Canad. Math. Bull. Vol. **16** (2), 1973

EXTENSIONS OF CONTRACTIVE MAPPINGS AND EDELSTEIN'S ITERATIVE TEST

BY
JACK BRYANT AND L. F. GUSEMAN, JR,

To Professor E. C. Klipple on his sixty-fifth birthday

1. **Introduction.** A mapping f from a metric space (X, d) into itself is said to be *contractive* if $x \neq y$ implies $d(f(x), f(y)) < d(x, y)$. Theorems of Edelstein [2] state that *a contractive selfmap f of a metric space X has a fixed point if, for some x_0, the sequence $\{f^n(x_0)\}$ of iterates at x_0 has a convergent subsequence; moreover, the sequence $\{f^n(x_0)\}$ converges to the unique fixed point of f.* Nadler [3] observes that, from the point of view of applications, it is usually as difficult to verify the condition (for some $x_0 \cdots$) as it is to find the fixed point directly. He introduces the following terminology: the *iterative test* (for contractive maps) *is conclusive* (ITC) *for* (X, d) provided for each contractive selfmap f, if f has a fixed point, then the sequence $\{f^n(x)\}$ converges for each x (necessarily to the fixed point).

Nadler proves [3, Theorem 1] that *if (X, d) is locally compact and connected then* ITC *for X.* Since finite dimensional Banach spaces are locally compact and connected, it is natural to ask for what subsets of a finite dimensional Banach space is ITC. If Y is a dense subset of a metric space X for which each contractive selfmap $f: Y \to Y$ has a contractive (instead of the expected nonexpansive) extension to all of the metric space X, we say Y has the *contractive extension property* (CEP). A subset D of a linear space X is called *line segment dense* (LSD) if for each x, y in X the line segment between x and y contains a point of D distinct from x and y. If X is a dense subset of a finite dimensional Banach space, then LSD$\Rightarrow$CEP$\Rightarrow$ITC (Theorem 2 and Theorem 3 of [3]). Nadler poses the following problems:

A. For what spaces is ITC a topological invariant? That is, if under every metric for X with the given topology, X has ITC, what can be said about X?

B. If the iterative test is conclusive for a dense subset A of a finite dimensional Banach space X, then is A line segment dense in X?

C. Is the iterative test conclusive for the planar set Q^2?

In §2, we give an example of a selfmap of the subset Q^2 of E^2 which shows that Q^2 does not have ITC, and subsequently show that ITC is not a topological invariant of the space Q of rationals (with the usual topology). In §3, we give a negative answer to Problem B. The methods introduced there furnish a slight extension of

Received by the editors April 15, 1971 and, in revised form, August 3, 1971.

Nadler's Theorem 2 and Theorem 3 to spaces which need not be subsets of finite dimensional Banach spaces. In §4, two problems which are suggested by our solution of Nadler's Problem B are stated.

2. **A selfmap of Q^2.** We construct a contractive selfmap of the set Q^2 of points in E^2 with both coordinates rational (Q^2 having the relative metric) with a fixed point and without the sequence of iterates converging at each point.

(2.1) THEOREM *The iterative test is not conclusive for Q^2.*

Proof. Before giving the construction, we describe in a general way how the function is defined and what has to be done to show it works. Consider three half-infinite lines L_1, L_2, L_3 in the first quadrant with one endpoint 0 and with slopes $0 \leq s_1 < s_2 < s_3$, respectively. The fundamental unit in the construction is a mapping f defined as follows: let z_1 be any nonzero element of L_1. Let $z_2 = \cos \alpha\, e^{i\alpha} z_1$ (α the angle between L_1 and L_2) be the projection of z_1 onto L_2 and let z_3 be the projection of z_2 onto L_3. If z is between L_1 and L_2, write (uniquely with real a, b) $z = az_1 + bz_2$ and define $f(z) = az_2 + bz_3$. This definition seems to make f depend on z_1, but actually f depends only on the lines. We will see that, if the slopes s_i are rational, then f takes Q^2 into Q^2 (as far as it goes). The first part of the construction is to pick lines which tend to a line with irrational slope and define the map piecewise as above. We then pick lines from the other side which tend to the limiting line and finally extend this function to the whole plane.

