THE RADICAL OF THE GROUP ALGEBRA OF A SUBGROUP, OF A POLYCYCLIC GROUP AND OF A RESTRICTED *SN*-GROUP

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1. Introduction and notation

Let G be a group and let K be an algebraically closed field of characteristic $p>0$. The twisted group algebra $K^t(G)$ of G over K is defined as follows: let G have elements $a, b, c, \ldots$ and let $K^t(G)$ be a vector space over K with basis elements $\bar{a}, \bar{b}, \bar{c}, \ldots$; a multiplication is defined on this basis of $K^t(G)$ and extended by linearity to $K^t(G)$ by letting

$$\bar{x}\bar{y} = \alpha(x, y)\overline{xy} \ (x, y \in G),$$

where $\alpha(x, y)$ is a non-zero element of K, subject to the condition that

$$\alpha(x, y)\alpha(xy, z) = \alpha(y, z)\alpha(x, yz) \ (x, y, z \in G)$$

which is both necessary and sufficient for associativity. If, for all $x, y \in G$, $\alpha(x, y)$ is the identity of K then $K^t(G)$ is the usual group algebra $K(G)$ of G over K. We denote the Jacobson radical of $K^t(G)$ by $JK^t(G)$. We are interested in the relationship between $JK^t(G)$ and $JK^t(H)$ where H is a normal subgroup of G. In § **2** we show, among other results, that if certain centralising conditions are satisfied and if $JK(H)$ is locally nilpotent then $JK(H)K(G)$ is also locally nilpotent and thus contained in $JK(G)$. It is observed that in the absence of some centralising conditions these conclusions are false. We show, in particular, that if H and $G/C(H)$ are locally finite, $C(H)$ being the centraliser of H, and if G/H has no non-trivial elements of order p, then $JK(G)$ coincides with the locally nilpotent ideal $JK(H)K(G)$. The latter, and probably more significant, part of this paper is concerned with particular types of groups. We introduce the notion of a restricted *SN*-group and show that if G is such a group and if G has no non-trivial elements of order p then $JK^t(G) = \{0\}$. It is also shown that if G is polycyclic then $JK^t(G)$ is nilpotent.

We let e be the identity of G and we denote the index of a subgroup B of G by $|G{:}B|$ and its centraliser in G by $C(B)$. If X is a non-empty subset of $K(G)$, Supp X denotes the subset of elements of G appearing with non-zero coefficients in the representation of the elements of X as linear combinations of the elements of G.

2. Centralising conditions on subgroups

Throughout this section we assume that A is a subgroup of G, that H is a normal subgroup of G and that $G = HA$ (we do not assume $H \cap A = \{e\}$). For convenience we make the following definition.

Definition. *If B is a subgroup of G then a non-empty subset S of $K(G)$ is called B-invariant if $b^{-1}Sb = S$ for all $b \in B$.*

Thus a subgroup S of G is B-invariant if and only if S is normal in the subgroup SB.

The next lemma is similar to Theorem 4.1 of (**5**).

Lemma 2.1. *Let B be a subgroup of G and let S be a B-invariant subalgebra of $K(G)$. Then $SK(B)$ is a subalgebra of $K(G)$ and*

$$[SK(B)]^\rho = S^\rho K(B) \quad (\rho = 1, 2, \ldots).$$

Proof. Since S is B-invariant we have $SK(B) = K(B)S$ and this implies the result.

Lemma 2.2. *Let C be a subgroup of G centralising H and let $| A\colon A \cap C|$ be finite. Let S be a finitely generated subalgebra of $K(H)$ and let T be the subalgebra of $K(G)$ generated by $\{a^{-1}sa\colon\ a \in A,\ s \in S\}$. Then T is a finitely generated A-invariant subalgebra of $K(H)$.*

Proof. T is clearly an A-invariant subalgebra of $K(H)$. Let

$$A = (A \cap C)a_1 \cup (A \cap C)a_2 \cup \ldots \cup (A \cap C)a_n$$

be a coset decomposition of $A \cap C$ in A. Let S be generated by $\{s_1, s_2, \ldots, s_r\}$. Then we assert that T is generated by

$$\{a_i^{-1}s_ja_i\colon\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, r\}.$$

We observe first that T is certainly generated by $\{a^{-1}s_ja\colon a \in A, j = 1, 2, \ldots, r\}$. But for all $a \in A$ there exists k, $1 \leqq k \leqq n$, and $c \in A \cap C$ such that $a = ca_k$. Hence, as $S \subseteq K(H)$,

$$a^{-1}s_ja = a_k^{-1}c^{-1}s_jca_k = a_k^{-1}s_ja_k$$

and this establishes the lemma.

