Glasgow Mathematical Journal (2024), 1–7
doi:10.1017/S0017089524000272

Glasgow Mathematical Journal

RESEARCH ARTICLE

Semiabelian varieties and transcendence on Weierstrass sigma functions

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Received: 19 August 2023; **Revised:** 9 May 2024; **Accepted:** 12 September 2024

Keywords: p-adic; semiabelian varieties; transcendence; Weierstrass sigma functions

2020 Mathematics Subject Classification: *Primary* - 11J95; *Secondary* - 11J72, 11S99, 14K12

Abstract
We establish new results on complex and p-adic linear independence on a class of semiabelian varieties. As applications, we obtain transcendence results concerning complex and p-adic Weierstrass sigma functions associated with elliptic curves.

1. Introduction

Let G be a commutative algebraic group defined over the field of algebraic numbers $\overline{\mathbb{Q}}$ of positive dimension d, and Lie(G) denote the Lie algebra of G. By fixing a choice of $\overline{\mathbb{Q}}$-basis for Lie(G), one can identify Lie(G) with the $\overline{\mathbb{Q}}$-vector space $\overline{\mathbb{Q}}^d$. The set of complex points $G(\mathbb{C})$ of G has naturally the structure of a complex Lie group whose Lie algebra Lie($G(\mathbb{C})$) is the complex vector space $\mathrm{Lie}(G) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. This is identified with the $\mathbb{C}$-vector space $\mathbb{C}^d$, and one has an analytic homomorphism, the so-called (complex) exponential map $\exp_{G(\mathbb{C})} : \mathbb{C}^d \to G(\mathbb{C})$. A vector $u \in \mathbb{C}^d$ is called a *logarithm of an algebraic point of* G if $\exp_{G(\mathbb{C})}(u)$ is an algebraic point on G, that is $\exp_{G(\mathbb{C})}(u) \in G(\overline{\mathbb{Q}})$. Transcendence result concerning the coordinates of a logarithm of algebraic points of abelian varieties was first given by S. Lang in 1962. Namely, he proved that if A is an abelian variety defined over $\overline{\mathbb{Q}}$ and u is a non-zero logarithm of an algebraic point of A, then at least one of the coordinates of u is transcendental (see [7, Theorem 2]). The p-adic analogue of Lang's result is due to D. Bertrand in 1977. Let $\mathcal{A}_p \subseteq \mathbb{C}_p^d$ denote the p-adic domain of the p-adic exponential map $\exp_{A(\mathbb{C}_p)}$ of A (see [3]). If $\mathfrak{u}$ is a non-zero vector in $\mathcal{A}_p$ such that $\exp_{A(\mathbb{C}_p)}(\mathfrak{u}) \in A(\overline{\mathbb{Q}})$, then there is at least one coordinate of $\mathfrak{u}$ is transcendental (see [2, Proposition 2]). There are some extensions of these results in both complex and p-adic cases, among which D. H. Pham has recently obtained results on complex and p-adic linear independence concerning the coordinates of abelian logarithms of algebraic points (see [11]).

The first aim of this note is to establish results on complex and p-adic linear independence on semiabelian varieties, which are determined by extensions of simple abelian varieties by the multiplicative group $\mathbb{G}_m$. To state the results, let S be a semiabelian variety defined over $\overline{\mathbb{Q}}$ given by the exact sequence

$$1 \to \mathbb{G}_m \to S \to A \to 1,$$

where A is an abelian variety defined over $\overline{\mathbb{Q}}$ (not necessarily simple). It follows from [1, Section 2] that the extension S corresponds to an algebraic point P in $A^*(\overline{\mathbb{Q}})$ via the canonical isomorphism $\mathrm{Ext}^1(A, \mathbb{G}_m) \cong \mathrm{Pic}^0(A) = A^*$. Note that if P is not a torsion point of A^*, the extension S is non-trivial

Dedicated to my Father-in-law, Dr. Ngoc Quang Le on the occasion of his 65th birthday.