The difficult part of the construction is to select the angles so that the map f will be contractive. To motivate the rather complex induction, let $\alpha_1, \alpha_2, \ldots$ be a decreasing sequence (which we actually define recursively below) with $\sum \alpha_i < 2$; let $z_1 = 1$, $z_{k+1} = \cos \alpha_k\, e^{i\alpha_k} z_k$. Let $\theta_1 = 0$, $\theta_k = \sum_{i=1}^{k-1} \alpha_i$. If $\theta_n \leq \theta < \theta_{n+1}$, $z = re^{i\theta}$, $r > 0$, write (with real s and t) $z = sz_n + tz_{n+1}$, and define $f(z) = sz_{n+1} + tz_{n+2}$; note $s \geq 0$ and $t \geq 0$. Let $w = uz_m + vz_{m+1}$ and suppose $m \geq n$. Direct calculation gives

$$\begin{aligned} |z-w|^2 - |f(z)-f(w)|^2 &= s^2 A_n + 2st A_{n+1} + t^2 A_{n+1} - 2su B_{n\,m} - 2sv B_{n\,m+1} - 2tu B_{n+1\,m} \\ &\qquad - 2tv B_{n+1\,m+1} + u^2 A_m + 2uv A_{m+1} + v^2 A_{m+1}, \end{aligned}$$

where (with $(z \mid w)$ denoting the inner product of complex numbers z and w)

$$\begin{aligned} A_n &= |z_n|^2 - |z_{n+1}|^2 \\ &= \sin^2 \alpha_n \prod_{i=1}^{n} \cos^2 \alpha_i \\ B_{n\,m} &= (z_n \mid z_m) - (z_{n+1} \mid z_{m+1}). \end{aligned}$$

(We note $B_{n\,n+1} = A_{n+1}$ since $(z_n \mid z_{n+1}) = |z_{n+1}|^2$.) To show f is contractive, we must show that the quadratic form

$$F_{n\,m}(s,t,u,v)=|z-w|^2-|f(z)-f(w)|^2$$

is positive when $s, t, u, v \geq 0$ and not all are zero. Let $x=(s,t,u,v)^T$ (T=transpose) and note $F_{n\,m}(s,t,u,v)=(x \mid \mathfrak{A}_{n\,m}x)$, with

$$\mathfrak{A}_{n\,m}=\begin{pmatrix} A_n & A_{n+1} & -B_{n\,m} & -B_{n\,m+1} \\ A_{n+1} & A_{n+1} & -B_{n+1\,m} & -B_{n+1\,m+1} \\ -B_{n\,m} & -B_{n+1\,m} & A_m & A_{m+1} \\ -B_{n\,m+1} & -B_{n+1\,m+1} & A_{m+1} & A_{m+1} \end{pmatrix}$$

The best thing that can happen to this matrix is that all its elements be positive. However, that is not always possible, the impossible cases corresponding to values of $m=n$, $m=n+1$, and $m=n+2$. Our choice of the angles α_i will make every element of $\mathfrak{A}_{n\,m}$ positive when $m>n+2$. The case $n=m$ is easier and is taken care of by requiring the sequence $\{\alpha_n\}$ to be decreasing. The cases $m=n+1$ and $m=n+2$ require special treatment and account for most of the complication that follows.

