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Local regularity for nonlocal double phase equations in the Heisenberg group

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We prove interior boundedness and Hölder continuity for the weak solutions of nonlocal double phase equations in the Heisenberg group $\mathbb{H}^n$. This solves a problem raised by Palatucci and Piccinini et al. in 2022 and 2023 for the nonlinear integro-differential problems in Heisenberg setting. Our proof of the a priori estimates bases on De Giorgi–Nash–Moser theory, where the important ingredients are Caccioppoli-type inequality and Logarithmic estimate. To achieve this goal, we establish a new and crucial Sobolev–Poincaré type inequality in local domain, which may be of independent interest and potential applications.

Keywords: energy inequalities; Heisenberg group; nonlocal double phase equation; regularity; Sobolev–Poincaré type inequality

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1. Introduction

In this paper, we are interested in local behaviour of the weak solutions to nonlocal double phase problem in the Heisenberg group $\mathbb{H}^n$, whose prototype is

$$\text{P.V.} \int_{\mathbb{H}^n} \left[\frac{|u(\xi) - u(\eta)|^{p-2}(u(\xi) - u(\eta))}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} + a(\xi, \eta) \frac{|u(\xi) - u(\eta)|^{q-2}(u(\xi) - u(\eta))}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+tq}} \right] d\eta = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $1 < p \leq q < \infty$, $s, t \in (0,1)$, $a(\cdot,\cdot) \geq 0$, $Q = 2n+2$ is the homogeneous dimension and Ω is an open bounded subset of $\mathbb{H}^n$ ($n \geq 1$). In the display above, $\|\cdot\|_{\mathbb{H}^n}$ and P.V. mean the standard Heisenberg norm and "in the principal value sense", respectively. The main feature of the integro-differential equation (1.1) is that the leading operator could change between two different fractional elliptic phases according to whether the modulating coefficient a is zero or not.

We observe that, if the coefficient $a \equiv 0$, equation (1.1) is reduced to the p-fractional subLaplace equation arising in many diverse contexts, such as quantum mechanics, image segmentation models, ferromagnetic analysis and so on. Let us pay attention to the linear scenario first, i.e., $p=2$. This kind of problems can be regarded as an extension of the conformally invariant fractional subLaplacian $(-\Delta_{\mathbb{H}^n})^s$ in $\mathbb{H}^n$ proposed initially in [2] by the spectral formula

$$(-\Delta_{\mathbb{H}^n})^s := 2^s |T|^s \frac{\Gamma\left(-\frac{1}{2}\Delta_H |T|^{-1} + \frac{1+s}{2}\right)}{\Gamma\left(-\frac{1}{2}\Delta_H |T|^{-1} + \frac{1-s}{2}\right)}, \quad s \in (0,1),$$

where $s \in (0,1)$, $\Gamma(\cdot)$ is the Euler Gamma function, T is the vertical vector field, and $\Delta_{\mathbb{H}^n}$ is the typical Kohn–Spencer subLaplacian on $\mathbb{H}^n$. Subsequently, Roncal and Thangavelu [36] demonstrated the representation as below

$$(-\Delta_{\mathbb{H}^n})^s u(\xi) := C(n,s) \text{P.V.} \int_{\mathbb{H}^n} \frac{u(\xi) - u(\eta)}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+2s}} d\eta, \quad \xi \in \mathbb{H}^n, \tag{1.2}$$

holds true for $C(n,s) > 0$ depending only on n, s. During the last decade, several aspects of the fractional operator of the type (1.2) have been investigated, such as Hardy and uncertainty inequalities on stratified Lie groups [6], Sobolev and Morrey-type embedding theory for fractional Sobolev space $H^s(\mathbb{H}^n)$ [1], Harnack and Hölder estimates in Carnot groups [18], Liouville-type theorem [7]. One can refer to [19–22] and references therein for more results on the linear case. Regarding the nonlinear analogue to (1.2), the p-growth scenario is considered ($p \neq 2$). For what concerns the regularity properties of weak solutions to the fractional p-subLaplace equations on the Heisenberg group, Manfredini et al. [31] established the interior boundedness and Hölder continuity via employing the De Giorge–Nash–Moser iteration; see also [32] for the nonlocal Harnack inequality, where the asymptotic behaviour of fractional linear operator was proved as well. In addition, as for the obstacle problems connected with the nonlocal p-subLaplacian, we refer to [34] in which Piccinini studied systematically solvability, semicontinuity, boundedness and Hölder regularity up to the boundary for weak solutions. More interesting estimates or fundamental functional inequalities can be found in [27, 28, 33]. To some extent, we can see that the results mentioned above extended the counterparts of the fractional Euclidean setting in [13, 14, 26, 29, 30] to the Heisenberg framework.

Equation (1.1) could be viewed naturally as the nonlocal version of the classical double phase problem of the following type

$$-\text{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = 0 \quad \text{in } \Omega. \tag{1.3}$$

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Within the Euclidean context, the regularity theory of weak solutions to (1.3) or minimizers of the corresponding functionals has been developed extensively, beginning with the pioneering papers of Colombo and Mingione [8, 9]. Under $a \in L^\infty_{\rm loc}(\Omega)$ and, $p \le q \le \frac{np}{n-p}$ if $p < n$, or $p \le q < \infty$ if $p \ge n$, local boundedness for u was shown; and further under $u \in L^\infty_{\rm loc}(\Omega)$, $a \in C^{0,\alpha}_{\rm loc}(\Omega)$ and $p \le q \le p + \alpha$, Hölder continuity of u was obtained as well, see, e.g. [9, 10].

Very recently, the investigation of nonlocal problems with nonstandard growth, especially of those with (p, q)-growth condition, has been attracting increasing attention, however only in the fractional Euclidean spaces. In this respect, De Filippis and Palatucci [12] introduced nonlocal double phase equations of the form (1.1) in the Euclidean spaces, and established Hölder continuity for bounded viscosity solutions. Weak theory on this class of nonlocal equations was rapidly explored in hot pursuit, for example, [37] for self-improving inequalities on bounded weak solutions, [17] for Hölder regularity and relationship between weak and viscosity solutions in the differentiability exponents $s \ge t$, [4] for Hölder property with weaker assumption on solutions in the case $s < t$, [24] for the sharp Hölder index and the parabolic version. Concerning more regularity and related results for nonlocal problems possessing nonuniform growth, one can see [3, 5, 16, 23, 35] and references therein.

In particular, we would like to mention that Palatucci, Piccinini, et al. in a series of papers [31–33] proposed the open problems about the regualrity of solutions to the so-called nonlocal double phase equation in the Heisenberg group $\mathbb{H}^n$. In this paper, influenced by the works [4, 14] we answer this question and develop the local regularity theory for the weak solutions of such equations in the Heisenberg group $\mathbb{H}^n$, including the boundedness and Hölder continuity of solutions. The main difficulties which are different from the previous ones are mainly two parts. One is that equation (1.1) not only possesses the nonlocal feature of the embraced integro-differential operators and the noneuclidean geometrical structure of the Heisenberg group, but also inherits the typical characteristics exhibited by the (local) double phase problems due to the (p, q)-growth condition and the presence of the non-negative variable coefficient a. We need to find some appropriate assumptions on the summability exponents $p, q \in (1, \infty)$ and differentiability exponents $s, t \in (0, 1)$ together with the variable coefficient a in order to locally rebalance the non-uniform ellipticity of the operator. The other one is that the existing Sobolev embedding theorem, lemma 2.2, cannot be applied to our setting directly. To overcome this point, we have to establish a suitable Sobolev–Poincaré type inequality on balls in the Heisenberg group $\mathbb{H}^n$. It may be of independent interest and potential applications when investigating regularity properties for some other nonlocal equations in the Heisenberg group. These difficulties make the current study more challenging than the fractional p-subLaplacian case.