(i.e. the semiabelian variety S is not the direct product $\mathbb{G}_m \times A$ of $\mathbb{G}_m$ and A). It is known that the exact sequence

$$1 \to \mathbb{G}_m \to S \to A \to 1$$

induces naturally an exact sequence of $\overline{\mathbb{Q}}$-vector spaces

$$0 \to \mathrm{Lie}(\mathbb{G}_m) \to \mathrm{Lie}(S) \to \mathrm{Lie}(A) \to 0.$$

In particular, this gives $\mathrm{Lie}(S) \cong \mathrm{Lie}(\mathbb{G}_m) \times \mathrm{Lie}(A)$. Hence, by a choice of $\overline{\mathbb{Q}}$-basis for $\mathrm{Lie}(A)$, one can identify the Lie algebra $\mathrm{Lie}(S)$ with the $\overline{\mathbb{Q}}$-vector space $\overline{\mathbb{Q}} \times \overline{\mathbb{Q}}^g = \overline{\mathbb{Q}}^{g+1}$, here $g = \dim A$. Our theorem now reads as follows:

Theorem 1.1. *Let S be a semiabelian variety defined over $\overline{\mathbb{Q}}$ given by an extension $1 \to \mathbb{G}_m \to S \to A \to 1$ with A a simple abelian variety defined over $\overline{\mathbb{Q}}$ of dimension g. Let $\exp_{S(\mathbb{C})}$ and $\exp_{S(\mathbb{C}_p)}$ be the complex and p-adic exponential maps of S, respectively.*

(i) Let $u = (u_0, u_1, \ldots, u_g)$ be a vector in $\mathbb{C}^{g+1}$ with $u_0 \neq 0$ and $(u_1, \ldots, u_g) \neq (0, \ldots, 0)$ such that $\exp_{S(\mathbb{C})}(u) \in S(\overline{\mathbb{Q}})$. Then $1, u_0, u_1, \ldots, u_g$ are linearly independent over $\overline{\mathbb{Q}}$.

(ii) Let $\mathfrak{u} = (\mathfrak{u}_0, \mathfrak{u}_1, \ldots, \mathfrak{u}_g)$ be a vector in $\mathcal{S}_p$ with $\mathfrak{u}_0 \neq 0$ and $(\mathfrak{u}_1, \ldots, \mathfrak{u}_g) \neq (0, \ldots, 0)$ such that $\exp_{S(\mathbb{C}_p)}(\mathfrak{u}) \in S(\overline{\mathbb{Q}})$, where $\mathcal{S}_p$ is the p-adic domain of $\exp_{S(\mathbb{C}_p)}$. Then $1, \mathfrak{u}_0, \mathfrak{u}_1, \ldots, \mathfrak{u}_g$ are linearly independent over $\overline{\mathbb{Q}}$.

2. Semiabelian varieties and algebraic subgroups

This section involves the structure of algebraic subgroups concerning semiabelian varieties defined over $\overline{\mathbb{Q}}$, which plays a crucial point for the proofs in the next section. The following lemma describes the form of a connected algebraic subgroup of the direct product of the additive group $\mathbb{G}_a$ with a semiabelian variety defined over $\overline{\mathbb{Q}}$.

Lemma 2.1. *Let S be a semiabelian variety defined over $\overline{\mathbb{Q}}$ and H a connected algebraic subgroup of the algebraic group $\mathbb{G}_a \times S$ defined over $\overline{\mathbb{Q}}$. Then H is of the form $H_a \times R$, where H_a and R are connected algebraic subgroups of $\mathbb{G}_a$ and S defined over $\overline{\mathbb{Q}}$, respectively.*

Proof. Let π_a and π be the projections of $\mathbb{G}_a \times S$ on $\mathbb{G}_a$ and S, respectively. Put

$$H_a = \pi_a\big(H \cap (\mathbb{G}_a \times \{e\})\big), \quad R = \pi\big(H \cap (\{0\} \times S)\big),$$

where e is the identity element of S. Then H_a and R are connected algebraic subgroups of $\mathbb{G}_a$ and S defined over $\overline{\mathbb{Q}}$, respectively. Let I be the image of H under the projection

$$\mathbb{G}_a \times S \to (\mathbb{G}_a \times S)/(H_a \times R) \cong (\mathbb{G}_a/H_a) \times (S/R).$$

