ABSOLUTE RIESZ SUMMABILITY FACTORS FOR FOURIER SERIES

by PREM CHANDRA

(Received 24th February 1969)

1. Definitions and notation

Let Σa_n be a given infinite series and $\{\lambda_n\}$ a non-negative, strictly increasing, monotonic sequence, tending to infinity with n. We write, for $w > \lambda_0$,

$$A_\lambda(w) = A_\lambda^0(w) = \sum_{\lambda_n \leqq w} a_n;$$

and, for $r > 0$, we write

$$\begin{aligned} A_\lambda^r(w) &= \sum_{\lambda_n \leqq w} (w-\lambda_n)^r a_n \\ &= r\int_{\lambda_0}^{w} A_\lambda(\tau)(w-\tau)^{r-1}d\tau \\ &= \int_{\lambda_0}^{w} (w-\tau)^r dA_\lambda(\tau). \end{aligned}$$

$A_\lambda^r(w)$ is known as the *Riesz sum* of "*type*" λ_n and "*order*" r, and

$$R_\lambda^r(w) = A_\lambda^r(w)/w^r$$

is called the Riesz mean of *type* λ_n and *order* r.

The series Σa_n is said to be summable by Riesz means of type λ_n and order r, or summable (R, λ_n, r), $r \geqq 0$, to sum s (finite), if $R_\lambda^r(w) \to s$ as $w \to \infty$ (see Riesz (5)).

The series Σa_n is said to be absolutely summable (R, λ_n, r), or summable $| R, \lambda_n, r |$, $r > 0$, if $R_\lambda^r(w)$ is of bounded variation in (h, ∞), where h is some finite positive number (see Obrechkoff (3), (4)).

By definition, summability $| R, \lambda_n, 0 |$ is equivalent to absolute convergence.

Throughout this paper $\delta > 0$ and we use the following notation:

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\}. \tag{1.1}$$

$$R(w, t) = \sum_{\exp(n^\alpha) \leqq w} n^{\beta-1} \exp(n^\alpha) \sin nt. \tag{1.2}$$

$$P(w, t) = \int_0^t u^\delta \frac{\partial}{\partial u} R(w, u)du. \tag{1.3}$$

$$Q(w, t) = \int_t^\pi u^\delta \frac{\partial}{\partial u} R(w, u)du. \tag{1.4}$$

2. Introduction

In 1951, Mohanty (2) proved the following:

Theorem A. *If* $\phi(t) \in BV(0, \pi)$, *then the series* $\sum_{n=1}^{\infty} A_n(x)/\log(n+1)$ *is summable* $|R, \exp(n^{\alpha}), 1|$ $(0<\alpha<1)$.

Here "$f(x) \in BV(a, b)$" means that $f(x)$ is of bounded variation in (a, b).

Theorem B. *If* $t^{-\delta}\phi(t) \in BV(0, \pi)$, *then at* $t = x$ *the Fourier series of* $f(t)$ *is summable* $|R, \exp\{w(\log w)^{-\beta}\}, 1|$, *where* $\delta>0$ *and* $\beta = 1+\delta^{-1}$.

The purpose of this paper is to further investigate absolute Riesz summability factors of Fourier series, by taking a hypothesis like that of Theorem B and the absolute Riesz summability process of the kind used in Theorem A.

We establish the following result.

Theorem. *Let* $\alpha>0, \beta>0, 1>\alpha+\beta$, *and* $\delta = \beta/(1-\alpha)$. *If* $t^{-\delta}\phi(t) \in BV(0, \pi)$, *then* $\sum_{n=1}^{\infty} A_n(x)n^{\beta}$ *is summable* $|R, \exp(n^{\alpha}), 1|$.

3. We require the following order-estimates for large w, uniformly in $0<t \leqq \pi$:

$$\sum_{\exp(n^{\alpha}) \leqq w} n^{\beta-1}\exp(n^{\alpha}) = O\{w(\log w)^{\beta/\alpha-1}\}. \tag{3.1}$$

$$R(w, t) = O\{t^{-1}w(\log w)^{(\beta-1)/\alpha}\}. \tag{3.2}$$

$$P(w, t) = O\{t^{\delta}w(\log w)^{\beta/\alpha-1}\}. \tag{3.3}$$

$$Q(w, t) = O\{t^{\delta-1}w(\log w)^{(\beta-1)/\alpha}\}. \tag{3.4}$$

For the proof of (3.1) we have

$$R(w, t) = O\{\sum_{\exp(n^{\alpha}) \leqq w} n^{\beta-1}\exp(n^{\alpha})\}.$$

The case $\beta = 0$ has been established in Mohanty (1), and the general case can be proved in a similar way.