Now we are in a position to state our main contributions. We first collect some notations, definitions as well as assumptions. Let s, t and p, q satisfy

$$1 < p \le q < \infty, \quad 0 < s \le t < 1, \tag{1.4}$$

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and the coefficient $a : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}^+$ fulfil

$$0 \le a(\xi,\eta) = a(\eta,\xi) \le \|a\|_{L^\infty}, \quad \xi,\eta \in \mathbb{H}^n, \tag{1.5}$$

and

$$|a(\xi,\eta) - a(\xi',\eta')| \le [a]_\alpha \Big(\|{\xi'}^{-1} \circ \xi\|_{\mathbb{H}^n} + \|{\eta'}^{-1} \circ \eta\|_{\mathbb{H}^n} \Big)^\alpha, \tag{1.6}$$

for $(\xi,\eta), (\xi',\eta') \in \mathbb{H}^n \times \mathbb{H}^n$ and $\alpha \in (0,1]$.

For convenience, we introduce the following notations:

$$H(\xi,\eta,\tau) := \frac{\tau^p}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{sp}} + a(\xi,\eta) \frac{\tau^q}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{tq}}, \quad \xi,\eta \in \mathbb{H}^n \text{ and } \tau > 0,$$

and

$$J_l(\tau_1 - \tau_2) = |\tau_1 - \tau_2|^{l-2}(\tau_1 - \tau_2),$$

with $\tau_1, \tau_2 \in \mathbb{R}$ and $l \in \{p,q\}$, and

$$\rho(u;\Omega) = \int_\Omega \int_\Omega H(\xi,\eta,|u(\xi) - u(\eta)|) \frac{d\xi d\eta}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q}},$$

for every measurable set $\Omega \subset \mathbb{H}^n$ and $u : \Omega \to \mathbb{R}$. A function space related to weak solutions to (1.1) is defined as

$$\begin{aligned} \mathcal{A}(\Omega) : = \{ u : \mathbb{H}^n \to \mathbb{R} : u|_\Omega \in L^p(\Omega) \ \text{ and} \\ \iint_{\mathcal{C}_\Omega} H(\xi,\eta,|u(\xi) - u(\eta)|) \frac{d\xi d\eta}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q}} < \infty \Big\}, \end{aligned}$$

where

$$\mathcal{C}_\Omega := (\mathbb{H}^n \times \mathbb{H}^n) \setminus ((\mathbb{H}^n \backslash \Omega) \times (\mathbb{H}^n \backslash \Omega)).$$

Additionally, in view of the nonlocal nature of this problem, we need define a tail space

$$L_{sp}^{q-1}(\mathbb{H}^n) := \left\{ u \in L_{\text{loc}}^{q-1}(\mathbb{H}^n) : \int_{\mathbb{H}^n} \frac{|u(\xi)|^{q-1}}{(1 + \|\xi\|_{\mathbb{H}^n})^{Q+sp}} d\xi < \infty \right\},$$

and the nonlocal tail

$$T(u;\xi_0,r) := \int_{\mathbb{H}^n \backslash B_r(\xi_0)} \left(\frac{|u(\xi)|^{p-1}}{\|\xi_0^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} + \|a\|_{L^\infty} \frac{|u(\xi)|^{q-1}}{\|\xi_0^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+tq}} \right) d\xi.$$

We can notice that the quantity T is finite if $u \in L_{sp}^{q-1}(\mathbb{H}^n)$.

We now give the definition of weak solutions to (1.1).

Definition 1.1. weak solution *If $u \in \mathcal{A}(\Omega)$ satisfies*

$$\iint_{\mathcal{C}_\Omega} \left[\frac{J_p(u(\xi)-u(\eta))(\varphi(\xi)-\varphi(\eta))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} + a(\xi,\eta)\frac{J_q(u(\xi)-u(\eta))(\varphi(\xi)-\varphi(\eta))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}} \right] d\xi d\eta = 0, \tag{1.7}$$

for every $\varphi \in \mathcal{A}(\Omega)$ with $\varphi = 0$ a.e. in $\mathbb{H}^n \backslash \Omega$, then we call u a weak solution to (1.1).

Note that $u \in \mathcal{A}(\Omega)$ implies $u \in HW^{s,p}(\Omega)$, i.e., $\mathcal{A}(\Omega) \subset HW^{s,p}(\Omega)$. Hence in this work, we only consider the case $sp \le Q$. Otherwise, the complementary scenario $sp > Q$ ensures the local boundedness and Hölder continuity directly because of the fractional Morrey embedding in the Heisenberg group [1].

Our main results are stated as follows. The first one is the local boundedness of weak solutions.

Theorem 1.2 *Let the conditions (1.4) and (1.5) be in force. If*

$$\begin{cases} p \le q \le \frac{Qp}{Q-sp} & \text{when } sp < Q, \\ p \le q < \infty & \text{when } sp \ge Q, \end{cases} \tag{1.8}$$

then every weak solution $u \in \mathcal{A}(\Omega) \cap L^{q-1}_{sp}(\mathbb{H}^n)$ to (1.1) is locally bounded in Ω.

The second one is about the Hölder regularity of weak solutions to (1.1) via supposing $a(\cdot,\cdot)$ is Hölder continuous and the distance between q and p is small. For simplicity, we denote

$$\mathbf{data} := \mathbf{data}(n, p, q, s, t, \alpha, [a]_\alpha, \|a\|_{L^\infty}),$$

as the set of basic parameters intervening in the problem.

Theorem 1.3 *Let the conditions (1.4)–(1.6) with*

$$tq \le sp + \alpha, \tag{1.9}$$

be in force. If weak solution $u \in \mathcal{A}(\Omega) \cap L^{q-1}_{sp}(\mathbb{H}^n)$ to (1.1) has local boundedness in Ω, then it is locally Hölder continuous as well, that is, for any subset $\Omega' \subset\subset \Omega$, u belongs to $C^{0,\beta}_{\mathrm{loc}}(\Omega')$ with some $\beta \in \left(0, \frac{sp}{q-1}\right)$ depending on **data** *and $\|u\|_{L^\infty(\Omega')}$.*

Putting these two theorems above, Hölder continuity is immediately obtained without local boundedness assumption under the intersecting conditions.

Remark 1.4. For the case $s > t$, local boundedness can be obtained under (1.5), (1.8) by checking the proof of theorem 1.2. Meanwhile, following the proof of theorem 1.3 and making a few slight modifications, we can deduce, under the same preconditions of theorem 1.3, that weak solutions are also of the class $C^{0,\beta}_{\mathrm{loc}}(\Omega')$ with some $\beta \in \left(0, \frac{\min\{sp,tq\}}{q-1}\right)$.

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This paper is organized as follows. In § 2, we introduce the Heisenberg group and function spaces, and then deduce some needful Sobolev embedding theorems. Section 3 is dedicated to proving local boundedness of weak solutions by the Caccioppoli-type estimate. At last, we shall show that the locally bounded weak solutions to (1.1) are Hölder continuous via establishing Logarithmic-type inequality in § 4.

