Notes on Everett's Interpolation Formula.

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1. Writing of his now well-known Interpolation Formula, Professor Everett said,* "The only novelty in the formula is the simplicity of its form. . . . The best known formulae for interpolation by central differences are difficult to carry in the memory on account of their unsymmetrical aspect, one law being applicable to the odd and another to the even terms. . . . This disadvantage does not apply to the formula proposed," viz., that which now goes by Everett's name.

2. Professor Everett does not appear to have been aware, and so far as the writer can ascertain it has not been previously pointed out, that Laplace gave † a formula in the same symmetrical form, and indeed immediately transformable into Everett's by a change of sign and origin. Laplace's formula is

$$\begin{aligned} y_{x+i} = (i+1)\{y_x &+ [(i+1)^2-1]\Delta^2 y_{x-1}/3\,! \\ &+ [(i+1)^2-1][(i+1)^2-4]\Delta^4 y_{x-2}/5\,!\ldots\} \\ - i\,\{y_{x-1} &+ [i^2-1]\,\Delta^2 y_{x-2}/3\,! + [i^2-1][i^2-4]\,\Delta^4 y_{x-3}/5\,!\ldots\} \end{aligned}$$

Putting $q = -i$, $p = 1 - q = i + 1$, and advancing the origin by unity we may at once transform Laplace's result to Everett's formula in its usual shape.

3. Laplace proves the formula by his method of Generating Functions, but it is easy to put the proof into the modern shape of

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* *Brit. Ass. Rep.*, 1900, p. 648.

† *Mémoire sur les Suites; (Histoire de l'Acad.* . . . Paris, 1779 (published 1782), pp. 217–221). The demonstration is repeated in the *Théorie Anal. des Probs.*, p. 15–17, and is substantially reproduced in English in the *Encyl. Metropolitana* (Article, "Finite Differences"), Vol. II., pp. 286–7. De Morgan also gives it in his own fashion, *Diff. and Int. Calc.*, p. 545–6.

operators, and so to arrange it as to avoid the final change of sign and origin. He speaks of "finding the value of $1/t^i$ [our E^i] in a series arranged according to powers of z" (our Ω); but in the strict sense this is impossible,* and his final result in our notation takes the form $E^n = F(n, \Omega) - F(n-1, \Omega)E$; *i.e.* it involves E as well as powers of Ω, hence the form of the resulting formula which involves both u_0 and u_1 and their even central differences.

4. Put $\Omega u_0 \equiv \Delta^2 u_{-1} = \dfrac{(E-1)^2}{E} u_0$ or $\Omega = \dfrac{(E-1)^2}{E} = E + E^{-1} - 2.$

Then if α is an "umbral" symbol, whose only office is to carry coefficients of its powers, E^{1-n} will be the coefficient of α^{n-1} in the expansion of

$$\frac{1}{1-\alpha E^{-1}} = \frac{1-\alpha E}{1-\alpha[E+E^{-1}]+\alpha^2} = \frac{1-\alpha E}{1-\alpha[\Omega+2]+\alpha^2} = \frac{1-\alpha E}{(1-\alpha)^2-\alpha\Omega}$$
$$= (1-\alpha E)\left[(1-\alpha)^{-2} + \alpha\Omega(1-\alpha)^{-4} + \alpha^2\Omega^2(1-\alpha)^{-6} + \dots\right].$$

Thus if (m, r) denote the coefficient of α^{m-1} in $(1-\alpha)^{-r}$, the total coefficient of α^{n-1} in the above (*i.e.* the value of E^{1-n}) will be

$$\begin{cases} (n, 2) + (n-1, 4)\Omega + (n-2, 6)\Omega^2 + \dots \\ -E[(\overline{n-1}, 2) + (\overline{n-2}, 4)\Omega + (\overline{n-3}, 6)\Omega^2 + \dots \end{cases}$$

But $(m, r) = \dbinom{m+r-2}{r-1}$, and $\Omega u_0 = \Delta^2 u_{-1}$. Thus

$$E^{1-n} u_0 = u_{1-n} = nu_0 + \frac{(n+1)n(n-1)}{3!}\Delta^2 u_{-1} + \dots$$
$$-\left[(\overline{n-1}u_1) + \frac{n(n-1)(n-2)}{3!}\Delta^2 u_0 + \dots\right]$$

or if $p = 1 - n = -(n-1)$, $1 - p \equiv q = n$, changing the sign of the second line and the sign of each factor in it involving n:

$$u_p = qu_0 + \frac{q(q^2-1)}{3!}\Delta^2 u_{-1} + \dots$$
$$+ pu_1 + \frac{p(p^2-1)}{3!}\Delta^2 u_0 + \dots$$

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* The expansion of E^n in powers of Ω would involve fractional powers, as may be seen by solving the equation $\dfrac{(E-1)^2}{E} = \Omega$.

