{GL}(q,\mathbb{H})$ with the following group, we regard $\mathrm{GL}(q,\mathbb{H})$ as a subgroup of $\mathrm{Sp}(q,q)$:

$$\left\{\begin{pmatrix} A & 0\\ 0 & {}^{\mathrm{t}}\bar{A}\end{pmatrix}\;\middle|\; A\in\mathrm{GL}(q,\mathbb{H})\right\}.$$

We consider $\mathrm{U}(q,q)\cap\mathrm{GL}(q,\mathbb{H})$ as a subgroup of $\mathrm{Sp}(q,q)$. This group is

$$\left\{\begin{pmatrix} A & 0\\ 0 & {}^{\mathrm{t}}\bar{A}\end{pmatrix}\;\middle|\; A\in\mathrm{GL}(q,\mathbb{C})\right\}.$$

We identify it with $\mathrm{GL}(q,\mathbb{C})$ and obtain the following 'square quadruplet':

$$\begin{array}{ccc} \mathrm{GL}(q,\mathbb{H}) & \subseteq & \mathrm{Sp}(q,q)\\ \cup\!\shortmid & & \cup\!\shortmid\\ \mathrm{GL}(q,\mathbb{C}) & \subseteq & \mathrm{U}(q,q)\end{array} \tag{9}$$

In (9), each inclusion gives a symmetric pair. We easily see that $\mathrm{U}(q,q)$, $\mathrm{GL}(q,\mathbb{H})$, and $\mathrm{GL}(q,\mathbb{C})$ are the centralizers in $\mathrm{Sp}(q,q)$ of their centers, respectively. Since $\mathrm{GL}(q,\mathbb{C})$ has the same rank and the same real rank as $\mathrm{Sp}(q,q)$, we can choose θ-stable maximally split Cartan subgroup ${}^sH$ of $\mathrm{Sp}(q,q)$ which is contained in $\mathrm{GL}(q,\mathbb{C})$. We denote by ${}^s\mathfrak{h}$ the complexified Lie algebra of ${}^sH$. We may apply the notations on the root system for $\Delta(\mathfrak{g},{}^s\mathfrak{h})$.

We choose the standard Borel subalgebra $\mathfrak{b}_1(q)$ of $\mathfrak{g}=\mathfrak{sp}(n,\mathbb{C})$ corresponding to Δ^+ in § 3.1. We define a subset $S(q)=\{e_i-e_{i+1}\mid 1\leqslant i\leqslant n\}$ of Π. We denote by $\tilde{\mathfrak{p}}(q)$ the standard parabolic subalgebra corresponding to $S(q)$, namely $\mathfrak{b}_1(q)\subseteq\tilde{\mathfrak{p}}(q)$ and $\Delta(\tilde{\mathfrak{p}}(q),{}^s\mathfrak{h})=\Delta^+\cup(\mathbb{Z}S(q)\cap\Delta(\mathfrak{g},{}^s\mathfrak{h}))$. Then, we easily see that $\mathrm{GL}(q,\mathbb{H})$ is the θ-stable Levi subgroup for $\tilde{\mathfrak{p}}(q)$.

Next, we consider another simple system Π_u of $\Delta(\mathfrak{g},{}^s\mathfrak{h})$ as follows:

$$\Pi_u=\{e_i-e_{i+2}\mid 1\leqslant i\leqslant n-2\}\cup\{e_{n-1}+e_n\}\cup\{-2e_2\}.$$

We also put $S_u(q) = \Pi_u - \{-2e_2\}$. We choose the standard Borel subalgebra $\mathfrak{b}_2(q)$ of $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C})$ corresponding to Π_u and denote by $\tilde{\mathfrak{q}}(q)$ the parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}_2(q)$ and corresponding to $S_u(q)$. Since, $\theta(S_u(q)) = -S_u(q)$ and $\theta(-2e_2) \equiv -2e_2 \pmod{\mathbb{Z}S_u(q)}$, $\tilde{\mathfrak{q}}(q)$ is θ-stable. We easily see that $\mathrm{U}(q,q)$ is a Levi subgroup for $\tilde{\mathfrak{q}}(q)$. $\mathrm{U}(q,q)$, $\mathrm{GL}(q,\mathbb{H})$, and $\mathrm{GL}(q,\mathbb{C})$ are the centralizers of their centers in $\mathrm{Sp}(q,q)$. In fact, the Lie algebra of the center of $\mathrm{U}(q,q)$ (respectively $\mathrm{GL}(q,\mathbb{H})$) is spanned by $\sum_{i=1}^{q}(E_{2i-1} - E_{2i})$ (respectively $E_1 + \cdots + E_n$). Here, $n = 2q$ and we follow the notations in § 3.1. The center of $\mathrm{U}(q,q)$ (respectively $\mathrm{GL}(q,\mathbb{H})$) is compact (respectively real split) and θ-stable, and $\mathrm{U}(q,q)$ (respectively $\mathrm{GL}(q,\mathbb{H})$) is a Levi subgroup for a maximal θ-stable (respectively σ-stable) parabolic subalgebra (say $\tilde{\mathfrak{q}}(q)$ (respectively $\tilde{\mathfrak{p}}(q)$)) of $\mathfrak{sp}(2q,\mathbb{C}) = \mathfrak{sp}(q,q) \otimes_{\mathbb{R}} \mathbb{C}$.

Since ${}^s\mathfrak{h} \subseteq \tilde{\mathfrak{p}}(q) \cap \tilde{\mathfrak{q}}(q)$, $(\tilde{\mathfrak{p}}(q), \tilde{\mathfrak{q}}(q))$ forms a $\sigma\theta$-pair. Put $\mathfrak{p}(q) = \tilde{\mathfrak{p}}(q) \cap (\mathfrak{u}(q,q) \otimes_{\mathbb{R}} \mathbb{C})$ and $\mathfrak{q}(q) = \tilde{\mathfrak{q}}(q) \cap (\mathfrak{gl}(q,\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C})$.

Similarly, $\mathrm{GL}(q,\mathbb{C})$ is the centralizer of the split (respectively compact) part of its center in $\mathrm{U}(q,q)$ (respectively $\mathrm{GL}(q,\mathbb{H})$). $\mathrm{GL}(q,\mathbb{C})$ is a Levi subgroup for a maximal σ-stable (respectively θ-stable) parabolic subalgebra $\mathfrak{p}(q)$ (respectively $\mathfrak{q}(q)$) of $\mathfrak{gl}(2q,\mathbb{C}) = \mathfrak{u}(q,q) \otimes_{\mathbb{R}} \mathbb{C}$ (respectively $\mathfrak{gl}(2q,\mathbb{C}) = \mathfrak{gl}(q,\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$).

Usually $\mathfrak{p}(q)$ is called a Siegel parabolic subalgebra and $\mathfrak{q}(q)$ is the one defined in § 2.4, the unique (up to $\mathrm{Sp}(q)$-conjugacy) pure imaginary θ-stable parabolic subalgebra. We denote by $P(q)$ the Siegel parabolic subgroup of G corresponding to $\mathfrak{p}(q)$. For $\ell \in \mathbb{Z}$ and $t \in \sqrt{-1}\,\mathbb{R}$, we define a one-dimensional unitary representation $\xi_{\ell,t}$ of $\mathrm{GL}(n,\mathbb{C})$ as follows:

$$\xi_{\ell,t}(g) = \left(\frac{\det(g)}{|\det(g)|}\right)^{\ell} |\det(g)|^{t}.$$

We define the degenerate unitary principal series with respect to $P(q)$ as follows:

$$I_q(\ell,t) = \mathrm{Ind}_{P(q)}^{\mathrm{U}(q,q)}(\xi_{\ell,t}) \quad (\ell \in \mathbb{Z}, t \in \sqrt{-1}\,\mathbb{R}). \tag{10}$$

We introduce similar structure for $\mathrm{SO}^*(4q)$ as follows.

$$\begin{array}{ccc} \mathrm{GL}(q,\mathbb{H}) & \subseteq & \mathrm{SO}^*(4q) \\ \cup\!\mid & & \cup\!\mid \\ \mathrm{GL}(g,\mathbb{C}) & \subseteq & \mathrm{U}(q,q) \end{array} \tag{11}$$

In fact, as in the case of $\mathrm{Sp}(q,q)$, $\mathrm{U}(q,q)$ (respectively $\mathrm{GL}(q,\mathbb{H})$) above is the centralizer of $\sum_{i=1}^{q}(E_{2i-1}-E_{2i})$ (respectively $E_1+\cdots+E_n$) in $\mathrm{SO}^*(4q)$. (Here, $n = 2q$.) $\mathrm{GL}(q,\mathbb{C})$ is the intersection of $\mathrm{U}(q,q)$ and $\mathrm{GL}(q,\mathbb{H})$. For $q \geqslant 2$, we define

$$\Pi_u = \{e_i - e_{i+2} \mid 1 \leqslant i \leqslant n-2\} \cup \{e_{n-1} + e_n\} \cup \{-e_2 - e_4\},$$
$$S_u(q) = \Pi_u - \{-e_2 - e_4\}.$$

If $q = 1$, put $\Pi_u = \{e_1 + e_2, e_1 - e_2\}$ and $S_u(1) = \{e_1 + e_2\}$. We define $\tilde{\mathfrak{p}}(q)$ and $\tilde{\mathfrak{q}}(q)$ in a similar manner to the case of $G = \mathrm{Sp}(q,q)$. In this case, the situation is quite similar to the case of $\mathrm{Sp}(q,q)$.

3.3 Maximal parabolic subgroups

Let k be a positive integer such that $k \leqslant q$. If $G = \mathrm{Sp}(p,q)$, put $p' = p - k$ and $q' = q - k$. If $G = \mathrm{SO}^*(2n)$, put $r = n - 2k$. We put $A = \sum_{j=1}^{k} E_j$. Then we have $\theta(A) = -A$. We denote by $\mathfrak{a}_{(k)}$ the one-dimensional Lie subalgebra of ${}^s\mathfrak{h}$ spanned by A.

We define a subset $S(k)$ of Π as follows. If $G = \mathrm{Sp}(p,q)$, we define

$$S(k) = \begin{cases} \Pi - \{e_{2k} - e_{2k+1}\} & \text{if } p' > 0, \\ \Pi - \{2e_n\} & \text{if } p' = 0. \end{cases}$$

If $G = \mathrm{SO}^*(2n)$, we define

$$S(k) = \begin{cases} \Pi - \{e_{2k} - e_{2k+1}\} & \text{if } r > 0, \\ \Pi - \{e_{n-1} + e_n\} & \text{if } r = 0. \end{cases}$$

We denote by $M_{(k)}$ (respectively $\mathfrak{m}_{(k)}$) the standard maximal Levi subgroup (respectively subalgebra) of G (respectively $\mathfrak{g}$) corresponding to $S(k)$. Namely $M_{(k)}$ is the centralizer of $\mathfrak{a}_{(k)}$ in G. Let $P_{(k)}$ be a parabolic subgroup of G whose θ-invariant Levi part is $M_{(k)}$. We denote by $N_{(k)}$ the nilradical of $P_{(k)}$. We denote by $\mathfrak{p}_{(k)}$, $\mathfrak{m}_{(k)}$, and $\mathfrak{n}_{(k)}$ the complexified Lie algebra of $P_{(k)}$, $M_{(k)}$, and $N_{(k)}$, respectively. We choose $P_{(k)}$ so that $\{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subseteq \mathfrak{n}_{(k)}\} \subseteq \Delta^+$.

Formally, we denote by $\mathrm{Sp}(0,0)$ and $\mathrm{SO}^*(0)$ the trivial group $\{1\}$. Then, we have

$$M_{(k)} \cong \begin{cases} \mathrm{GL}(k,\mathbb{H}) \times \mathrm{Sp}(p',q') & \text{if } G = \mathrm{Sp}(p,q), \\ \mathrm{GL}(k,\mathbb{H}) \times \mathrm{SO}^*(2r) & \text{if } G = \mathrm{SO}^*(2n). \end{cases}$$

Often, we identify $\mathrm{GL}(k,\mathbb{H})$, $\mathrm{Sp}(p',q')$, and $\mathrm{SO}^*(2r)$ with subgroups of $M_{(k)}$ in obvious ways. We call such identifications the standard identifications. The Cartan involution θ induces Cartan involutions on $M_{(k)}$, $\mathrm{GL}(k,\mathbb{H})$, $\mathrm{Sp}(p',q')$, and $\mathrm{SO}^*(2r)$ and we denote them by the same letter θ. We put $M^\circ_{(k)} = \mathrm{Sp}(p',q')$ (respectively $\mathrm{SO}^*(2r)$), if $G = \mathrm{Sp}(p,q)$ (respectively $\mathrm{SO}^*(2n)$).

We denote by $\mathfrak{m}^\circ_k$ the complexified Lie algebra of M°_k.

Later, we treat various $\mathrm{Sp}(p,q)$'s and $\mathrm{SO}^*(2n)$'s at the same time. So, sometimes we write $P_{(k)}(p,q)$ (respectively $P^*_{(k)}(2n)$) for $P_{(k)}$ if $G = \mathrm{Sp}(p,q)$ (respectively $G = \mathrm{SO}^*(2n)$).

We define a basis $\Pi^{(k)}_u$ of $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$ as follows. If $2k = n$, then we put $\Pi^{(k)}_u = \Pi_u$, where Π_u is defined in § 3.2. If $2k < n$, then we put

$$\Pi^{(k)}_u = \{e_i - e_{i+2} \mid 1 \leqslant i \leqslant 2k-2\} \cup \{e_{2k-1} + e_{2k}, -e_2 - e_{2k+1}\} \cup \{\gamma \in \Pi \mid \gamma(E_i) = 0\ (1 \leqslant i \leqslant 2k)\}.$$

Here, Π is the basis of $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$ defined in § 3.1. We denote by $\mathfrak{b}_{(k)}$ the standard Borel subalgebra of $\mathfrak{g}$. Put $S^{(k)}_u = \Pi^{(k)}_u - \{-e_2 - e_{2k+1}\}$. Let $\mathfrak{q}_{(k)}$ be the parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}_{(k)}$ corresponding to $S^{(k)}_u$. Since, $\theta(\mathbb{Z}S^{(k)}_u) = \mathbb{Z}S^{(k)}_u$ and $\theta(-e_2 - e_{2k+1}) \equiv -e_2 - e_{2k+1} \pmod{\mathbb{Z}S^{(k)}_u}$, $\mathfrak{q}_{(k)}$ is θ-stable. We easily see that $\mathrm{U}(k,k) \times M^\circ_{(k)}$ is a Levi subgroup (say $L_{(k)}$) for $\mathfrak{q}_{(k)}$. We denote by $\mathfrak{l}_{(k)}$ the complexifed Lie algebra of $L_{(k)}$.

Since ${}^s\mathfrak{h} \subseteq \mathfrak{p}_{(k)} \cap \mathfrak{q}_{(k)}$, $(\mathfrak{p}_{(k)}, \mathfrak{q}_{(k)})$ is a $\sigma\theta$-pair.

We denote by $G_{(k)}$ the centralizer of $\{E_i \mid 2k < i \leqslant n\}$ in G. (If $2k = n$, we put $G_{(k)} = G$.) If $G = \mathrm{Sp}(p,q)$ (respectively $G = \mathrm{SO}^*(2n)$), then $G_{(k)}$ is isomorphic to $\mathrm{Sp}(k,k)$ (respectively $\mathrm{SO}^*(4k)$). We have the following diagram:

$$\begin{array}{ccc} M_{(k)} & \subseteq & G_{(k)} M^\circ_{(k)} \\ \cup\!\mid & & \cup\!\mid \\ M_{(k)} \cap L_{(k)} & \subseteq & L_{(k)} \end{array} \tag{12}$$

Taking the intersection of $G_{(k)}$ and each term of (12), we have a square quadruplet in the sense of § 3.2:

$$\begin{array}{ccc} \mathrm{GL}(k,\mathbb{H}) & \subseteq & G_{(k)} \\ \cup\!\mid & & \cup\!\mid \\ \mathrm{GL}(k,\mathbb{C}) & \subseteq & \mathrm{U}(k,k) \end{array} \tag{13}$$

Put $a_G = 1$ (respectively $a_G = -1$), if $G = \mathrm{Sp}(p,q)$ (respectively $G = \mathrm{SO}^*(2n)$). We have the following result. The proof is straightforward.

LEMMA 3.3.1. *Define $\xi_{\mathfrak{p}_{(k)},\mathfrak{q}_{(k)}}$ as in § 2.2. For $1 \leqslant i \leqslant n$, we have:*

$$\xi_{\mathfrak{p}_{(k)},\mathfrak{q}_{(k)}}(E_i) = \begin{cases} (-1)^{i+1}\dfrac{2n-3k+a_G}{2} & \text{if } 1 \leqslant i \leqslant 2k, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by ${}^s\mathfrak{h}_{(k)}$ (respectively ${}^s\mathfrak{h}^{(k)}$) the $\mathbb{C}$-linear span of $\{E_i \mid 2k < i \leqslant n\}$ (respectively $\{E_i \mid 1 \leqslant i \leqslant 2k\}$). Using the direct sum decomposition ${}^s\mathfrak{h} = {}^s\mathfrak{h}^{(k)} \oplus {}^s\mathfrak{h}_{(k)}$, we have ${}^s\mathfrak{h}^* = ({}^s\mathfrak{h}^{(k)})^* \oplus {}^s\mathfrak{h}^*_{(k)}$.

Let π be an irreducible unitary representation of $M^\circ_{(k)}$. Since ${}^s\mathfrak{h}_{(k)} = {}^s\mathfrak{h} \cap \mathfrak{m}^\circ_{(k)}$, ${}^s\mathfrak{h}_{(k)}$ is a Cartan subalgebra of $\mathfrak{m}^\circ_{(k)}$. Let $\lambda_\pi \in {}^s\mathfrak{h}^*_{(k)}$ be the infinitesimal character of π (λ_π is determined up to the Weyl group action).