Denote by p_a and p the projections of $(\mathbb{G}_a/H_a) \times (S/R)$ onto $\mathbb{G}_a/H_a$ and onto S/R, respectively. One can show that $I \cong p_a(I)$ and $I \cong p(I)$, hence $p_a(I) \cong p(I)$. On the other hand, H_a is either trivial or $\mathbb{G}_a$, and this leads to the algebraic subgroup $p_a(I)$ of $\mathbb{G}_a/H_a$ is either $\mathbb{G}_a$ or trivial. We claim that the first case cannot hold. In fact, if not, then this gives $p(I) \cong \mathbb{G}_a$. Since $p(I)$ is an algebraic subgroup of S/R defined over $\overline{\mathbb{Q}}$, there exists a connected algebraic subgroup Q of S containing R defined over $\overline{\mathbb{Q}}$ such that $p(I) \cong Q/R$, and then $Q/R \cong \mathbb{G}_a$. Furthermore, it follows from [10, Proposition 2.1.3] that Q and R are semiabelian varieties. We therefore get the short exact sequences

$$1 \to \mathbb{G}_m^q \to Q \to B \to 1$$

and

$$1 \to \mathbb{G}_m^r \to R \to C \to 1$$

of Q and R, respectively, for non-negative integers q, r and for abelian varieties B, C defined over $\overline{\mathbb{Q}}$. Since Q/R is isomorphic to $\mathbb{G}_a$, the composition of the homomorphism of algebraic groups $\mathbb{G}_m^q \to Q$ (from the first exact sequence above) with the projection $Q \to Q/R$ must be trivial. This means that $G_m^q \hookrightarrow R$. We now consider the cases:

If $q > r$, then

$$\mathbb{G}_m^{q-r} = \mathbb{G}_m^q/\mathbb{G}_m^r \hookrightarrow R/\mathbb{G}_m^r \cong C.$$

If $q < r$, then

$$\mathbb{G}_m^{r-q} = \mathbb{G}_m^r/\mathbb{G}_m^q \hookrightarrow Q/\mathbb{G}_m^q \cong B$$

(since $\mathbb{G}_m^r \hookrightarrow R$ from the second exact sequence and R is an algebraic subgroup of Q).

If $q = r$, then

$$\mathbb{G}_a \cong Q/R \cong (Q/\mathbb{G}_m^q)/(R/\mathbb{G}_m^r) \cong B/C.$$

But these cases cannot hold because of the fact that there is no non-trivial algebraic homomorphism from a linear algebraic group to an abelian variety defined over $\overline{\mathbb{Q}}$ (see [5, Lemma 2.3]). Hence, $p_a(I)$ must be trivial, and this shows that so is I. From this, we are able to conclude that $H = H_a \times R$, which completes the proof of the lemma. □

3. Proof of the theorem

(i) Let G be the commutative algebraic group defined by $G = \mathbb{G}_a \times S$. Then G is defined over $\overline{\mathbb{Q}}$ and its Lie algebra $\mathrm{Lie}(G)$ is identified with $\overline{\mathbb{Q}} \times \overline{\mathbb{Q}}^{g+1} = \overline{\mathbb{Q}}^{g+2}$. Suppose by contradiction that the numbers $1, u_0, u_1, \ldots, u_g$ are linearly dependent over $\overline{\mathbb{Q}}$, that is there is a non-zero linear form L in $g+2$ variables with coefficients in $\overline{\mathbb{Q}}$ such that $L(1, u_0, u_1, \ldots, u_g) = 0$. Let V be the $\overline{\mathbb{Q}}$-vector space defined by the zero set of L in $\overline{\mathbb{Q}}^{g+2}$. Then one has the vector $\tilde{u} := (1, u)$ lies in the $\mathbb{C}$-vector space $V \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$, and furthermore,

$$\exp_{G(\mathbb{C})}(\tilde{u}) = (1, \exp_{S(\mathbb{C})}(u)) \in G(\overline{\mathbb{Q}}).$$