It is obvious from considerations of symmetry that in any formula for u_n, made up of u_0 and u_1 and their respective even central differences, the coefficients of each pair of corresponding terms must be the same odd functions of n and $1-n$ respectively.

5. Mr D. C. Fraser, M.A., F.I.A. [*Jour. Inst. Actuaries*, vol. 50, p. 19], points out that "Everett's formula can be written

$$u_p = F(p)\,u_1 - F(p-1)\,u_0.$$

When p is an integer a meaning can be given to each of the functions on the right. $F(p)\,u_1$ is the sum of p alternate u's, of which u_1 is the central term; and $F(p-1)\,u_0$ is the sum of $(p-1)$ alternate u's, of which u_0 is the central term. Thus

$$\begin{aligned} F(p)\,u_1 &= u_p + u_{p-2} + u_{p-4} + \dots + u_{-p+2} \\ F(p-1)\,u_0 &= \phantom{u_p +{}} u_{p-2} + u_{p-4} + \dots + u_{-p+2}. \end{aligned}$$

Here $(p-1)$ of the values are common to the two functions, and disappear when the difference is taken, leaving as the result the value u_p."

No proof was given of this interesting result, which may be established as follows:

$F(p)\,.\,u_1 - F(p-1)\,.\,u_0 = u_p = E^p\,u_0$. Replacing u_1 by Eu_0, and then removing the operand u_0,

$$\begin{aligned} & F(p)\,E - F(p-1) &&= E^p \\ \text{or}\quad & F(p)\,E^p - F(p-1)\,E^{p-1} &&= E^{2p-1} \\ \text{say}\quad & U(p) - U(p-1) &&= E^{2p-1} \end{aligned}$$

$$\therefore\quad U(p) = \sum E^{2p+1} \qquad = E^{2p-1} + E^{2p-3} + \dots + E$$

where no constant of integration is required because both sides reduce to E when $p=1$.

$$\therefore\quad F(p) = U(p)\,E^{-p} = E^{p-1} + E^{p-3} + \dots + E^{-(p-1)}$$

$$\begin{aligned} \text{Thus}\quad F(p)\,u_1 &= \left[E^p + E^{p-2} + \dots + E^{-p+2}\right]u_0 \\ F(p-1)\,u_0 &= \left[\phantom{E^p +{}} E^{p-2} + \dots + E^{-p+2}\right]u_0 \end{aligned}$$

and these are equivalent to Fraser's results.

6. The work of the last paragraph suggests that Everett's formula may be deduced symbolically by means of a well-known trigonometric series, viz.,*

$$\sin m\,/\,\phi \cos\phi = m\sin\phi - \frac{m(m^2-2^2)}{3\,!}\sin^3\phi + \ldots$$

Divide both sides by $\sin\phi$; then put $2n$ for m, and $i\phi/2$ for ϕ give the trigonometric functions their exponential values and put $e^{i\phi}\equiv E$, and we get the algebraic result

$$\frac{E^n - E^{-n}}{E - E^{-1}},\ \text{say}\ \psi(n) = n + \frac{n(n^2-1)}{3\,!}(E-1)^2/E + \ldots =$$

But by common algebra $E^n = E\psi(n) + \psi(1-n)$, or when E is used as an operator $E^n u_0 = u_n = \psi(n)u_1 + \psi(1-n)u_0$; and from this Everett's formula follows at once, since $(E-1)^2/E \equiv \Delta^2 E^{-1}$ (see par. 4). This mode of derivation is thought to be new.

7. A simple algebraic transformation of Bessel's formula (so-called, but really Newton's) leads at once to Everett's,† since the latter taken to any order of even differences is identical with Bessel's taken to the next higher order of odd differences [*vide* Everett, *Jour. Inst. Act.*, XXXV., 456; D. C. Fraser, *idem* L, 19]. Bessel's formula may be written

$$u_n = \tfrac{1}{2}(u_0+u_1) + \left[\binom{n}{1}+\binom{n-1}{1}\right]\frac{\Delta u_0}{2} + \ldots + \binom{n'}{2r}\frac{\Delta^{2r}u_{-r}+\Delta^{2r}u_{-r+1}}{2}$$

$$+\tfrac{1}{2}\left[\binom{n'+1}{2r+1}+\binom{n'}{2r+1}\right]\left[\Delta^{2r+1}u_{-r} = \Delta^{2r}u_{-r+1}-\Delta^{2r}u_{-r}\right] + \ldots\ldots$$

where for brevity we write n' for $n+r-1$.

But
$$\binom{n'}{2r} = \binom{n'+1}{2r+1} - \binom{n'}{2r+1},$$

so that the general pair of terms reduces to

$$\binom{n'+1}{2r+1}\Delta^{2r}u_{-r+1} - \binom{n'}{2r+1}\Delta^{2r}u_{-r},$$

* See Hobson's *Plane Trig.* (4th Ed.), p. 276, eq. (8).