For $\ell \in \mathbb{Z}$ and $t \in \sqrt{-1}\,\mathbb{R}$, we consider the one-dimensional representation $\xi_{\ell,t}$ of $\mathrm{GL}(k,\mathbb{C})$ defined in § 3.2. We consider the representation $\xi_{\ell,t} \boxtimes \pi$ of $\mathrm{GL}(k,\mathbb{C}) \times M^\circ_{(k)}$. Let $\lambda_{\ell,t,\pi}$ be the infinitesimal character of $\xi_{\ell,t} \boxtimes \pi$. Then we have:

$$\begin{aligned} \lambda_{\ell,t,\pi}(E_{2i-1}) &= \frac{k-1+\ell+t}{2} - i + 1 \quad (1 \leqslant i \leqslant k), \\ \lambda_{\ell,t,\pi}(E_{2i}) &= \frac{k-1-\ell+t}{2} - i + 1 \quad (1 \leqslant i \leqslant k), \\ \lambda_{\ell,t,\pi}|_{{}^s\mathfrak{h}_{(k)}} &= \lambda_\pi. \end{aligned}$$

We define:

$$c_{\ell,t,\pi} = \max\left(\{0\} \cup \left\{ |\lambda_\pi(E_i)| \,\middle|\, n-2k < i \leqslant n, \left\{\pm\lambda(E_i) - \frac{\ell+t-1}{2}\right\} \cap \mathbb{Z} \neq \emptyset \right\}\right).$$

Applying part ii of Theorem 2.2.3 to the $\sigma\theta$-pair $(\mathfrak{p}_{(k)},\mathfrak{q}_{(k)})$, we obtain the following proposition.

PROPOSITION 3.3.2. *Let π be an irreducible unitary representation of $M^\circ_{(k)}$. Let $\ell \in \mathbb{N}$ and $t \in \sqrt{-1}\,\mathbb{R}$. We assume $\ell \geqslant 2c_{\ell,t,\pi} - 1$. Put $S = k(n-2k+1)$ (respectively $S = k(2n-3k)$), if $G = \mathrm{Sp}(p,q)$ (respectively if $G = \mathrm{SO}^*(2n)$).*

Then,

$$\mathrm{Ind}^G_{P_{(k)}}(A_k(\ell,t) \boxtimes \pi) \cong (\mathcal{R}^{\mathfrak{g},K}_{\mathfrak{q}_{(k)},K\cap L_{(k)}})^S (I_k(\ell+2n-k+a_G,t) \boxtimes \pi).$$

Here, $A_k(\ell,t)$ (respectively $I_k(\ell,t)$) is a quaternionic Speh representation (respectively a degenerate principal series representation) defined in § 2.4 (7) (respectively § 3.2 (10)).

3.4 θ-stable parabolic subalgebras

We retain the notations in §§ 3.1 and 3.3. The classifications of the K-conjugate class of θ-stable parabolic subalgebras with respect to real classical groups are more or less well known. Here, we review the classification for $G = \mathrm{U}(p,q)$, $\mathrm{Sp}(p,q)$, and $\mathrm{SO}^*(2n)$. First, we discuss θ-stable parabolic subalgebras with respect to $\mathrm{U}(p,q)$ (cf. [Vog97, Example 4.5]).

Let ℓ be a positive integer. Put

$$\mathbb{P}_\ell(p,q) = \left\{ ((p_1,\ldots,p_\ell),(q_1,\ldots,q_\ell)) \in \mathbb{N}^\ell \times \mathbb{N}^\ell \,\middle|\, \sum_{i=1}^{\ell} p_i = p, \ \sum_{i=1}^{\ell} q_i = q, \ p_j + q_j > 0 \text{ for all } 1 \leqslant j \leqslant \ell \right\}.$$

We also put $\mathbb{P}(p,q) = \bigcup_{\ell>0} \mathbb{P}_\ell(p,q)$ and $\mathbb{P}(0,0) = \mathbb{P}_0(0,0) = \{((\emptyset),(\emptyset))\}$. If $(\underline{\mathbf{p}},\underline{\mathbf{q}}) \in \mathbb{P}(p,q)$ satisfies $(\underline{\mathbf{p}},\underline{\mathbf{q}}) \in \mathbb{P}_\ell(p,q)$, we call ℓ the length of $(\underline{\mathbf{p}},\underline{\mathbf{q}})$. For $(\underline{\mathbf{p}},\underline{\mathbf{q}}) \in \mathbb{P}(p,q)$, we define

$I_{(\underline{\mathbf{p}},\underline{\mathbf{q}})} = \mathrm{diag}(I_{p_1,q_1},\dots,I_{p_\ell,q_\ell})$. Then we have

$$\mathrm{U}(p,q) = \{g \in \mathrm{GL}(p+q,\mathbb{C}) \mid {}^{\mathrm{t}}\bar{g} I_{(\underline{\mathbf{p}},\underline{\mathbf{q}})} g = I_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}\}.$$

Let θ be the Cartan involution given by the conjugation by $I_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}$. In this realization, we denote by $\mathfrak{q}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ the block-upper-triangular parabolic subalgebra of $\mathfrak{gl}(p+q,\mathbb{C}) = \mathfrak{u}(p,q)\otimes_{\mathbb{R}}\mathbb{C}$ with blocks of sizes $p_1+q_1,\dots,p_\ell+q_\ell$ along the diagonal. Then, $\mathfrak{q}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ is a θ-stable parabolic subalgebra. The corresponding Levi subgroup $\mathrm{U}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ consists of diagonal blocks

$$\mathrm{U}(\underline{\mathbf{p}},\underline{\mathbf{q}}) \cong \mathrm{U}(p_1,q_1)\times\cdots\times\mathrm{U}(p_\ell,q_\ell).$$

We denote by $\mathfrak{u}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ the Lie algebra of $\mathrm{U}(\underline{\mathbf{p}},\underline{\mathbf{q}})$. Via the above construction of $\mathfrak{q}(\underline{\mathbf{p}},\underline{\mathbf{q}})$, the K-conjugate class of θ-stable parabolic subalgebras with respect to $\mathrm{U}(p,q)$ is classified by $\mathbb{P}(p,q)$.

For $G = \mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$, we put

$$\mathbb{P}_G = \begin{cases} \displaystyle\bigcup_{\substack{p'\leqslant p\\ q'\leqslant q}} \mathbb{P}(p',q') & \text{if } G = \mathrm{Sp}(p,q),\\ \displaystyle\bigcup_{p'+q'\leqslant n} \mathbb{P}(p',q') & \text{if } G = \mathrm{SO}^*(2n). \end{cases}$$

The K-conjugate class of θ-stable parabolic subalgebras with respect to G is classified by $\mathbb{P}_G$. We give a construction of the θ-stable parabolic subalgebra $\tilde{\mathfrak{q}}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ for $(\underline{\mathbf{p}},\underline{\mathbf{q}}) \in \mathbb{P}_G$.

First, we assume $G = \mathrm{Sp}(p,q)$, $(\underline{\mathbf{p}},\underline{\mathbf{q}}) \in \mathbb{P}_\ell(p',q')$, $0\leqslant p'\leqslant p$, and $0\leqslant q'\leqslant q$. Put $p_0 = p-p'$ and $q_0 = q-q'$. Then we have a symmetric pair $(\mathrm{Sp}(p,q),\mathrm{Sp}(p',q')\times\mathrm{Sp}(p_0,q_0))$. Taking account of the realization of $\mathrm{Sp}(p',q')$ as the automorphism group of an indefinite Hermitian form on an $\mathbb{H}$-vector space (§ 3.2 (8)), we see that $\mathrm{U}(p',q')\subseteq\mathrm{Sp}(p',q')$. Hence we have $\mathrm{U}(p',q')\times\mathrm{Sp}(p_0,q_0)\subseteq\mathrm{Sp}(p,q)$. Put $L_{(p',q')}(p,q) = \mathrm{U}(p',q')\times\mathrm{Sp}(p_0,q_0)$. Since the centralizer in $\mathrm{Sp}(p,q)$ of the center of $\mathrm{U}(p',q')$ is $L_{(p',q')}(p,q)$, then $L_{(p',q')}(p,q)$ is a Levi subgroup of a θ-stable maximal parabolic subalgebra $\tilde{\mathfrak{q}}_{(p',q')}(p,q)$ of $\mathfrak{sp}(p+q,\mathbb{C})$. We denote by $\tilde{\mathfrak{u}}_{(p',q')}(p,q)$ the nilradical of $\tilde{\mathfrak{q}}_{(p',q')}(p,q)$. In fact there are two possibilities of the choice of $\tilde{\mathfrak{u}}_{(p',q')}(p,q)$. Our choice should be compatible with the construction in § 3.4. Namely, we should choose $\tilde{\mathfrak{u}}_{(p',q')}(p,q)$ so that $\tilde{\mathfrak{q}}_{(p',q')}(p,q) = \tilde{\mathfrak{q}}_{(k)}$, if $p'=q'=k$. Such a choice is determined as follows. For $\ell\in\mathbb{Z}$, we define a character η_ℓ of $\mathrm{U}(p',q')$ by

$$\eta_\ell(g) = \det(g)^\ell \quad \text{for } g\in\mathrm{U}(p',q').$$

Let π be any irreducible unitary representation of $\mathrm{Sp}(p_0,q_0)$. Then, we choose $\tilde{\mathfrak{u}}_{(p',q')}(p,q)$ so that $\eta_\ell\boxtimes\pi$ is good with respect to $\tilde{\mathfrak{q}}_{(p',q')}(p,q)$ for a sufficiently large ℓ.

We denote by $\mathfrak{l}_{(p',q')}(p,q)$ the complexified Lie algebra of $L_{(p',q')}(p,q)$. Let $\mathfrak{q}(\underline{\mathbf{p}},\underline{\mathbf{q}})$ be the θ-stable parabolic subgroup of $\mathfrak{u}(p',q')\otimes_{\mathbb{R}}\mathbb{C}$ defined as above. Since $L_{(p',q')}(p,q) = \mathrm{U}(p',q')\times\mathrm{Sp}(p_0,q_0)$, then $\mathfrak{q}(\underline{\mathbf{p}},\underline{\mathbf{q}})\oplus\mathfrak{sp}(p_0+q_0,\mathbb{C})$ is a θ-stable parabolic subalgebra of $\mathfrak{l}_{(p',q')}$. Define

$$\tilde{\mathfrak{q}}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(p,q) = (\mathfrak{q}(\underline{\mathbf{p}},\underline{\mathbf{q}})\oplus\mathfrak{sp}(p_0+q_0,\mathbb{C})) + \tilde{\mathfrak{u}}_{(p',q')}(p,q).$$

Then $\tilde{\mathfrak{p}}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(p,q)$ is a θ-stable parabolic subalgebra of $\mathfrak{sp}(p+q,\mathbb{C})$. The corresponding Levi subgroup is $L_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(p,q) = \mathrm{U}(\underline{\mathbf{p}},\underline{\mathbf{q}})\times\mathrm{Sp}(p_0,q_0)$.

Next, we consider the case $G=\mathrm{SO}^*(2n)$. Assume $(\underline{\mathbf{p}},\underline{\mathbf{q}})\in\mathbb{P}_\ell(p',q')$, $p'+q'\leqslant n$. Put $n' = p'+q'$ and $n_0 = n-n'$. Then we have a symmetric pair $(\mathrm{SO}^*(2n),\mathrm{SO}^*(2n')\times\mathrm{SO}^*(2n_0))$. There is a symmetric pair $(\mathrm{U}(p',q'),\mathrm{SO}^*(2n'))$. Put $L^*_{(p',q')}(2n) = \mathrm{U}(p',q')\times\mathrm{SO}^*(2n_0)$. Since the centralizer in $\mathrm{SO}^*(2n)$ of the center of $\mathrm{U}(p',q')$ is $L^*_{(p',q')}(2n)$, then $L^*_{(p',q')}(2n)$ is a Levi subgroup of a θ-stable maximal parabolic subalgebra $\tilde{\mathfrak{q}}^*_{(p',q')}(2n)$ of $\mathfrak{so}(2n,\mathbb{C})$. Now we can construct a θ-stable parabolic subalgebra $\tilde{\mathfrak{p}}^*_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(2n)$ of $\mathfrak{so}(2n,\mathbb{C})$ in the same way as for the case of $G=\mathrm{Sp}(p,q)$. In this case the Levi subgroup $L^*_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(2n)$ is isomorphic to $\mathrm{U}(\underline{\mathbf{p}},\underline{\mathbf{q}})\times\mathrm{SO}^*(2n_0)$.

3.5 A rearrangement formula

First, we consider the case of $G = \mathrm{Sp}(p,q)$. Let p' and q' be non-negative integers such that $p'+q' > 0$. Moreover, we assume that $p' \leqslant p$ and $q' \leqslant q$. Put $p_0 = p - p'$ and $q_0 = q - q'$. We consider θ-stable maximal parabolic subalgebra $\tilde{\mathfrak{q}}_{(p',q')}(p,q)$ defined in § 3.4.

Let $\mathfrak{h}(p_0,q_0)$ (respectively $\mathfrak{h}_{(p',q')}$) be a θ- and σ-stable compact Cartan subalgebra for $\mathrm{Sp}(p_0,q_0)$ (respectively $\mathrm{U}(p',q')$).

Taking account of $L_{(p',q')}(p,q) = \mathrm{U}(p',q') \times \mathrm{Sp}(p_0,q_0)$, we put

$$\mathfrak{h}(p,q) = \mathfrak{h}_{(p',q')} \oplus \mathfrak{h}(p_0,q_0) \subseteq \mathfrak{l}_{(p',q')}(p,q) \subseteq \mathfrak{sp}(p+q,\mathbb{C}).$$

Then, $\mathfrak{h}(p,q)$ is a θ- and σ-stable compact Cartan subalgebra for $\mathrm{Sp}(p,q)$. Using the above direct sum decomposition, we regard $\mathfrak{h}^*_{(p',q')}$ and $\mathfrak{h}(p_0,q_0)^*$ as subspaces of $\mathfrak{h}(p,q)^*$. We introduce an orthonormal basis $\{f_1,\dots,f_{p'+q'}\}$ (respectively $\{f_{p'+q'+1},\dots,f_{p+q}\}$) of $\mathfrak{h}^*_{(p',q')}$ (respectively $\mathfrak{h}(p_0,q_0)^*$) such that

$$\begin{aligned}
\Delta(\mathfrak{sp}(p+q,\mathbb{C}),\mathfrak{h}(p,q)) &= \{\pm f_i \pm f_j \mid 1 \leqslant i < j \leqslant p+q\} \cup \{\pm 2f_i \mid 1 \leqslant i \leqslant p+q\},\\
\Delta(\mathfrak{u}(p',q') \otimes_{\mathbb{R}} \mathbb{C},\mathfrak{h}_{(p',q')}) &= \{f_i - f_j \mid 1 \leqslant i,j \leqslant p'+q',\ i \neq j\},\\
\Delta(\mathfrak{sp}(p_0+q_0,\mathbb{C}),\mathfrak{h}(p_0,q_0)) &= \{\pm f_i \pm f_j \mid p'+q' < i < j \leqslant p+q\} \cup \{\pm 2f_i \mid p'+q' < i \leqslant p+q\},\\
\Delta(\tilde{\mathfrak{u}}_{(p',q')}(p,q),\mathfrak{h}(p,q)) &= \{f_i \pm f_j \mid 1 \leqslant i \leqslant p'+q' < j \leqslant p+q\}\\
&\quad \cup \{f_i + f_j \mid 1 \leqslant i \leqslant j \leqslant p'+q'\}.
\end{aligned}$$

We denote by $F_1,\dots,F_{p+q}$ the basis of $\mathfrak{h}(p,q)$ dual to $f_1,\dots,f_{p+q}$. We have

$$\delta(\tilde{\mathfrak{u}}_{(p',q')}(p,q))(F_i) = \begin{cases} \dfrac{2p+2q-p'-q'+1}{2} & \text{if } 1 \leqslant i \leqslant p'+q',\\ 0 & \text{otherwise.}\end{cases}$$

For $\ell \in \mathbb{Z}$, we consider the one-dimensional unitary representation η_ℓ of $\mathrm{U}(p',q')$ defined in § 3.5.

Let Z be any Harish-Chandra module for $\mathrm{Sp}(p_0,q_0)$ with an infinitesimal character $\lambda \in \mathfrak{h}(p_0,q_0)^* \subseteq \mathfrak{h}(p,q)^*$; λ is unique up to the Weyl group action. Put $\|\lambda\| = \max(\{0\} \cup \{|\lambda(F_i)| \mid p'+q' < i \leqslant p+q, \lambda(F_i) \in \mathbb{Z}\})$; $\|\lambda\|$ is invariant under the Weyl group action on λ, so we write $\|Z\| = \|\lambda\|$.

Then $\eta_\ell \boxtimes Z$ has an infinitesimal character $[\ell,\lambda] \in \mathfrak{h}(p,q)^*$ such that

$$[\ell,\lambda](F_i) = \begin{cases} \ell + \dfrac{p'+q'+1}{2} - i & \text{if } 1 \leqslant i \leqslant p'+q',\\ \lambda(F_i) & \text{if } p'+q' < i \leqslant p+q.\end{cases}$$

We denote by $\mathcal{H}(\mathrm{Sp}(p,q))_\mu$ the category of Harish-Chandra modules for $\mathrm{Sp}(p,q)$ with an infinitesimal character μ.

DEFINITION 3.5.1. For $\ell \in \mathbb{Z}$ and $Z \in \mathcal{H}(\mathrm{Sp}(p_0,q_0))_\lambda$, put

$$\mathcal{R}^{p,q}_{p',q'}(\ell)(Z) = (\mathcal{R}^{\mathfrak{sp}(p+q,\mathbb{C}),K}_{\tilde{\mathfrak{q}}_{(p',q')}(p,q),L_{(p',q')}(p,q)\cap K})^S((\eta_\ell \boxtimes Z) \otimes \mathbb{C}_{2\delta(\tilde{\mathfrak{u}}_{(p',q')}(p,q))}),$$

where $S = [p'(4p-3p'+1)+q'(4q-3q'+1)]/2$. If $\ell \geqslant \|\lambda\| - (p_0+q_0)$, then the above cohomological induction is in the good range and we have an exact functor

$$\mathcal{R}^{p,q}_{p',q'}(\ell) : \mathcal{H}(\mathrm{Sp}(p_0,q_0))_\lambda \to \mathcal{H}(\mathrm{Sp}(p,q))_{[\ell,\lambda]+\delta(\tilde{\mathfrak{u}}_{(p',q')}(p,q))}.$$

Next, we consider the following setting. Let k be a positive integer such that $k \leqslant p$ and $k \leqslant q$. Let p' and q' be non-negative integers such that $p'+q' > 0$. Moreover, we assume that $p'+k \leqslant p$ and $q'+k \leqslant q$. We consider θ-stable parabolic subalgebra $\tilde{\mathfrak{q}}_{(p',q')}(p-k,q-k)$ of $\mathfrak{m}^\circ_{(k)} = \mathrm{Sp}(p-k,q-k)$ defined in § 3.5.

