THE POSITIVE VALUES OF INHOMOGENEOUS TERNARY QUADRATIC FORMS

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1. Let $f(x, y, z)$ be an indefinite ternary quadratic form of signature (2,1) and determinant $d \neq 0$. Davenport [3] has shown that there exist integral x, y, z with

$$0 < f(x, y, z) \leqq (4|d|)^{\frac{1}{3}}, \tag{1.1}$$

the equality sign being necessary if and only if f is a positive multiple of $f_1(x, y, z) = x^2 + yz$.

I consider here the problem of finding the best possible result of this form,

$$0 < f(x, y, z) \leqq (\lambda|d|)^{\frac{1}{3}} \tag{1.2}$$

(the exponent $\frac{1}{3}$ being dictated by homogeneity) when x, y, z are congruent modulo 1 to arbitrarily assigned real numbers x_0, y_0, z_0.

Since the choice $x_0, y_0, z_0 = 0, 0, 0$ gives integral x, y, z, it is clear from Davenport's result that any result (1.2) must have $\lambda \geqq 4$. I prove that in fact (1.2) is always soluble when $\lambda = 4$, and that the equality sign is also necessary for a further class of forms.

THEOREM. *If $f(x, y, z)$ has signature (2,1) and determinant $d \neq 0$, then for any x_0, y_0, z_0, there exist $x, y, z \equiv x_0, y_0, z_0 \pmod 1$ satisfying*

$$0 < f(x, y, z) \leqq (4|d|)^{\frac{1}{3}}. \tag{1.3}$$

The equality sign is necessary if and only if f is equivalent to a positive multiple of either

$$f_1(x, y, z) = x^2 + yz \quad \textit{with} \quad x_0, y_0, z_0 \equiv 0, 0, 0 \pmod 1, \quad \textit{or} \tag{1.4}$$

$$f_2(x, y, z) = x^2 + y^2 - 2z^2 \quad \textit{with} \quad x_0, y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \pmod 1. \tag{1.5}$$

For the proof, we begin by choosing coprime integral x, y, z satisfying (1.1). By applying a suitable integral unimodular transformation to the variables (and also to x_0, y_0, z_0), we can ensure that

$$f(1, 0, 0) = a, \quad 0 < a \leqq (4|d|)^{\frac{1}{3}}. \tag{1.6}$$

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Completing the square, we write

$$g(x, y, z) = \frac{1}{a} f(x, y, z) = (x + \beta y + \gamma z)^2 + h(y, z), \tag{1.7}$$

say, where β, γ are real and h is indefinite.

Let h have discriminant $\Delta^2 (\Delta > 0)$, so that, by (1.6),

$$\Delta^2 = \frac{4|d|}{a^3} \geqq 1.$$

We write for convenience

$$k = \Delta^{\frac{2}{3}} \geqq 1, \tag{1.8}$$

$$\kappa = \begin{cases} [k] & \text{if } k \text{ is not integral} \\ k - 1 & \text{if } k \text{ is integral,} \end{cases} \tag{1.9}$$

so that $k - 1 \leqq \kappa < k$. We note for later reference that, by Davenport's result, we may suppose that $k > 1$ if f is not equivalent to a positive multiple of f_1.

Now we may satisfy (1.2) with strict inequality and $\lambda = 4$ if and only if we can satisfy

$$0 < g(x, y, z) < k \quad \text{with} \quad x, y, z \equiv x_0, y_0, z_0 \pmod 1. \tag{1.10}$$

2. The next step in the proof is to show that (1.10) is soluble if we can choose $y, z \equiv y_0, z_0$ so that $h(y, z)$ lies in a certain interval. We need the following simple result:

LEMMA 1. *For any real λ, there exists an integer u with*

$$0 < (u + \lambda)^2 - \alpha \leqq p(\alpha), \tag{2.1}$$

where

$$p(\alpha) = \begin{cases} \frac{1}{4} - \alpha & \textit{if} \quad \alpha < 0 & \text{(2.2a)} \\ \left(\frac{n}{2} + 1\right)^2 - \alpha & \textit{if} \quad \left(\frac{n}{2}\right)^2 \leqq \alpha < \left(\frac{n+1}{2}\right)^2 \quad (n = 0, 1, 2, \cdots). & \text{(2.2b)} \end{cases}$$

For each α, the equality sign is needed in (2.1) *if and only if*

$$\lambda \equiv \tfrac{1}{2} \pmod 1 \quad \textit{if} \ \text{(2.2a)} \ \textit{holds}, \tag{2.3a}$$

$$\lambda \equiv \frac{n}{2} \pmod 1 \quad \textit{if} \ \text{(2.2b)} \ \textit{holds}. \tag{2.3b}$$

PROOF. The result, with the gloss (2.3a), is obvious if $\alpha < 0$, since we may choose u with $|u + \lambda| \leqq \frac{1}{2}$.

Suppose then that $\alpha \geqq 0$, and let n be the non-negative integer satisfying (2.2b). We choose the integer u with

$$\tfrac{1}{2}(n + 1) \leqq |u + \lambda| \leqq \tfrac{1}{2}(n + 2)$$

and immediately obtain (2.1); there is inequality in (2.1) unless $|u + \lambda| = \frac{1}{2}n + 1$, i.e. unless (2.3b) holds.

Conversely, if (2.3b) holds, then for all integral u

$$\text{either } |u+\lambda| \leqq \tfrac{1}{2}n \qquad \text{or} \quad |u+\lambda| \geqq \tfrac{1}{2}n + 1,$$

whence

$$\text{either } (u+\lambda)^2 - \alpha \leqq 0 \quad \text{or} \quad (u+\lambda)^2 - \alpha \geqq p(\alpha).$$

The equality sign in (2.1) is therefore necessary when (2.3b) holds.

LEMMA 2. *Suppose that* $k \geqq 1$ *and that* α *satisfies*

$$\tfrac{1}{4} - k < \alpha < \tfrac{1}{4}\kappa^2, \tag{2.4}$$

where κ *is defined by* (1.9). *Then, in the notation of Lemma* 1,

$$p(\alpha) < k. \tag{2.5}$$

PROOF. If $\alpha < 0$, then $p(\alpha) = \frac{1}{4} - \alpha < k$.

If $\alpha \geqq 0$, choose the integer n to satisfy (2.2b). Then

$$\tfrac{1}{4}n^2 \leqq \alpha < \tfrac{1}{4}\kappa^2, \quad n < \kappa;$$

since n and κ are both integral, it follows that $n + 1 \leqq \kappa$. Hence, by (2.2b),

$$p(\alpha) = (\tfrac{1}{2}n + 1)^2 - \alpha \leqq (\tfrac{1}{2}n + 1)^2 - \tfrac{1}{4}n^2 = n + 1 \leqq \kappa;$$

this implies (2.5), since always $\kappa < k$.

The following result now reduces our problem to one involving inhomogeneous binary forms:

LEMMA 3. *The inequality* (1.10) *is soluble if there exist* $y, z \equiv y_0, z_0 \pmod 1$ *satisfying*

$$-\tfrac{1}{4}\kappa^2 < h(y, z) < k - \tfrac{1}{4}. \tag{2.6}$$

PROOF. We choose $y, z \equiv y_0, z_0$ to satisfy (2.6), and write

$$h(y, z) = -\alpha, \quad x = u + x_0.$$

Then α satisfies (2.4) and

$$g(x, y, z) = (x + \beta y + \gamma z)^2 + h(y, z) = (u + \lambda)^2 - \alpha,$$

where $\lambda = x_0 + \beta y + \gamma z$ and u is integral when $x \equiv x_0 \pmod 1$. It now follows at once from Lemmas 1 and 2 that we may choose u so that (1.10) holds.