Hence, thanks to the analytic subgroup theorem (see [16] or [17]), there exists a connected algebraic subgroup H of G of positive dimension defined over $\overline{\mathbb{Q}}$ for which $\mathrm{Lie}(H)$ is contained in V and $\tilde{u}$ lies in $\mathrm{Lie}(H(\mathbb{C}))$. The above lemma tells us that H must be of the form $H_a \times R$ with H_a an algebraic subgroup of $\mathbb{G}_a$ and R an algebraic subgroup of S (defined over $\overline{\mathbb{Q}}$). Since

$$(1, u_0, u_1, \ldots, u_g) = \tilde{u} \in \mathrm{Lie}(H(\mathbb{C})) = \mathrm{Lie}((H_a \times R)(\mathbb{C})) = \mathrm{Lie}(H_a(\mathbb{C})) \times \mathrm{Lie}(R(\mathbb{C})),$$

it follows that $\mathrm{Lie}(H_a(\mathbb{C}))$ is not trivial, and this leads to H_a must be $\mathbb{G}_a$. In particular, $\mathrm{Lie}(H_a) = \overline{\mathbb{Q}}$ and since $\mathrm{Lie}(H) \subseteq V$, the linear form L is given by $L = a_0X_0 + a_1X_1 + \cdots + a_gX_g$ with coefficients $a_0, a_1, \ldots, a_g \in \overline{\mathbb{Q}}$ not all zero. Let W denote the $\overline{\mathbb{Q}}$-vector space defined by

$$W = \{(w_0, w_1, \ldots, w_g) \in \overline{\mathbb{Q}}^{g+1} : a_0w_0 + a_1w_1 + \cdots + a_gw_g = 0\}.$$

Then one has $\mathrm{Lie}(R) \subseteq W$. Let $\pi : S \to A$ denote the projection from the extension $1 \to \mathbb{G}_m \to S \to A \to 1$. Then we get an isomorphism of algebraic groups $R/R \cap \mathbb{G}_m \cong \pi(R)$. On the other hand, since A is a simple abelian variety, it follows that either $\pi(R)$ is trivial or $\pi(R)$ is A. In the first case, $R = Ker\pi \cong \mathbb{G}_m$, and therefore $\mathrm{Lie}(R)$ is identified with the $\overline{\mathbb{Q}}$-vector subspace $\overline{\mathbb{Q}} \times \{0\} \times \cdots \times \{0\}$ of $\mathrm{Lie}(S) = \overline{\mathbb{Q}} \times \overline{\mathbb{Q}}^g$. Since $\mathrm{Lie}(R) \subseteq W$, it follows that $a_0\alpha = 0$ for all $\alpha \in \overline{\mathbb{Q}}$. This implies that $a_0 = 0$. By the expression of the exponential map $\exp_{S(\mathbb{C})}$ given in [1, Section 2.3], we obtain $\exp_{A(\mathbb{C})}(u_1, \ldots, u_g) \in A(\overline{\mathbb{Q}})$, and therefore by the analytic subgroup theorem again, one can find an algebraic subgroup B of A defined over $\overline{\mathbb{Q}}$ of positive dimension such that $\mathrm{Lie}(B)$ is contained in the $\overline{\mathbb{Q}}$-vector space $\{(z_1, \ldots, z_g) \in \overline{\mathbb{Q}}^g : a_1z_1 + \cdots + a_gz_g = 0\}$. Note that A is simple, this gives B must be equal to A. But this cannot happen since $\dim B = \dim_{\overline{\mathbb{Q}}} \mathrm{Lie}(B) \leq g - 1$. Thus, $\pi(R) = A$ and this means that $R/R \cap \mathbb{G}_m \cong A$. It is clear that $R \cap G_m$ is either trivial or $\mathbb{G}_m$, one can show that $R \cong A$ (since R is a proper algebraic subgroup of S). In this

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case, similarly, we obtain L is of the form $L = a_0X_0$. In particular, $u_0 = 0$ which is a contradiction, and this completes the proof for the first part of Theorem 1.1.