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So, the Levi subgroup $L_{(p',q')}(p-k,q-k)$ of $\tilde{\mathfrak{q}}_{(p',q')}(p-k,q-k)$ is written as $\mathrm{U}(p',q')\times\mathrm{Sp}(p-p'-k,q-q'-k)$. Put ${}^sH^{(k)}=\exp({}^s\mathfrak{h}^{(k)})\cap G$. Then ${}^sH^{(k)}$ is a maximally split Cartan subgroup of $\mathrm{GL}(k,\mathbb{H})$. (Here, we consider the decomposition $M_{(k)}=\mathrm{GL}(k,\mathbb{H})\times M^{\circ}_{(k)}$.) We fix a compact Cartan subgroup ${}^uH_{(k,p',q')}$ of $L_{(p',q')}(p-k,q-k)$ and put $H(k,p',q')={}^sH^{(k)}\times{}^uH_{(k,p',q')}$. We denote by $\mathfrak{h}(k,p',q')$ the complexified Cartan subalgebra of $H(k,p',q')$. We denote by $L'_{(k,p',q')}$ the centralizer of the center of $\mathrm{U}(p',q')$ in G. Then, we have $L'(k,p',q')\cong\mathrm{U}(p',q')\times\mathrm{Sp}(p-p',q-q')$. Let $\tilde{\mathfrak{q}}'(k,p',q')$ be a θ-stable parabolic subalgebra of $\mathrm{Sp}(p,q)$ with the Levi subgroup $L'_{(k,p',q')}$. Let ϕ be any irreducible unitary representation of $\mathrm{Sp}(p-p',q-q')$. We choose $\tilde{\mathfrak{q}}'(k,p',q')$ so that $\eta_\ell\boxtimes\phi$ is good with respect to $\tilde{\mathfrak{q}}'(k,p',q')$ for sufficiently large ℓ. Then $\tilde{\mathfrak{q}}'(k,p',q')$ is K-conjugate to $\tilde{\mathfrak{q}}_{(p',q')}(p,q)$ defined in § 3.4.

Since $\mathfrak{h}(k,p',q')\subseteq\tilde{\mathfrak{q}}'(k,p',q')\cap\mathfrak{p}_{(k)}(p,q)$, then $(\mathfrak{p}_{(k)}(p,q),\tilde{\mathfrak{q}}'(k,p',q'))$ is a $\sigma\theta$-pair.

Since ${}^s\mathfrak{h}^{(k)}\subseteq{}^s\mathfrak{h}$ has a basis $E_1,\dots,E_{2k}$, for any $\lambda\in({}^s\mathfrak{h}^{(k)})^*$, we define

$$\|\lambda\|=\max(\{0\}\cup\{|\lambda(E_i)|\mid 1\leqslant i\leqslant 2k,|\lambda(E_i)|\in\mathbb{Z}\}).$$

For any Harish-Chandra module V with an infinitesimal character $\lambda\in({}^s\mathfrak{h}^{(k)})^*$, we put $\|V\|=\|\lambda\|$. This is well defined, since $\|\lambda\|$ is invariant under the Weyl group action. For example, we easily have our next lemma.

Lemma 3.5.2.

i) *If χ is a one-dimensional unitary representation of* $\mathrm{GL}(k,\mathbb{H})$, *then* $\|\chi\|=0$.
ii) *For $\ell\in\mathbb{Z}$ and $t\in\sqrt{-1}\,\mathbb{R}$, we have*

$$\|A_k(\ell,t)\|=\begin{cases}\dfrac{2k+\ell-1}{2} & \textit{if } \ell \textit{ is odd and } t=0,\\ 0 & \textit{otherwise.}\end{cases}$$

Applying Theorem 2.2.3 to the $\sigma\theta$-pair $(\mathfrak{p}_{(k)}(p,q),\tilde{\mathfrak{q}}'(k,p',q'))$, we get the following.

Theorem 3.5.3 (Rearrangement formula for $\mathrm{Sp}(p,q)$). *Let k be a positive integer such that $k\leqslant p$ and $k\leqslant q$. Let p' and q' be non-negative integers such that $p'+q'>0$. Moreover, we assume that $p'+k\leqslant p$ and $q'+k\leqslant q$. Let V (respectively Z) be a Harish-Chandra module with an infinitesimal character for* $\mathrm{GL}(k,\mathbb{H})$ *(respectively* $\mathrm{Sp}(p-p'-k,q-q'-k)$*). Let ℓ be an integer such that $\ell\geqslant\max\{\|V\|,\|Z\|\}-(p-p'-k)-(q-q'-k)$. Then we have*

$$[\mathrm{Ind}_{P_{(k)}(p,q)}^{\mathrm{Sp}(p,q)}(V\boxtimes\mathcal{R}_{p',q'}^{p-k,q-k}(\ell)(Z))]=[\mathcal{R}_{p',q'}^{p,q}(\ell-2k)(\mathrm{Ind}_{P_{(k)}(p-p',q-q')}^{\mathrm{Sp}(p-p',q-q')}(V\boxtimes Z))].$$

The above cohomological inductions are in the good region.

Next, we consider the case of $\mathrm{SO}^*(2n)$.

Put $n_0=n-p'-q'$. We consider θ-stable maximal parabolic subalgebra $\tilde{\mathfrak{q}}^*_{(p',q')}(2n)$ defined in § 3.5.

Let $\mathfrak{h}(2n_0)$ (respectively $\mathfrak{h}_{(p',q')}$) be a θ- and σ-stable compact Cartan subalgebra for $\mathrm{SO}^*(2n_0)$ (respectively $\mathrm{U}(p',q')$).

Taking account of $L^*_{(p',q')}(2n)=\mathrm{U}(p',q')\times\mathrm{SO}^*(2n_0)$, we put

$$\mathfrak{h}(2n)=\mathfrak{h}_{(p',q')}\oplus\mathfrak{h}(2n_0)\subseteq\mathfrak{l}^*_{(p',q')}(2n)\subseteq\mathfrak{so}(2n,\mathbb{C}).$$

Then, $\mathfrak{h}(2n)$ is a θ- and σ-stable compact Cartan subalgebra for $\mathrm{SO}^*(2n)$. Using the above direct sum decomposition, we regard $\mathfrak{h}^*_{(p',q')}$ and $\mathfrak{h}(2n_0)^*$ as subspaces of $\mathfrak{h}(2n)^*$. We introduce an orthonormal

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basis $\{f_1,\dots,f_{p'+q'}\}$ (respectively $\{f_{p'+q'+1},\dots,f_{2n}\}$) of $\mathfrak{h}^*_{(p',q')}$ (respectively $\mathfrak{h}(2n_0)^*$) such that

$$\begin{aligned}\Delta(\mathfrak{so}(2n,\mathbb{C}),\mathfrak{h}(2n)) &= \{\pm f_i \pm f_j \mid 1 \leqslant i < j \leqslant p+q\},\\ \Delta(\mathfrak{u}(p',q')\otimes_{\mathbb{R}}\mathbb{C},\mathfrak{h}_{(p',q')}) &= \{f_i f_j \mid 1 \leqslant i,j \leqslant p'+q',\ i\neq j\},\\ \Delta(\mathfrak{so}(2n_0,\mathbb{C}),\mathfrak{h}(2n_0)) &= \{\pm f_i \pm f_j \mid p'+q' < i < j \leqslant p+q\},\\ \Delta(\tilde{\mathfrak{u}}^*_{(p',q')}(2n),\mathfrak{h}(2n)) &= \{f_i \pm f_j \mid 1 \leqslant i \leqslant p'+q' < j \leqslant p+q\}\cup\{f_i+f_j \mid 1\leqslant i<j\leqslant p'+q'\}.\end{aligned}$$

We denote by $F_1,\dots,F_{p+q}$ the basis of $\mathfrak{h}(2n)$ dual to $f_1,\dots,f_{2n}$.

We have

$$\delta(\tilde{\mathfrak{u}}^*_{(p',q')}(2n))(F_i) = \begin{cases}\dfrac{2n-p'-q'-1}{2} & \text{if } 1\leqslant i\leqslant p'+q',\\ 0 & \text{otherwise.}\end{cases}$$

Let Z be any Harish-Chandra module for $\mathrm{SO}^*(2n_0)$ with an infinitesimal character $\lambda\in\mathfrak{h}(2n_0)^*\subseteq\mathfrak{h}(2n)^*$; λ is unique up to the Weyl group action. Put $\|\lambda\| = \max(\{0\}\cup\{|\lambda(F_i)| \mid p'+q' < i \leqslant 2n, \lambda(F_i)\in\mathbb{Z})\})$; $\|\lambda\|$ is invariant under the Weyl group action on λ, so we write $\|Z\| = \|\lambda\|$.

Then $\eta_\ell\boxtimes Z$ has an infinitesimal character $[\ell,\lambda]\in\mathfrak{h}(2n)^*$ such that

$$[\ell,\lambda](F_i) = \begin{cases}\ell + \dfrac{p'+q'+1}{2} - i & \text{if } 1\leqslant i\leqslant p'+q',\\ \lambda(F_i) & \text{if } p'+q' < i \leqslant p+q.\end{cases}$$

We denote by $\mathcal{H}(\mathrm{SO}^*(2n))_\mu$ the category of Harish-Chandra modules for $\mathrm{SO}^*(2n)$ with an infinitesimal character μ.

DEFINITION 3.5.4. For $\ell\in\mathbb{Z}$ and $Z\in\mathcal{H}(\mathrm{SO}^*(2n_0))_\lambda$, put

$$\mathcal{R}^{2n}_{p',q'}(\ell)(Z) = (\mathcal{R}^{\mathfrak{so}(2n,\mathbb{C}),K}_{\tilde{\mathfrak{q}}^*_{(p',q')}(2n),L^*_{(p',q')}(2n)\cap K})^S((\eta_\ell\boxtimes Z)\otimes\mathbb{C}_{2\delta(\tilde{\mathfrak{u}}^*_{(p',q')}(2n))}),$$

where $S=(p'+q')(n-p'-q')+p'q'$. If $\ell\geqslant\|\lambda\|-n_0+1$, then the above cohomological induction is in the good range and we have an exact functor

$$\mathcal{R}^{2n}_{p',q'}(\ell):\mathcal{H}(\mathrm{SO}^*(2n_0))_\lambda\to\mathcal{H}(\mathrm{SO}^*(2n))_{[\ell,\lambda]+\delta(\tilde{\mathfrak{u}}^*_{(p',q')}(2n))}.$$

In a similar way to the case of $\mathrm{Sp}(p,q)$, we have the next theorem.

THEOREM 3.5.5 (Rearrangement formula for $\mathrm{SO}^*(2n)$). *Let k be a positive integer such that $k\leqslant p$ and $k\leqslant q$. Let p' and q' be non-negative integers such that $p'+q'>0$. Moreover, we assume that $p'+q'+2k\leqslant n$. Let V (respectively Z) be a Harish-Chandra module with an infinitesimal character for $\mathrm{GL}(k,\mathbb{H})$ (respectively $\mathrm{SO}^*(2(n-p'-q'-2k))$). Let ℓ be an integer such that $\ell\geqslant\max\{\|V\|,\|Z\|\}-(n-p'-q'-2k)-1$. Then we have*

$$[\mathrm{Ind}^{\mathrm{SO}^*(2n)}_{P^*_{(k)}(2n)}(V\boxtimes\mathcal{R}^{2(n-2k)}_{p',q'}(\ell)(Z))] = [\mathcal{R}^{2n}_{p',q'}(\ell-2k)(\mathrm{Ind}^{\mathrm{SO}^*(2(n-p'-q'))}_{P^*_{(k)}(2(n-p'-q'))}(V\boxtimes Z))].$$

The above cohomological inductions are in the good region.

3.6 Decomposition formulas

DEFINITION 3.6.1. Let k be a positive integer and ℓ be an integer such that $\ell+k\in2\mathbb{Z}$. Let i be an integer such that $0\leqslant i\leqslant k$. We define the following derived functor module for $U(k,k)$:

$$B^{(i)}_k(\ell) = ({}^u\mathcal{R}^{\mathfrak{u}(k,k)\otimes_{\mathbb{R}}\mathbb{C},\mathrm{U}(k)\times\mathrm{U}(k)}_{\mathfrak{q}((i,k-i),(k-i,i)),\mathrm{U}(i)\times\mathrm{U}(k-i)\times\mathrm{U}(k-i)\times\mathrm{U}(i)})^{2i(k-i)}(\eta_{(\ell+k)/2}\boxtimes\eta_{(\ell-k)/2}).$$

$B^{(i)}_k(\ell)$ is not in the good region. In fact, it is an irreducible unitary representation located at the end of the weakly fair region in the sense of [Vog88].

We quote the following reducibility result of the degenerate principal series.

Theorem 3.6.2 (Kashiwara–Vergne, Johnson, ...). *Let $\ell \in \mathbb{Z}$ and $t \in \sqrt{-1}\,\mathbb{R}$.*

i) *If $\ell + k \in 2\mathbb{Z}$, then*

$$I_k(\ell, 0) = \bigoplus_{i=0}^{k} B_k^{(i)}(\ell).$$

ii) *If $t \neq 0$ or $\ell + k + 1 \in 2\mathbb{Z}$, then $I_k(\ell, t)$ is irreducible.*

Some remarks are in order. The reducibility of $I_k(\ell, 0)$ is established by [KV79]. The irreducibility result is due to [Joh90]. Identifying irreducible components in part i as derived functor modules is an easy conclusion from [BV83] and it has been more or less well known by experts. For example, a proof is given in [Mat96, 3.4].

Combining Theorem 3.6.2 and Proposition 3.3.2, we obtain the following proposition.

Proposition 3.6.3.

i) *Let p, q be positive integers such that $q \leqslant p$. Let $G = \mathrm{Sp}(p, q)$ and let k be a positive integer such that $k \leqslant q$. Let V be an irreducible unitary representation of $\mathrm{Sp}(p - k, q - k)$. Let m be an integer such that $m \geqslant \|V\| + k - 1$. Then we have*

$$\mathrm{Ind}_{P_{(k)}(p,q)}^{\mathrm{Sp}(p,q)}(A_k(2m+1, 0) \boxtimes V) \cong \bigoplus_{i=0}^{k} \mathcal{R}_{i,k-i}^{p,q}(m - n + k)(\mathcal{R}_{k-i,i}^{p-i,q-k+i}(m - n + 2k)(V)).$$

ii) *Let n be a positive integer. Let $G = \mathrm{SO}^*(2n)$ and let k be a positive integer such that $2k \leqslant n$. Let V be an irreducible unitary representation of $\mathrm{SO}^*(2(n - 2k))$. Let m be an integer such that $m \geqslant \|V\| + k - 1$. Then we have*

$$\mathrm{Ind}_{P^*_{(k)}(2n)}^{\mathrm{SO}^*(2n)}(A_k(2m+1, 0) \boxtimes V) \cong \bigoplus_{i=0}^{k} \mathcal{R}_{i,k-i}^{2n}(m - n + k + 1)(\mathcal{R}_{k-i,i}^{2(n-k)}(m - n + 2k + 1)(V)).$$

We introduce notations for derived functor modules.

First, we assume $G = \mathrm{Sp}(p, q)$, $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_m(p', q')$, $0 \leqslant p' \leqslant p$, and $0 \leqslant q' \leqslant q$. Put $p_0 = p - p'$ and $q_0 = q - q'$. We consider the derived functor modules with respect to $\tilde{\mathfrak{p}}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(p, q)$. For $1 \leqslant i \leqslant m$, we put $p_i^* = p_1 + \cdots + p_i$ and $q_i^* = q_1 + \cdots + q_i$. Let $\ell, \ldots, \ell_m$ be integers and put

$$\begin{aligned} A_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}^{p,q}(\ell_1, \ldots, \ell_m) = \mathcal{R}_{p_1,q_1}^{p,q}(\ell_1)(\mathcal{R}_{p_2,q_2}^{p-p_1,q-q_1}(\ell_2)(\cdots(\mathcal{R}_{p_i,q_i}^{p-p_{i-1}^*,q-q_{i-1}^*}(\ell_i)& \\ (\cdots(\mathcal{R}_{p_m,q_m}^{p_0+p_m,q_0+q_m}(\ell_m)(1_{\mathrm{Sp}(p_0,q_0)}))\cdots))\cdots)).& \end{aligned}$$

Here, $1_{\mathrm{Sp}(p_0,q_0)}$ is the trivial representation of $\mathrm{Sp}(p_0, q_0)$. In this setting, we define

$$\begin{aligned} \delta_i &= p + q - p_i^* - q_i^* - \frac{p_i + q_i - 1}{2} \quad (1 \leqslant i \leqslant m), \\ \tilde{\ell}_i &= \ell_i + \delta_i \quad (1 \leqslant i \leqslant m). \end{aligned}$$

Then, $A_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}^{p,q}(\ell_1, \ldots, \ell_m)$ is in the good (respectively weakly fair) region if and only if $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_m \geqslant 0$ (respectively $\tilde{\ell}_1 \geqslant \tilde{\ell}_2 \geqslant \cdots \geqslant \tilde{\ell}_m \geqslant 0$).