Let (when the α_i involved are defined)

$$C_n = A_{n+1}-A_{n+1}^2/A_n$$

$$D_n = A_{n+2}(C_n-A_{n+2})/C_n.$$

Let (beginning the construction of the α_i) $\alpha_1=\pi/4$ and choose α_2 so that $\tan\theta_3=\tan\theta_2+2^{-2}=1+2^{-2}$. Let $p_2=2$. Let $\pi/2>\alpha$, $\beta>0$ and consider the function

$$g(\delta)=\cos(\alpha+\beta)-\cos\alpha\cos\delta\cos(\delta+\beta).$$

Since $g(0)=-\sin\alpha\sin\beta<0$, $g(\delta)<0$ if δ is sufficiently small (depending on α and β). Note also that, since $A_2<A_1$, we have $C_1>0$. Choose α_3 so that

(i) $A_3=\cos^2\alpha_1\cos^2\alpha_2\sin^2\alpha_3<C_1$;

(ii) $0<\alpha_3<\alpha_2$;

(iii) $\tan\theta_4=\tan\theta_3+2^{-k_3}$, where $p_3>2p_2$;

(iv) $\cos(\alpha_1+\alpha_2)-\cos\alpha_1\cos\alpha_3\cos(\alpha_2+\alpha_3)<0$.

By (i), $D_1>0$. By (ii), $C_2>0$. Choose α_4 so that

(i) $A_4<C_2$ and $A_4<D_1$

(ii) $0<\alpha_4<\alpha_3$

(iii) $\tan\theta_5=\tan\theta_4+2^{-p_4}$ where $p_4>2p_3$

(iv) for $n=1, 2$

$$\cos\left(\sum_{i=n}^{3}\alpha_i\right)-\cos\alpha_n\cos\alpha_4\cos\left(\sum_{i=n+1}^{4}\alpha_i\right)<0.$$

https://doi.org/10.4153/CMB-1973-033-9 Published online by Cambridge University Press

Continuing by induction, we obtain a sequence $\{\alpha_i\}$ such that for $k=1, 2, \ldots,$ (the $k=1$ case being directly above)

(i) $A_{k+3}<C_{k+1}$ and $A_{k+3}<D_k$,

(ii) $0<\alpha_{k+4}<\alpha_{k+3}$,

(iii) $\tan\theta_{k+4}=\tan\theta_{k+3}+2^{-p_{k+3}}$, $p_{k+3}>2p_{k+1}$,

(vi) for $n=1, \ldots, k+1$

$$\cos\left(\sum_{j=n}^{k+2}\alpha_j\right)-\cos\alpha_n\cos\alpha_{k+3}\cos\left(\sum_{j=n+1}^{k+3}\alpha_j\right)<0.$$

(The induction is straightforward, taking into account the fact about $g(\delta)<0$ noted above and the formula $A_{k+3}=\sin^2\alpha_{k+3}\sum_{j=1}^{k+2}\cos^2\alpha_j$ which shows A_{k+3} can be made small.) Note the series $\sum_{j=1}^{\infty}\alpha_j=\theta_\infty$ converges and $\tan\theta_\infty=1+\sum_{i=2}^{\infty}2^{-p_i}$ is irrational. Let $W=\{re^{i\theta}:0\le\theta<\theta_\infty\}$ and let $f: W\to W$ be defined as outlined above. Since $m_i=\tan\theta_i$ is rational for each i, z_i has both (rectangular) coordinates rational, for $x_{i+1}=(1+m_i\,m_{i+1})x_i/(1+m_{i+1}^2)$, $y_{i+1}=m_{i+1}\,x_{i+1}$. Thus a point z with both coordinates rational has its representation as $z=sz_n+tz_{n+1}$ with s and t both rational. Thus $f(z)\in Q^2$. Note $f^n(z_1)=z_{n+1}$, and clearly $\{z_n\}$ does not converge in Q^2. We note that each point of the line $z=re^{i\theta_\infty}$, $r\ge0$, is a fixed point of the extension f^- of f to the closure W^- of W. We now show f is contractive.

(1) $m=n$: It is better in this case to analyze the form directly. If $z\ne w$, then

$$F_{n\,m}(x)=(s-u)^2(A_n-A_{n+1})+((s-u)+(t-v))^2A_{n+1}>0$$

(2) $m=n+1$: Consider the matrix

$$\mathfrak{A}^*=\begin{pmatrix} a & b & -b & 0\\ b & b & -b & -c\\ -b & -b & b & c\\ 0 & -b & c & c\end{pmatrix}$$

where $a=A_n$, $b=A_{n+1}$, $c=A_{n+2}$.