3. Turning now to the question of the solubility of (2.6) we prove

LEMMA 4. *The inequality* (2.6), *where* $h(y, z)$ *is an indefinite form of discriminant* $k^3 (k \geqq 1)$, *is soluble unless either* $k = 1$ *or*

$$k = 2 \quad \textit{and} \quad h(y, z) \sim y^2 - 2z^2 \quad \textit{with} \quad y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1. \tag{3.1}$$

PROOF. We may suppose that $k > 1$, so that $\kappa \geqq 1$. We use Theorems 1 and 2 of Blaney [2], which state (in a slightly different notation) that the inequality

$$(3.2) \qquad -1 < \phi(y, z) < m, \quad y, z \equiv y_0, z_0 \pmod 1,$$

where $\phi(y, z)$ has discriminant $\Delta^2 > 0$, is soluble if

$$(3.3) \qquad m > 0, \quad \Delta^2 < 16m$$

or if

$$(3.4) \qquad m \geqq 3, \quad \Delta^2 \leqq (m+1)(m+9),$$

unless $m = 4n - 1$ for some integer $n \geqq 1$, $\Delta^2 = (m+1)(m+9)$, and $\phi(y, z)$ is equivalent to

$$(3.5) \qquad 2ny^2 - 2(n+2)z^2 \quad \text{with} \quad y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1.$$

Now (2.6) may be written in the form (3.2) with

$$\phi(y, z) = \frac{4}{\kappa^2} h(y, z), \qquad m = \frac{4k - 1}{\kappa^2};$$

and, since h has discriminant k^3, ϕ has discriminant $\Delta^2 = 16k^3/\kappa^4$.

Using (3.3), we see that therefore (2.6) is soluble if

$$(3.6) \qquad k^3 < \kappa^2(4k - 1).$$

If $2 < k \leqq 3$, then $\kappa = 2$, and (3.6) is easily verified; if $k > 3$, the stronger inequality $k^3 < (k-1)^2(4k-1)$ is easily found to hold. Thus (2.6) is certainly soluble if $k > 2$.

If $1 < k \leqq 2$, we have

$$(3.7) \qquad \kappa = 1, \quad m = 4k - 1, \quad \phi(y, z) = 4h(y, z).$$

Since now $m > 3$, we may use the criterion (3.4), which reduces to

$$16k^3 \leqq 4k(4k + 8),$$

i.e.

$$k^2 \leqq k + 2, \quad (k-2)(k+1) \leqq 0,$$

and this holds, with strict inequality, unless $k = 2$. Hence (2.6) is also soluble if $1 < k \leqq 2$ unless $k = 2$ and, from (3.5) with $n = k = 2$,

$$\phi(y, z) \sim 4y^2 - 8z^2$$

$$h(y, z) \sim y^2 - 2z^2 \quad \text{with} \quad y_0, z_0 \equiv \tfrac{1}{2}, \tfrac{1}{2} \pmod 1.$$

This completes the proof of the lemma.

4. Proof of the Theorem. Lemmas 3 and 4 assert that, apart from the two exceptional cases stated, we may satisfy (1.10), and so (1.1) with strict inequality. It remains to show that the exceptional cases arise only when f is equivalent to a positive multiple of one of the forms (1.4), (1.5), and that then (1.3) may be satisfied (but not with strict inequality).

Suppose first that $k = 1$; then, as noted in § 1, $f(x, y, z)$ is equivalent to a positive multiple of $f_1(x, y, z) = x^2 + yz$. Without loss of generality, we may suppose that

$$f(x, y, z) = f_1(x, y, z) = x^2 + yz,\ d = -\tfrac{1}{4}.$$

For any y_0, z_0, we may choose $y, z \equiv y_0, z_0 \pmod 1$ so that

$$-\tfrac{1}{2} < y \leqq \tfrac{1}{2}, \quad -\tfrac{1}{2} < z \leqq \tfrac{1}{2}$$