Next, we assume $G = \mathrm{SO}^*(2n)$, $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_m(p', q')$, and $0 \leqslant p' + q' \leqslant n$. Put $n_0 = n - p' - q'$. We consider the derived functor modules with respect to $\tilde{\mathfrak{p}}^*_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(2n)$. For $1 \leqslant i \leqslant m$, we put $p_i^* = p_1 + \cdots + p_i$ and $q_i^* = q_1 + \cdots + q_i$. Let $\ell, \ldots, \ell_m$ be integers and put

$$\begin{aligned} A_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}^{2n}(\ell_1, \ldots, \ell_m) = \mathcal{R}_{p_1,q_1}^{2n}(\ell_1)(\mathcal{R}_{p_2,q_2}^{2n-p_1-q_1}(\ell_2)(\cdots(\mathcal{R}_{p_i,q_i}^{2n-p_{i-1}^*-q_{i-1}^*}(\ell_i)& \\ (\cdots(\mathcal{R}_{p_m,q_m}^{n_0+p_m+q_m}(\ell_m)(1_{\mathrm{SO}^*(2n_0)}))\cdots))\cdots)).& \end{aligned}$$

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Here, $1_{\mathrm{SO}^*(2n_0)}$ is the trivial representation of $\mathrm{SO}^*(2n_0)$. In this setting, we define

$$\delta_i = p+q-p_i^*-q_i^*-\frac{p_i+q_i-1}{2}-1 \quad (1\leqslant i\leqslant m),$$
$$\tilde{\ell}_i = \ell_i+\delta_i \quad (1\leqslant i\leqslant m).$$

Then, $A^{2n}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell_1,\dots,\ell_m)$ is in the good (respectively weakly fair) region if and only if $\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_m\geqslant 0$ (respectively $\tilde{\ell}_1\geqslant\tilde{\ell}_2\geqslant\cdots\geqslant\tilde{\ell}_m\geqslant 0$).

Combining Theorem 3.5.3, Theorem 3.5.5, and Proposition 3.6.3, we have the next theorem.

Theorem 3.6.4.

i) *Let p,q be positive integers such that $q\leqslant p$. We consider the setting of $G=\mathrm{Sp}(p,q)$. We assume $(\underline{\mathbf{p}},\underline{\mathbf{q}})\in\mathbb{P}_m(p',q')$, $0\leqslant p'\leqslant p$, and $0\leqslant q'\leqslant q$. Let k be a positive integer. Put $n=p+q$ and put $n'_j=(p_j+q_j)+\cdots+(p_m+q_m)+2k$ for $1\leqslant i\leqslant m$. Let s be a non-negative integer.*
Let $\ell_1,\dots,\ell_m$ be integers such that $\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_m\geqslant 0$. Moreover, we assume there is some $1\leqslant j\leqslant m$ such that $\ell_{j-1}\geqslant s-n'_j+3k$ and $s-n'_j+2k\geqslant\ell_j$. (Here, we put, formally, $\ell_0=+\infty$.) Put $\underline{\mathbf{p}}'_i=(p_1,\dots,p_{j-1},i,k-i,p_j,\dots,p_m)$ and $\underline{\mathbf{q}}'_i=(q_1,\dots,q_{j-1},k-i,i,q_j,\dots,q_m)$ for $1\leqslant i\leqslant k$. Then we have
$$\begin{aligned}&\mathrm{Ind}^{\mathrm{Sp}(p+k,q+k)}_{P_{(k)}(p+k,q+k)}(A_k(2s+1)\boxtimes A^{p,q}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell_1,\dots,\ell_m))\\ &\quad\cong\bigoplus_{i=0}^{k}A^{p+k,q+k}_{(\underline{\mathbf{p}}'_i,\underline{\mathbf{q}}'_i)}(\ell_1-2k,\dots,\ell_{j-1}-2k,s-n'_j+k,s-n'_j+2k,\ell_j,\dots,\ell_1).\end{aligned}\tag{14}$$

ii) *Let n be a positive integer and we consider the setting of $G=\mathrm{SO}^*(2n)$. We assume $(\underline{\mathbf{p}},\underline{\mathbf{q}})\in\mathbb{P}_m(p',q')$, $0\leqslant p'+q'\leqslant n$. Let k be a positive integer. Put $n'_j=(p_j+q_j)+\cdots+(p_m+q_m)+2k$ for $1\leqslant i\leqslant m$. Let s be a non-negative integer.*
Let $\ell_1,\dots,\ell_m$ be integers such that $\ell_1\geqslant\ell_2\geqslant\cdots\geqslant\ell_m\geqslant 0$. Moreover, we assume there is some $1\leqslant j\leqslant m$ such that $\ell_{j-1}\geqslant s-n'_j+3k+1$ and $s-n'_j+2k+1\geqslant\ell_j$. (Here, we put, formally, $\ell_0=+\infty$.) Put $\underline{\mathbf{p}}'_i=(p_1,\dots,p_{j-1},i,k-i,p_j,\dots,p_m)$ and $\underline{\mathbf{q}}'_i=(q_1,\dots,q_{j-1},k-i,i,q_j,\dots,q_m)$ for $1\leqslant i\leqslant k$. Then we have
$$\begin{aligned}&\mathrm{Ind}^{\mathrm{SO}^*(2(n+2k))}_{P^*_{(k)}(2(n+2k))}(A_k(2s+1)\boxtimes A^{2n}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell_1,\dots,\ell_m))\\ &\quad\cong\bigoplus_{i=0}^{k}A^{2(n+2k)}_{(\underline{\mathbf{p}}'_i,\underline{\mathbf{q}}'_i)}(\ell_1-2k,\dots,\ell_{j-1}-2k,s-n'_j+k+1,s-n'_j+2k+1,\ell_j,\dots,\ell_1).\end{aligned}\tag{15}$$

iii) *The derived functor modules on the right-hand side of (14) and (15) are all non-zero and irreducible. (Actually, they are good-range cohomological induction form non-zero irreducible modules.)*

Here, we apply the translation principle in weakly fair range in [Vog88] to the above result and obtain our next theorem.

Theorem 3.6.5.

i) *Let p,q be positive integers such that $q\leqslant p$. We consider the setting of $G=\mathrm{Sp}(p,q)$. We assume $(\underline{\mathbf{p}},\underline{\mathbf{q}})\in\mathbb{P}_m(p',q')$, $0\leqslant p'\leqslant p$, and $0\leqslant q'\leqslant q$. Let k be a positive integer. Put $n=p+q$ and put $n'_j=(p_j+q_j)+\cdots+(p_m+q_m)+2k$ for $1\leqslant i\leqslant m$. Let s be an integer such that $2s+1\geqslant -k$. Let $\ell_1,\dots,\ell_m$ be integers such that $\tilde{\ell}_1\geqslant\tilde{\ell}_2\geqslant\cdots\geqslant\tilde{\ell}_m\geqslant 0$. We choose any $1\leqslant j\leqslant m$ such that $\tilde{\ell}_{j-1}\geqslant s+(k+1)/2\geqslant\tilde{\ell}_j$. (Here, we put, formally, $\ell_0=+\infty$.)*

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Put $\underline{\mathbf{p}}'_i = (p_1, \ldots, p_{j-1}, i, k-i, p_j, \ldots, p_m)$ *and* $\underline{\mathbf{q}}'_i = (q_1, \ldots, q_{j-1}, k-i, i, q_j, \ldots, q_m)$ *for* $1 \leqslant i \leqslant k$. *Then we have*

$$\begin{aligned}&\mathrm{Ind}_{P_{(k)}(2(p+k,q+k))}^{\mathrm{Sp}(p+k,q+k)}(A_k(2s+1) \boxtimes A^{p,q}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell_1, \ldots, \ell_m)) \\ &\quad\cong \bigoplus_{i=0}^{k} A^{p+k,q+k}_{(\underline{\mathbf{p}}'_i,\underline{\mathbf{q}}'_i)}(\ell_1 - 2k, \ldots, \ell_{j-1} - 2k, s - n'_j + k, s - n'_j + 2k, \ell_j, \ldots, \ell_1). \qquad (16)\end{aligned}$$

ii) *Let* n *be a positive integer and we consider the setting of* $G = \mathrm{SO}^*(2n)$. *We assume* $(\underline{\mathbf{p}}, \underline{\mathbf{q}}) \in \mathbb{P}_m(p', q')$, $0 \leqslant p' + q' \leqslant n$. *Let* k *be a positive integer. Put* $n'_j = (p_j + q_j) + \cdots + (p_m + q_m) + 2k$ *for* $1 \leqslant i \leqslant m$.

Let s *be an integer such that* $2s + 1 \geqslant -k$. *Let* $\ell_1, \ldots, \ell_m$ *be integers such that* $\tilde{\ell}_1 \geqslant \tilde{\ell}_2 \geqslant \cdots \geqslant \tilde{\ell}_m \geqslant 0$. *We choose any* $1 \leqslant j \leqslant m$ *such that* $\tilde{\ell}_{j-1} \geqslant s + (k+1)/2 \geqslant \tilde{\ell}_j$. *(Here, we put, formally,* $\ell_0 = +\infty$.*) Put* $\underline{\mathbf{p}}'_i = (p_1, \ldots, p_{j-1}, i, k-i, p_j, \ldots, p_m)$ *and* $\underline{\mathbf{q}}'_i = (q_1, \ldots, q_{j-1}, k-i, i, q_j, \ldots, q_m)$ *for* $1 \leqslant i \leqslant k$. *Then we have*

$$\begin{aligned}&\mathrm{Ind}_{P^*_{(k)}(2(n+2k))}^{\mathrm{SO}^*(2(n+2k))}(A_k(2s+1) \boxtimes A^{2n}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell_1, \ldots, \ell_m)) \\ &\quad\cong \bigoplus_{i=0}^{k} A^{2(n+2k)}_{(\underline{\mathbf{p}}'_i,\underline{\mathbf{q}}'_i)}(\ell_1 - 2k, \ldots, \ell_{j-1} - 2k, s - n'_j + k + 1, s - n'_j + 2k + 1, \ell_j, \ldots, \ell_1). \qquad (17)\end{aligned}$$

Proof. The proof is similar to the arguments in [Mat96, § 3.3]. We consider the case of $G = \mathrm{Sp}(p, q)$. (The case of $G = \mathrm{SO}^*(2n)$ is similar.) For an integer a, we denote by η_a the one-dimensional representation of $\mathrm{GL}(h, \mathbb{C})$ defined by $\eta_a(g) = \det(g)^a$. Let $a_1, \ldots, a_m$ and b be non-negative integers and consider a one-dimensional representation $\eta = \eta_{a_1} \boxtimes \cdots \boxtimes \eta_{a_{j-1}} \boxtimes \eta_b \boxtimes \eta_b \boxtimes \eta_{a_j} \boxtimes \cdots \boxtimes \eta_{a_m} \boxtimes 1_{\mathrm{Sp}(p_0+q_0,\mathbb{C})}$ of $\mathrm{GL}(p_1 + q_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(p_{j-1} + q_{j-1}, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C}) \times \mathrm{GL}(p_j + q_j, \mathbb{C}) \times \cdots \times \mathrm{GL}(p_m + q_m, \mathbb{C}) \times \mathrm{Sp}(p_0 + q_0, \mathbb{C})$. If $\eta_1 \geqslant \cdots \geqslant a_{j-1} \geqslant b \geqslant a_j \geqslant \cdots \geqslant a_m$, then there is an irreducible finite-dimensional representation V of $G_{\mathbb{C}}$ which contains η as the highest weight space. If we choose $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{j-1} \gg b \gg a_j \geqslant \cdots \geqslant a_m$ suitably, we have that $s' = s + b$, $\ell'_r = \ell_r + a_r$ $(1 \leqslant r \leqslant m)$ satisfy the regularity assumption in Theorem 3.6.4. So, we have:

$$\begin{aligned}&\mathrm{Ind}_{P_{(k)}(p+k,q+k)}^{\mathrm{Sp}(p+k,q+k)}(A_k(2s'+1) \boxtimes A^{p,q}_{(\underline{\mathbf{p}},\underline{\mathbf{q}})}(\ell'_1, \ldots, \ell'_m)) \\ &\quad\cong \bigoplus_{i=0}^{k} A^{p+k,q+k}_{(\underline{\mathbf{p}}'_i,\underline{\mathbf{q}}'_i)}(\ell'_1 - 2k, \ldots, \ell'_{j-1} - 2k, s' - n'_j + k, s' - n'_j + 2k, \ell'_j, \ldots, \ell'_1). \qquad (18)\end{aligned}$$

Let T be the translation functor from the infinitesimal character of the modules in (18) to that of (16). If we apply T to both sides of (18), we obtain (16) above. The argument is the same as [Mat96, Lemma 3.3.3]. The main ingredient is [Vog88, Proposition 4.7]. (We may apply similar argument to non-elliptic cohomological induction by [Vog82a, Lemma 7.2.9(b)].) □

Remark. In Theorem 3.6.5, the choice of j need not be unique. So, depending on the choice of j, we have apparently different formulas. Their compatibility is assured by [Mat96, Theorem 3.3.4], which is an easy conclusion of [BV83, Theorem 4.2]. The derived functor modules on the right-hand sides of (16) and (17) are all in the weakly fair region.

4. Reduction of irreducibilities

4.1 Comparison of Hecke algebra module structures

Let G be a connected real reductive linear Lie group as in § 1.2. Moreover, we assume that all the Cartan subgroups of G are connected. This assumption is satisfied for the groups $\mathrm{Sp}(p, q)$, $\mathrm{SO}^*(2n)$,

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and their Levi subgroups. It will allow us to simplify the description of coherent families, which we now recall.

Under this assumption, we may write the regular character (H,Γ,λ) as (H,λ), since Γ is uniquely determined by λ. We fix a regular weight ${}^s\lambda \in {}^s\mathfrak{h}^*$. Put $\Lambda = {}^s\lambda + \mathcal{P}_G$.

We denote by $W_{{}^s\lambda}$ (respectively $\Delta_{{}^s\lambda}$) the integral Weyl group (respectively the integral root system) for λ. Namely, we put

$$W_{{}^s\lambda} = \{w \in W \mid w{}^s\lambda - {}^s\lambda \in \mathcal{Q}\}, \quad \Delta_{{}^s\lambda} = \left\{\alpha \in \Delta \;\middle|\; \frac{\langle \alpha, {}^s\lambda\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}\right\}.$$

We put $\Delta^+_{{}^s\lambda} = \{\alpha \in \Delta_{{}^s\lambda} \mid \langle \alpha, {}^s\lambda\rangle > 0\}$. Then, $\Delta^+_{{}^s\lambda}$ is a positive system for $\Delta_{{}^s\lambda}$. We denote by $\Pi_{{}^s\lambda}$ the set of simple roots in $\Delta^+_{{}^s\lambda}$.

A map Θ of Λ to the space of invariant eigendistributions on G is called a coherent family on Λ if it satisfies the following conditions. (Our assumption that all the Cartan subgroups are connected makes the definition of a coherent family much simpler. For the formulation in the general setting, see [Vog82a].)

C1) For all $\eta \in \Lambda$, $\Theta(\eta)$ is a complex linear combination of the distribution characters of Harish-Chandra modules with infinitesimal character η.

C2) For any finite-dimensional representation E, we have

$$[E]\Theta(\eta) = \sum_{\mu \in \mathcal{P}_G} [\mu : E]\Theta(\eta+\mu) \quad (\eta \in \Lambda).$$

Here, $[\mu : E]$ means the multiplicity of the weight μ in E.

We denote by $\mathcal{C}(\Lambda)$ the set of coherent families on Λ. For $w \in W_{{}^s\lambda}$ and $\Theta \in \mathcal{C}(\Lambda)$, we define $w\cdot\Theta$ by $(w\cdot\Theta)(\eta) = \Theta(w^{-1}\eta)$. We see that $\mathcal{C}(\Lambda)$ is a $W_{{}^s\lambda}$-representation. This representation is called the coherent continuation representation for Λ.

For any Harish-Chandra $(\mathfrak{g},K)$-module V with an infinitesimal character ${}^s\lambda$, there is a unique coherent family Θ_V such that $\Theta_V({}^s\lambda) = [V]$. For a regular character $\gamma = (H,\lambda)$ such that $\chi_\lambda = \chi_{{}^s\lambda}$, we put $\Theta^G_\gamma = \Theta_{\pi_G(\gamma)}$ and $\bar{\Theta}^G_\gamma = \Theta_{\bar{\pi}_G(\gamma)}$. If $\eta \in \Lambda$ is regular and dominant (with respect to $\Delta^+_{{}^s\lambda}$), then $(H, \mathbf{i}_{{}^s\lambda,\lambda}(\eta))$ is a regular character and we have $\Theta^G_\gamma(\eta) = [\pi_G(H,\mathbf{i}_{{}^s\lambda,\lambda}(\eta))]$ and $\bar{\Theta}^G_\gamma(\eta) = [\bar{\pi}_G(H,\mathbf{i}_{{}^s\lambda,\lambda}(\eta))]$. Put $\mathrm{St}_G({}^s\lambda) = \{\Theta^G_\gamma \mid \gamma \in R_G({}^s\lambda)\}$ and $\mathrm{Irr}_G({}^s\lambda) = \{\bar{\Theta}^G_\gamma \mid \gamma \in R_G({}^s\lambda)\}$. We define a bijection $\Theta \rightsquigarrow \bar{\Theta}$ of $\mathrm{St}_G({}^s\lambda)$ onto $\mathrm{Irr}_G({}^s\lambda)$ by $\bar{\Theta^G_\gamma} = \bar{\Theta}^G_\gamma$ for $\gamma \in R_G({}^s\lambda)$. $\mathrm{St}_G({}^s\lambda)$ forms a basis of $\mathcal{C}(\Lambda)$ and so does $\mathrm{Irr}_G({}^s\lambda)$.

We write $\bar{\Theta}^G_\gamma = \sum_{\Theta\in\mathrm{St}_G({}^s\lambda)} M_G(\gamma,\Theta)\Theta$ and $M_G(\gamma,\delta) = M_G(\gamma,\Theta_\delta) \in \mathbb{C}$. For $\gamma = (H,\lambda) \in R_G({}^s\lambda)$ and $w \in W_{{}^s\lambda}$, the cross-product is defined as follows:

$$w \times \gamma = (H, \mathbf{i}_{{}^s\lambda,\lambda}(w)^{-1}\lambda).$$

Then, we have $w \times \gamma \in R_G({}^s\lambda)$. Moreover, for any $\gamma,\gamma' \in R_G({}^s\lambda)$ such that $\Theta^G_\gamma = \Theta^G_{\gamma'}$, we have $\Theta^G_{w\times\gamma} = \Theta^G_{w\times\gamma'}$ for all $w \in W_{{}^s\lambda}$. So, we put $w \times \Theta^G_\gamma = \Theta^G_{w\times\gamma}$.

