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# CONTINUOUS RINGS WITH ACC ON ANNIHILATORS

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ABSTRACT. It is shown that a two-sided continuous ring with ascending chain condition on left annihilators is quasi-Frobenius.

A well-known result of C. Faith asserts that a left (or right) self-injective ring with ascending chain condition on left annihilators is a quasi-Frobenius ring [3]. It is natural to ask whether this result can be extended to continuous rings. In [4] examples are given which show that one-sided continuity and one-sided chain conditions may not necessarily yield the continuity or chain conditions on the opposite side. In this paper we show that a two-sided continuous ring with ascending chain condition on left annihilators is indeed a quasi-Frobenius ring.

Throughout this paper all rings considered are associative with identity and all modules are unitary $R$-modules. We write $J(M)$, $Z(M)$ and $\mathrm{Soc}(M)$ for the Jacobson radical, the singular submodule and the socle of ${}_RM$ respectively. Also, for any subset $X$ of $R$, $\ell_R(X)$ (resp. $r_R(X)$) represents the left (resp. right) annihilator of $X$ in $R$.

According to Utumi [8], a ring $R$ is called a *left continuous ring* if:

(i) every left ideal of $R$ is essential in a direct summand of $R$ and

(ii) every left ideal isomorphic to a direct summand of $R$ is itself a direct summand.

In this paper we establish the following result:

THEOREM 1. *Let $R$ be a left and right continuous ring. If $R$ has* ACC *on left annihilators then $R$ is a quasi-Frobenius ring.*

Before we begin the proof we need some lemmas.

LEMMA 1. *If $R$ has* ACC *on left annihilators, then $Z({}_RR)$ is nilpotent.*

PROOF. This is a well known result and we refer the reader to [7, p. 56].

LEMMA 2. *If $R$ is a left continuous ring, then $Z({}_RR) = J(R)$, $R/J(R)$ is a regular left continuous ring, and idempotents modulo $J(R)$ can be lifted.*

PROOF. See [8].

LEMMA 3. *If $R$ is a left continuous ring with* ACC *on left annihilators, then $R$ is a direct sum of indecomposable uniform left ideals. In particular $R$ is a semiperfect ring.*

PROOF. A module $M$ satisfying the first condition in the definition of continuity is called an extending module (or a CS-module). In [6, Lemma 3], Okado proved the

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following: let $M$ be a CS-module such that $R$ has ACC on annihilators of elements of $M$, then $M$ is a direct sum of uniform submodules of $M$. Now an application of Lemma 2 ensures that $R$ is a semiperfect ring.

LEMMA 4. *Let $M = \bigoplus_{i=1}^{k} M_i$. Then $M$ is continuous if and only if each $M_i$ is continuous and $M_j$-injective for $j \neq i$.*

PROOF. See [5, Theorem 13].

LEMMA 5. *Suppose $R$ is a two-sided continuous, two-sided artinian ring. Then $R$ is a quasi-Frobenius ring.*

PROOF. See [8, Theorem 7.10].

The next lemma is a key result for the proof of Theorem 1.

LEMMA 6. *Let $R$ be a semiprimary ring with* ACC *on left annihilators such that* $\mathrm{Soc}({}_RR) = \mathrm{Soc}(R_R)$ *is finite dimensional as a right $R$-module. Then $R$ is right artinian.*