(ii) The proof of the second part follows directly from the p-adic analytic subgroup theorem (see [6] or [9]) and from the same argument as in the proof of the first part above. □

4. Applications to complex and p-adic Weierstrass sigma functions

This last section is devoted to discuss applications concerning the complex and p-adic Weierstrass sigma functions associated with elliptic curves. To begin with, let E be an elliptic curve defined over $\mathbb{C}$ characterised by the Weierstrass model

$$Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3 = 0$$

with $g_2, g_3 \in \mathbb{C}$ such that $g_2^3 - 27g_3^2 \neq 0$. Let Λ denote the period lattice of E, and $\wp$ the Weierstrass elliptic function associated with E (or relative to Λ), which is defined as:

$$\wp(z) = \frac{1}{z^2} + \sum_{w\in\Lambda^*}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right),$$

where $\Lambda^* = \Lambda \setminus \{0\}$. There are two more Weierstrass functions associated with E, which are auxiliary to the function $\wp$, called Weierstrass sigma and zeta functions, respectively. They play important roles in studying elliptic curves and are considered elliptic analogues of the classical trigonometric functions. In detail, the Weierstrass sigma function σ associated with E is defined as:

$$\sigma(z) = z\prod_{w\in\Lambda^*}\left(1 - \frac{z}{w}\right)e^{\frac{z}{w}+\frac{z^2}{2w^2}},$$

and the Weierstrass zeta function ζ associated with E is defined as:

$$\zeta(z) = \frac{1}{z} + \sum_{w\in\Lambda^*}\left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right).$$

These functions are related by:

$$\frac{d}{dz}\log\sigma(z) = \zeta(z); \quad \frac{d}{dz}\zeta(z) = -\wp(z).$$

It is convenient to recall that the corresponding Laurent expansions at the origin of these Weierstrass functions are given by:

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}(\Lambda)z^{2k};$$

$$\zeta(z) = \frac{1}{z} - \sum_{k=1}^{\infty}G_{2k+2}(\Lambda)z^{2k+1},$$

where $G_k(\Lambda) = \sum_{w\in\Lambda^*} w^{-k}$ for $k \geq 3$ are the Eisenstein series (of weight k), and

$$\sigma(z) = \sum_{m,n=0}^{\infty} a_{m,n}\left(\frac{1}{2}g_2\right)^m (2g_3)^n \frac{z^{4m+6n+1}}{(4m+6n+1)!}$$

where the sequence $(a_{m,n})_{m,n\geq 0}$ is defined by the recurrence relation

$$a_{m,n} = 3(m+1)a_{m+1,n+1} + \frac{16}{3}(n+1)a_{m-2,n+1} - \frac{1}{3}(2m+3n-1)(4m+6n-1)a_{m-1,n}$$

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with $a_{0,0} = 1$ and $a_{m,n} = 0$ for either m or n is negative. One can show that $g_2 = 60G_4(\Lambda)$ and $g_3 = 140G_6(\Lambda)$, and by induction, $G_{2k+2}(\Lambda)$ can be represented as polynomials P_{2k+2} in terms of g_2, g_3 with rational coefficients (see [4, Chapter IV]).

Now, let E be an elliptic curve defined over $\overline{\mathbb{Q}}$ and G an extension of E by $\mathbb{G}_m$ defined over $\overline{\mathbb{Q}}$. Recall that the group $\mathrm{Ext}^1(E, \mathbb{G}_m)$ of extension classes of E by $\mathbb{G}_m$ is isomorphic to the Picard group $\mathrm{Pic}^0(E) = E^*$ of divisors on E modulo principal divisors. This means that the extension G corresponds to an algebraic point P in $E(\overline{\mathbb{Q}})$ (here we identify E with E^*). Let $q \in \mathbb{C}$ be an elliptic logarithm of P, that is $\exp_{E(\mathbb{C})}(q) = P$, here the exponential map $\exp_{E(\mathbb{C})}$ of the elliptic curve E is given by:

$$\begin{aligned} \exp_{E(\mathbb{C})} : \mathbb{C} &\to E(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C}) \\ z &\mapsto [\wp(z) : \wp'(z) : 1]. \end{aligned}$$