An elementary calculation gives $\mathfrak{A}^*=\mathfrak{B}^T\mathfrak{B}$ where

$$\mathfrak{B}=\begin{pmatrix} \sqrt{a} & b/\sqrt{a} & b/\sqrt{a} & 0\\ 0 & e & -e & -c/e\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & f\end{pmatrix}$$

where

$$e=(b-b^2/a)^{1/2}\quad\text{and}\quad f=(c-c^2/e^2)^{1/2}.$$

If $\mathfrak{B}$ is real, then $\mathfrak{A}^*$ is nonnegative i.e. $(x\mid\mathfrak{A}^*x)\ge0$ for each $x\in E^4$. But $b-b^2/a=C_n>0$, and, using $A_{n+2}<C_n$, we obtain $ce^2-c^2=A_{n+2}(C_n-A_{n+2})>0$. Hence $\mathfrak{B}$ is real. But still more calculation shows that

$$\begin{aligned}(x\mid\mathfrak{A}_{n\,m}x)&=(x\mid\mathfrak{A}^*x)-2sv\,B_{n\,m+1}\\&\ge-2sv\,B_{n\,n+2}.\end{aligned}$$

https://doi.org/10.4153/CMB-1973-033-9 Published online by Cambridge University Press

We will see below that $B_{n\ n+2}<0$. If $s=0$ or $v=0$ then z and w are in the same "wedge" $\{\theta_n \le \theta \le \theta_{n+1}\}$, which is Case 1. Hence $(x \mid \mathfrak{A}_{n\ m}x)>0$, which finishes Case 2.

(3) $m=n+2$: Consider the matrix

$$\mathfrak{A}^+ = \begin{pmatrix} a & b & 0 & 0 \\ b & b & -c & 0 \\ 0 & -c & c & d \\ 0 & 0 & d & d \end{pmatrix}$$

where a, b, and c are as above and $d=A_{n+3}$. We have $\mathfrak{A}^+=\mathfrak{C}^T\mathfrak{C}$ where

$$\mathfrak{C} = \begin{pmatrix} \sqrt{a} & b/\sqrt{a} & 0 & 0 \\ 0 & e & -c/e & 0 \\ 0 & 0 & f & d/f \\ 0 & 0 & 0 & g \end{pmatrix}$$

with e and f as above and $g=(d-d^2/f^2)^{1/2}$. We already saw e and f are real; we have, by the definition of D_n, $df^2-d^2=A_{n+3}(D_n-A_{n+3})>0$ by condition (i).Thus $\mathfrak{C}$ is real and so $\mathfrak{A}^+$ is nonnegative. We have

$$(x \mid \mathfrak{A}_{n\ m}x) = (x \mid \mathfrak{A}^+x)-2su\, B_{n\ n+2}-2sv\, B_{n\ n+3}-2tv\, B_{n+1\ n+3}.$$

Below we see the three B's are negative, giving $(x \mid \mathfrak{A}_{n\ m}x) \ge -2sv\, B_{n\ n+3}>0$ since $sv\neq 0$ (otherwise we are in Case 2).

(4) $m>n+2$: Here we show each $B_{i\ j}$ in $\mathfrak{A}_{n\ m}$ is negative, so that each element of the matrix is positive. Let $j\ge i+2$ and compute:

$$\begin{aligned} B_{i\ j} &= (z_i \mid z_j)-(z_{i+1} \mid z_{j+1}) \\ &= |z_i|\,|z_j| \left\{ \cos\left(\sum_{n=i}^{j-1} \alpha_n\right) - \cos\alpha_i \cos\alpha_j \cos\left(\sum_{n=i+1}^{j} \alpha_n\right) \right\}. \end{aligned}$$

By condition (iv) of the induction, the term in curley brackets is negative when there are at least two terms in the sum, i.e. when $j\ge i+2$. Thus $B_{i\ j}<0$ when $j\ge i+2$. This completes the proof that f is contractive in W.

