least two decimal places, and the approximation given by using equation (5) is correct to at least four places.

For example, when $n = 4$, equation (4) gives $A = 24{\cdot}975\ldots$ the correct value being $24{\cdot}977\ldots$.

The Wallace line and the Wallace point.

By J. S. MACKAY, LL.D.

In what follows I propose to give the history of two theorems and to state some of the consequences that have been developed from them.

The first theorem is :

If a triangle be inscribed in a circle, and from any point in the circumference perpendiculars be drawn to the sides, the feet of these perpendiculars lie in a straight line.

This straight line is sometimes called the pedal line of the triangle, but it is much more frequently named the Simson line, from the belief that Robert Simson of Glasgow was the discoverer of the theorem. This belief is erroneous, for the theorem is not to be found in any of Simson's published works ; I have searched every one of them for it in vain. It may be worth mentioning also that no writer who has used the appellation Simson line has ever given a reference to any passage of Simson's works where the theorem is either stated or implied. How then has this appellation arisen ? The first time that the theorem is attributed to Simson is about 1814 in an article by F. J. Servois in Gergonne's *Annales de Mathématiques*, IV. 250. Servois merely says he believes (*je crois*) the theorem is Simson's. Poncelet in his *Propriétés Projectives*, published in 1822, remarks (§ 468) that Servois attributes the theorem to Simson, and it is, I conjecture, this reproduction of Servois's belief by Poncelet on which succeeding geometers have relied when they bestowed the name Simson line.

If the credit of the discovery of the line may not then be given to Simson, to whom does it belong ? In the *Proceedings of the Edinburgh Mathematical Society*, III. 104 (1885) Dr Thomas Muir mentions the fact that the theorem in question occurs in an article by William Wallace in Leybourn's *Mathematical Repository* (old series), II. 111. Apart from the circumstance that I have not met

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with the theorem in any writer anterior to Wallace, the following are the reasons which induce me to think that Wallace is the discoverer of the line. Wallace's article in the second volume of the *Mathematical Repository* was preceded by another of his in the first volume (p. 309) in which he states the theorem that if three straight lines touch a parabola, the circle described through their intersections passes through the focus. In his proof of this theorem (II. 54–5) he draws perpendiculars from the focus on the three tangents and shows that the feet of the perpendiculars lie on the tangent at the vertex; in other words, that the tangent at the vertex is the so-called Simson line of the triangle formed by the three other tangents. The supposition here is that the triangle varies according to a certain law, and that the point from which perpendiculars are drawn to its sides is fixed. Change the supposition, and assume that the triangle is fixed and that the point from which perpendiculars are drawn to its sides varies according to a certain law, and the theorem whose origin we are seeking emerges at once. The close connection of the parabola theorem and the so-called Simson line theorem with each other, the latter being readily suggested by the former, and their corresponding proximity in Wallace's articles in the *Repository* lead one to the belief that the so-called Simson line discovery came after the discovery of the parabola property. Poncelet, who doubtless never saw Wallace's articles, treats the two theorems in the same way. See *Propriétés Projectives*, §§ 466–8.

Two pertinent facts remain to be adduced. First, that Simson in his *Sectiones Conicae* does not mention the parabola property; second, that Lambert, who anticipated* Wallace in his discovery of the parabola property, does not give the so-called Simson line property. The discovery of the Wallace line therefore dates back only to about the year 1799 or 1800.

The first generalisation of Wallace's theorem is due to Poncelet,† who states that the perpendiculars on the sides of the triangle may be replaced by obliques making, in cyclical order, equal angles with the sides.

The same generalisation is also made by Steiner in an article ‡

* *Insignjores Orbitae Cometarum Proprietates*, Sectio I. § 15 (1761).

† *Propriétés Projectives*, § 468 (1822).

‡ See Steiner's *Gesammelte Werke*, I. 197.

in Gergonne's *Annales*, XIX. 37–64 (1828). Steiner adds that all the straight lines obtained in varying the angle of the obliques will envelope a parabola whose focus is the point from which the obliques are drawn.

Poncelet's generalisation is sometimes attributed to Chasles, who gives it in his *Géométrie Supérieure*, § 395, with an interesting proof by involution.