PROOF. We prove our result by induction on the index of nilpotency of the Jacobson radical of $R$. Suppose $J^{n-1} \neq 0$ and $J^n = 0$ for some positive integer $n$. If $n = 1$, $R$ is semisimple artinian. Suppose the result is true for every $k < n$. Since ACC on left annihilators is equivalent to DCC on right annihilators, we can write $r_R(J) = r_R(\{j_1, \ldots, j_m\})$ for some finite subset $\{j_1, \ldots, j_m\}$ of elements of $J$. Since $R/J$ is semisimple, $\mathrm{Soc}({}_RR) = r_R(J)$. Write $-: R \longrightarrow R/\mathrm{Soc}({}_RR)$ for the canonical quotient map. Clearly $\bar{R}$ is a semiprimary ring with $J^{n-1}(\bar{R}) = 0$. Since $\bar{R}$ is a quotient of $R$ by a right annihilator, $\bar{R}$ retains the DCC on right annihilators and hence the ACC on left annihilators. Now let $\bar{x} \in \mathrm{Soc}(\bar{R}_{\bar{R}})$. Then $xJ \subseteq r_R(J)$ and so $(Jx)J = 0$. Since $\mathrm{Soc}({}_RR) = \mathrm{Soc}(R_R)$, it follows that $J(Jx) = 0$ and so $Jx \subseteq r_R(J) = \mathrm{Soc}(R_R)$. Thus $\bar{x} \in r_{\bar{R}}(\bar{J}) = \mathrm{Soc}({}_{\bar{R}}\bar{R})$, i.e., $\mathrm{Soc}(\bar{R}_{\bar{R}}) \subseteq \mathrm{Soc}({}_{\bar{R}}\bar{R})$. By symmetry, since $r_R(J) = \ell_R(J)$, $\mathrm{Soc}({}_{\bar{R}}\bar{R}) \subseteq \mathrm{Soc}(\bar{R}_{\bar{R}})$. Thus $\mathrm{Soc}({}_{\bar{R}}\bar{R}) = \mathrm{Soc}(\bar{R}_{\bar{R}})$. Now, since $r_R(J) = r_R(\{j_1, \ldots, j_m\})$ there is a right $R$-monomorphism:

$$f: \bar{R} \longrightarrow \bigoplus_{i=1}^{m} j_i R$$

given by $f(\bar{x}) = (j_1x, \ldots, j_nx)$. Since $f\big(\mathrm{Soc}(\bar{R}_R)\big) \subseteq \mathrm{Soc}(R_R)$ which is finite dimensional, it follows that $\mathrm{Soc}(\bar{R}_R)$ is finite dimensional and hence $\mathrm{Soc}(\bar{R}_{\bar{R}})$ is finite dimensional. Now, by induction hypotheses, it follows that $\bar{R}$ is right artinian. Now from the exactness of the sequence

$$0 \longrightarrow \big(\mathrm{Soc}({}_RR)\big)_R \longrightarrow R_R \longrightarrow \big(R/\mathrm{Soc}({}_RR)\big)_R \longrightarrow 0$$

it follows that $R$ is right artinian.

We now prove Theorem 1.

PROOF OF THEOREM 1. Since $R$ is left and right continuous, it follows from Lemmas 1, 2 and 3 that $R$ is a semiprimary ring and $Z({}_RR) = Z(R_R) = J(R)$. Now, since $\mathrm{Soc}({}_RR)$ is the intersection of all the essential left ideals of $R$, we infer that

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$\big(\mathrm{Soc}({}_RR)\big) \cdot Z({}_RR) = 0$ and hence $\big(\mathrm{Soc}({}_RR)\big) \cdot J(R) = 0$. Thus $\mathrm{Soc}({}_RR) \subseteq \mathrm{Soc}(R_R)$. Similarly $\mathrm{Soc}(R_R) \subseteq \mathrm{Soc}({}_RR)$ and hence $\mathrm{Soc}({}_RR) = \mathrm{Soc}(R_R)$. Since $R$ is semiprimary, we can write $R = \bigoplus_{i=1}^{m} e_iR$ as a direct sum of indecomposable right ideals. Since $R$ is right continuous, it follows from Lemma 4 that each $e_iR$ is continuous as a right $R$-module and hence uniform. Thus $R$ is right finite dimensional and so $\mathrm{Soc}(R_R)$ is finite dimensional as a right $R$-module. By Lemma 6, it follows that $R$ is right artinian. Now by Hopkin's theorem, $R$ is right noetherian and hence has ACC on right annihilators. By symmetry $R$ is left artinian. Thus $R$ is a quasi-Frobenius ring.

COROLLARY 7. *Suppose R is a two-sided continuous ring with* ACC *on essential left ideals. Then R is a quasi-Frobenius ring.*

PROOF. By [4, p. 4], $R$ is a left artinian ring and by Theorem 1, $R$ is a quasi-Frobenius ring.

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