For (3.2) let $\exp(m^{\alpha}) \leqq w<\exp\{(m+1)^{\alpha}\}$. We have

$$R(w, t) = \sum_{n=1}^{p-1} n^{\beta-1}\exp(n^{\alpha})\sin nt + \sum_{n=p}^{m} n^{\beta-1}\exp(n^{\alpha})\sin nt$$
$$= U+V, \text{ say,}$$

where p is an integer such that $n^{\beta-1}\exp(n^{\alpha})$ is monotonic increasing for $n \geqq p$, for example, $p = [\{(1-\beta)/\alpha\}^{1/\alpha}]+1$. Then we have

$$U = O(1),$$

and

$$V = \sum_{n=p}^{m} n^{\beta-1}\exp(n^{\alpha})\sin nt$$
$$\leqq (\exp(m^{\alpha})/m^{1-\beta}) \max_{p \leqq m' \leqq m} \left|\sum_{n=m'}^{m} \sin nt\right|$$

by Abel's Lemma,

$$= O\{w(\log w)^{(\beta-1)/\alpha}/t\}.$$

Hence, finally, we have

$$R(w, t) = O\{w(\log w)^{(\beta-1)/\alpha}/t\}.$$

To prove (3.3) we have by the second mean value theorem

$$\begin{aligned} P(w, t) &= t^{\delta} \int_{\eta}^{t} \frac{\partial}{\partial u} R(w, u)du \\ &= t^{\delta}[R(w, t) - R(w, \eta)] \\ &= O\{t^{\delta} \sum_{\exp(n^{\alpha}) \leqq w} n^{\beta} \exp(n^{\alpha})/n\} \\ &= O\{t^{\delta} w(\log w)^{\beta/\alpha - 1}\}, \end{aligned}$$

by (3.1).

Finally for (3.4) we have, integrating by parts,

$$Q(w, t) = [u^{\delta}R(w, u)]_t^{\pi} - \delta \int_t^{\pi} u^{\delta-1}R(w, u)du$$

$$= O\{t^{\delta-1}w(\log w)^{(\beta-1)/\alpha}\} + O\{w(\log w)^{(\beta-1)/\alpha} \int_t^{\pi} u^{\delta-2}du\}$$

by (3.2)

$$= O\{w(\log w)^{(\beta-1)/\alpha}t^{\delta-1}\},$$

uniformly in $0 < t \leqq \pi$.

4. For the proof of the theorem we shall require the following lemmas. For Lemma 1, see Obrechkoff (3), (4).

Lemma 1. *If* Σa_n *is summable* $|R, \lambda_n, r|$, $r \geqq 0$ *then it is also summable* $|R, \lambda_n, r'|$, $r' > r$.

Lemma 2. *The Fourier series of the special function* $|t|^r$ $(r \geqq 0)$, *defined outside* $(-\pi, \pi)$ *by periodicity, is absolutely convergent at* $t = 0$.

We use this lemma only for the case $0 \leqq r < 1$.

Proof. The proof is trivial for $r = 0$, therefore we prove it for $r > 0$.

Let
$$|t|^r \sim \sum_{n=1}^{\infty} \alpha_n \cos nt,$$

where
$$\alpha_n = \frac{2}{\pi} \int_0^{\pi} t^r \cos nt dt.$$

https://doi.org/10.1017/S0013091500009202 Published online by Cambridge University Press

Then, on integrating by parts, we obtain

$$\alpha_n = -\frac{2r}{n\pi}\int_0^{\pi} t^{r-1}\sin ntdt$$

$$= -\frac{2r}{n\pi}\left(\int_0^{n^{-1}} + \int_{n^{-1}}^{\pi}\right) t^{r-1}\sin ntdt$$

$$= -\frac{2r}{n\pi}(I_1+I_2), \text{ say.}$$

Since $|\sin nt| \leqq nt$, we have

$$I_1 = O\left(n\int_0^{n^{-1}} t^r dt\right) = O(n^{-r}).$$

Again, since t^{-1+r} decreases in (n^{-1}, π) in the case $0<r<1$ we have, by the second mean value theorem,

$$I_2 = n^{1-r}\int_{n^{-1}}^{t'} \sin ntdt \quad (n^{-1}<t'<\pi)$$

$$= O(n^{-r}).$$

For $r \geqq 1$ we have, by the second mean value theorem,

$$I_2 = \pi^{r-1}\int_{\eta}^{\pi} \sin ntdt \quad (n^{-1}<\eta<\pi)$$

$$= O(n^{-1}).$$

Thus, finally, we have $\alpha_n = O(n^{-1-r})$ in the case $0<r<1$, and in the case $r \geqq 1$

$$\alpha_n = O(n^{-1-r})+O(n^{-2}) = O(n^{-2}).$$

This proves the lemma.