The second generalisation of Wallace's theorem occurs in Gergonne's *Annales*, XIV. 28 (1823) and is probably due to Gergonne himself. It is

If from any point in the circumference of a circle concentric with the circumscribed circle of a triangle perpendiculars be drawn to the sides of the triangle, and their feet be joined, there is formed a triangle of constant area. When this circle becomes the circumscribed circle the area vanishes.

If two circles concentric with the circumscribed circle of a triangle be such that the sum of the squares of their radii is equal to twice the square of the circumscribed radius, and two triangles be formed from them as above, those two triangles are equivalent.

An analytical demonstration is given in the same volume, pp. 280–5, by M. Querret, St. Malo.

The next generalisation of Wallace's theorem is found in *The Mathematician*, I. 70–73 (1845). It is

If from any point in the plane of a triangle perpendiculars be let fall upon the sides, and the extremities of these perpendiculars be joined two and two; if the triangle thus formed be denoted by Δ', the original triangle by Δ, the radius of the circumscribed circle by R, and the distance between the assumed point and the centre of the circumscribed circle by R′, then

$$\frac{\Delta'}{\Delta} = \pm \frac{R^2 - R'^2}{4R^2}$$

according as the assumed point is within or without the circumscribed circle.

When the perpendiculars are replaced by obliques making the same angle ϕ with the sides

$$\frac{\Delta'}{\Delta} = \pm \frac{R^2 - R'^2}{4R^2} \operatorname{cosec}^2\phi$$

The author of this generalisation, whose demonstration is analytical, appends only his initials W. F. [From the list of subscribers to the first volume of *The Mathematician* I conjecture that the name is William Finley] and dates his communication, Oct. 12th, 1843. A geometrical demonstration of the theorem as stated by Finley is given by Mr John Davis, Skibbereen, in *The Mathematician*, II. 37–8 (Nov. 1845). Two others, by Mr J. M'Dowell and Mr Matthew Collins, will be found in *Mathematical Questions with their Solutions from the "Educational Times,"* XVIII. 95–7 (1872). A trigonometrical proof with some applications is given by M. Vuibert in his *Journal de Mathématiques Élémentaires*, IV. 49–52 (1880).

The next generalisation occurs in the *Revue des Sociétés Savantes*, V. 203–233 (1870), where M. Combette has a memoir entitled *Etude d'un lieu géométrique dont il n'a pas été fait mention.* This memoir, the title of which is not quite accurate, and the researches preceding it, establish the following results :

(1) If P be an assumed point, and D, E, F its projections on the sides of triangle ABC, the locus of P when the area of triangle DEF is constant is the circumference of a circle concentric with the circumscribed circle of ABC.

(2) When P coincides with O the circumscribed centre, the area of triangle DEF is one-fourth of the area of ABC.

(3) When OP increases from zero to R, the area of DEF diminishes from one-fourth of ABC to zero, the locus becomes the circumscribed circle itself, and the triangle DEF becomes the Wallace line.

(4) When OP increases from R indefinitely, the area of DEF increases from zero indefinitely.

(5) Hence for every value of the area of DEF included between zero and one-fourth of ABC, the locus of P will consist of two circumferences concentric with the circumscribed circle, the one interior and the other exterior to it.

(6) The sum of the squares of the radii of these two circumferences will be double the square of the circumscribed radius.

(7) The locus of P does not change its nature or its centre when the perpendiculars let fall on the sides become straight lines all making equal angles with the sides.

(8) The relation between the radius R′ of the circumference which corresponds to a given value Δ′ of the variable area, the radius R of the circumscribed circle, the surface Δ of the given triangle,

and the angle ϕ which the lines drawn from a point of the locus make with the sides is

$$\frac{\Delta'}{\Delta} = \pm \frac{R^2 - R'^2}{4R^2 \sin^2\phi}.$$

(9) If the triangle ABC is replaced by a plane polygon, and the projections on its sides of a point P in the plane are joined, the locus of P when this area is constant is still a circle, which has always the same centre whatever be the value of the area.

(10) When the point P is taken in space and projected on the sides of a plane polygon, the locus of P when the area obtained by joining the projections is constant, becomes a cylindrical surface of revolution whose axis perpendicular to the plane of the polygon is always the same for all values of the area.

A large number of the theorems connected with the Wallace line will be found in an article by Mr John Alison in the *Proceedings of the Edinburgh Mathematical Society*, III. 77–93 (1885).

The second theorem is

If four straight lines intersect each other and form four triangles, circles circumscribed about each of these triangles will pass through one and the same point.

