Canad. Math. Bull. Vol. **44** (3), 2001 pp. 335–336

Inductive Limit Toral Automorphisms of Irrational Rotation Algebras

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Abstract. Irrational rotation C^*-algebras have an inductive limit decomposition in terms of matrix algebras over the space of continuous functions on the circle and this decomposition can be chosen to be invariant under the flip automorphism. It is shown that the flip is essentially the only toral automorphism with this property.

In [3] Elliott and Evans proved that the irrational rotation C^*-algebra A_θ, where $0 < \theta < 1$, is an inductive limit of algebras of the form $M_q\big(C(S^1)\big) \oplus M_{q'}\big(C(S^1)\big)$ where q, q' are denominators in successive convergents of the continued fraction expansion of θ. A simpler proof was subsequently given by Elliott and Lin in [4]. Following the approach of [3], Walters showed in [6] that the flip automorphism α determined by $\alpha(U) = U^*$ and $\alpha(V) = V^*$, where U and V are the generators of A_θ, leaves invariant an appropriately chosen Elliott-Evans decomposition. Subsequently Boca obtained an alternative proof in [1], based on the methods used in [4].

The flip is the image of $-I_2$ under the action of $\mathrm{SL}(2,\mathbb{Z})$ on A_θ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : U \mapsto e^{\pi i a c \theta} U^a V^c$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : V \mapsto e^{\pi i b d \theta} U^b V^d$$

which was introduced by Brenken [2] and Watatani [7]. In [6] Walters posed the question if every finite order automorphism σ of A_θ arising from a matrix in $\mathrm{SL}(2,\mathbb{Z})$ is an inductive limit automorphism with respect to some choice of the basic building blocks of Elliott and Evans. The purpose of the present short note is to answer this question in the negative by showing that the only such inductive limit automorphisms of A_θ, other than the identity, are conjugate to the flip.

Note that if p/q and p'/q' are successive convergents in the continued fraction expansion of θ then $|pq' - qp'| = 1$, so q and q' are coprime.

Theorem *Let $0 < \theta < 1$ be irrational, let A_θ be the associated rotation algebra, with generators U, V satisfying $VU = e^{2\pi i\theta} UV$, and let σ be the automorphism of A_θ determined by $\sigma(U) = e^{\pi i a c\theta} U^a V^c$ and $\sigma(V) = e^{\pi i b d \theta} U^b V^d$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$. If σ leaves invariant each of a sequence of sub algebras A_n with inductive limit A_θ, and each A_n is isomorphic to $M_{q_n}\big(C(S^1)\big) \oplus M_{q_n'}\big(C(S^1)\big)$ for coprime q_n, q_n' then σ is either the identity or is conjugate to the flip α determined by $\alpha(U) = U^*$ and $\alpha(V) = V^*$.*

Received by the editors July 29, 1999.
AMS subject classification: 46L40, 46L35.

Proof Observe firstly that from the condition that q_n and q_n' are coprime, σ must fix rather than interchange the non-trivial central projections $(I_{q_n}, 0)$ and $(0, I_{q_n'})$ in A_n. Hence, when $K_1(A_n)$ is identified with $\mathbb{Z}^2$ using the identification of $K_1\big(C(S^1)\big)$ with $\mathbb{Z}$ then $\sigma_*\colon K_1(A_n) \to K_1(A_n)$ is of the form $(n, m) \mapsto (an, bm)$ for some $a, b \in \mathbb{Z}$. Indeed, since σ_* is invertible, σ_* is of the form $(n, m) \mapsto (\pm n, \pm m)$. From the fact that $K_1(A_\theta)$ is isomorphic to $\mathbb{Z}^2$, with generators corresponding to $[U]$ and $[V]$, it follows that for sufficiently large n the embedding of A_n in A_θ corresponds to an isomorphism $\beta\colon K_1(A_n) \to K_1(A_\theta)$. Hence there is a commuting diagram

$$\begin{array}{ccc} K_1(A_n) & \xrightarrow{\ \sigma_*\ } & K_1(A_n) \\ \beta\big\downarrow & & \big\downarrow\beta \\ K_1(A_\theta) & \xrightarrow[\ \sigma_*\ ]{} & K_1(A_\theta) \end{array}$$

where, from the definition of σ, $\sigma_*\colon K_1(A_\theta) \to K_1(A_\theta)$ is given by the action of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{Z}^2$. It follows that $\sigma_*\colon K_1(A_n) \to K_1(A_n)$ corresponds to an element with determinant 1, so it is given by the action of $\pm I_2$ on $\mathbb{Z}^2$, and that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is conjugate to $\pm I_2$ in $\mathrm{GL}(2, \mathbb{Z})$. The conjugacy can be implemented in $\mathrm{SL}(2, \mathbb{Z})$. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is conjugate to $-I_2$ in $\mathrm{SL}(2, \mathbb{Z})$ then, by the proof of Proposition 3 and Lemma 4 of [5], σ is conjugate to the flip. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is conjugate, and hence equal, to I_2 then $\sigma = \mathrm{id}$. ■

If σ is conjugate to the flip then, by the results of [1] and [6], A_θ possesses a σ-invariant decomposition. Hence the theorem gives necessary and sufficient conditions for this to happen.

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https://doi.org/10.4153/CMB-2001-033-6 Published online by Cambridge University Press