whence

$$-\tfrac{1}{4} < yz \leqq \tfrac{1}{4}.$$

If now $-\frac{1}{4} < yz \leqq 0$, we choose $x \equiv x_0 \pmod 1$ with $\frac{1}{2} \leqq |x| \leqq 1$, and

$$0 < f \leqq 1;$$

if $0 < yz \leqq \frac{1}{4}$, we choose $x \equiv x_0 \pmod 1$ with $|x| \leqq \frac{1}{2}$, and then

$$0 < f \leqq \tfrac{1}{2};$$

thus in either case we may satisfy

$$0 < f \leqq 1 = (4|d|)^{\frac{1}{3}}.$$

Clearly the equality sign is necessary only if $yz = 0$, $x = 1$, i.e. when $x_0, y_0, z_0 \equiv 0, 0, 0 \pmod 1$. But in this case x, y, z are integral, so that clearly (1.3) cannot be satisfied with strict inequality. This disposes of the exceptional case (1.4) of the theorem.

Suppose next that (3.1) holds so that, after a suitable equivalence transformation, we may take

(4.1) $f(x, y, z) = (x + \beta y + \gamma z)^2 + y^2 - 2z^2,\ y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2} \pmod 1,\ d = -2.$

By a further transformation of the type $x \to x + ky + lz$ (k, l integral), we may suppose that

(4.2) $$0 \leqq \beta < 1, \quad 0 \leqq \gamma < 1.$$

Choosing $y = \frac{3}{2}$, $z = \frac{1}{2}$ and $x \equiv x_0 \pmod 1$ with $|x + \frac{3}{2}\beta + \frac{1}{2}\gamma| \leqq \frac{1}{2}$, we obtain at once

$$\tfrac{7}{4} \leqq f(x, y, z) \leqq 2 = (4|d|)^{\frac{1}{3}},$$

so that (1.3) may certainly be satisfied. Also, the choices

$$y = \pm\tfrac{1}{2}, \quad z = \pm\tfrac{1}{2}, \quad \tfrac{1}{2} \leqq |x \pm \tfrac{1}{2}\beta \pm \tfrac{1}{2}\gamma| \leqq 1$$

(for all combinations of $\pm$) give

$$0 \leqq f(x, y, z) = (x \pm \tfrac{1}{2}\beta \pm \tfrac{1}{2}\gamma)^2 - \tfrac{1}{4} \leqq \tfrac{3}{4} < (4|d|)^{\frac{1}{3}};$$

it follows that, if (1.3) cannot be satisfied with strict inequality,

$$x_0 \pm \tfrac{1}{2}\beta \pm \tfrac{1}{2}\gamma \equiv \tfrac{1}{2} \pmod 1.$$

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From these four relations, with (4.2), we obtain at once

$$\beta = \gamma = 0, \quad x_0 \equiv \tfrac{1}{2} \pmod 1,$$

so that we are led to the exceptional case (1.5) of the theorem.

Finally, if f is the form f_2 of (1.5) with $x_0, y_0, z_0 \equiv \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \pmod 1$, we have

$$4f = X^2 + Y^2 - 2Z^2 \text{ with } X, Y, Z \text{ odd integers;}$$

thus $4f$ is integral and

$$4f \equiv 1 + 1 - 2 \equiv 0 \pmod 8,$$

so that $4f \leqq 0$ or $4f \geqq 8$ for all choices of x, y, z. Hence (1.3) cannot be satisfied with strict inequality.

This completes the proof of the theorem.

5. Conclusion. The result of the theorem represents a first step (possibly the easiest one) in the general problem of finding best-possible asymmetric inequalities for inhomogeneous ternary quadratic forms. The corresponding problem for binary forms is essentially solved (see for example [1]); and by using the general methods above, together with more precise techniques for binary forms, one may expect to establish further results. In particular, it is not difficult to show (by using sharper forms of (1.1) and (3.4)) that the result (1.3) is isolated.

References

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