Lemma 3. *The integral* $I = \int_1^{\infty} w^{-2}\,|P(w, \pi)|\,dw$ *is convergent.*

Proof. We have

$$I = \int_1^{\infty} w^{-2}\left|\sum_{\exp(n^{\alpha}) \leqq w} \exp(n^{\alpha})B_n\right| dw,$$

where $$B_n = n^{\beta}\int_0^{\pi} u^{\delta}\cos nudu = O(n^{-1-\delta-\beta}),$$

by using the arguments of Lemma 2. Since ΣB_n is absolutely convergent, the lemma follows by Lemma 1.

5. Proof of the theorem

We have

$$A_n(x) = \frac{2}{\pi}\int_0^\pi \phi(t)\cos ntdt$$
$$= \frac{2}{\pi}\phi(\pi)\pi^{-\delta}\int_0^\pi u^\delta \cos nudu - \frac{2}{\pi}\int_0^\pi d\{\phi(t)t^{-\delta}\}\int_0^t u^\delta \cos nudu,$$

integrating by parts.

The series $\sum_{n=1}^{\infty} A_n(x)n^\beta$ is summable $|R, \exp(n^\alpha), 1|$ if

$$I = \int_1^\infty w^{-2}\left|\sum_{\exp(n^\alpha)\leqq w} A_n(x)\exp(n^\alpha)n^\beta\right| dw < \infty.$$

Now $I \leqq I_1 + I_2$ where

$$I_1 = \frac{2}{\pi}\phi(\pi)\pi^{-\delta}\int_1^\infty w^{-2}|P(w,\pi)|\,dw;$$
$$I_2 = \frac{2}{\pi}\int_0^\pi |d\{\phi(t)t^{-\delta}\}|\int_1^\infty w^{-2}|P(w,t)|\,dw.$$

Since $\frac{2}{\pi}\phi(\pi).\pi^{-\delta}$ and $\int_0^\pi |d(\phi(t)t^{-\delta})|$ are finite by hypothesis it is enough, for the proof of the theorem, to prove that

$$J_1 = \int_1^\infty w^{-2}|P(w,\pi)|\,dw < \infty$$

and

$$J_2 = \int_1^\infty w^{-2}|P(w,t)|\,dw = O(1),$$

uniformly in $0<t<\pi$.

Using the fact that $P(w,t) = P(w,\pi) - Q(w,t)$ we have

$$J_2 \leqq \int_1^\tau w^{-2}|P(w,t)|\,dw + \int_1^\infty w^{-2}|P(w,\pi)|\,dw + \int_\tau^\infty w^{-2}|Q(w,t)|\,dw,$$

where $\tau = \exp\{t^{-\alpha/(1-\alpha)}\}$.

It is therefore sufficient to prove that

$$K_1 = \int_1^\infty w^{-2}|P(w,\pi)|\,dw < \infty; \tag{5.1.1}$$

$$K_2 = \int_1^\tau w^{-2}|P(w,t)|\,dw = O(1); \tag{5.1.2}$$

$$K_3 = \int_\tau^\infty w^{-2}|Q(w,t)|\,dw = O(1), \tag{5.1.3}$$

uniformly in $0<t<\pi$.

https://doi.org/10.1017/S0013091500009202 Published online by Cambridge University Press

The proof of (5.1.1) has been accomplished in Lemma 3 and the boundedness of (5.1.3) can be observed immediately by using (3.4). For the proof of (5.1.2) we have by (3.3)

$$K_2 = O\left\{t^{\delta}\int_1^{\tau} w^{-1}(\log w)^{\beta/\alpha-1}dw\right\}$$

$$= O\{t^{\delta}(\log \tau)^{\beta/\alpha}\} = O(1),$$

uniformly in $0<t<\pi$.

This paper is based on Chapter V of the author's Ph.D. Thesis entitled " Absolute summability " submitted in 1968 to the Department of Post-graduate Studies and Research in Mathematics, University of Jabalpur.

The author is very thankful to the referee for his valuable suggestion for the statement of the theorem in a general form which combines the two separate theorems of the original version.

REFERENCES

(1) R. Mohanty, A criterion for the absolute convergence of a Fourier series, *Proc. London Math. Soc.* (2) **51** (1950), 186-196.

(2) R. Mohanty, On the absolute Riesz summability of Fourier series and allied series, *Proc. London Math. Soc.* (2) **52** (1951), 259-320.

(3) N. Obrechkoff, Sur la sommation absolue des séries de Dirichlet, *C. R. Acad. Sci. Paris* **186** (1928), 215-217.

(4) N. Obrechkoff, Über die absolute Summierung der Dirichletschen Reihen, *Math. Zeitschrift* **30** (1929), 375-386.

(5) M. Riesz, Sur les séries de Dirichlet et les séries entières, *C. R. Acad. Sci. Paris* **149** (1909), 909-912.

Department of Mathematics
Government Science College,
Jabalpur (India)