This theorem was proposed for proof in 1804 in the first volume of Leybourn's *Mathematical Repository* (new series), p. 22 (pagination of the Questions) by Scoticus. In reference to Scoticus, I make the following quotation from an article by Mr T. T. Wilkinson in the *Mechanics' Magazine*, Vol. LV., p. 447 (1851). "This fictitious signature is one of those adopted by Dr Wallace, as appears from a manuscript note in the late Professor Davies's copy of the *Mathematical Repository*, on the authority of Professor Leybourn, the editor."

Two solutions of the theorem are given in the *Repository*, Vol. I., Part I., p. 170, but they are not essentially different. No additional properties connected with this figure, which is that of a complete quadrilateral, seem to have appeared in the *Repository* till about 1832, when Professor Davies pointed out four (see Vol. VI., Part I., pp. 124–7, and Question 555). One of these he had proposed in the *Leeds Correspondent* in 1821; another, which forms question 555, is the second of Steiner's series of theorems.

In Gergonne's *Annales*, XVIII. 302 (1828) Steiner enunciates without proof the following theorems regarding four intersecting straight lines :

1°. These four straight lines, taken three by three, form four triangles such that their circumscribed circles pass all through the same point P.

2°. The centres of these four circles and the point P are on the circumference of a fifth circle.

3°. The feet of the perpendiculars* let fall from the point P on the four straight lines are situated on one straight line R, and this property belongs exclusively to the point P.

4°. The orthocentres of the four triangles (1°) are situated on one straight line R′.

5°. The straight lines R and R′ are parallel, and the straight line R passes through the middle of the perpendicular let fall from P on R′.

6°. The middle points of the diagonals of the complete quadrilateral formed by the four straight lines (1°) are situated on one straight line† R″.

7°. The straight line R″ is perpendicular to the two straight lines R, R′.

8°. For each of the four triangles (1°) there are an inscribed circle and three escribed circles, making in all sixteen circles, whose centres are four by four on one circumference so as to give rise to eight new circles.

9°. These eight new circles are divided into two groups such that each of the four circles of one of these groups cuts orthogonally all the circles of the other group ; whence it follows that the centres of the circles of the two groups are on two straight lines perpendicular to each other.

10°. Finally, these last two straight lines intersect at the point P mentioned above.

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* This property is extended to equally inclined obliques by Mr J. W. Elliott in the *Lady's and Gentleman's Diary* for 1852, p. 71.

† Steiner attributes this theorem to Newton, but gives no reference to the passage where it occurs. It is usually ascribed to Gauss under the date 1810 (Zach, *monatl. Korresp.* 22 p. 115) ; but it was given by Mr J. T. Connor, Lewes Academy, in the *Ladies' Diary* for 1795.

Professor W. K. Clifford, in discussing* Wallace's theorem states that it is the third of a series.

"If we take any two straight lines they determine a point, viz., their point of intersection.

"If we take three straight lines we get three such points of intersection; and these three determine a circle, viz., the circle circumscribing the triangle formed by the three lines.

"Four straight lines determine four sets of three lines by leaving out each in turn; and the four circles belonging to these sets of three meet in a point.

"In the same way five lines determine five sets of four, and each of these sets of four gives rise, by the proposition just proved, to a point. It has been shown by Miquel† that these five points lie on the same circle.

"And this series of theorems has been shown‡ to be endless. Six straight lines determine six sets of five by leaving them out one by one. Each set of five has, by Miquel's theorem, a circle belonging to it. These six circles meet in the same point, and so on for ever. Any even number ($2n$) of straight lines determines a point as the intersection of the same number of circles. If we take one line more, this odd number ($2n+1$) determines as many sets of $2n$ lines, and to each of these sets belongs a point; these $2n+1$ points lie on a circle."

In connection with Wallace's two theorems, mention may be made of the principle of what M. De Longchamps calls Recurrent Geometry,§ and of his application of it.

Imagine three straight lines d_1, d_2, d_3 in a plane. Let P be a remarkable and well defined point of this triangle; for example, the centroid, the orthocentre, the circumscribed centre, and so on. To mark that P is a well determined point of the triangle $d_1\, d_2\, d_3$ let it be denoted by P_{123}.

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* Clifford's *Common Sense of the Exact Sciences*, pp. 30-1 (1885).