Let H be a θ-stable Cartan subgroup of G and let $\mathfrak{h}$ be its complexified Lie algebra. For a non-compact imaginary root $\beta \in \Delta(\mathfrak{g},\mathfrak{h})$, we denote by $c_\beta \in \mathrm{Ad}(G_{\mathbb{C}})$ the Cayley transform associated with β (see [Kna86, page 419]). For a real root $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$, we denote by c^α the (inverse)Cayley transform associated with α. (In [Kna86, page 420], Knapp denotes c^α by d_α.)

We recall the Cayley transforms of regular characters (cf. [Vog83b]). Fix $\gamma = (H,\lambda) \in R_G(H,{}^s\lambda)$, and choose $\alpha \in \Delta_{{}^s\lambda}$ such that α is non-compact imaginary with respect to γ. We put $c_\alpha(\gamma) = (\mathrm{Ad}(c_{\mathbf{i}_{{}^s\lambda,\lambda}(\alpha)})(H), \lambda\cdot\mathrm{Ad}(c_{\mathbf{i}_{{}^s\lambda,\lambda}(\alpha)})^{-1})$. Then, we have $c_\alpha(\gamma) \in R_G({}^s\lambda)$ and α is real with respect to $c_\alpha(\gamma)$. It is easy to see that $c_\alpha(\Theta^G_\gamma) = \Theta^G_{c_\alpha(\gamma)}$ is well defined.

Conversely, consider $\gamma \in R_G({}^s\lambda)$ and $\alpha \in \Delta_{{}^s\lambda}$ which is real with respect to γ. We say that α satisfies the parity condition with respect to γ, if there is some $\gamma' \in R_G({}^s\lambda)$ such that α is noncompact imaginary with respect to γ' and $\gamma = c_\alpha(\gamma')$. If α satisfies the parity condition with respect to γ, there are just two regular characters in $R_G(\mathrm{Ad}(c^{\mathbf{i}_{{}^s\lambda,\lambda}(\alpha)})(H), {}^s\lambda)$, say $c_+^\alpha(\gamma)$ and $c_-^\alpha(\gamma)$, in the preimage of γ with respect to c_α. Since we assume that all the Cartan subgroups of G are connected, $c_\pm^\alpha(\gamma)$ are not K-conjugate to each other. It is easy to see that $c_\pm^\alpha(\Theta_\gamma^G) = \Theta^G_{c_\pm^\alpha(\gamma)}$ is well defined.

We denote by $H(W_{{}^s\lambda})$ the Iwahori–Hecke algebra for $W_{{}^s\lambda}$. We denote by q the indeterminant appearing in the definition of $H(W_{{}^s\lambda})$.

Put $\mathcal{C}(\Lambda)_q = \mathcal{C}(\Lambda) \otimes_{\mathbb{C}} \mathbb{C}[q]$. We introduce $H(W_{{}^s\lambda})$-module structure on $\mathcal{C}(\Lambda)_q$ as in [Vog83b, page 239]. The important thing is that the Hecke algebra module structure is completely determined by the action of cross-product and Cayley transforms on the K-conjugacy classes of regular characters in $R_G({}^s\lambda)$.

If we consider the specialization at $q = 1$ of this Hecke algebra module $\mathcal{C}(\Lambda)_q$, then we have a $W_{{}^s\lambda}$-representation on $\mathcal{C}(\Lambda)$. The relation to the coherent continuation representation is given as follows.

Theorem 4.1.1 ([Vog82a, Vog82b]). *We have an isomorphism*

$$(\textit{Specialization of } \mathcal{C}(\Lambda)_q \textit{ at } q=1) \cong (\textit{Coherent continuation representation}) \otimes \mathrm{sgn},$$

where sgn *means the signature representation of* $W_{{}^s\lambda}$. *This isomorphism preserves the basis* $\mathrm{St}_G({}^s\lambda)$.

The following result is crucial in our proof.

Theorem 4.1.2 (see [Vog83b] and [ABV92, Chapter 16]). *For $\gamma, \delta \in R_G({}^s\lambda)$, the complex number $M(\gamma,\delta)$ is computed from an algorithm (the Kazhdan–Lusztig type algorithm) which depends only on the Hecke algebra structure on $\mathcal{C}(\Lambda)_q$.*

Let P be a parabolic subgroup of G with θ-stable Levi part L such that ${}^sH \subseteq L$. (We remark that all the Cartan subgroups of L are connected.) We fix a regular character ${}^s\lambda \in {}^s\mathfrak{h}^*$ as above. Put $\Lambda_L = {}^s\lambda + \mathcal{P}_L$ and $\Lambda_G = {}^s\lambda + \mathcal{P}_G$. Then, we easily see $\Lambda_G \subseteq \Lambda_L$. Let Θ be a coherent family on Λ_L. For fixed $\nu \in \Lambda_G$, we write $\Theta(\nu) = \sum_{i=1}^n a_i[V_i]$, where V_i are certain Harish-Chandra $(\mathfrak{l}, K\cap L)$-modules with infinitesimal character ν and a_i are complex numbers. We write $\mathrm{Ind}_L^G(\Theta)(\nu) = \sum_{i=1}^n a_i[\mathrm{Ind}_P^G(V_i)]$. The above definition is independent of the choice of the linear combination, since the parabolic induction is exact. From a property of induction, the above definition depends only on L and does not depend on P. Moreover, $\nu \rightsquigarrow \mathrm{Ind}_L^G(\Theta)(\nu)$ forms a coherent family on Λ_G, thanks to a version of Mackey's tensor product theorem [SV80, Lemma 5.8] for induction and the exactness of the induction.

Let H be a θ-stable Cartan subgroup of L. Hence H is also a Cartan subgroup of G. Let $\gamma = (H,\lambda)$ be a regular character for L with an infinitesimal character ${}^s\lambda$. Then γ is also a regular character for G. We easily see $\mathrm{Ind}_L^G(\Theta_\gamma^L) = \Theta_\gamma^G$.

Next we describe a result on the comparison of Hecke module structures. Besides G we also consider another real reductive linear Lie group G' whose Cartan subgroups are all connected. We denote the objects with respect to G' by attaching the 'prime' to the notations for the corresponding objects for G. For example, we fix a Cartan involution θ' for G' and fix a θ'-invariant maximally split Cartan subgroup ${}^sH'$, etc. We fix a regular weight ${}^s\lambda \in {}^s\mathfrak{h}^*$ and put $\Lambda = {}^s\lambda + \mathcal{P}_G$. Moreover, we assume the following conditions on G and G':

C1) There is a linear isomorphism $\psi : {}^s\mathfrak{h}^* \to ({}^s\mathfrak{h}')^*$ such that $\psi(\Delta_{{}^s\lambda}) = \Delta'$. Here, Δ' means the root system with respect to $(\mathfrak{g}', {}^s\mathfrak{h}')$. Moreover, $\psi({}^s\lambda)$ is regular integral with respect to Δ' and $\psi(\mathcal{P}_G) \subseteq \mathcal{P}_{G'}$; and ψ induces an isomorphism $\psi_\sharp : W_{{}^s\lambda} \to W'$. Here, W' is the Weyl group for Δ'.

C2) There is a bijection Ψ of the K-conjugacy classes of ${}^s\lambda$-integral θ-invariant Cartan subgroups of G to the K'-conjugacy classes of $\psi({}^s\lambda)$-integral θ'-invariant Cartan subgroups of G'.

C3) There is a bijection $\tilde{\Psi} : \mathrm{St}_G({}^s\lambda) \to \mathrm{St}_{G'}(\psi({}^s\lambda))$ which is compatible with Ψ in condition C2.

C4) For $\Theta \in \mathrm{St}_G({}^s\lambda)$, we have $\psi \circ \theta_\Theta = \theta_{\tilde{\Theta}} \circ \psi$. Hence, for $\alpha \in \Delta_{{}^s\lambda}$, we have α is imaginary, complex, real with respect to Θ if and only if $\psi(\alpha)$ is imaginary, complex, real, respectively, with respect to $\tilde{\Psi}(\Theta)$.

C5) Let $\alpha \in \Delta_{{}^s\lambda}$ and $\Theta \in \mathrm{St}_G({}^s\lambda)$. If α is imaginary, we have α is compact with respect to Θ if and only if $\psi(\alpha)$ is compact with respect to $\tilde{\Psi}(\Theta)$. If α is real, we have α satisfying the parity condition with respect to Θ if and only if $\psi(\alpha)$ satisfies the parity condition with respect to $\tilde{\Psi}(\Theta)$.

C6) $\tilde{\Psi}$ is compatible with the cross-actions. Namely, for $w \in W_{{}^s\lambda}$ and $\Theta \in \mathrm{St}_G({}^s\lambda)$ we have $\psi_\sharp(w) \times \tilde{\Psi}(\Theta) = \tilde{\Psi}(w \times \Theta)$.

C7) $\tilde{\Psi}$ is compatible with the Cayley transform. Namely, if $\Theta \in \mathrm{St}_G({}^s\lambda)$ and if $\alpha \in \Delta_{{}^s\lambda}$ is noncompact imaginary with respect to Θ, then we have $\tilde{\Psi}(c_\alpha(\Theta)) = c_{\psi(\alpha)}(\tilde{\Psi}(\Theta))$. Moreover, if $\Theta \in \mathrm{St}_G({}^s\lambda)$ and if $\alpha \in \Delta_{{}^s\lambda}$ is real and satisfies the parity condition with respect to Θ, we have $\tilde{\Psi}(c^\alpha_\pm(\Theta)) = c^{\psi(\alpha)}_\pm(\tilde{\Psi}(\Theta))$.

Put $\Lambda' = \psi({}^s\lambda) + \mathcal{P}_{G'}$. Since $\mathrm{St}_G({}^s\lambda)$ (respectively $\mathrm{St}_{G'}(\psi({}^s\lambda))$) forms a basis of $\mathcal{C}(\Lambda)$ (respectively $\mathcal{C}(\Lambda')$), $\tilde{\Psi}$ in condition C3 extends to a linear (respectively $\mathbb{C}[q]$-module) isomorphism of $\mathcal{C}(\Lambda)$ (respectively $\mathcal{C}(\Lambda)_q$) onto $\mathcal{C}(\Lambda')$ (respectively $\mathcal{C}(\Lambda')$). We denote these isomorphisms of complex vector spaces and $\mathbb{C}[q]$-modules by the same letter $\tilde{\Psi}$. If we identify $W_{{}^s\lambda}$ and W' via the isomorphism $\psi_\sharp$ in condition C1 above, we can regard $\mathcal{C}(\Lambda')$ (respectively $\mathcal{C}(\Lambda')_q$) as a $W_{{}^s\lambda}$-representation (respectively a $H(W_{{}^s\lambda})$-module).

Examining the definition of the Hecke algebra module structures [Vog83b, page 239], we easily see that conditions C4–C7 imply that $\tilde{\Psi}$ is an $H(W_{{}^s\lambda})$-module isomorphism of $\mathcal{C}(\Lambda)_q$ onto $\mathcal{C}(\Lambda')_q$. From Theorem 4.1.1, we also see that $\tilde{\Psi} : \mathcal{C}(\Lambda) \to \mathcal{C}(\Lambda')$ is an isomorphism between coherent continuation representations.

From Theorem 4.1.2 (the Kazhdan–Lusztig type algorithm for Harish-Chandra modules), we see $\overline{\tilde{\Psi}(\Theta)} = \tilde{\Psi}(\bar{\Theta})$ for all $\Theta \in \mathrm{St}_G({}^s\lambda)$. Here, $\Theta \rightsquigarrow \bar{\Theta}$ is a bijection of $\mathrm{St}_G({}^s\lambda)$ (respectively $\mathrm{St}_{G'}(\psi({}^s\lambda))$) onto $\mathrm{Irr}_G({}^s\lambda)$ (respectively $\mathrm{Irr}_{G'}(\psi({}^s\lambda))$).

Lemma 4.1.3. *In the setting above, let $\eta \in \Lambda$ and let $\Xi \in \mathcal{C}(\Lambda)$. Assume that there exists an irreducible Harish-Chandra $(\mathfrak{g}', K')$-module V' such that $\tilde{\Psi}(\Xi)(\psi(\eta)) = [V']$. Then, there is some irreducible Harish-Chandra $(\mathfrak{g}, K)$-module V with the infinitesimal character η such that $\Xi(\eta) = [V]$.*

Proof. There is some $w \in W_{{}^s\lambda}$ such that $\langle \alpha, w\eta \rangle \geqslant 0$ for all $\alpha \in \Delta^+_{{}^s\lambda}$. We write $w\Xi = \sum_{\bar{\Theta} \in \mathrm{Irr}_G({}^s\lambda)} c_{\bar{\Theta}} \bar{\Theta}$. Since $\tilde{\Psi}(\Xi)(\psi(\eta)) = \tilde{\Psi}(w\Xi)(\psi(w\eta))$, we have $[V'] = \sum_{\bar{\Theta} \in \mathrm{Irr}_G({}^s\lambda)} c_{\bar{\Theta}} \tilde{\Psi}(\bar{\Theta})(\psi(w\eta))$. It is known that there is a unique $\bar{\Upsilon}_0 \in \mathrm{Irr}_{G'}({}^s\lambda)$ such that $\bar{\Upsilon}_0(\psi(w\eta)) = [V']$ (cf. [Vog82a, Theorem 7.2.7]). Put $\bar{\Theta}_0 = \tilde{\Psi}^{-1}(\bar{\Upsilon})$. For any $\bar{\Theta} \in \mathrm{Irr}_{G'}({}^s\lambda)$ either $\bar{\Theta}(\psi(w\eta)) = 0$ or $\bar{\Theta}(\psi(w\eta)) = [X]$ for some irreducible Harish-Chandra module X (cf. [Vog83b, Theorem 7.6]). Hence, we have $c_{\bar{\Theta}_0} = 1$ and if $c_{\bar{\Theta}} \neq 0$ and $\bar{\Theta} \neq \bar{\Theta}_0$ then $\tilde{\Psi}(\bar{\Theta})(\psi(w\eta)) = 0$. From [Vog83b, Theorem 7.6] (also see [Vog83b, Definition 5.3]), the above conditions C1–C7 imply that $\tilde{\Psi}(\bar{\Theta})(\psi(w\eta)) = 0$ if and only if $\bar{\Theta}(w\eta) = 0$ for all $\bar{\Theta} \in \mathrm{Irr}_G({}^s\lambda)$. Hence, we have $\bar{\Theta}(w\eta) = 0$ if $c_{\bar{\Theta}} \neq 0$ and $\bar{\Theta} \neq \bar{\Theta}_0$. Moreover, there is an irreducible Harish-Chandra $(\mathfrak{g}, K)$-module V such that $\bar{\Theta}_0(w\lambda) = [V]$. Therefore $\Xi(\eta) = (w\Xi)(w\eta) = \sum_{\bar{\Theta} \in \mathrm{Irr}_G({}^s\lambda)} c_{\bar{\Theta}} \bar{\Theta}(\psi(w\eta)) = \bar{\Theta}_0(w\eta) = [V]$. ◻

4.2 Standard parabolic subgroups

In this section, let G be either $\mathrm{Sp}(n-q, q)$ with $2q \leqslant n$ or $\mathrm{SO}^*(2n)$. Fix θ, ${}^sH$, etc. as in § 3.1.

We also fix some particular orthonormal basis $e_1, \ldots, e_n$ of ${}^s\mathfrak{h}^*$, as in § 3.1. We fix a simple system Π of $\Delta(\mathfrak{g}, {}^s\mathfrak{h})$ as in § 3.1.

Let $\kappa = (k_1, \ldots, k_s)$ be a finite sequence of positive integers such that

$$k_1 + \cdots + k_s \leqslant \begin{cases} q & \text{if } G = \mathrm{Sp}(p,q), \\ n/2 & \text{if } G = \mathrm{SO}^*(2n). \end{cases}$$

We put $k_i^* = k_1 + \cdots + k_i$ for $1 \leqslant i \leqslant s$ and $k_0^* = 0$. If $G = \mathrm{Sp}(p,q)$, put $p' = p - k_s^*$ and $q' = q - k_s^*$. If $G = \mathrm{SO}^*(2n)$, put $r = n - 2k_s^*$. We put $A_i = \sum_{j=1}^{2k_i} E_{k_{i-1}^*+j} (1 \leqslant i \leqslant s)$. Then we have $\theta(A_i) = -A_i$ for $1 \leqslant i \leqslant s$. We denote by $\mathfrak{a}_\kappa$ the Lie subalgebra of ${}^s\mathfrak{h}$ spanned by $\{A_i \mid 1 \leqslant i \leqslant s\}$. We define a subset $S(\kappa)$ of Π as follows. If $G = \mathrm{Sp}(p,q)$, we define

$$S(\kappa) = \begin{cases} \Pi - \{e_{2k_i^*} - e_{2k_i^*+1} \mid 1 \leqslant i \leqslant s\} & \text{if } p' > 0, \\ \Pi - (\{e_{2k_i^*} - e_{2k_i^*+1} \mid 1 \leqslant i \leqslant s-1\} \cup \{2e_n\}) & \text{if } p' = 0. \end{cases}$$

If $G = \mathrm{SO}^*(2n)$, we define

$$S(\kappa) = \begin{cases} \Pi - \{e_{2k_i^*} - e_{2k_i^*+1} \mid 1 \leqslant i \leqslant s\} & \text{if } r > 0, \\ \Pi - (\{e_{2k_i^*} - e_{2k_i^*+1} \mid 1 \leqslant i \leqslant s-1\} \cup \{e_{n-1} + e_n\}) & \text{if } r = 0. \end{cases}$$

We denote by M_κ (respectively $\mathfrak{m}_\kappa$) the centralizer of $\mathfrak{a}_\kappa$ in G (respectively $\mathfrak{g}$). M_κ is a Levi subgroup of G. Let P_κ be a parabolic subgroup of G whose θ-invariant Levi part is M_κ. We denote by N_κ the nilradical of P_κ. We denote by $\mathfrak{p}_\kappa$, $\mathfrak{m}_\kappa$, and $\mathfrak{n}_\kappa$ the complexified Lie algebra of P_κ, M_κ, and N_κ, respectively. We choose P_κ so that $\{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subseteq \mathfrak{n}_\kappa\} \subseteq \Delta^+$. Formally, we denote by $\mathrm{Sp}(0,0)$ and $\mathrm{SO}^*(0)$ the trivial group $\{1\}$ and we denote by $\mathrm{GL}(\kappa, \mathbb{H})$ a product group $\mathrm{GL}(k_1, \mathbb{H}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{H})$. Then, we have

$$M_\kappa \cong \begin{cases} \mathrm{GL}(\kappa, \mathbb{H}) \times \mathrm{Sp}(p', q') & \text{if } G = \mathrm{Sp}(p,q), \\ \mathrm{GL}(\kappa, \mathbb{H}) \times \mathrm{SO}^*(2r) & \text{if } G = \mathrm{SO}^*(2n). \end{cases}$$

Often, we identify $\mathrm{GL}(\kappa, \mathbb{H})$, $\mathrm{Sp}(p', q')$, and $\mathrm{SO}^*(2r)$ with subgroups of M_κ in obvious ways. We call such identifications the standard identifications. The Cartan involution θ induces Cartan involutions on M_κ, $\mathrm{GL}(\kappa, \mathbb{H})$, $\mathrm{Sp}(p', q')$, and $\mathrm{SO}^*(2r)$ and we denote them by the same letter θ. We put $M_\kappa^\circ = \mathrm{Sp}(p', q')$ if $G = \mathrm{Sp}(p,q)$ and put $M_\kappa^\circ = \mathrm{SO}^*(2r)$ if $G = \mathrm{SO}^*(2n)$. We denote by $\mathfrak{m}_\kappa^\circ$ the complexified Lie algebra of M_κ°.