2. Functional setting

In this section, we introduce the Heisenberg group $\mathbb{H}^n$ and some function spaces, and establish several important Sobolev embedding results. The Euclidean space $\mathbb{R}^{2n+1}$ $(n \geq 1)$ with the group multiplication

$$\xi \circ \eta = \left(x_1 + y_1, x_2 + y_2, \cdots, x_{2n} + y_{2n}, \tau + \tau' + \frac{1}{2} \sum_{i=1}^{n} \left(x_i y_{n+i} - x_{n+i} y_i \right) \right),$$

where $\xi = (x_1, x_2, \cdots, x_{2n}, \tau)$, $\eta = (y_1, y_2, \cdots, y_{2n}, \tau') \in \mathbb{R}^{2n+1}$, leads to the Heisenberg group $\mathbb{H}^n$. The left invariant vector field on $\mathbb{H}^n$ is of the form

$$X_i = \partial_{x_i} - \frac{x_{n+i}}{2} \partial_\tau, \; X_{n+i} = \partial_{x_{n+i}} + \frac{x_i}{2} \partial_\tau, \quad 1 \leq i \leq n,$$

and a non-trivial commutator is

$$T = \partial_\tau = [X_i, X_{n+i}] = X_i X_{n+i} - X_{n+i} X_i, \; 1 \leq i \leq n.$$

We call that $X_1, X_2, \cdots, X_{2n}$ are the horizontal vector fields on $\mathbb{H}^n$ and T the vertical vector field. Denote the horizontal gradient of a smooth function u on $\mathbb{H}^n$ by

$$\nabla_H u = (X_1 u, X_2 u, \cdots, X_{2n} u).$$

The Haar measure in $\mathbb{H}^n$ is equivalent to the Lebesgue measure in $\mathbb{R}^{2n+1}$. We denote the Lebesgue measure of a measurable set $E \subset \mathbb{H}^n$ by $|E|$. For $\xi = (x_1, x_2, \cdots, x_{2n}, \tau)$, we define its module as

$$\|\xi\|_{\mathbb{H}^n} = \left(\left(\sum_{i=1}^{2n} {x_i}^2 \right)^2 + \tau^2 \right)^{\frac{1}{4}}.$$

The Carnot-Carathéodary metric between two points ξ and η in $\mathbb{H}^n$ is the shortest length of the horizontal curve joining them, denoted by $d(\xi, \eta)$. The C-C metric is equivalent to the Korànyi metric, i.e., $d(\xi, \eta) \sim \|\xi^{-1} \circ \eta\|_{\mathbb{H}^n}$. The ball

$$B_r(\xi_0) = \{\xi \in \mathbb{H}^n : d(\xi, \xi_0) < r\},$$

is defined by the C-C metric d. When not important or clear from the context, we will omit the center as follows: $B_r := B_r(\xi_0)$.

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Let $1 \leq p < \infty$, $s \in (0,1)$, and $v : \mathbb{H}^n \to \mathbb{R}$ be a measurable function. The Gagliardo semi-norm of v is defined as

$$[v]_{HW^{s,p}(\mathbb{H}^n)} = \left(\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{|v(\xi) - v(\eta)|^p}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta \right)^{\frac{1}{p}},$$

and the fractional Sobolev spaces $HW^{s,p}(\mathbb{H}^n)$ on the Heisenberg group are defined as

$$HW^{s,p}(\mathbb{H}^n) = \left\{ v \in L^p(\mathbb{H}^n) : [v]_{HW^{s,p}(\mathbb{H}^n)} < \infty \right\},$$

endowed with the natural fractional norm

$$\|v\|_{HW^{s,p}(\mathbb{H}^n)} = \left(\|v\|_{L^p(\mathbb{H}^n)}^p + [v]_{HW^{s,p}(\mathbb{H}^n)}^p \right)^{\frac{1}{p}}.$$

For any open set $\Omega \subset \mathbb{H}^n$, we can define similarly fractional Sobolev spaces $HW^{s,p}(\Omega)$ and fractional norm $\|v\|_{HW^{s,p}(\Omega)}$. The space $HW_0^{s,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $HW^{s,p}(\Omega)$. Throughout this paper, we denote a generic positive constant as c or C. If necessary, relevant dependencies on parameters will be illustrated by parentheses, i.e., $c = c(n,p)$ means that c depends on n, p. Now we recall the fractional Poincaré type inequality and Sobolev embedding in the Heisenberg group $\mathbb{H}^n$; see [34, proposition 2.7] and [28, theorem 2.5], respectively.

LEMMA 2.1. Poincaré type inequality *Let $p \geq 1$, $s \in (0,1)$ and $v \in HW^{s,p}(B_r)$. Then we have*

$$\int_{B_r} |v - (v)_r|^p \, d\xi \leq c r^{sp} \int_{B_r} \int_{B_r} \frac{|v(\xi) - v(\eta)|^p}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta,$$

where $c = c(n,p) > 0$, $(v)_r = \fint_{B_r} v d\xi$.

LEMMA 2.2. *Let $1 < p < \infty$, $s \in (0,1)$ such that $sp < Q$. Let also $v : \mathbb{H}^n \to \mathbb{R}$ be a measurable compactly supported function. Then there is a positive constant $c = c(n,p,s)$ such that*

$$\|v\|_{L^{p_s^*}(\mathbb{H}^n)}^p \leq c [v]_{HW^{s,p}(\mathbb{H}^n)}^p,$$

with $p_s^* = \frac{Qp}{Q-sp}$ being a critical Sobolev exponent.

Now we also give the following result, a truncation lemma near $\partial\Omega$.

LEMMA 2.3. *Let $p \geq 1$, $s \in (0,1)$ and $v \in HW^{s,p}(B_r)$. If $\varphi \in C^{0,1}(B_r) \cap L^\infty(B_r)$, then it holds that $\varphi v \in HW^{s,p}(B_r)$ and $\|\varphi v\|_{HW^{s,p}(B_r)} \leq c \|v\|_{HW^{s,p}(B_r)}$ with $c > 0$ depending on n, p, s, r and φ.*

The proof of this lemma is very similar to that of [15, lemma 5.3], so we omit it here. Based on lemmas 2.1–2.3, we could conclude a Sobolev–Poincaré inequality on balls in the Heisenberg group, which plays a crucial role in proving regularity of solutions.

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Proposition 2.4. Sobolev–Poincaré type inequality *Let* $1 < p < \infty$, $s \in (0,1)$ *fulfil* $sp < Q$. *Suppose that* $v \in HW^{s,p}(B_R(\xi_0))$ *and* $B_r(\xi_0) \subset B_R(\xi_0)$ $(0 < r < R)$ *are concentric balls. Then there exists a positive constant* $c = c(n,p,s)$ *such that*

$$\left(\fint_{B_r} |v - (v)_r|^{p_s^*} \, d\xi\right)^{\frac{p}{p_s^*}} \leq cD_1(R,r) \fint_{B_R} \int_{B_R} \frac{|v(\xi) - v(\eta)|^p}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta,$$

where

$$D_1(R,r) := r^{sp} \left(\frac{R}{r}\right)^{2Q} \left[\left(\frac{R}{R-r}\right)^p + \left(\frac{R}{R-r}\right)^{Q+sp}\right].$$