† Laplace adopted the converse process and deduced Bessel's and Stirling's formulae from his own form of Everett's, discussed *supra*.

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which takes Everett's form if the coefficients are evaluated, and p and q substituted for n and $(1-n)$ respectively.

8. In this same way it may be shewn * that where the given u's correspond to *unequal* intervals of the variable, the divided-difference interpolation formula corresponding to Bessel's for equal intervals may be put into Everett's form, *i.e.* in terms of two u's and their even divided differences. Let the given values of u be

$$\ldots u_C,\ u_B,\ u_A,\ u_a,\ u_b,\ u_c\ \ldots,$$

where u_x is to be expressed in terms of u_A and u_a and their divided differences, which we shall denote by ∇u, $\nabla^2 u$, ... using the advancing-difference notation for the subscripts. Then we have

$$\begin{aligned} u_x &= u_A + (x-A)\nabla u_A + (x-A)(x-a)\nabla^2 u_B \\ &\qquad\qquad + (x-B)(x-A)(x-a)\nabla^3 u_B + \ldots \\ &= u_a + (x-a)\nabla u_A + (x-A)(x-a)\nabla^2 u_A \\ &\qquad\qquad + (x-A)(x-a)(x-b)\nabla^3 u_B + \ldots \end{aligned}$$

where the odd ∇'s fall on the line between A and a, and the even ∇'s fall on the lines of u_A and u_a.

These formulae can be written down at once by means of W. F. Sheppard's excellent general rules. [*Enc. Britt.*, 11th edition, Vol. XIV., p. 710 (article on "Interpolation"), quoted *Jour. Inst. Act.*, L. 136]. These rules, slightly expanded, are as follow, and they deserve to be better known:

"(i) We start with any tabulated value of u.

"(ii) We pass to the successive differences by steps, each of which may be either upward or downward" [each step involving a new u whose subscript will be numerically the next lower / higher if the step is up / down and the u's are arranged in the numerical sequence of the variables].

"(iii) The new suffix [of u] which is introduced at each step determines the new factor (involving x) for use in the *next* term" [*i.e.* each divided-difference of the nth order has for its coefficient the product of n factors of the form $(x-a_k)$ when a_k represents a value of the variable, and is to be given all the n values that were involved in determining the last *preceding* difference].

These rules apply whether the intervals are equal or unequal; if they are equal the divided-differences are of the form $\Delta^n u / n!$

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* This has been otherwise shewn by R. Todhunter, *Jour. Inst Act.*, L., 136–7, who calls attention to the alternative method here adopted.

9. The formula corresponding to Bessel's will be the mean of the last two results, viz.,

$$u_n = \tfrac{1}{2}(u_A + u_a) + \tfrac{1}{2}(2x - A - a)\,\nabla u_A + \tfrac{1}{2}(x-A)(x-a)\,(\nabla^2 u_B + \nabla^2 u_A)$$
$$+ \tfrac{1}{2}(x-A)(x-a)(2x - B - b)\,\nabla^3 u_B + \dots$$

In the first pair of terms substitute $(u_a - u_A)/(a - A)$ for ∇u_A and we get

$$\frac{a-x}{a-A}u_A + \frac{x-A}{a-A}u_a.$$

In the next pair, note that the factors $(x-A)(x-a)$ are common, and substitute $(\nabla^2 u_A - \nabla^2 u_B)/(b-B)$ for $\nabla^3 u_B$ and we get

$$\frac{(x-A)(x-a)}{b-B}\left\{(b-x)\,\nabla^2 u_B + (x-B)\,\nabla^2 u_A\right\},$$

or
$$\frac{(A-x)(a-x)(b-x)}{b-B}\nabla^2 u_B + \frac{(x-B)(x-A)(x-a)}{b-B}\nabla^2 u_A.$$

Evidently each pair can be similarly treated, and we shall find

$$u_x = \frac{a-x}{a-A}u_A + \frac{(A-x)(a-x)(b-x)}{b-B}\nabla^2 u_B$$
$$+\frac{(B-x)(A-x)(a-x)(b-x)(c-x)}{c-C}\nabla^4 u_C + \dots$$
$$+\frac{x-A}{a-A}u_a + \frac{(x-B)(x-A)(x-a)}{b-B}\nabla^2 u_A$$
$$+\frac{(x-C)(x-B)(x-A)(x-a)(x-b)}{c-C}\nabla^4 u_B + \dots$$

where the law of formation corresponds to Everett's and is obvious.

10. Tables of the coefficients of Δ^2, Δ^4 and Δ^6 in Everett's equal-interval formula have been published [*Tracts for Computers*, No. V.; C. U. Press]. The tables, which are evidently reduced photographically from carefully prepared MS., are given to 10 decimal places for values of the variable from ·001 to ·999 progressing by ·001. The second central differences of the tabular values are also given to enable the coefficients for intermediate values to be calculated readily by means of a subsidiary application of Everett's formula.