For $\tau \in \mathfrak{S}_s$ and $\kappa = (k_1, \ldots, k_s)$, we define $\kappa^\tau = (k_{\tau(1)}, \ldots, k_{\tau(s)})$. Let ξ be an irreducible unitary representation of M_κ. Then ξ can be written as $\xi = \xi_1 \boxtimes \cdots \boxtimes \xi_s \boxtimes \xi_0$, where for $1 \leqslant i \leqslant s$ (respectively for $i = 0$) ξ_i is an irreducible unitary representation of $\mathrm{GL}(k_i, \mathbb{H})$ (respectively M_κ°). For $\tau \in \mathfrak{S}_s$, we denote by ξ^τ an irreducible unitary representation of L_{κ^τ}, $\xi_{\tau(1)} \boxtimes \cdots \boxtimes \xi_{\tau(s)} \boxtimes \xi_0$. The following is a special case of a well-known result of Harish-Chandra.

Lemma 4.2.1 (Harish-Chandra). *Let $\kappa = (k_1, \ldots, k_s)$ and $\tau \in \mathfrak{S}_s$ be as above. Let ξ be an irreducible unitary representation of M_κ. Then we have $\mathrm{Ind}_{P_\kappa}^G(\xi) \cong \mathrm{Ind}_{P_{\kappa^\tau}}^G(\xi^\tau)$.*

Let $A_k(\ell, t)$ $(\ell \in \{\ell' \in \mathbb{Z} \mid \ell' \geqslant -k\} \cup \{-\infty\})$ be the representation of $\mathrm{GL}(n, \mathbb{H})$ as in Definition 2.4.3. If $\ell \geqslant -k$, $A_k(\ell, t)$ is a quaternionic Speh representation in the weakly fair range. $A_k(-\infty, t)$ is a unitary one-dimensional representation.

Any derived functor module is a parabolic induction from an external tensor product of some $A_k(\ell, t)$'s. So, the unitarily induced module from a derived functor module (in the weakly fair range) can be written as

$$\mathrm{Ind}_{P_\kappa}^G(A_{k_1}(\ell_1, t_1) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s, t_s) \boxtimes Z). \tag{19}$$

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Here, Z is a derived functor module of M°_κ in the weakly fair range. Moreover, $\ell_i \in \{\ell \in \mathbb{Z} \mid \ell \geqslant -k_i\} \cup \{-\infty\}$, and $t_i \in \sqrt{-1}\,\mathbb{R}$ for $1 \leqslant i \leqslant s$. Using the well-known Harish-Chandra result, we may assume $\sqrt{-1}\,t_i \geqslant 0$ for all $1 \leqslant i \leqslant s$.

We assume that $\ell_i + 1 \in 2\mathbb{Z}$ and $t_i = 0$ for some $1 \leqslant i \leqslant s$. Then, using Lemma 4.2.1, we may assume $i = s$. Let $\kappa' = (k_1, \dots, k_{s-1})$. Then from induction by stages, we have

$$\begin{aligned}&\mathrm{Ind}^G_{P_\kappa}(A_{k_1}(\ell_1,t_1) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s,t_s) \boxtimes Z)\\ &\qquad \cong \mathrm{Ind}^G_{P'_\kappa}(A_{k_1}(\ell_1,t_1) \boxtimes \cdots \boxtimes A_{k_{s-1}}(\ell_{s-1},t_{s-1}) \boxtimes \mathrm{Ind}^{M^\circ_{\kappa'}}_{P_{(k_s)}}(A_{k_s}(\ell_s,0) \boxtimes Z)).\end{aligned}$$

Applying the decomposition formula, Theorem 3.6.5, we see that the above induced module is a direct sum of the induced modules of the form like

$$\mathrm{Ind}^G_{P'_\kappa}(A_{k_1}(\ell_1,t_1) \boxtimes \cdots \boxtimes A_{k_{s-1}}(\ell_{s-1},t_{s-1}) \boxtimes Z').$$

Here, Z' is a derived functor module of $M^\circ_{\kappa'}$ in the weakly fair range. Assume that we understand the reducibility of Z''s. Then, applying the above argument, we can reduce the irreducible decomposition of (19) to the following:

$$\mathrm{Ind}^G_{P_\kappa}(A_{k_1}(\ell_1,0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h,0) \boxtimes A_{k_{h+1}}(\ell_{h+1},t_{h+1}) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s,t_s) \boxtimes Z). \tag{20}$$

Here, ℓ_i is not an odd integer if $1 \leqslant i \leqslant h$, $\sqrt{-1}\,t_i > 0$ if $h < i \leqslant s$, and Z is an irreducible representation of M°_κ whose infinitesimal character is in $\mathcal{P}_{M^\circ_\kappa}$. Put $\tau = (k_1, \dots, k_h)$ and $\tau' = (k_{h+1}, \dots, k_s)$. Also put $a = k_1 + \cdots + k_h$ and $b = k_{h+1} + \cdots + k_s$.

We now state the main result of § 4.

Theorem 4.2.2. *The following are equivalent:*

i) *The above (20) is irreducible.*

ii) *The following induced module (21) is irreducible:*

$$\mathrm{Ind}^{\mathrm{SO}^*(4a)}_{P_\tau}(A_{k_1}(\ell_1,0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h,0)). \tag{21}$$

Remark. Under an appropriate regularity condition on $\ell_1, \dots, \ell_h$, we may apply Proposition 3.3.2 to (21) successively, and we obtain that (21) is a good-range elliptic cohomological induction from an irreducible module like $I_{k_1}(\ell'_1,0) \boxtimes \cdots \boxtimes I_{k_h}(\ell'_h,0)$. Hence (21) is irreducible for such parameters.

In § 5, we show (21) is irreducible if $\ell_1, \dots, \ell_h$ are all $-\infty$.

4.3 Proof of Theorem 4.2.2

We denote by ${}^s\mathfrak{h}_\kappa$ (respectively ${}^s\mathfrak{h}^\kappa$) the $\mathbb{C}$-linear span of $E_1, \dots, E_{2k^*_s}$ (respectively $E_{2k^*_s+1}, \dots, E_n$). Then, we can regard ${}^s\mathfrak{h}_\kappa$ (respectively ${}^s\mathfrak{h}^\kappa$) as the complexified Lie algebra of a θ-invariant maximally split Cartan subgroup of $\mathrm{GL}(\kappa,\mathbb{H})$ (respectively $\mathrm{Sp}(p',q')$ or $\mathrm{SO}^*(2r)$) via the standard identification. We have a direct sum decomposition ${}^s\mathfrak{h} = {}^s\mathfrak{h}_\kappa \oplus {}^s\mathfrak{h}^\kappa$ and it induces ${}^s\mathfrak{h}^* = {}^s\mathfrak{h}^*_\kappa \oplus ({}^s\mathfrak{h}^\kappa)^*$. Namely, we identify ${}^s\mathfrak{h}^*_\kappa$ (respectively ${}^s\mathfrak{h}^*_r$) the $\mathbb{C}$-linear span of $e_1, \dots, e_{2k^*_s}$ (respectively $e_{2k^*_s+1}, \dots, e_n$).

We denote by ρ the half-sum of the roots in Δ^+. Let $\eta \in {}^s\mathfrak{h}^*$ be the infinitesimal character of $A_{k_1}(\ell_1,0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h,0) \boxtimes A_{k_{h+1}}(\ell_{h+1},t_{h+1}) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s,t_s) \boxtimes Z$. We may (and do) assume that $\mathrm{Re}(\eta)$ is in the closed Weyl chamber with respect to $\Delta^+ \cap \Delta(\mathfrak{m}_\kappa, {}^s\mathfrak{h})$.

We fix a sufficiently large integer N and we put ${}^s\lambda = 2N\rho + \eta$ and $\Lambda = {}^s\lambda + \mathcal{P}_G$. Then, we have $\eta \in \Lambda$. Hence, $\Delta_\eta = \Delta_{{}^s\lambda}$. Moreover, we have ${}^s\lambda$ is regular and $\Delta^+_{{}^s\lambda} = \Delta^+ \cap \Delta_{{}^s\lambda}$.

We construct a surgroup G' of M_κ as follows. As a Lie group, G' is a product group $\mathrm{SO}^*(4a) \times \mathrm{GL}(b,\mathbb{H}) \times M^\circ_\kappa$. The embedding of $M_\kappa = \mathrm{GL}(\tau,\mathbb{H}) \times \mathrm{GL}(\tau',\mathbb{H}) \times M^\circ_\kappa$ into G' is induced from the inclusions $\mathrm{GL}(\tau,\mathbb{H}) \subseteq \mathrm{SO}^*(4a)$ and $\mathrm{GL}(\tau',\mathbb{H}) \subseteq \mathrm{GL}(b,\mathbb{H})$. We easily see that we may fix a Cartan involution whose restriction to M_κ is θ. We denote such a Cartan involution on G' by the same letter θ and denote by K' the corresponding maximal compact subgroup. Since M_κ is a Levi subgroup of both G and G', ${}^sH$ is a θ-stable maximally split Cartan subgroup of G' as well as G.

We denote by $\mathfrak{g}'$ the complexified Lie algebra of G' and denote by Δ' the root system for $(\mathfrak{g}', {}^s\mathfrak{h})$. From the construction of G', we have that the integral root system $\Delta_{{}^s\lambda}$ coincides with $\Delta'_{{}^s\lambda}$.

We want to apply Lemma 4.1.3 to G, G', and ${}^s\lambda$ above. In our setting, we put ${}^sH' = {}^sH$ and ${}^s\mathfrak{h}' = {}^s\mathfrak{h}$ and put ψ in condition C1 to be the identity map. Hereafter, we denote by $G^\sharp$ any of G and G'. Similarly, we write $K^\sharp$, etc.

In order to define Ψ and $\tilde{\Psi}$, we describe conjugacy classes of Cartan subgroups in G and G'.

First, we remark that there is one-to-one correspondence between $G^\sharp$-conjugacy classes of Cartan subgroups in $G^\sharp$ and $K^\sharp$-conjugacy classes of θ-stable Cartan subgroups in $G^\sharp$ [Mat79]. Second, a G-conjugacy class of Cartan subgroups of G is determined by the dimension of the split part and $\mathrm{GL}(k, \mathbb{H})$ has a unique G-conjugacy class of Cartan subgroups (cf. [Sug59]). Hence, we see that a K-conjugacy class (respectively a K'-conjugacy class) of θ-stable (respectively θ'-stable) Cartan subgroups of G (respectively G') is determined by the dimension of the split part. We also see that the same statement holds for M_κ.

Since there is obvious one-to-one correspondence between the conjugacy classes of Cartan subgroups and the conjugacy classes of the Cartan subalgebras which are stable with respect to the complex conjugation, hereafter we consider Cartan subalgebras rather than Cartan subgroups. In order to understand the Cayley transforms on Cartan subalgebras, we examine some particular Cartan subalgebras as follows. Let m be the greatest positive integer which is equal to or less than $h/2$. For $1 \leqslant i \leqslant m$, we put $\alpha_i = e_{2i-1} + e_{2i}$. Then, $\{\alpha_1, \ldots, \alpha_m\}$ is the entire collection of real roots in Δ^+. We define $c^{\alpha_i} \in G_{\mathbb{C}}$ as in § 4.1. Since $\alpha_1, \ldots, \alpha_m$ are mutually orthogonal, we may regard α_i as a real root for $\mathrm{Ad}(c^{\alpha_j})({}^s\mathfrak{h})$. So, we can regard $\mathrm{Ad}(c^{\alpha_i})(\mathrm{Ad}(c^{\alpha_j})({}^s\mathfrak{h}))$ as a result of successive applications of Cayley transforms to ${}^s\mathfrak{h}$. Because of the orthogonality of α_i and α_j, we see $\mathrm{Ad}(c^{\alpha_i})(\mathrm{Ad}(c^{\alpha_j})({}^s\mathfrak{h})) = \mathrm{Ad}(c^{\alpha_j})(\mathrm{Ad}(c^{\alpha^i})({}^s\mathfrak{h}))$.

Let $J = \{\alpha_{r_1}, \ldots, \alpha_{r_k}\} \subseteq \{\alpha_1, \ldots, \alpha_m\}$. Here, we assume $r_i \neq r_j$ for $i \neq j$. Similarly as above, we can define successive applications of Cayley transforms as follows:

$$\mathfrak{h}_J = \mathrm{Ad}(c^{\alpha_{r_k}})(\mathrm{Ad}(c^{\alpha_{r_{k-1}}})(\cdots(\mathrm{Ad}(c^{\alpha_{r_1}})({}^s\mathfrak{h})\cdots))).$$

Here $\mathfrak{h}_J$ depends only on J and it is σ- and θ-stable. We denote by H_J the corresponding Cartan subgroup of G to $\mathfrak{h}_J$.

Put $J_0 = \{\alpha_{k_s^*+1}, \ldots, \alpha_m\}$. If $J \subseteq J_0$, then $H_J \subseteq M_\kappa$ and H_J is a θ-stable Cartan subgroup of M_κ.

Since a $K^\sharp$-conjugacy (respectively $K \cap M_\kappa$-conjugacy) class of θ-stable Cartan subgroups of $G^\sharp$ (respectively M_κ) is determined by the dimension of the split part, for $J_1, J_2 \subseteq J_0$, the following statements are equivalent:

a) H_{J_1} is K-conjugate to H_{J_2}.

b) H_{J_1} is K'-conjugate to H_{J_2}.

c) H_{J_1} is $K \cap M_\kappa$-conjugate to H_{J_2}.

d) Card $J_1 =$ card J_2.

If $J \subseteq J_0$, H_J is ${}^s\lambda$-integral with respect to both G and G'. Conversely, it is easy to check that any ${}^s\lambda$-integral θ-stable Cartan subgroup of $G^\sharp$ is $K^\sharp$-conjugate to H_J for some $J \subseteq J_0$. (For example, using a criterion for the parity condition [Vog82a], we may check that α_i satisfies the parity condition with respect to ${}^s\lambda$ if and only if $\alpha_i \in J_0$. The statement is deduced from this fact.) We also remark that any θ-stable Cartan subgroup of M_κ is $K \cap M_\kappa$-conjugate to some H_J with $J \subseteq J_0$.

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Hence, there is a bijection Φ (respectively Φ') of the set of the $K \cap M_\kappa$-conjugacy classes of θ-stable Cartan subgroups of M_κ to the set of K-conjugacy (respectively K'-conjugacy) classes of ${}^s\lambda$-integral θ-stable Cartan subgroups of G (respectively G'). In fact Φ (respectively Φ') is defined such that the image of the K-conjugacy class of H_J under Ψ is the K'-conjugacy class of H_J for any $J \subseteq J_0$. We put $\Psi = \Phi' \circ \Phi^{-1}$. Ψ is a bijection of the set of the K-stable conjugacy classes of ${}^s\lambda$-integral θ-stable Cartan subgroups of G to the set of the K'-conjugacy classes of ${}^s\lambda$-integral θ'-stable Cartan subgroups of G'. Ψ is compatible with Cayley transforms on (conjugacy classes of) Cartan subgroups, since Φ and Φ' are.

Next, we consider the lift of Ψ to the standard coherent families.

We put $J(i) = \{\alpha_m, \alpha_{m-1}, \dots, \alpha_{m-i+1}\}$ for $1 \leqslant i \leqslant m - k_s^*$ and $J(0) = \emptyset$. Put $H_i = H_{J(i)}$ for $0 \leqslant i \leqslant m - k_s^*$. Then, we easily see $H_1, \dots, H_{m-k_s^*}$ form a complete system of representatives of the $K^\sharp$-conjugacy (respectively $K \cap M_\kappa$-conjugacy) classes of θ-stable Cartan subgroups of $G^\sharp$ (respectively M_κ). We denote by $\mathfrak{h}_i$ the complexified Lie algebra of H_i and by $W(\mathfrak{g}^\sharp, \mathfrak{h}_i)$ the Weyl group for $(\mathfrak{g}^\sharp, \mathfrak{h}_i)$. We denote by $W(G^\sharp; H_i)$ the subgroup of $W(\mathfrak{g}^\sharp, \mathfrak{h}_i)$ consisting of the elements of $W(\mathfrak{g}^\sharp, \mathfrak{h}_i)$ whose representatives can be chosen in $G^\sharp$. We collect some of the useful facts below.