Proof. Take $\varphi(\xi) \in C_0^\infty(B_R(\xi_0))$ as a cut-off function such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_r(\xi_0)$, $\operatorname{supp} \varphi \subset B_{\frac{R+r}{2}}(\xi_0)$ and $|\nabla_H \varphi| \leq \frac{c}{R-r}$ in $B_R(\xi_0)$. Then $(v - (v)_r)\varphi \in HW_0^{s,p}(B_R)$ and further $(v - (v)_r)\varphi \in HW_0^{s,p}(\mathbb{H}^n)$ by zero extension. We split $\mathbb{H}^n \times \mathbb{H}^n$ into

$$(B_R \times B_R) \cup (\mathbb{H}^n \backslash B_R \times B_R) \cup (B_R \times \mathbb{H}^n \backslash B_R) \cup (\mathbb{H}^n \backslash B_R \times \mathbb{H}^n \backslash B_R).$$

By virtue of lemma 2.2 and the definition of φ, we get

$$\begin{aligned}
\left(\int_{B_r} |v - (v)_r|^{p_s^*} \, d\xi\right)^{\frac{p}{p_s^*}} &\leq \left(\int_{\mathbb{H}^n} |(v - (v)_r)\varphi|^{p_s^*} \, d\xi\right)^{\frac{p}{p_s^*}} \\
&\leq c \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{|(v(\xi) - (v)_r)\varphi(\xi) - (v(\eta) - (v)_r)\varphi(\eta)|^p}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta \\
&\leq c \int_{B_R} \int_{B_R} \frac{|(v(\xi) - (v)_r)\varphi(\xi) - (v(\eta) - (v)_r)\varphi(\eta)|^p}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta \\
&\quad + c \int_{B_R} \int_{H^n \backslash B_R} \frac{|(v(\eta) - (v)_r)\varphi(\eta)|^p}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta \\
&=: J_1 + J_2.
\end{aligned}$$

Note that

$$\begin{aligned}
J_1 &\leq c \int_{B_R} \int_{B_R} \frac{|v(\xi) - v(\eta)|^p}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta + c \int_{B_R} \int_{B_R} \frac{|\varphi(\xi) - \varphi(\eta)|^p |(v(\eta) - (v)_r)|^p}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta \\
&=: c \int_{B_R} \int_{B_R} \frac{|v(\xi) - v(\eta)|^p}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta + J_{11}.
\end{aligned}$$

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We first evaluate J_{11} as

$$
\begin{aligned}
J_{11} &\leq \frac{c}{(R-r)^p}\int_{B_R}\int_{B_R}\frac{|v(\eta)-(v)_r|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+p(s-1)}}\,d\xi d\eta \\
&\leq \frac{c}{(R-r)^p}\int_{B_R}|v(\eta)-(v)_r|^p\int_{B_{2R}(\eta)}\frac{1}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+p(s-1)}}\,d\xi d\eta \\
&\leq c\left(\frac{R}{R-r}\right)^p R^{-sp}\int_{B_R}|v(\eta)-(v)_r|^p\,d\eta \\
&\leq c\left(\frac{R}{R-r}\right)^p R^{-sp}\left(\int_{B_R}|v(\eta)-(v)_R|^p\,d\eta+\int_{B_R}|(v)_R-(v)_r|^p\,d\eta\right) \\
&\leq c\left(\frac{R}{R-r}\right)^p R^{-sp}\left(R^{sp}\int_{B_R}\int_{B_R}\frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\,d\xi d\eta+|(v)_R-(v)_r|^p\,|B_R|\right),
\end{aligned}
$$

where in the last line we have utilized lemma 2.1. On the other hand,

$$
\begin{aligned}
|(v)_R-(v)_r|^p\,|B_R| &= |B_R|\left|\fint_{B_r}(v-(v)_R)\,d\xi\right|^p \\
&\leq |B_R|\fint_{B_r}|v-(v)_R|^p d\xi \\
&\leq \frac{|B_R|}{|B_r|}\int_{B_R}|v-(v)_R|^p d\xi \\
&\leq c\left(\frac{R}{r}\right)^Q R^{sp}\int_{B_R}\int_{B_R}\frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}d\xi d\eta.
\end{aligned}
$$

Thus

$$
\begin{aligned}
J_1 &\leq c\left(1+\left(\frac{R}{R-r}\right)^p+\left(\frac{R}{r}\right)^Q\left(\frac{R}{R-r}\right)^p\right)\int_{B_R}\int_{B_R}\frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}d\xi d\eta \\
&\leq c\left(\frac{R}{r}\right)^Q\left(\frac{R}{R-r}\right)^p\int_{B_R}\int_{B_R}\frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^nn}^{Q+sp}}d\xi d\eta.
\end{aligned}
$$

Moreover, for $\xi\in\mathbb{H}^n\backslash B_R$, $\eta\in B_{\frac{R+r}{2}}$, owing to the triangle inequality [11] there holds that

$$
\begin{aligned}
\|\xi^{-1}\circ\xi_0\|_{\mathbb{H}^n} &\leq \left(1+\frac{\|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n}}{\|\xi^{-1}\circ\eta\|_{\mathbb{H}^n}}\right)\|\xi^{-1}\circ\eta\|_{\mathbb{H}^n} \\
&\leq \left(1+\frac{(R+r)/2}{(R-r)/2}\right)\|\xi^{-1}\circ\eta\|_{\mathbb{H}^n}=\frac{2R}{R-r}\|\xi^{-1}\circ\eta\|_{\mathbb{H}^n}.
\end{aligned}
$$

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From this, it follows that

$$\begin{aligned}
J_2 &\le c\int_{B_{\frac{R+r}{2}}}\int_{\mathbb{H}^n\setminus B_R}\frac{|v(\eta)-(v)_r|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\,d\xi d\eta\\
&\le c\left(\frac{R}{R-r}\right)^{Q+sp}\int_{\mathbb{H}^n\setminus B_R}\frac{1}{\|\xi^{-1}\circ\xi_0\|_{\mathbb{H}^n}^{Q+sp}}d\xi\int_{B_{\frac{R+r}{2}}}|v(\eta)-(v)_r|^p\,d\eta\\
&\le c\frac{R^Q}{(R-r)^{Q+sp}}\int_{B_R}|v(\eta)-(v)_r|^p\,d\eta\\
&\le c\frac{R^Q}{(R-r)^{Q+sp}}\left(R^{sp}+\frac{R^{Q+sp}}{r^Q}\right)\int_{B_R}\int_{B_R}\frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\,d\xi d\eta\\
&\le c\left(\frac{R}{r}\right)^Q\left(\frac{R}{R-r}\right)^{Q+sp}\int_{B_R}\int_{B_R}\frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n n}^{Q+sp}}d\xi d\eta,
\end{aligned}$$

the procedure of which is analogous to J_1. Eventually, we obtain

$$\begin{aligned}
&\left(\int_{B_r}|v-(v)_r|^{p_s^*}\,d\xi\right)^{\frac{p}{p_s^*}}\\
&\le c\left(\frac{R}{r}\right)^Q\left[\left(\frac{R}{R-r}\right)^p+\left(\frac{R}{R-r}\right)^{Q+sp}\right]\int_{B_R}\int_{B_R}\frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n n}^{Q+sp}}d\xi d\eta,
\end{aligned}$$

which implies the statement. □

If we let $R=2r$ in the preceding Sobolev–Poincaré inequality, then we can get the very simple version below.

Corollary 2.5. *Let $1<p<\infty, s\in(0,1)$ fulfil $sp<Q$. Suppose that $v\in HW^{s,p}(B_{2r})$ and $B_r\subset B_{2r}$ are concentric balls. Then there exists a positive constant $c(n,p,s)$ such that*

$$\left(\fint_{B_r}|v-(v)_r|^{p_s^*}\,d\xi\right)^{\frac{p}{p_s^*}}\le cr^{sp}\fint_{B_{2r}}\int_{B_{2r}}\frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\,d\xi d\eta.$$

The following result shows an embedding relation between the fractional Sobolev spaces $HW^{t,q}(\Omega)$ and $HW^{s,p}(\Omega)$.