† Liouville's *Journal*, III. 486-7 (1838).

‡ Clifford in the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, V. 124-141 (1871).

§ See *Journal de Mathématiques Élémentaires* (2nd series), II. 3-10, 25-33, 49-56, 73-8, 121-6 (1883). The substance of these papers had appeared in 1877 in the *Nouvelle Correspondance Mathématique*, III. 306-312, 340-7, and in the *Annales scientifiques de l'Ecole normale*, III. 321-341 (1866).

Now take four straight lines d_1, d_2, d_3, d_4. If one of them, d_4 for example, be removed, there remains a triangle $d_1d_2d_3$ to which corresponds the point P_{123} previously defined, and on which it is proposed to make a study of recurrent geometry. If one of the four given straight lines be successively removed, four points are obtained

$$P_{123}, \ P_{234}, \ P_{341}, \ P_{412}.$$

These four points possess, whether from their respective positions with regard to each other, or from their situation with regard to the given straight lines, geometrical properties which may in certain cases be remarkable. From a geometrical study of them there may result the discovery of a point which will be denoted by P_{1234}. This point may possess with regard to the straight lines d_1, d_2, d_3, d_4, or with regard to the points P_{123}, P_{234}, P_{341}, P_{412} geometrical properties which recall those which connected the point P_{123} to the triangle $d_1d_2d_3$.

Let the point P_{123}, defined by the triangle, be called a P point of the 3rd order, and the analogous point P_{1234}, which corresponds to four straight lines, be called a P point of the 4th order.

Now take five straight lines, and successively remove one of them, d_5 for instance. We shall have to consider a complete quadrilateral formed by the four other straight lines, and to this quadrilateral corresponds a P point of the 4th order, which will be denoted by P_{1234}. Thus five points of the 4th order are obtained

$$P_{1234}, \ P_{2345}, \ P_{3451}, \ P_{4512}, \ P_{5123}.$$

If with these five points there can be obtained a P point which is deduced from them as the P point of the 4th order was deduced from the P points of the 3rd order, this point will be a P point of the 5th order, and will be denoted by P_{12345}.

M. De Longchamps has applied his principle of recurrent geometry to the circumscribed centre, to the centroid, to the nine point circle, and, as follows, to the Wallace line.

If from a point P (which will be called the fundamental point) taken on a circle O, perpendiculars be drawn to the sides of a triangle whose vertices 1, 2, 3 are situated on O, the feet of these perpendiculars are in one straight line d_{123}.

Take now four points 1, 2, 3, 4 on the circle O. If one of these points be successively removed, four triangles will be obtained, and to each of them along with the point P will correspond a Wallace line of the 3rd order,

$$d_{123},\ d_{234},\ d_{341},\ d_{412}.$$

If from the fundamental point P perpendiculars be drawn to these four Wallace lines of the 3rd order, the four points thus obtained are situated on a straight line. This straight line will be a Wallace line of the 4th order, and will be denoted by d_{1234}.

Similarly if five points 1, 2, 3, 4, 5 be taken on a circle O, the projections of the fundamental point P on the five Wallace lines of the 4th order will be situated on a straight line. This straight line will be a Wallace line of the 5th order, and will be denoted by d_{12345}.

Hence the following general theorem:

If n points be taken on a circle, and one of them be removed, $(n-1)$ points are obtained to which correspond a Wallace line of the $(n-1)^{th}$ order. Thus there are n Wallace lines of the $(n-1)^{th}$ order. The projections of the fundamental point P on these n lines are situated on one straight line, called a Wallace line of the n^{th} order, and denoted by $d_{12\ldots n}$.

Note on an Equation of Motion.

By A. J. Pressland, M.A.

It can be shown by means of relative motion that if two bodies A and B move with velocities u and v in the same straight line, and a third body C move with velocity $u+v$ also in the same straight line, the space passed over by C is equal to the sum of the spaces passed over by A and by B in the same time.

Let A move with an initial velocity u and an acceleration f for an interval t.

Its velocity at the end of the interval will be $u+ft$ which call v.

Then $$u+ft=v \text{ or } u=v-ft.$$

Now let B move with velocity v and acceleration $-f$. Its velocity at the end of the interval t will be $v-ft$, that is u.

Hence the motion of B is the exact counterpart, or reverse, of that of A. Therefore each passes over the same space s.

Hence C passes over a space $2s$.