Theorem 2.3. *Let C be a subgroup of G centralising H and let $| A\colon A \cap C|$ be finite. Let I be a locally nilpotent G-invariant ideal of $K(H)$. Then $IK(G)$ is a locally nilpotent ideal of $K(G)$.*

Proof. Since I is G-invariant, $IK(G)$ is an ideal of $K(G)$, and, since $G = HA$, $IK(G) = IK(A)$. Let $u_1, u_2, \ldots, u_r \in IK(G)$ and let U be the subalgebra generated by $\{u_1, u_2, \ldots, u_r\}$. For suitable $x_1, x_2, \ldots, x_s \in A$ and

$$h_{\lambda\mu} \in I\ (\lambda = 1, 2, \ldots, r;\ \mu = 1, 2, \ldots, s)$$

we have

$$u_\lambda = \sum_{\mu=1}^{s} h_{\lambda\mu}x_\mu \quad (\lambda = 1, 2, \ldots, r).$$

Let S be the subalgebra generated by $\{h_{\lambda\mu}: \lambda = 1, 2, ..., r; \mu = 1, 2, ..., s\}$ and let T be the subalgebra generated by $\{a^{-1}sa: a \in A, s \in S\}$. By Lemma 2.2, T is a finitely generated A-invariant subalgebra of I and so T is nilpotent. But $U \subseteq TK(A)$ and thus, by Lemma 2.1, U is nilpotent. The theorem is now proved.

Remark. If, in the above theorem, $| A: A \cap C |$ is not finite then the theorem is false. To see this we consider the example on p. 294 of (5). In this example H is a normal abelian p-subgroup of G, A is an infinite cyclic subgroup generated by an element g and $C = \{e\}$. $JK(H)$ is locally nilpotent, yet, as is shown, $JK(H)K(G) \nsubseteq JK(G)$. Indeed, we now know that $JK(G) = \{0\}$.

Theorem 2.4. *Suppose that, for all non-trivial finitely generated subgroups H_0 and A_0, $H_0 \subseteq H$, $A_0 \subseteq A$ respectively, $| A_0; C(H_0) \cap A_0 |$ is finite. Let I be a locally nilpotent G-invariant ideal of $K(H)$. Then $IK(G)$ is a locally nilpotent ideal of $K(G)$.*

Proof. Let U be the subalgebra of $IK(G)$ generated by $u_1, u_2, ..., u_r$; as in the previous theorem we require to show U is nilpotent. For suitable $x_1, x_2, ..., x_r \in A$ and $h_{\lambda\mu} \in I$ ($\lambda = 1, 2, ..., r$; $\mu = 1, 2, ..., s$) we have

$$u_\lambda = \sum_{\mu=1}^{s} h_{\lambda\mu} x_\mu \quad (\lambda = 1, 2, ..., r).$$

Let $W = \text{Supp}\{h_{\lambda\mu}: \lambda = 1, 2, ..., r ; \mu = 1, 2, ..., s\}$. Then

$$W = \{w_1, w_2, ..., w_t\}, \text{ say.}$$

Let A_0 be the subgroup generated by $\{x_1, x_2, ..., x_s\}$ and let H_0 be the subgroup generated by W^* where $W^* = \{a^{-1}w_i a: a \in A_0; i = 1, 2, ..., t\}$. Then W^* is finite since $| A_0: C(w_i) \cap A_0 |$ is finite ($i = 1, 2, ..., t$). Thus H_0 is a finitely generated A_0-invariant subgroup of G and also

$$U \subseteq [I \cap K(H_0)]K(A_0).$$

Let $G_0 = H_0 A_0$. Then $I \cap K(H_0)$ is a locally nilpotent G_0-invariant ideal of $K(G_0)$. Hence, by Theorem 2.3, $[I \cap K(H_0)]K(A_0)$ is locally nilpotent and so U is nilpotent. This proves the theorem.

We now make some applications of the above theorems.

Theorem 2.5. *Let $G/C(H)$ be locally finite and let $JK(H)$ be locally nilpotent. Then $JK(H)K(G)$ is locally nilpotent.*

Proof. Let H_0 and A_0 be finitely generated subgroups of H and $G(= A)$ respectively. Then $C(H) \subseteq C(H_0)$ and so

$$| A_0: C(H_0) \cap A_0 | \leqq | A_0: C(H) \cap A_0 |.$$

But

$$A_0/(C(H) \cap A_0) \cong A_0 C(H)/C(H)$$

which, being a finitely generated subgroup of $G/C(H)$, is finite. Hence

$$| A_0 : C(H_0) \cap A_0 |$$

is finite and the result now follows from Theorem 2.4.