Lemma 2.6. *Let $1<p\le q$ and $0<s<t<1$. Let also Ω be a bounded measurable subset of $\mathbb{H}^n$. Then there holds that, for each $v\in HW^{t,q}(\Omega)$,*

$$\left(\int_\Omega\int_\Omega\frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}d\xi d\eta\right)^{\frac{1}{p}}\le c|\Omega|^{\frac{q-p}{pq}}(\operatorname{diam}(\Omega))^{t-s}\left(\int_\Omega\int_\Omega\frac{|v(\xi)-v(\eta)|^q}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}d\xi d\eta\right)^{\frac{1}{q}},$$

where $c>0$ depends upon n,p,q,s,t.

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Proof. For $p<q$, we first utilize the Hölder inequality to get

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\,d\xi d\eta \\
&= \int_\Omega\int_\Omega \frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{(Q+tq)\frac{p}{q}}}\frac{1}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q\frac{q-p}{q}+(s-t)p}}\,d\xi d\eta \\
&\le \left(\int_\Omega\int_\Omega \frac{|v(\xi)-v(\eta)|^q}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\,d\xi d\eta\right)^{\frac{p}{q}}\left(\int_\Omega\int_\Omega \frac{1}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+\frac{(s-t)pq}{q-p}}}\,d\xi d\eta\right)^{\frac{q-p}{q}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_\Omega\int_\Omega \frac{1}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+\frac{(s-t)pq}{q-p}}}\,d\xi d\eta &\le \int_\Omega\int_{B_d(\eta)} \frac{1}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+\frac{(s-t)pq}{q-p}}}\,d\xi d\eta \\
&\le Q\,|B_1|\int_\Omega\int_0^d \rho^{\frac{(t-s)pq}{q-p}-1}\,d\rho d\eta \\
&= \frac{Q\,|B_1|\,(q-p)}{(t-s)pq}d^{\frac{(t-s)pq}{q-p}}\,|\Omega|\,,
\end{aligned}$$

with $d:=\operatorname{diam}(\Omega)$. The combination of preceding inequalities implies the desired display.

If $q=p$, noting $\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}\le \operatorname{diam}(\Omega)$ for $\xi,\eta\in\Omega$ and $s<t$, we can readily obtain

$$\begin{aligned}
\left(\int_\Omega\int_\Omega \frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\,d\xi d\eta\right)^{\frac{1}{p}} &= \left(\int_\Omega\int_\Omega \frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tp}}\frac{1}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{(s-t)p}}\,d\xi d\eta\right)^{\frac{1}{p}} \\
&\le (\operatorname{diam}(\Omega))^{t-s}\left(\int_\Omega\int_\Omega \frac{|v(\xi)-v(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tp}}\,d\xi d\eta\right)^{\frac{1}{p}}.
\end{aligned}$$

Now, we complete the proof. □

The forthcoming two lemmas are the consequences of these results above, which will be exploited in the proof of boundedness and Hölder continuity for solutions.

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LEMMA 2.7. *Assume that $s,t\in(0,1)$, $1<p\le q$ and (1.8) hold. Then for every $f\in HW^{s,p}(B_r)$ we infer that*

$$\fint_{B_r}\left(\left|\frac{f}{r^s}\right|^p+a_0\left|\frac{f}{r^t}\right|^q\right)d\xi\le ca_0\frac{D_1^{\frac{q}{p}}(R,r)}{r^{tq}}\left(\fint_{B_R}\int_{B_R}\frac{|f(\xi)-f(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}d\xi d\eta\right)^{\frac{q}{p}}$$
$$+c\frac{D_1(R,r)}{r^{sp}}\left(\frac{|\operatorname{supp}f|}{|B_r|}\right)^{\frac{sp}{Q}}\fint_{B_R}\int_{B_R}\frac{|f(\xi)-f(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}d\xi d\eta$$
$$+c\left(\frac{R}{r}\right)^Q\left(\frac{|\operatorname{supp}f|}{|B_r|}\right)^{p-1}\fint_{B_R}\left(\left|\frac{f}{r^s}\right|^p+a_0\left|\frac{f}{r^t}\right|^q\right)d\xi,$$

where $\operatorname{supp}f:=\{B_r: f\neq 0\}$, *and $c>0$ depends only upon n,p,q,s,t, and a_0 is any positive constant.*

Proof. By the Hölder inequality and proposition 2.4, we obtain

$$\fint_{B_r}\left|\frac{f}{r^s}\right|^p d\xi\le c\fint_{B_r}\left|\frac{f-(f)_r}{r^s}\right|^p\chi_{\{f\neq0\}}\,d\xi+c\left|\frac{(f)_r}{r^s}\right|^p$$
$$\le c\left(\frac{|\operatorname{supp}f|}{|B_r|}\right)^{\frac{sp}{Q}}\left(\fint_{B_r}\left|\frac{f-(f)_r}{r^s}\right|^{p_s^*}d\xi\right)^{\frac{p}{p_s^*}}+c\left|\frac{(f)_r}{r^s}\right|^p$$
$$\le c\frac{D_1(R,r)}{r^{sp}}\left(\frac{|\operatorname{supp}f|}{|B_r|}\right)^{\frac{sp}{Q}}\fint_{B_R}\int_{B_R}\frac{|f(\xi)-f(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}d\xi d\eta$$
$$+c\left(\frac{|\operatorname{supp}f|}{|B_r|}\right)^{p-1}\fint_{B_r}\left|\frac{f}{r^s}\right|^p d\xi,$$

where we used the inequality below,

$$\left|\frac{(f)_r}{r^s}\right|^p=r^{-sp}\left|\fint_{B_r}f\chi_{\{f\neq0\}}\,d\xi\right|^p\le\left(\frac{|\operatorname{supp}f|}{|B_r|}\right)^{p-1}\fint_{B_r}\left|\frac{f}{r^s}\right|^p d\xi.$$

On the other hand, via the Hölder inequality and proposition 2.4 again,

$$\fint_{B_r}\left|\frac{f}{r^t}\right|^q d\xi\le c\left(\fint_{B_r}\left|\frac{f-(f)_r}{r^t}\right|^{p_s^*}d\xi\right)^{\frac{q}{p_s^*}}+c\left|\frac{(f)_r}{r^t}\right|^q$$
$$\le c\frac{D_1^{\frac{q}{p}}(R,r)}{r^{tq}}\left(\fint_{B_R}\int_{B_R}\frac{|f(\xi)-f(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}d\xi d\eta\right)^{\frac{q}{p}}+c\left|\frac{(f)_r}{r^t}\right|^q$$
$$\le c\frac{D_1^{\frac{q}{p}}(R,r)}{r^{tq}}\left(\fint_{B_R}\int_{B_R}\frac{|f(\xi)-f(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}d\xi d\eta\right)^{\frac{q}{p}}$$
$$+c\left(\frac{|\operatorname{supp}f|}{|B_r|}\right)^{p-1}\fint_{B_r}\left|\frac{f}{r^t}\right|^q d\xi,$$