We turn now to the problem of extending f. Reflect the region W about the line $L_\infty=\{re^{i\theta_\infty}: r\ge 0\}$. Denote by L_k the reflection of the line $\{re^{i\theta_k}: r\ge 0\}$; note that the angle between L_k and L_{k+1} is α_{k+1}, and that the slope of L_k is $\tan(\theta_\infty+\sum_{i=k}^\infty \alpha_i)=\tan(2\theta_\infty-\theta_k)$. (Thus L_0 has negative slope.) To be sure, none of the lines L_i has rational slope. We modify the angles α_k so that all the slopes are rational and such that the inequalities $A_{n+2}<C_n$, $A_{n+3}<D_n$, $B_{n\ m}<0$ when $m\ge n+2$ and $0<\alpha_{n+1}<\alpha_n$ are true. Then f defined in the same way will be contractive and map Q^2 into Q^2. We note first that $A_{n+2}<C_n$ if and only if $\sin^2\alpha_n \sin^2\alpha_{n+1}-\sin^4\alpha_{n+1}\cos^4\alpha_n-\cos^2\alpha_{n+1}\sin^2\alpha_{n+2}\sin^2\alpha_n>0$. This condition only involves α_n, α_{n+1} and α_{n+2}. A similar but more complicated continuous function of α_n, α_{n+1}, α_{n+2}, and α_{n+3} is positive if and only if $A_{n+3}<D_n$. The significance of these observations is that the

truth of, e.g., $A_{n+2}<C_n$ is independent of the value of α_k for $k<n$ or $k>n+2$. Consider now the inequalities $B_{n\ m}<0$ when $m\geq n+2$. The sign of $B_{1\ m}$ is the same as that of

$$\cos(\alpha_1+\beta)-\cos\alpha_1\cos\alpha_m\cos(\beta+\alpha_m) = g(\alpha_1,\beta,\alpha_m),$$

where $\beta=\sum_{i=2}^{m-1}\alpha_i$. It is easy to see that

$$g(\alpha,\beta,\delta) = -\sin\alpha\sin\beta_0+O(\delta)+O(\beta-\beta_0)$$

as $\beta\to\beta_0$, $\delta\to 0$, uniformly in $\alpha\in[.9\alpha_1, 1.1\alpha_1]$. Using this, we see we need only modify α_1 to satisfy a finite number of inequalities, each one true for sufficiently small change in α_1. Consider now $n=k>1$. Here we change α_k and α_{k+1} keeping the sum $\alpha_k+\alpha_{k+1}$ constant (thereby not changing the slope of any line but L_k). The only real difference is in the uniform estimate on $B_{k\ m}$ in which $-\sin\alpha_k\sin\beta_0$ is replaced by $-\sin\alpha_k'\sin(\alpha_{k+1}'+\beta_0)$, uniformly for α_k' near α_k, α_{k+1}' near α_{k+1}. (Here $\beta_0=\sum_{i=k+2}^{\infty}\alpha_i$.)

Consider now $z\in W$, $w\in W^*$ (the reflection (modified as above) of W on the other side of L_∞), and note that since each point of L_∞ is fixed and since

$$|f^-(s)-f(z)| < |s-z|$$

for $s\in L_\infty$, we have f contractive over $W\cup W^*$. Since the slope of L_0 is negative, $W\cup W^*$ contains the points in Q^2 which are in the first quadrant. If $h(x,y)$ (using E^2 notation again) is defined by $h(x,y)=((|x|+x)/2, (|y|+y)/2)$, then h is nonexpansive and takes Q^2 into the points of Q^2 in the first quadrant. Thus the composition $f\circ h$ (which we denote again by f) takes Q^2 into Q^2, is contractive, has fixed point 0, but with $\{f^n(z_1)\}$ not convergent in Q^2.

It is well known that the space Q of rationals is homeomorphic to the space Q^2. We may thus move (2.1) down to Q and obtain the following result:

(2.2) Corollary. *The iterative test is not conclusive under some equivalent remetrization of Q.*

(Since the iterative test *is* conclusive for Q in its usual metric, this means that ITC is not a topological invariant for Q.)