Corollary. *Let H and $G/C(H)$ be locally finite. Then $JK(H)K(G)$ is locally nilpotent.*

Proof. The local finiteness of H implies easily that $JK(H)$ is locally nilpotent.

Theorem 2.6. *Let G/H be locally finite. Then*

(i) *$JK(H)K(G) \subseteq JK(G)$ and*
(ii) *$JK(G)/JK(H)K(G)$ is locally nilpotent.*
(iii) *If G/H has no non-trivial elements of order p then $JK(H)K(G) = JK(G)$.*

Proof (i) $JK(H)K(G)$ is an ideal of $K(G)$ and we therefore require to show that all elements of $JK(H)K(G)$ have quasi-inverses. Let $x \in JK(H)K(G)$. Then

$$x = h_1g_1 + h_2g_2 + \ldots + h_rg_r$$

where $h_i \in JK(H)$, $g_i \in G$ $(i = 1, 2, \ldots, r)$. Let G_0 be the subgroup generated by $H \cup \{g_1, g_2, \ldots, g_r\}$. Then G_0/H is finitely generated and so is finite. Hence ((4), Proposition 1.3) $JK(H)K(G_0) \subseteq JK(G_0)$ and thus $x \in JK(G_0)$. Thus x has a quasi-inverse in $K(G_0)$ and so in $K(G)$.

(ii) Let U be the subalgebra of $JK(G)$ generated by $\{u_1, u_2, \ldots, u_n\}$. We require to show that for some $\rho > 0$ $U^\rho \subseteq JK(H)K(G)$. Let G_0 be the subgroup generated by $H \cup \text{Supp}\{u_1, u_2, \ldots, u_n\}$. Then G_0/H is finite and so ((4), Proposition 1.3) there exists $\rho > 0$ such that $[JK(G_0)]^\rho \subseteq JK(H)K(G_0)$. Now $U \subseteq JK(G) \cap K(G_0) \subseteq JK(G_0)$ and therefore $U^\rho \subseteq JK(H)K(G_0) \subseteq JK(H)K(G)$.

(iii) If G/H has no non-trivial elements of order p then G/H belongs to a JK-class ((6), p. 54-55), which implies that $JK(G) \subseteq JK(H)K(G)$. This fact, together with (i), proves (iii).

By combining Theorems 2.5 and 2.6 the following is immediate.

Theorem 2.7. *Let G/H and $G/C(H)$ be locally finite. Then JK/G is locally nilpotent if and only if $JK(H)$ is locally nilpotent.*

This result is derivable by other means on observing that our group-theoretical conditions are equivalent to the assertion that $G/(H \cap C(H))$ is locally finite.

3. Polycyclic groups and restricted *SN*-groups

In (6) we were concerned with conditions under which $JK(G) \subseteq JK(H)K(G)$, utilising the previously known result that if G/H is finite and has no non-trivial elements of order p then this relation holds. Our arguments on directed systems, etc., were essentially group-theoretical and require trifling modifications in order to apply to twisted group algebras once Proposition 1.3 of (**4**) is known. Thus we can, in particular, assert the following.

Lemma 3.1. *Let G/H be a finitely generated abelian group with no non-trivial elements of order p. Then $JK^t(G) \subseteq JK^t(H)K^t(G)$.*

Combining this with Proposition 1.3 of (**4**) we obtain easily the next lemma.

Lemma 3.2. *Let G/H be a finitely generated abelian group. Then there exists $\rho>0$ such that*

$$[JK^t(G)]^\rho \subseteq JK^t(H)K^t(G).$$

This result yields the following important lemma.

Lemma 3.3.† *Let G/H be an abelian group and let G have no non-trivial elements of order p. Then $JK^t(H) = \{0\}$ implies that $JK^t(G) = \{0\}$.*

Proof. Suppose $x \in JK^t(G)$, $x \neq 0$. Let N be generated by $H \cup \mathrm{Supp}\,(x)$. Then $x \in JK^t(N)$ and N/H is a finitely generated abelian group. By Lemma 3.1 there exists $\rho>0$ such that

$$[JK^t(N)]^\rho \subseteq JK^t(H)K^t(N) = \{0\}.$$

But N has no non-trivial elements of order p and so $K^t(N)$ has no proper nilpotent ideals ((**4**), Theorem 3.2). Thus we derive a contradiction if $x \neq 0$.

We apply these results first to polycyclic groups.

Theorem 3.4. *Let G be a polycyclic group. Let M be the subgroup generated by those elements of G, of orders a power of p, having at most a finite number of conjugates. Then M is a finite normal subgroup of G and $JK^t(G)$ coincides with the nilpotent ideal $JK^t(M)K^t(G)$.*

Proof. By Dietzmann's Lemma ((**3**), p. 154) M is locally finite and normal. Since G is polycyclic, M is finitely generated and thus M is finite. Consequently $JK^t(M)K^t(G)$ is nilpotent ((**4**), Lemma 1.2; cf. Lemma 2.1).