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where we can see that

$$\left|\frac{(f)_r}{r^t}\right|^q \leq \left(\frac{|\operatorname{supp} f|}{|B_r|}\right)^{q-1} \fint_{B_r} \left|\frac{f}{r^t}\right|^q d\xi \leq \left(\frac{|\operatorname{supp} f|}{|B_r|}\right)^{p-1} \fint_{B_r} \left|\frac{f}{r^t}\right|^q d\xi.$$

We finally observe the plain relation that

$$\fint_{B_r} \left|\frac{f}{r^s}\right|^p + a_0 \left|\frac{f}{r^t}\right|^q d\xi \leq c \left(\frac{R}{r}\right)^Q \fint_{B_R} \left|\frac{f}{r^s}\right|^p + a_0 \left|\frac{f}{r^t}\right|^q d\xi.$$

In summary, we combine all the previous inequalities to arrive at the desired display. □

Now denote

$$a_R^+ := \sup_{B_R \times B_R} a(\cdot,\cdot) \quad \text{and} \quad a_R^- := \inf_{B_R \times B_R} a(\cdot,\cdot).$$

LEMMA 2.8. *Let $s, t \in (0,1)$, $1 < p \leq q$ and $a(\cdot,\cdot)$ satisfy (1.6) and (1.9). Assume $f \in HW^{t,q}(B_{\bar{R}}) \cap L^\infty(B_{\bar{R}})$ with $\bar{R} \leq 1$. Then for $\gamma := \min\left\{\frac{p_s^*}{p}, \frac{q_t^*}{q}\right\} > 1$, we have*

$$\begin{aligned}
&\left[\fint_{B_r} \left(\left|\frac{f}{r^s}\right|^p + a_{\bar{R}}^+ \left|\frac{f}{r^t}\right|^q\right)^\gamma d\xi\right]^{\frac{1}{\gamma}} \\
&\leq c\left(1 + \|f\|_{L^\infty(B_r)}^{q-p}\right)\left(\frac{D_1(R,r)}{r^{sp}} + \frac{\widetilde{D}_1(R,r)}{r^{tq}}\right) \fint_{B_R} \int_{B_R} \frac{H(\xi,\eta,|f(\xi)-f(\eta)|)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^Q} d\xi d\eta \\
&\quad + c\left(1 + \|f\|_{L^\infty(B_r)}^{q-p}\right) \fint_{B_R} \left(\left|\frac{f}{r^s}\right|^p + a_{\bar{R}}^- \left|\frac{f}{r^t}\right|^q\right) d\xi,
\end{aligned}$$

where $B_r \subset B_R \subseteq B_{\bar{R}}$ are concentric balls with $\frac{1}{2}\bar{R} \leq r < R \leq \bar{R}$, and $c > 0$ depends only on n, p, q, s, t and $[a]_\alpha$. Here $\widetilde{D}_1(R,r)$ is the corresponding $D_1(R,r)$ defined in proposition 2.4 with sp replaced by tq.

Proof. In view of Hölder continuity of a, we have

$$a_{\bar{R}}^+ \leq a_{\bar{R}}^- + 4[a]_\alpha \bar{R}^\alpha \leq a_{\bar{R}}^- + 8[a]_\alpha r^\alpha.$$

Then we by employing $tq \leq sp + \alpha$, $r \leq 1$ have

$$a_{\bar{R}}^+ \left|\frac{f}{r^t}\right|^q \leq a_{\bar{R}}^- \left|\frac{f}{r^t}\right|^q + c r^{\alpha - tq + sp} |f|^{q-p} \left|\frac{f}{r^s}\right|^p.$$

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Thus

$$\begin{aligned}
&\left[\fint_{B_r}\left(\left|\frac{f}{r^s}\right|^p+a_{\bar{R}}^+\left|\frac{f}{r^t}\right|^q\right)^\gamma d\xi\right]^{\frac{1}{\gamma}}\\
&\le c\left(1+\|f\|_{L^\infty(B_r)}^{q-p}\right)\left[\fint_{B_r}\left(\left|\frac{f}{r^s}\right|^p+a_{\bar{R}}^-\left|\frac{f}{r^t}\right|^q\right)^\gamma d\xi\right]^{\frac{1}{\gamma}}\\
&\le c\left(1+\|f\|_{L^\infty(B_r)}^{q-p}\right)\left[\fint_{B_r}\left(\left|\frac{f-(f)_r}{r^s}\right|^p+a_{\bar{R}}^-\left|\frac{f-(f)_r}{r^t}\right|^q\right)^\gamma d\xi\right]^{\frac{1}{\gamma}}\\
&\quad+c\left(1+\|f\|_{L^\infty(B_r)}^{q-p}\right)\left(\left|\frac{(f)_r}{r^s}\right|^p+a_{\bar{R}}^-\left|\frac{(f)_r}{r^t}\right|^q\right).
\end{aligned}$$

Observe that

$$\left|\frac{(f)_r}{r^s}\right|^p+a_{\bar{R}}^-\left|\frac{(f)_r}{r^t}\right|^q\le\fint_{B_r}\left(\left|\frac{f}{r^s}\right|^p+a_{\bar{R}}^-\left|\frac{f}{r^t}\right|^q\right)d\xi.$$

Moreover, it follows from proposition 2.4 that

$$\begin{aligned}
\left[\fint_{B_r}\left|\frac{f-(f)_r}{r^s}\right|^{p\gamma}d\xi\right]^{\frac{1}{\gamma}}&\le\left(\fint_{B_r}\left|\frac{f-(f)_r}{r^s}\right|^{p_s^*}d\xi\right)^{\frac{p}{p_s^*}}\\
&\le\frac{cD_1(R,r)}{r^{sp}}\fint_{B_R}\int_{B_R}\frac{|f(\xi)-f(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}d\xi d\eta,
\end{aligned}$$

and

$$\begin{aligned}
\left[\fint_{B_r}\left|\frac{f-(f)_r}{r^t}\right|^{q\gamma}d\xi\right]^{\frac{1}{\gamma}}&\le\left(\fint_{B_r}\left|\frac{f-(f)_r}{r^t}\right|^{q_t^*}d\xi\right)^{\frac{q}{q_t^*}}\\
&\le\frac{c\widetilde{D}_1(R,r)}{r^{tq}}\fint_{B_R}\int_{B_R}\frac{|f(\xi)-f(\eta)|^q}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}d\xi d\eta.
\end{aligned}$$

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Merging the last four inequalities leads to

$$\begin{aligned}
&\left[\fint_{B_r}\left(\left|\frac{f}{r^s}\right|^p+a_{\bar{R}}^+\left|\frac{f}{r^t}\right|^q\right)^\gamma d\xi\right]^{\frac{1}{\gamma}}\\
&\leq c\left(1+\|f\|_{L^\infty(B_r)}^{q-p}\right)\left(\frac{D_1(R,r)}{r^{sp}}+\frac{\widetilde{D}_1(R,r)}{r^{tq}}\right)\\
&\quad\cdot\fint_{B_R}\int_{B_R}\left(\frac{|f(\xi)-f(\eta)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}+a_{\bar{R}}^-\frac{|f(\xi)-f(\eta)|^q}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\right)d\xi d\eta\\
&\quad+c\left(1+\|f\|_{L^\infty(B_r)}^{q-p}\right)\fint_{B_R}\left(\left|\frac{f}{r^s}\right|^p+a_{\bar{R}}^-\left|\frac{f}{r^t}\right|^q\right)d\xi\\
&\leq c\left(1+\|f\|_{L^\infty(B_r)}^{q-p}\right)\left(\frac{D_1(R,r)}{r^{sp}}+\frac{\widetilde{D}_1(R,r)}{r^{tq}}\right)\fint_{B_R}\int_{B_R}\frac{H(\xi,\eta,|f(\xi)-f(\eta)|)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}d\xi d\eta\\
&\quad+c\left(1+\|f\|_{L^\infty(B_r)}^{q-p}\right)\fint_{B_R}\left(\left|\frac{f}{r^s}\right|^p+a_{\bar{R}}^-\left|\frac{f}{r^t}\right|^q\right)d\xi.
\end{aligned}$$

We now finish the proof. □

3. Local boundedness

This section is devoted to showing the interior boundedness of weak solutions to equation (1.1) by means of the key ingredient, a Caccioppoli-type inequality in the nonlocal framework. The forthcoming lemma indicates the multiplication of each function in $\mathcal{A}(\Omega)$ and a cut-off function also belongs to $\mathcal{A}(\Omega)$.