Proof. Let $h:Q\to Q^2$ be a homeomorphism and let f be the contractive selfmap of Q^2 constructed in the proof of (2.1). For $x,y\in Q$ define $d(x,y)=|h(x)-h(y)|$. The metric d is equivalent to the usual one since h is a homeomorphism. Define $g:Q\to Q$ by $g=h^{-1}\circ f\circ h$. We have

$$\begin{aligned} d(g(x),g(y)) &= |h(g(x))-h(g(y))| = |f(h(x))-f(h(y))| \\ &< |h(x)-h(y)| = d(x,y), \end{aligned}$$

so that g is contractive. Clearly g has a fixed point and $\{g^n(x)\}$ fails to converge for many x.

3. **Extensions of densely defined contractive maps.** A subset S of a metric space (X, d) is said to be *metric segment dense* (MSD) if for each pair x, y in X there exists z in S such that $d(x, z)+d(z, y)=d(x, y)$. If X is a strictly convex Banach space (i.e. if $\|u\|=\|v\|=1$ and $u\neq v$ implies $\|u+v\|<2$), then every MSD subset is LSD.

(3.1) THEOREM *If $g: S\to S$ is a contractive selfmap of a metric segment dense subset of X then the continuous extension $f: X\to X$ of g is contractive.*

Proof. Let $x, y \in X$ and get $z, w \in S$ such that $d(x, y)=d(x, z)+d(z, w)+d(w, y)$. We have

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f(z))+d(f(z), f(w))+d(f(w), f(y)) \\ &< d(x, z)+d(z, w)+d(w, y) = d(x, y). \end{aligned}$$

(3.2) COROLLARY. *If the iterative test is conclusive for X then the iterative test is conclusive for each metric segment dense subset of X.*

Proof. In view of (3.1), this is exactly Nadler's Theorem 2 and Theorem 3 restated for metric (instead of Banach) spaces.

We return now to Problem B, stated in §1, and note that if the finite dimensional Banach space X is not strictly convex, the possibility exists that a subset may be metric segment dense without being line segment dense. This idea leads us to the following negative answer to Problem B:

(3.3) THEOREM *There is a finite dimensional Banach space X and a dense subset S of X such that the iterative test is conclusive for S but S is not line segment dense.*

Proof. Let X be the plane with $\|(x_1, x_2)\|=|x_1|+|x_2|$. Let S denote the set of lines with slope 1 passing through points $(r, 0)$ with r rational. The dense subset S is not LSD, (for example, no point on the line joining $(\pi, 0)$ to $(\pi+1, 1)$ is in S), but S is MSD. (In fact, for this space X we completely characterize the metric segment dense subsets S as those for which $S \cap L$ is dense in L for each line L parallel to the coordinate axes.) Since S is MSD, (3.2) applies and the iterative test is conclusive for S.

4. **Problems.** Let S be a dense subset of a finite dimensional Banach space. The implications (between properties of S)

$$\text{LSD} \Rightarrow \text{MSD} \Rightarrow \text{CEP} \Rightarrow \text{ITC}$$

follow from Nadler [3], (3.1) and (3.2) above. The proof of (3.3) shows that the first implication is not reversible. (For strictly convex spaces, the first two are

https://doi.org/10.4153/CMB-1973-033-9 Published online by Cambridge University Press

equivalent.) We expand Nadler's Problem B as follows (for dense subsets of finite dimensional Banach spaces):

PROBLEM I. *Does* CEP *imply* MSD?

PROBLEM II. *Does* ITC *imply* CEP?

Added in proof. The solution of Problem II is "no" (J. L. Solomon, Proc. Amer. Math. Soc., (to appear)).

REFERENCES

1. L. M. Blumenthal, *Theory and application of distance geometry*, Clarendon Press, Oxford, 1953.

2. M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74–79.

3. Sam B. Nadler, Jr., *A note on an iterative test of Edelstein*, Canad, Math. Bull. (3) **15** (1972), 381–386.

TEXAS A & M UNIVERSITY,
COLLEGE STATION, TEXAS