Lemma 4.3.1. *For $1 \leqslant i \leqslant m - k_s^*$, we have*

i) $W(\mathfrak{m}_\kappa, \mathfrak{h}_i) \subseteq W(\mathfrak{g}', \mathfrak{h}_i) \subseteq W(\mathfrak{g}, \mathfrak{h}_i)$,

ii) $W(G'; H_i) = W(\mathfrak{g}', \mathfrak{h}_i) \cap W(G; H_i)$,

iii) $R_{M_\kappa}(H_i, {}^s\lambda) \subseteq R_{G'}(H_i, {}^s\lambda) \subseteq R_G(H_i, {}^s\lambda)$.

Part i is easy to see from our construction of G'. Part ii is easily checked using [Vog82b, Proposition 4.16]. Part iii follows from part i.

We define $\tilde{\Omega} : \mathrm{St}_{G'}({}^s\lambda) \to \mathrm{St}_G({}^s\lambda)$ by $\tilde{\Omega}(\Theta_\gamma^{G'}) = \Theta_\gamma^G$ for $\gamma \in R_{G'}(H_i, {}^s\lambda)$ for $1 \leqslant i \leqslant m - k_s^*$.

We have remarked in § 1.3, case iii, that for $\gamma_1 = (H_i, \lambda_1),\ \gamma_2 = (H_i, \lambda_2) \in R_{G^\sharp}(H_i, {}^s\lambda)$, the following statements are equivalent:

a) γ_1 and γ_2 are $K^\sharp$-conjugate.

b) There is some $w \in W(G^\sharp; H_i)$ such that $\lambda_1 = w\lambda_2$.

c) $\Theta_{\gamma_1}^{G^\sharp} = \Theta_{\gamma_2}^{G^\sharp}$.

Hence, from parts ii and iii of Lemma 4.2.1, we see that $\tilde{\Omega}$ is well defined.

We have our next result.

Lemma 4.3.2. *$\tilde{\Omega}$ is bijective.*

Proof. From Lemma 4.3.1, part ii, and the above remark, we see that the regularity of ${}^s\lambda$ implies the injectivity of $\tilde{\Omega}$. So, we show the surjectivity.

First, we fix some $1 \leqslant i \leqslant m - k_s^*$. Then $\mathrm{Ad}(c^{\alpha_{k_s^*+i}}) \circ \mathrm{Ad}(c^{\alpha_{k_s^*+i-1}}) \circ \cdots \circ \mathrm{Ad}(c^{\alpha_{k_s^*+1}})$ induces a linear isomorphism of $\mathfrak{h}$ onto $\mathfrak{h}_i$. So, we also have an isomorphism $\mathfrak{h}^* \cong \mathfrak{h}_i^*$. We denote by $\bar{e}_1, \dots, \bar{e}_n \in \mathfrak{h}_i^*$ the image of $e_1, \dots, e_n \in \mathfrak{h}^*$ under this isomorphism. Then the Cartan involution acts on $\bar{e}_1, \dots, \bar{e}_n$ as follows:

$$\theta(\bar{e}_{2i-1}) = -\bar{e}_{2i}, \theta(\bar{e}_{2i}) = -\bar{e}_{2i-1} \quad (1 \leqslant i \leqslant m - i),$$
$$\theta(\bar{e}_i) = \bar{e}_i \quad (2(m-i) < i \leqslant n).$$

We also denote by $\lambda \in \mathfrak{h}_i^*$ the image of ${}^s\lambda$ under this isomorphism. Write $\lambda = \sum_{j=1}^n \ell_j \bar{e}_j$. Let $w \in W(\mathfrak{g}, \mathfrak{h}_i)$ and write $\lambda = \sum_{j=1}^n \bar{\ell}_j \bar{e}_j$. Then $\bar{\ell}_1, \dots, \bar{\ell}_n$ is made from $\ell_1, \dots, \ell_n$ by a permutation of their indices and sign flips. We assume that $\gamma_w = (H_i, w\lambda) \in R_G({}^s\lambda)$. Then, it should satisfy the condition R5 in § 1.3. So, we easily see that:

d1) $\bar{\ell}_j \in \mathbb{Z}$ $(2(m-i) < j \leqslant n)$,

d2) $\bar{\ell}_{2j-1} - \bar{\ell}_{2j} \in \mathbb{Z}$ $(1 < j \leqslant m-i)$.

We write $\sum_{k=1}^{n} a_i \bar{e}_i \in \mathfrak{h}_i^*$ by $(a_1, \ldots, a_n)$.

From [Vog82b, Proposition 4.86], we easily see that the following elements in $W(\mathfrak{g}, \mathfrak{h}_i)$ are contained in $W(G; H_i)$:

$$w_j(a_1, \ldots, a_n) = (a_1, \ldots, a_{2j-2}, -a_{2j-1}, -a_{2j}, a_{2j+2}, \ldots, a_n) \quad (1 \leqslant j \leqslant m-i),$$

$$w_{b,c}(a_1, \ldots, a_n) = (a_1, \ldots, a_{2b-2}, a_{2c-1}, a_{2c}, a_{2b+1}, \ldots, a_{2c-2}, a_{2b-1}, a_{2b}, a_{2c+1}, \ldots, a_n) \qquad (1 \leqslant b < c \leqslant m-i).$$

If we choose the product w^* of suitable w_j's and $w_{b,c}$'s above, we may have that $w^*\lambda = (d_1, \ldots, d_n)$ satisfies:

e1) $d_j \in \mathbb{Z}$ for all $k_s^* < j \leqslant n$;

e2) $d_j \notin \mathbb{R}$ for all $k_h^* < j \leqslant k_s^*$;

e3) $d_j - \frac{1}{2} \in \mathbb{Z}$ for all $1 \leqslant j \leqslant k_h^*$.

This means that $\gamma' = (H_i, w^* w\lambda) \in R_{G'}({}^s\lambda)$ and $\Theta_{\gamma_w}^G = \Theta_{\gamma'}^G$. Hence $\tilde{\Omega}$ is surjective. □

We define $\tilde{\Psi} : \mathrm{St}_{G'}({}^s\lambda) \to \mathrm{St}_G({}^s\lambda)$ by the inverse of $\tilde{\Omega}$. From the above constructions, we easily obtain the following lemma.

Lemma 4.3.3. *ψ, Ψ, and $\tilde{\Psi}$ defined above satisfy conditions C1–C7 in § 4.1.*

Now, we finish the proof of Theorem 4.2.2. If $\gamma \in R_{M_\kappa}({}^s\lambda)$, then, taking account of $\gamma \in R_{G^\sharp}({}^s\lambda)$, we easily see the following:

f1) $\Theta_\gamma^{G'} = \mathrm{Ind}_{M_\kappa}^{G'}(\Theta_\gamma^{M_\kappa})$,

f2) $\Theta_\gamma^{G} = \mathrm{Ind}_{M_\kappa}^{G}(\Theta_\gamma^{M_\kappa})$,

f3) $\tilde{\Psi}(\Theta_\gamma^G) = \Theta_\gamma^{G'}$.

Taking account of the additivity of induction, we see that for all $\bar{\Theta} \in \mathrm{Irr}_{M_\kappa}({}^s\lambda)$ we have $\tilde{\Psi}(\mathrm{Ind}_{M_\kappa}^{G}(\bar{\Theta})) = \mathrm{Ind}_{M_\kappa}^{G'}(\bar{\Theta})$. It is easy to see that there is some $\bar{\Theta} \in \mathrm{Irr}_{M_\kappa}({}^s\lambda)$ such that $\bar{\Theta}(\eta) = [A_{k_1}(\ell_1, 0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h, 0) \boxtimes A_{k_{h+1}}(\ell_{h+1}, t_{h+1}) \boxtimes \cdots \boxtimes A_{k_s}(\ell_s, t_s) \boxtimes Z]$. Hence, Lemma 4.1.3 implies that the irreducibility of (20) is reduced to the irreducibility of a Harish-Chandra module which is the external product of the following:

g1) Z, which is an irreducible Harish-Chandra module for M_κ°;

g2) $\mathrm{Ind}_{P_\tau}^{\mathrm{SO}^*(4a)}(A_{k_1}(\ell_1, 0) \boxtimes \cdots \boxtimes A_{k_h}(\ell_h, 0))$;

g3) Harish-Chandra modules for $\mathrm{GL}(b, \mathbb{H})$ induced from irreducible unitary representations of their parabolic subgroups.

The irreducibilities of g3 are found in [Vog86, page 502]. This completes the proof of Theorem 4.2.2.

5. Irreducibility representations $\mathrm{SO}^*(2n)$ and $\mathrm{Sp}(p,q)$ parabolically induced from one-dimensional unitary representations

5.1 Some induced representations of $\mathrm{SO}^*(4m)$

In this section we retain the notations in §§ 3.1 and 4.2, and consider the case of $G = \mathrm{SO}^*(2n)$. Moreover, we assume n is even. So, we write $n = 2m$.

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Since the universal covering group of $G_{\mathbb{C}}$ is a double cover, $\mathcal{P}_G$ (cf. § 1.2) is a subgroup of $\mathcal{P}$ of index 2. Put $\Lambda = \mathcal{P} - \mathcal{P}_G$ (set theoretical difference). Λ is the $\mathcal{P}_G$ coset in $\mathcal{P}$ other than $\mathcal{P}_G$ itself. We fix a regular weight ${}^s\lambda \in \Lambda$ as follows:

$$ {}^s\lambda = \sum_{i=1}^{n} \frac{2n-2i+1}{2} e_i. $$

Hereafter, we simply write $W = W(\mathfrak{g}, {}^s\mathfrak{h})$ and $\Delta = \Delta(\mathfrak{g}, {}^s\mathfrak{h})$. We have $W = W_{{}^s\lambda}$, $\Delta = \Delta_{{}^s\lambda}$, and

$$ \Pi_{{}^s\lambda} = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}. $$

Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ such that ${}^s\mathfrak{h} \subseteq \mathfrak{b}$ and the nilradical of $\mathfrak{b}$ is the sum of the root spaces corresponding to the roots in $\Delta^+_{{}^s\lambda}$. We denote by ρ the half-sum of the positive roots in $\Delta^+_{{}^s\lambda}$.

We consider a partition $\pi = (p_1, \ldots, p_k)$ of a positive integer m such that $0 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k$ and $p_1 + p_2 + \cdots + p_k = m$. Let $\mathrm{PT}(m)$ be the set of partitions of m. As in § 4.2, we consider the standard parabolic subgroup P_π and its Levi subgroup M_π of G corresponding to π.

Let $(\sigma^\pi_\lambda, \mathbb{C}^\pi_\lambda)$ be a one-dimensional unitary representation of M_π (or $\mathfrak{m}_\pi$) such that the restriction to ${}^s\mathfrak{h}$ of the differential of σ^π_λ is $\lambda \in {}^s\mathfrak{h}^*$.

We denote by ρ_π the half-sum of all the positive roots whose root space is in $\mathfrak{m}_\pi$. We put $\rho^\pi = \rho - \rho_\pi$. The infinitesimal character of $\mathrm{Ind}^G_{P_\pi}(\mathbb{C}^\pi_\lambda)$ is $\rho_\pi + \lambda$.

It is easy to construct a non-degenerate $\mathfrak{g}$-invariant pairing between $\mathrm{Ind}^G_{P_\pi}(\mathbb{C}^\pi_\lambda)$ and a generalized Verma module $M_{\mathfrak{p}_\pi}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\pi)} \mathbb{C}^\pi_{-\lambda-\rho^\pi}$.

We are going to show the following as our main result.

Lemma 5.1.1. *Let π be any partition of m. Then, $\mathrm{Ind}^G_{P_\pi}(\mathbb{C}^\pi_0)$ is irreducible.*

We prove this lemma in § 5.3.

Combining Lemma 5.1.1 and Theorem 4.2.2, we have the following corollary.

Corollary 5.1.2. *Representations of $\mathrm{SO}^*(2n)$ and $\mathrm{Sp}(p,q)$ induced from one-dimensional unitary representations of their parabolic subgroups are irreducible.*

5.2 Coherent continuation representation for $\mathrm{SO}^*(4m)$ with respect to Λ

We retain the notations in § 5.1.

For a partition $\pi = (p_1, \ldots, p_k) \in \mathrm{PT}(m)$ of m, put $p^*_i = \sum_{j=1}^{i} p_j$ for $1 \leqslant i \leqslant k$ and define a subset S_π of $\Pi = \Pi_{{}^s\lambda}$ (P_π is the standard parabolic subgroup corresponding to S_π) as follows:

$$ S_\pi = \Pi - (\{e_{2p^*_i} - e_{2p^*_i+1} \mid 1 \leqslant i \leqslant k-1\} \cup \{e_{2m-1} + e_{2m}\}). $$

For $\pi \in \mathrm{PT}(m)$, we denote by σ_π the MacDonald representation (cf. [Car85, page 368]) of W with respect to $S_\pi \subseteq \Pi$. From [LS79], σ_π is a special representation ([Lus79], [Lus82] also see [Car85, page 374]), which corresponds to the Richardson orbit in $\mathfrak{g}$ with respect to the parabolic subalgebra $\mathfrak{p}_\pi$ via the Springer correspondence.

There is another description of σ_π. Since W is the Weyl group of type D_{2m}, it is embedded into the Weyl group W' of type B_{2m}. It is well known that the irreducible representations of W' is parameterized by the pairs of partitions (κ, ω) such that $\kappa \in \mathrm{PT}(k)$ and $\omega \in \mathrm{PT}(2m-k)$ for some $0 \leqslant k \leqslant 2m$. Here, we regard $\mathrm{PT}(0)$ as consisting of the empty partition $\emptyset$. If $\kappa \neq \omega$, then the restriction of the representation corresponding to (κ, ω) is irreducible. However, the restriction of the irreducible W'-representation corresponding to (π, π) ($\pi \in \mathrm{PT}(m)$) to W is decomposed into two irreducible W-representations, which are equidimensional. From [Car85, page 423, lines 11–33], σ_π is one of the irreducible constituents.

For each partition $\kappa \in \mathrm{PT}(k)$, we denote by $\dim(\kappa)$ the dimension of the irreducible representation of $\mathfrak{S}_k$ corresponding to κ. It is well known that the dimension of the irreducible W'-representation corresponding to (κ, ω) ($\kappa \in \mathrm{PT}(k)$ and $\omega \in \mathrm{PT}(2m-k)$) is $(2m)! \dim(\kappa) \dim(\omega)/[k!(2m-k)!]$. (For example, see [Ker71].) So, we have the following lemma.

Lemma 5.2.1. *For $\pi \in \mathrm{PT}(m)$,*

$$\dim(\sigma_\pi) = \frac{(2m)! \dim(\pi)^2}{2(m!)^2}.$$

We shall show the following theorem.

Theorem 5.2.2. *As a W-module the coherent continuation representation $\mathcal{C}(\Lambda)$ is decomposed as*

$$\mathcal{C}(\Lambda) \cong \bigoplus_{\pi \in \mathrm{PT}(m)} \sigma_\pi.$$

First, we prove the next result.

Lemma 5.2.3. *For each $\pi \in \mathrm{PT}(m)$, the multiplicity of σ_π in $\mathcal{C}(\Lambda)$ is at least one.*

Proof. We have only to show that there is an irreducible Harish-Chandra $(\mathfrak{g}, K)$-module V such that the infinitesimal character of V is in Λ and the character polynomial of V [Kin81] generates a W-representation isomorphic to σ_π. First, we remark that $\mathrm{Ind}_{P_\pi}^G(\mathbb{C}^\pi_{{}^s\lambda-\rho_\pi})$ has a non-degenerate pairing with an irreducible generalized Verma module $M_{\mathfrak{p}_\pi}(-{}^s\lambda - \rho_\pi)$ with the infinitesimal character $-{}^s\lambda$. We easily see that there is at least one irreducible constituent V of $\mathrm{Ind}_{P_\pi}^G(\mathbb{C}^\pi_{{}^s\lambda-\rho_\pi})$ whose annihilator I in $U(\mathfrak{g})$ is the dual of the annihilator of the generalized Verma module. So the associated variety of I is the closure of the Richardson orbit (say $\mathcal{O}_\pi$) corresponding to $\mathfrak{p}_\pi$. The character polynomial with respect to V is proportional to the Goldie rank polynomial of I [Kin81] and the W-representation generated by the Goldie rank polynomial is σ_π. So, we obtain the lemma. □

Proof of Theorem 5.2.2. From Lemma 5.2.3, it suffices to show that

$$\dim \mathcal{C}(\Lambda) = \sum_{\pi \in \mathrm{PT}(m)} \dim(\sigma_\pi).$$

From Lemma 5.2.1, the right-hand side is

$$\sum_{\pi \in \mathrm{PT}(m)} \frac{(2m)!(\dim(\pi))^2}{2(m!)^2} = \frac{(2m)!}{2 \cdot m!},$$

since we have

$$\sum_{\pi \in \mathrm{PT}(m)} (\dim(\pi))^2 = \mathrm{card}\, \mathfrak{S}_m = m!.$$

So, we have to show that $\dim \mathcal{C}(\Lambda) = (2m)!/(2 \cdot m!)$.

Clearly, $\dim \mathcal{C}(\Lambda)$ is the number of K-conjugacy classes in the regular characters in $R_G({}^s\lambda)$. Since only maximally split Cartan subgroups are ${}^s\lambda$-integral, each K-conjugacy class has a representative in $R_G({}^sH, {}^s\lambda)$. We denote by $W(G; {}^sH)$ the subgroup of W consisting of the elements w of W such that some representative of w in $G_{\mathbb{C}}$ is in G (or equivalently in K) and normalizes ${}^sH$. Examining elements in K which preserve ${}^sH$, we easily see that $\dim \mathcal{C}(\Lambda) = \mathrm{card}(W/W(G; {}^sH))$. From [Kna75] (also see [Vog82b, Proposition 4.16]), $W(G; {}^sH)$ is generated by the following elements in W:

i) $s_{e_{2i-1}-e_{2i}} (1 \leqslant i \leqslant m)$: reflections with respect to compact imaginary roots in $\Delta = \Delta(\mathfrak{g}, {}^s\mathfrak{h})$;
ii) $s_{e_{2i-1}+e_{2i}} (1 \leqslant i \leqslant m)$: reflections with respect to real roots in Δ;
iii) $s_{e_{2i-1}-e_{2j-1}} s_{e_{2i}-e_{2j}} (1 \leqslant i < j \leqslant m)$.