LEMMA 3.1. *Let s,t,p and q satisfy (1.4) and $\varphi\in HW_0^{1,\infty}(B_r), v\in\mathcal{A}(\Omega)$. If one of the following two conditions holds:*

(i) The inequality (1.8) holds and $v\in L^p(B_{2r})$ satisfies $\rho(v;B_{2r})<\infty$;
(ii) $v\in L^q(B_{2r})$ satisfies $\rho(v;B_{2r})<\infty$,

then $\rho(v\varphi;\mathbb{H}^n)<\infty$. In particular, $v\varphi\in\mathcal{A}(\Omega)$ whenever $B_{2r}\subset\Omega$.

Proof. By $v\in\mathcal{A}(\Omega)$, proposition 2.4 and (1.8), we get $v\in L^q\left(B_{3r/2}\right)$ in (i). Thus, we just consider condition (ii). By the definition of $\rho(v\varphi;\mathbb{H}^n)$, we have

$$\begin{aligned}
\rho(v\varphi;\mathbb{H}^n)&=2\int_{\mathbb{H}^n\setminus B_{3r/2}}\int_{B_{3r/2}}H(\xi,\eta,|v(\xi)\varphi(\xi)-v(\eta)\varphi(\eta)|)\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^Q}\\
&\quad+\int_{B_{3r/2}}\int_{B_{3r/2}}H(\xi,\eta,|v(\xi)\varphi(\xi)-v(\eta)\varphi(\eta)|)\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^Q}\\
&=:2I_1+I_2. \qquad (3.1)
\end{aligned}$$

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Owing to $\varphi \in HW_0^{1,\infty}(B_r)$, we find

$$I_1 \le \left(\|\varphi\|_{L^\infty(B_r)}+1\right)^q \int_{\mathbb{H}^n\setminus B_{3r/2}}\int_{B_r}\left(\frac{|v(\xi)|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}+\|a\|_{L^\infty}\frac{|v(\xi)|^q}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\right)d\xi d\eta$$
$$\le c\left(\|\varphi\|_{L^\infty(B_r)}+1\right)^q\left(r^{-sp}\int_{B_r}|v(\xi)|^p\,d\xi+\|a\|_{L^\infty}r^{-tq}\int_{B_r}|v(\xi)|^q\,d\xi\right)<\infty. \quad (3.2)$$

The term I_2 is estimated as

$$\begin{aligned}
I_2 &\le c\int_{B_{3r/2}}\int_{B_{3r/2}} H(\xi,\eta,|(v(\xi)-v(\eta))\varphi(\eta)|)\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\\
&\quad + c\int_{B_{3r/2}}\int_{B_{3r/2}} H(\xi,\eta,|v(\xi)(\varphi(\xi)-\varphi(\eta))|)\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\\
&\le c\left(\|\varphi\|_{L^\infty(B_r)}+1\right)^q\int_{B_{3r/2}}\int_{B_{3r/2}} H(\xi,\eta,|v(\xi)|)\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\\
&\quad + c\|\nabla_H\varphi\|^p_{L^\infty(B_r)}\int_{B_{3r/2}}|v(\xi)|^p\int_{B_{3r}}\frac{d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+(s-1)p}}\,d\xi\\
&\quad + c\|\nabla_H\varphi\|^q_{L^\infty(B_r)}\|a\|_{L^\infty}\int_{B_{3r/2}}|v(\xi)|^q\int_{B_{3r}}\frac{d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+(t-1)p}}\,d\xi\\
&\le c\left(\|\varphi\|_{L^\infty(B_r)}+1\right)^q\rho(v;B_{2r})+c\|\nabla_H\varphi\|^p_{L^\infty(B_r)}r^{(1-s)p}\int_{B_{2r}}|v(\xi)|^p\,d\xi\\
&\quad + c\|\nabla_H\varphi\|^q_{L^\infty(B_r)}\|a\|_{L^\infty}r^{(1-t)q}\int_{B_{2r}}|v(\xi)|^q\,d\xi\\
&< \infty. \qquad (3.3)
\end{aligned}$$

Thus, it follows $\rho(v\varphi;\mathbb{H}^n)<\infty$ by combining (3.2), (3.3) with (3.1). □

Next, we prove a nonlocal Caccioppoli-type inequality. Define

$$h(\xi,\eta,\tau):=\frac{\tau^{p-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{sp}}+a(\xi,\eta)\frac{\tau^{q-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{tq}},\qquad \xi,\eta\in\mathbb{H}^n \text{ and } \tau>0. \quad (3.4)$$

The numerical inequality below, to be exploited frequently, is from [14, lemma 3.1].

Lemma 3.2. *Let $p\ge 1$ and $a,b\ge 0$. Then we have*

$$a^p-b^p\le pa^{p-1}|a-b|,$$

and

$$a^p-b^p\le \varepsilon b^p+c\varepsilon^{1-p}|a-b|^p,$$

for any $\varepsilon\in(0,1)$ and some $c=c(p)>0$.

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LEMMA 3.3. Caccioppoli-type inequality *Let $B_{2r}(\xi_0) \subset\subset \Omega$, $1 < p \le q$, (1.5) and (1.8) hold. Assume $u \in \mathcal{A}(\Omega)$ is a weak solution to (1.1). Then for any $\phi \in C_0^\infty(B_r)$ with $0 \le \phi \le 1$, we have*

$$\begin{aligned}
&\int_{B_r}\int_{B_r} H\left(\xi,\eta,|w_\pm(\xi)-w_\pm(\eta)|\right)\left(\phi^q(\xi)+\phi^q(\eta)\right)\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\\
\le& c\int_{B_r}\int_{B_r} H\left(\xi,\eta,|(\phi(\xi)-\phi(\eta))(w_\pm(\xi)+w_\pm(\eta))|\right)\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\\
&+c\left(\sup_{\xi\in\operatorname{supp}\phi}\int_{\mathbb{H}^n\setminus B_r} h\left(\xi,\eta,|w_\pm(\eta)|\right)\frac{d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\right)\int_{B_r} w_\pm(\xi)\phi^q(\xi)\,d\xi, \quad (3.5)
\end{aligned}$$

for some $c := c(n,s,t,p,q) > 0$, where $w_\pm := (u-k)_\pm$ with $k \ge 0$.