The exponential map of G is given by:

$$\begin{aligned} \exp_{G(\mathbb{C})} : \mathbb{C}^2 &\to G(\mathbb{C}) \subseteq (\mathbb{P}^2 \times \mathbb{P}^1)(\mathbb{C}) \\ (z, t) &\mapsto \big([\wp(z) : \wp'(z) : 1], [e^t f(z) : 1]\big), \end{aligned}$$

where $f(z) = \dfrac{\sigma(z+q)}{\sigma(z)\sigma(q)} e^{-\zeta(q)z}$ is Serre's function (see [1, Section 6]). We now obtain the following theorem, which is an improvement of a previous result in transcendence given by M. Waldschmidt (see [13, Theorem 1] or [14, Theorem 3.2.10]).

Theorem 4.1. *Let E be an elliptic curve defined over $\overline{\mathbb{Q}}$ and $\wp, \zeta, \sigma$ the Weierstrass elliptic, zeta, sigma functions associated with E, respectively. Let α, β be algebraic numbers and u, u_0 complex numbers such that $\wp(u), \wp(u_0)$ are algebraic numbers. Suppose that u_0 is not a torsion point and that $u, u + u_0$ are not in the lattice of periods of $\wp$. Then, the number*

$$\frac{\sigma(u+u_0)}{\sigma(u)\sigma(u_0)} e^{(\alpha - \zeta(u_0))u + \beta}$$

is transcendental.

Proof. Consider the algebraic point

$$\exp_{E(\mathbb{C})}(u_0) = [\wp(u_0) : \wp'(u_0) : 1] \in E(\overline{\mathbb{Q}}).$$

This algebraic point corresponds to a semiabelian variety G in $\mathrm{Ext}^1(E, \mathbb{G}_m)$. Then one has

$$\exp_{G(\mathbb{C})}(u, \alpha u + \beta) = ([\wp(u) : \wp'(u) : 1], [e^{\alpha u + \beta} f(u) : 1]),$$

with $f(u) = \dfrac{\sigma(u+u_0)}{\sigma(u)\sigma(u_0)} e^{-\zeta(u_0)u}$, and this gives

$$e^{\alpha u + \beta} f(u) = \frac{\sigma(u+u_0)}{\sigma(u)\sigma(u_0)} e^{(\alpha - \zeta(u_0))u + \beta}.$$

Hence, if this number is algebraic then $\exp_{G(\mathbb{C})}(u, \alpha u + \beta) \in G(\overline{\mathbb{Q}})$. Applying the first part of Theorem 1.1, we deduce that the elements $1, u, \alpha u + \beta$ are linearly independent over $\overline{\mathbb{Q}}$. This contradiction proves the theorem. □

It is natural to obtain a p-adic analogue of the above theorem, and in order to express such a result in the p-adic setting, we first recall the p-adic Weierstrass elliptic, zeta and sigma functions, respectively. By definition, the p-adic Weierstrass elliptic function $\wp_p$ is the *(Lutz-Weil) p-adic elliptic function* associated with the elliptic curve E (see [8] and [15]). This function satisfies the relation (as the complex one) $\wp_p'(z) = 4\wp_p^3(z) - g_2\wp_p - g_3$, but only on the neighbourhood of the origin

$$\mathcal{E}_p := \left\{ z \in \mathbb{C}_p; |1/4|_p \max\{|g_2|_p^{1/4}, |g_3|_p^{1/6}\} z \in B(r_p) \right\},$$

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where $B(r_p)$ is the set of all p-adic numbers x in $\mathbb{C}_p$ with $|x|_p < r_p := p^{-\frac{1}{p-1}}$. Note that $\wp_p$ is analytic on $\mathcal{E}_p \setminus \{0\}$ and expressed by:

$$\wp_p(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)P_{2k+2}(g_2, g_3)z^{2k}.$$

Let ζ_p and σ_p be the p-adic Weierstrass zeta and sigma functions, respectively. By definition, ζ_p and σ_p are meromorphic functions on the domain $\mathcal{E}_p \setminus \{0\}$ satisfying the followings:

$$\frac{d}{dz}\log\sigma_p(z) = \zeta_p(z); \quad \frac{d}{dz}\zeta_p(z) = -\wp_p(z),$$

and their corresponding p-adic expansions are also given by:

$$\zeta_p(z) = \frac{1}{z} - \sum_{k=1}^{\infty} P_{2k+2}(g_2, g_3)z^{2k+1}$$

and

$$\sigma_p(z) = \sum_{m,n=0}^{\infty} a_{m,n}\left(\frac{1}{2}g_2\right)^m (2g_3)^n \frac{z^{4m+6n+1}}{(4m+6n+1)!}.$$

We obtain the following theorem.