We now establish, by induction on the length of the derived series, that $JK^t(G)$ is nilpotent. This assertion is true for finitely generated abelian groups. Let now G' be the derived group of G. Then $JK^t(G')$ is nilpotent and hence $JK^t(G')K^t(G)$ is nilpotent ((**4**) Lemma 1.2; cf. Lemma 2.1). By Lemma 3.2 there exists $\rho>0$ such that

$$[JK^t(G)]^\rho \subseteq JK^t(G')K^t(G)$$

and therefore $JK^t(G)$ is nilpotent. It follows now that $JK^t(G) = JK^t(M)K^t(G)$ ((**4**), Theorem 3.7).

We wish now to establish semi-simplicity of $K^t(G)$ in the case of a particular generalisation of a soluble group.

† Results similar to Lemma 3.3 and also Theorem 3.5 below have been obtained independently by D. S. Passman in a preprint entitled "On the Semisimplicity of Twisted Group Algebras".

Definition. *Let G be a non-trivial group and let τ be an ordinal. Let G have, for each ordinal σ, $\sigma \leqq \tau$, a pair of subgroups U_σ, V_σ such that*

(i) *U_σ is a normal subgroup of V_σ and V_σ/U_σ is abelian;*
(ii) *$\rho<\sigma$ implies that $V_\rho \subseteq U_\sigma$;*
(iii) *$U_0 = \{e\}$, $V_\tau = G$ and*
(iv) $\bigcup_{0 \leqq \sigma \leqq \tau} (V_\sigma \backslash U_\sigma) = G\backslash\{e\}$.

Then we call G a restricted SN-group.

Our definition is motivated by that of Kurosh for SN-groups ((**3**), p. 171 and p. 182) but we have made the assumption, additional to the usual definition of *SN*-groups, that the total ordering, under inclusion, of the subgroups of the family is also a well-ordering.

Theorem 3.5. *Let G be a restricted SN-group as above and let G have no non-trivial elements of order p, then $JK^t(G) = \{0\}$.*

Proof. If $\tau = 1$ then $G = V_1$ and G is abelian, hence, in this case,

$$JK^t(G) = \{0\}.$$

We therefore argue by transfinite induction and assume that the result is true for all subgroups V_ρ, $\rho<\sigma$ (say) and we then prove that the result for V_σ.

We begin by showing that $JK^t(U_\sigma) = \{0\}$. Suppose therefore that

$$x \in JK^t(U_\sigma),\ x \neq 0.$$

Then $x = \sum_{i=1}^{n} \lambda_i \bar{g}_i$ where $\lambda_j \in K$, $g_j \in U_\sigma (j = 1, 2, \ldots, n)$. By (iv) of the definition there exists ρ_j such that

$$g_j \in V_{\rho_j} \backslash U_{\rho_j} \quad (j = 1, 2, \ldots, n).$$

Now $\rho_j<\sigma$ for $\sigma \neq \rho_j$ and $\sigma<\rho_j$ implies that $V_\sigma \subseteq U_{\rho_j} \subset V_{\rho_j}$ from which we would derive a contradiction $(j = 1, 2, \ldots, n)$. Let $\rho = \max\{\rho_1, \rho_2, \ldots, \rho_n\}$, then $\rho<\sigma$ and furthermore ((**4**), Lemma 1.9).

$$x \in JK^t(G) \cap K^t(V_\rho) \subseteq JK^t(V_\rho).$$

But $JK^t(V_\rho) = \{0\}$ and so we cannot have $x \neq 0$. Hence we have shown that $JK^t(U_\sigma) = \{0\}$. By Lemma 3.3, $JK^t(V_\sigma) = \{0\}$ and this completes the transfinite induction argument. Consequently we can assert that $JK^t(G) = \{0\}$.

Remarks. It is worth observing that if G is polycyclic then $K^t(G)$ has ascending chain condition on left and right ideals ((**2**), p. 429). [Strictly this is proved for $K(G)$ but the same arguments work for $K^t(G)$.] Therefore, by Levitzki's Theorem ((**1**), p. 51) every nil or locally nilpotent ideal of $K^t(G)$ is nilpotent. This suggests that if G is soluble perhaps $JK^t(G)$ is nil or locally nilpotent.

We have assumed that $p>0$ throughout this paper. There is a long-standing conjecture that if $p = 0$ then $JK(G) = \{0\}$ for any group G; a proof of this for an *SN*-group has been given by Villamayor ((**7**), p. 31).

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