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So, we can easily see $W(G;{}^sH)$ is isomorphic to $\mathfrak{S}_m \times ((\mathbb{Z}/2\mathbb{Z})^m \times (\mathbb{Z}/2\mathbb{Z})^m)$.

So, we have

$$\dim \mathcal{C}(\Lambda) = \frac{\operatorname{card} W}{\operatorname{card} W(G;{}^sH)} = \frac{(2m)! \cdot 2^{2m-1}}{m! \cdot 2^m \cdot 2^m} = \frac{(2m)!}{2 \cdot m!}$$

as desired. □

We can interpret this in terms of cell structure of the coherent continuation representation $\mathcal{C}(\Lambda)$ (see [BV83, McG98, Vog82b]).

A $W_{^s\lambda}$-subrepresentation of $\mathcal{C}(\Lambda)$ is called basal, if it is generated by a subset of $\mathrm{Irr}_G({}^s\lambda)$ as a $\mathbb{C}$-vector space. For $\gamma \in R_G({}^s\lambda)$, we denote by $\mathrm{Cone}(\gamma)$ the smallest basal subrepresentation of $\mathcal{C}(\Lambda)$ which contains $\bar{\Theta}^G_\gamma$. For $\gamma, \eta \in R_G({}^s\lambda)$, we write $\gamma \sim \eta$ (respectively $\gamma \leqslant \eta$) if $\mathrm{Cone}(\gamma) = \mathrm{Cone}(\eta)$ (respectively $\mathrm{Cone}(\gamma) \supseteq \mathrm{Cone}(\eta)$). Obviously $\sim$ is an equivalence relation on $R_G({}^s\lambda)$. For $\gamma \in R_G({}^s\lambda)$ let $s(\gamma)$ be the set of regular characters $\eta \in R_G({}^s\lambda)$ such that $\lambda \leqslant \eta$ and $\lambda \not\sim \eta$. We define $\mathrm{Cell}(\gamma) = \mathrm{Cone}(\gamma)/\sum_{\eta \in s(\gamma)} \mathrm{Cone}(\eta)$. A cell (respectively cone) for $\mathcal{C}(\Lambda)$ is a subquotient (respectively subrepresentation) of $\mathcal{C}(\Lambda)$ of the form $\mathrm{Cell}(\gamma)$ (respectively $\mathrm{Cone}(\gamma)$) for some $\gamma \in R_G({}^s\lambda)$.

For each cell, we can associate a nilpotent orbit in $\mathfrak{g}$ as follows. For $\mathrm{Cell}(\gamma)$, we consider an irreducible Harish-Chandra $(\mathfrak{g}, K)$-module $\bar{\pi}(\gamma)$. The annihilator (say I) of $\bar{\pi}(\gamma)$ in $U(\mathfrak{g})$ is a primitive ideal of $U(\mathfrak{g})$ and its associated variety is the closure of a single nilpotent orbit in $\mathfrak{g}$. The nilpotent orbit constructed above is independent of the choice of γ and we call it the associated nilpotent orbit for $\mathrm{Cell}(\gamma)$. For each $\mathrm{Cone}(\gamma)$, there is a canonical (up to scalar factor) $W_{^s\lambda}$-homomorphism (say ϕ_γ) of $\mathrm{Cone}(\gamma)$ to the realization as a Goldie rank polynomial representation of the special $W_{^s\lambda}$-representation corresponding to the associated nilpotent orbit via the Springer correspondence. In fact this ϕ_γ factors to $\mathrm{Cell}(\gamma)$. An important fact is that $\phi_\gamma(\bar{\Theta}^G_\eta)$ is non-zero and proportional to the Goldie rank polynomial of the annihilator of $\bar{\pi}(\eta)$ in $U(\mathfrak{g})$ for all $\eta \sim \gamma$ [Jos80, Kin81]. Hence, the multiplicity in $\mathrm{Cell}(\gamma)$ of the special W-representation corresponding to the associated nilpotent orbit via the Springer correspondence is at least one. McGovern proved that if G is a classical group then the multiplicity of the special representation is exactly one (cf. [McG98, Theorem 1]).

In our particular setting, the proof of Lemma 5.2.3 tells us that for each $\pi \in \mathrm{PT}(m)$ there is at least one cell whose associated nilpotent orbit is $\mathcal{O}_\pi$.

From Theorem 5.2.2, we obtain a corollary.

Corollary 5.2.4.

i) *There is a one-to-one correspondence between the set of cells for $\mathcal{C}(\Lambda)$ and $\mathrm{PT}(m)$ induced from the above association of nilpotent orbits to cells.*
ii) *Each cell for $\mathcal{C}(\Lambda)$ is irreducible and isomorphic to the special representation corresponding to the associated nilpotent orbit via the Springer correspondence.*

Harish-Chandra cells for classical groups are precisely studied by McGovern [McG98]. Almost all cases are treated in his paper. For type D groups there are some exceptions (cf. [McG98, page 224]). Our result can be regarded as a supplement of his result.

From Corollary 5.2.4, we have yet another corollary.

Corollary 5.2.5. *Let $\lambda \in \Lambda$ and let V_i $(i = 1, 2)$ be irreducible Harish-Chandra $(\mathfrak{g}, K)$-modules with an infinitesimal character λ. Assume that the annihilator of V_1 in $U(\mathfrak{g})$ coincides with that of V_2. Then, V_1 is isomorphic to V_2.*

Proof. We may assume that $\langle \lambda, \alpha \rangle \geqslant 0$ for all $\alpha \in \Delta^+_{^s\lambda}$. It is known that for each $i = 1, 2$ there is a unique coherent family $\bar{\Theta}^G_{\gamma_i} \in \mathrm{Irr}_G({}^s\lambda)$ such that $[V_i] = \bar{\Theta}^G_{\gamma_i}(\lambda)$. We show $\bar{\Theta}^G_{\gamma_1} = \bar{\Theta}^G_{\gamma_2}$.

First, we remark that the Goldie rank polynomial and the associated variety of the annihilator of V_i in $U(\mathfrak{g})$ coincide with those of $\bar{\pi}(\gamma_i)$ for each i. Hence, we have $\gamma_1 \sim \gamma_2$ since there is at most one cell whose associated nilpotent orbit is the unique dense orbit in the associated variety of V_i. We consider the homomorphism $\phi_{\gamma_1}(=\phi_{\gamma_2})$ mentioned above. Since $\phi_{\gamma_1}(\bar{\Theta}^G_{\gamma_i})$ is non-zero and proportional to the Goldie rank polynomial of the annihilator of V_i in $U(\mathfrak{g})$ for each $i=1,2$, $\bar{\Theta}^G_{\gamma_1}$ is proportional to $\bar{\Theta}^G_{\gamma_2}$ modulo the kernel of ϕ_{γ_1}. Since $\mathrm{Cell}(\gamma_1)=\mathrm{Cell}(\gamma_2)$ is irreducible, ϕ_{γ_1} induces an isomorphism of $\mathrm{Cell}(\gamma_1)$ to the corresponding Goldie rank polynomial representation. This means that $\bar{\Theta}^G_{\gamma_1}$ is proportional to $\bar{\Theta}^G_{\gamma_2}$ modulo the subspace of $\mathrm{Cone}(\gamma_1)$ generated as a $\mathbb{C}$-vector space by $\bar{\Theta}^G_{\eta}$ such that $\eta \geqslant \gamma_1$ and $\eta \not\sim \gamma_1$. Since $\mathrm{Irr}_G({}^s\lambda)$ is a basis of $\mathcal{C}(\Lambda)$, we have $\bar{\Theta}^G_{\gamma_1}=\bar{\Theta}^G_{\gamma_2}$ as desired. □

5.3 Proof of Lemma 5.1.1

We need the following two lemmas.

Lemma 5.3.1. *The annihilator of* $\mathrm{Ind}^G_{P_\pi}(\mathbb{C}^\pi_0)$ *in* $U(\mathfrak{g})$ *is a maximal ideal for all* $\pi \in PT(m)$.

Remark. In fact, a more general result holds. So, we consider the more general setting temporally. Let G be any connected real semisimple Lie group and P be any parabolic subgroup of G. We denote by M a Levi subgroup of P. We denote by $\mathfrak{g}$, $\mathfrak{m}$, and $\mathfrak{p}$ the complexified Lie algebras of G, M, and P, respectively. We denote by $\mathfrak{n}$ the nilradical of $\mathfrak{p}$. We denote by 1_M the trivial representation of M.

Lemma 5.3.2. *The annihilator of* $\mathrm{Ind}^G_P(1_M)$ *in* $U(\mathfrak{g})$ *is a maximal ideal.*

As far as I know, such a result has not been published but is known by experts (at least including D. A. Vogan). For the convenience of readers, we give a proof here.

Proof. We denote by $\mathbb{C}_{-\rho^P}$ a one-dimensional representation of $\mathfrak{p}$ defined by $\mathfrak{p} \ni X \rightsquigarrow -\frac{1}{2}\mathrm{tr}$ $(\mathrm{ad}(X)|_{\mathfrak{n}})$. From the existence of non-degenerate pairing, it suffices to show that the annihilator of a generalized Verma module $M_{\mathfrak{p}}(0)=U(\mathfrak{g})\otimes_{U(\mathfrak{p})}\mathbb{C}_{-\rho^P}$ is maximal. We denote by I the annihilator of $M_{\mathfrak{p}}(0)$ in $U(\mathfrak{g})$. We define

$$L(M_{\mathfrak{p}}(0),M_{\mathfrak{p}}(0))=\{\phi\in \mathrm{End}_{\mathbb{C}}(M_{\mathfrak{p}}(0)) \mid \dim \mathrm{ad}(U(\mathfrak{g}))\phi<\infty\}.$$

$L(M_{\mathfrak{p}}(0),M_{\mathfrak{p}}(0))$ has an obvious $U(\mathfrak{g})$-bimodule structure. Then, $L(M_{\mathfrak{p}}(0),M_{\mathfrak{p}}(0))$ is isomorphic to a Harish-Chandra module of an induced representation of $G_{\mathbb{C}}$ from a unitary one-dimensional representation of $P_{\mathbb{C}}$. Hence, $L(M_{\mathfrak{p}}(0),M_{\mathfrak{p}}(0))$ is completely reducible as a $U(\mathfrak{g})$-bimodule. Considering the action of $U(\mathfrak{g})$ on $M_{\mathfrak{p}}(0)$, we have an embedding of a $U(\mathfrak{g})$-bimodule $U(\mathfrak{g})/I \hookrightarrow L(M_{\mathfrak{p}}(0),M_{\mathfrak{p}}(0))$. Hence, $U(\mathfrak{g})/I$ is also completely reducible as a $U(\mathfrak{g})$-bimodule. We consider the unit element 1 of $U(\mathfrak{g})/I$. Then, $\mathbb{C}1$ is the unique trivial $\mathrm{ad}(U(\mathfrak{g}))$-type in $U(\mathfrak{g})/I$. So, the unit 1 must be contained in some irreducible component of $U(\mathfrak{g})/I$. Since $U(\mathfrak{g})/I$ is generated by 1 as a $U(\mathfrak{g})$-bimodule, $U(\mathfrak{g})/I$ is irreducible as a $U(\mathfrak{g})$-bimodule. This means that I is maximal. □

Proof of Lemma 5.1.1. From Corollary 5.2.5 and Lemma 5.3.1, we see that all the irreducible constituents of $\mathrm{Ind}^G_{P_\pi}(\mathbb{C}^\pi_0)$ are isomorphic to each other. However, the multiplicity of the trivial K-representation in $\mathrm{Ind}^G_{P_\pi}(\mathbb{C}^\pi_0)$ is just one. Hence $\mathrm{Ind}^G_{P_\pi}(\mathbb{C}^\pi_0)$ is irreducible as we desired. □

References

ABV92 J. Adams, D. Barbasch and D. A. Vogan Jr., *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics, vol. 104 (Birkhäuser, Boston, 1992).

BV83 D. Barbasch and D. A. Vogan Jr., *Weyl group representations and nilpotent orbits*, in *Representation theory of reductive groups*, Progress in Mathematics, vol. 40, ed. P. C. Trombi (Birkhäuser, Boston, 1983), 21–33.

Car85 R. W. Carter, *Finite groups of Lie type*, Wiley Series in Pure and Applied Mathematics (John Wiley and Sons, Chichester, 1985).

Enr79 T. J. Enright, *Relative Lie algebra cohomology and unitary representations of complex Lie groups*, Duke Math. J. **46** (1979), 513–525.

Hel70 S. Helgason, *A duality for symmetric spaces with applications to group representations*, Adv. Math. **5** (1970), 1–154.

HT93 R. Howe and E. Tan, *Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations*, Bull. Amer. Math. Soc. **28** (1993), 1–74.

Joh90 K. Johnson, *Degenerate principal series and compact groups*, Math. Ann. **287** (1990), 703–718.

Jos80 A. Joseph, *Goldie rank in the enveloping algebra of a semisimple Lie algebra II*, J. Algebra **65** (1980), 284–306.

KV79 M. Kashiwara and M. Vergne, *Functions on the Shilov boundary of the generalized half plane*, in *Non-commutative harmonic analysis*, Lecture Notes in Mathematics, vol. 728 (Springer-Verlag, Berlin, 1979).

Ker71 A. Kerber, *Representations of permutation group 1,2*, Lecture Notes in Mathematics, vols 240 and 495 (Springer-Verlag, Berlin, 1971, 1975).

Kin81 D. King, *The character polynomial of the annihilator of an irreducible Harish-Chandra module*, Amer. J. Math. **103** (1981), 1195–1240.

Kna75 Knapp, A. W. *Weyl group of a cuspidal parabolic*, Ann. Sci. École Norm. Sup. **8** (1975), 275–294.

Kna86 A. W. Knapp, *Representation theory of semisimple groups, an overview based on examples*, Princeton Mathematical Series, vol. 36 (Princeton University Press, Lawrenceville, NJ, 1986).

KV95 A. W. Knapp and D. A. Vogan Jr., *Cohomological induction and unitary representations*, Princeton Mathematical Series, vol. 45 (Princeton University Press, Lawrenceville, NJ, 1995).

Kob92 T. Kobayashi, *Singular unitary representations and discrete series for indefinite Stiefel manifolds* $U(p,q;\mathbb{F})/U(p-m,q;\mathbb{F})$, Memoirs of Amer. Math. Soc., vol. 95, no. 462 (American Mathematical Society, Providence, RI, 1992).

Kos69 B. Kostant, *On the existence and irreducibility of certain series of representations*, Bull. Amer. Math. Soc. **75** (1969), 627–642.

Lus79 G. Lusztig, *A class of irreducible representations of a Weyl group*, Indag. Math. **41** (1979), 323–335.

Lus82 G. Lusztig, *A class of irreducible representations of a Weyl group II*, Indag. Math. **44** (1982), 219–226.

LS79 G. Lusztig and N. Spaltenstein, *Induced unipotent classes*, J. London Math. Soc. **19** (1979), 41–52.

Mat79 T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), 331–357.

Mat96 H. Matumoto, *On the representations of* $U(m,n)$ *unitarily induced from derived functor modules*, Compositio Math. **100** (1996), 1–39.

McG98 W. M. McGovern, *Cells of Harish-Chandra modules for real classical groups*, Amer. J. Math. **120** (1998), 211–228.

Sah93 S. Sahi, *Unitary representations on the Shilov boundary of a symmetric tube domain*, in *Representation theory of groups and algebras*, Contemporary Mathematics, vol. 145 (American Mathematical Society, Providence, RI, 1993), 275–286.

Sch88 W. Schmid, *Geometric constructions of representations*, Adv. Stud. Pure Math. **14** (1988), 349–368.

Spe83 B. Speh, *Unitary representations of* $\mathrm{GL}(n,\mathbb{R})$ *with non-trivial* $(\mathfrak{g},K)$*-cohomology*, Invent. Math. **71** (1983), 443–465.

SV80 B. Speh and D. A. Vogan Jr., *Reducibility of generalized principal series representations*, Acta Math. **145** (1980), 227–299.

Sug59 M. Sugiura, *Conjugate classes of Cartan subalgebras in real semisimple Lie algebras*, J. Math. Soc. Japan **11** (1959), 374–434.

Tra01 P. E. Trapa, *Annihilators and associated varieties of $A_{\mathfrak{q}}(\lambda)$ modules for $U(p,q)$*, Compositio Math. **129** (2001), 1–45.

Vog82a D. A. Vogan Jr., *Representations of real reductive Lie groups*, Progress in Mathematics, vol. 15 (Birkhäuser, Boston, 1982).

Vog82b D. A. Vogan Jr., *Irreducible characters of semisimple Lie groups IV: character-multiplicity duality*, Duke Math. J. **49** (1982), 943–1073.

Vog83a D. A. Vogan Jr., *Irreducible characters of semisimple Lie groups III: proof of Kazhdan–Lusztig conjectures in the integral case*, Invent. Math. **71** (1983), 381–417.

Vog83b D. A. Vogan Jr., *The Kazhdan–Lusztig conjecture for real reductive groups*, in *Representation theory of reductive groups*, Progress in Mathematics, vol. 40 (Birkhäuser, 1983), 223–264.

Vog84 D. A. Vogan Jr., *Unitarizability of certain series of representations*, Ann. of Math. (2) **120** (1984), 141–187.

Vog86 D. A. Vogan Jr., *The unitary dual of $GL(n)$ over an archimedean field*, Invent. Math. **83** (1986), 449–505.

Vog88 D. A. Vogan Jr., *Irreducibilities of discrete series representations for semisimple symmetric spaces*, Adv. Stud. Pure Math. **14** (1988), 381–417.

Vog97 D. A. Vogan Jr., *Cohomology and group representations*, in *Representation theory and automorphic forms (Edinburgh, 1996)*, Proc. Symp. Pure Math., vol. 61 (American Mathematical Society, Providence, RI, 1997), 219–243.

VZ84 D. A. Vogan Jr. and G. Zuckerman, *Unitary representations with non-zero cohomology*, Compositio Math. **53** (1984), 51–90.

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