Theorem 4.2. *Let E be an elliptic curve defined over $\overline{\mathbb{Q}}$ and $\wp_p, \zeta_p, \sigma_p$ the p-adic Weierstrass elliptic, zeta, sigma functions associated with E, respectively. Let α and β be algebraic numbers and $\mathfrak{u}, \mathfrak{u}_0$ non-zero p-adic numbers in $\mathcal{E}_p$ such that $\wp_p(\mathfrak{u}), \wp_p(\mathfrak{u}_0)$ are algebraic numbers. If $\alpha\mathfrak{u} + \beta \in B(r_p)$, then the number*

$$\frac{\sigma_p(\mathfrak{u}+\mathfrak{u}_0)}{\sigma_p(\mathfrak{u})\sigma_p(\mathfrak{u}_0)} e_p^{(\alpha-\zeta_p(\mathfrak{u}_0))\mathfrak{u}+\beta}$$

is transcendental.

Proof. As in the complex case, we first consider the algebraic point

$$\exp_{E(\mathbb{C}_p)}(\mathfrak{u}_0) = [\wp_p(\mathfrak{u}_0) : \wp_p'(\mathfrak{u}_0) : 1] \in E(\overline{\mathbb{Q}}),$$

and this algebraic point gives the corresponding semiabelian variety G_p in $\mathrm{Ext}^1(E, \mathbb{G}_m)$. Since we have seen from above that the p-adic functions $\wp_p$, ζ_p and σ_p are represented by the same power series as the complex functions $\wp$, ζ and σ, respectively, one can show that the p-adic exponential map of G_p is also expressed as the same type as the complex exponential map. More precisely, the map $\exp_{G(\mathbb{C}_p)}$ is given by:

$$\exp_{G(\mathbb{C}_p)} : \mathcal{G}_p \to G_p(\mathbb{C}_p) \subseteq (\mathbb{P}^2 \times \mathbb{P}^1)(\mathbb{C}_p)$$

$$(z,t) \mapsto \big([\wp_p(z) : \wp_p'(z) : 1], [e_p^t f_p(z) : 1]\big),$$

with $\mathcal{G}_p = \mathcal{E}_p \times B(r_p)$ and $f_p(z) = \dfrac{\sigma_p(z+\mathfrak{u}_0)}{\sigma_p(z)\sigma_p(\mathfrak{u}_0)} e_p^{-\zeta_p(\mathfrak{u}_0)z}$ with e_p the usual p-adic exponential function, and prolonged to a function on the whole $\mathbb{C}_p$ (thanks to [12, Section 5.4.4]). Assume by contradiction as the complex case that the number

$$\frac{\sigma_p(\mathfrak{u}+\mathfrak{u}_0)}{\sigma_p(\mathfrak{u})\sigma_p(\mathfrak{u}_0)} e_p^{(\alpha-\zeta_p(\mathfrak{u}_0))\mathfrak{u}+\beta}$$

is algebraic, then we also get $\exp_{G_p(\mathbb{C}_p)}(\mathfrak{u}, \alpha\mathfrak{u}+\beta) \in G_p(\overline{\mathbb{Q}})$, and therefore we are able to deduce from the second part of Theorem 1.1 that $1, \mathfrak{u}, \alpha\mathfrak{u}+\beta$ are linearly independent over $\overline{\mathbb{Q}}$ (a contradiction). This completes the proof of the theorem. □

https://doi.org/10.1017/S0017089524000272 Published online by Cambridge University Press

Acknowledgements. The author would like to thank the editor and the anonymous referee for careful reading of this manuscript and useful comments.

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https://doi.org/10.1017/S0017089524000272 Published online by Cambridge University Press