Proof. We just consider the estimate for w_+, since the estimate for w_- can be proved similarly. By lemma 3.1, it follows that $w_+\phi^q \in \mathcal{A}(\Omega)$ from $u \in \mathcal{A}(\Omega)$ and $\phi \in C_0^\infty(B_r) \subset HW_0^{1,\infty}(B_r)$, so we can take the testing function $\varphi = w_+\phi^q$ in (1.7). Then we have

$$\begin{aligned}
0 =& \int_{B_r}\int_{B_r}\left[\frac{J_p(u(\xi)-u(\eta))(w_+(\xi)\phi^q(\xi)-w_+(\eta)\phi^q(\eta))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\right.\\
&\left.+a(\xi,\eta)\frac{J_q(u(\xi)-u(\eta))(w_+(\xi)\phi^q(\xi)-w_+(\eta)\phi^q(\eta))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\right]d\xi d\eta\\
&+2\int_{\mathbb{H}^n\setminus B_r}\int_{B_r}\left[\frac{J_p(u(\xi)-u(\eta))w_+(\xi)\phi^q(\xi)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\right.\\
&\left.+a(\xi,\eta)\frac{J_q(u(\xi)-u(\eta))w_+(\xi)\phi^q(\xi)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\right]d\xi d\eta\\
=:& J_1+J_2. \qquad (3.6)
\end{aligned}$$

We first estimate J_1. Since J_1 is symmetry for ξ and η, we may suppose without loss of generality that $u(\xi) \ge u(\eta)$. Then for $l \in \{p,q\}$, it yields

$$\begin{aligned}
&J_l\left(u(\xi)-u(\eta)\right)\left(w_+(\xi)\phi^q(\xi)-w_+(\eta)\phi^q(\eta)\right)\\
=&\begin{cases}
\left(w_+(\xi)-w_+(\eta)\right)^{l-1}\left(w_+(\xi)\phi^q(\xi)-w_+(\eta)\phi^q(\eta)\right), & \text{if } u(\xi)\ge u(\eta)\ge k\\
\left(u(\xi)-u(\eta)\right)^{l-1}w_+(\xi)\phi^q(\xi), & \text{if } u(\xi)\ge k\ge u(\eta)\\
0, & \text{if } k\ge u(\xi)\ge u(\eta)
\end{cases}\\
\ge& J_l\left(w_+(\xi)-w_+(\eta)\right)\left(w_+(\xi)\phi^q(\xi)-w_+(\eta)\phi^q(\eta)\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&w_+(\xi)\phi^q(\xi)-w_+(\eta)\phi^q(\eta)\\
=&\frac{w_+(\xi)-w_+(\eta)}{2}\left(\phi^q(\xi)+\phi^q(\eta)\right)+\frac{w_+(\xi)+w_+(\eta)}{2}\left(\phi^q(\xi)-\phi^q(\eta)\right),
\end{aligned}$$

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which implies

$$\begin{aligned}&J_l\left(w_+\left(\xi\right)-w_+\left(\eta\right)\right)\left(w_+\left(\xi\right)\phi^q\left(\xi\right)-w_+\left(\eta\right)\phi^q\left(\eta\right)\right)\\ \geq &|w_+(\xi)-w_+(\eta)|^l\frac{\phi^q(\xi)+\phi^q(\eta)}{2}-|w_+(\xi)-w_+(\eta)|^{l-1}\frac{w_+(\xi)+w_+(\eta)}{2}|\phi^q(\xi)-\phi^q(\eta)|.\end{aligned}$$

Since

$$\begin{aligned}\left|\phi^q\left(\xi\right)-\phi^q\left(\eta\right)\right|&\leq q\left(\phi^{q-1}\left(\xi\right)+\phi^{q-1}\left(\eta\right)\right)\left|\phi\left(\xi\right)-\phi\left(\eta\right)\right|\\&\leq c\left(q\right)\left(\phi^q\left(\xi\right)+\phi^q\left(\eta\right)\right)^{\frac{q-1}{q}}\left|\phi\left(\xi\right)-\phi\left(\eta\right)\right|,\end{aligned}$$

from lemma 3.2, we use Young's inequality, $0\leq\phi\leq 1$ and $\frac{q-1}{q}>0$ to deduce that

$$\begin{aligned}&\left|w_+\left(\xi\right)-w_+\left(\eta\right)\right|^{l-1}\frac{w_+\left(\xi\right)+w_+\left(\eta\right)}{2}\left|\phi^q\left(\xi\right)-\phi^q\left(\eta\right)\right|\\ &\leq c\left(q\right)\left|w_+\left(\xi\right)-w_+\left(\eta\right)\right|^{l-1}\left(w_+\left(\xi\right)+w_+\left(\eta\right)\right)\left(\phi^q\left(\xi\right)+\phi^q\left(\eta\right)\right)^{\frac{l-1}{l}+\frac{q-l}{ql}}\left|\phi\left(\xi\right)-\phi\left(\eta\right)\right|\\ &\leq \varepsilon\left|w_+\left(\xi\right)-w_+\left(\eta\right)\right|^l\left(\phi^q\left(\xi\right)+\phi^q\left(\eta\right)\right)\\ &\quad+c\left(\varepsilon,q\right)\left(\phi^q\left(\xi\right)+\phi^q\left(\eta\right)\right)^{\frac{q-l}{q}}\left|\phi\left(\xi\right)-\phi\left(\eta\right)\right|^l\left(w_+\left(\xi\right)+w_+\left(\eta\right)\right)^l\\ &\leq \varepsilon\left|w_+\left(\xi\right)-w_+\left(\eta\right)\right|^l\left(\phi^q\left(\xi\right)+\phi^q\left(\eta\right)\right)+c\left(\varepsilon,q\right)\left|\phi\left(\xi\right)-\phi\left(\eta\right)\right|^l\left(w_+\left(\xi\right)+w_+\left(\eta\right)\right)^l.\end{aligned}$$

Then, by choosing ε small enough, we have

$$\begin{aligned}&J_l\left(w_+\left(\xi\right)-w_+\left(\eta\right)\right)\left(w_+\left(\xi\right)\phi^q\left(\xi\right)-w_+\left(\eta\right)\phi^q\left(\eta\right)\right)\\ \geq&\left|w_+\left(\xi\right)-w_+\left(\eta\right)\right|^l\frac{\phi^q\left(\xi\right)+\phi^q\left(\eta\right)}{4}-c|\phi(\xi)-\phi\left(\eta\right)|^l\left(w_+\left(\xi\right)+w_+\left(\eta\right)\right)^l.\end{aligned}$$

Thus, we get

$$\begin{aligned}J_1\geq&\int_{B_r}\int_{B_r}\left[\frac{|w_+(\xi)-w_+(\eta)|^p(\phi^q(\xi)+\phi^q(\eta))/4-c|\phi(\xi)-\phi(\eta)|^p(w_+(\xi)+w_+(\eta))^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\right.\\ &+a(\xi,\eta)\frac{|w_+\left(\xi\right)-w_+\left(\eta\right)|^q(\phi^q(\xi)+\phi^q(\eta))/4}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\\ &\left.-\frac{c|\phi(\xi)-\phi(\eta)|^q(w_+\left(\xi\right)+w_+\left(\eta\right))^q}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\right]d\xi d\eta\\ \geq&\int_{B_r}H\left(\xi,\eta,\left|w_+\left(\xi\right)-w_+\left(\eta\right)\right|\right)\left(\phi^q\left(\xi\right)+\phi^q\left(\eta\right)\right)\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\\ &-c\int_{B_r}\int_{B_r}H\left(\xi,\eta,|\phi(\xi)-\phi(\eta)|\left(w_+\left(\xi\right)+w_+\left(\eta\right)\right)\right)\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}.\end{aligned}\tag{3.7}$$

Now we estimate J_2. Note that

$$J_l\left(u\left(\xi\right)-u\left(\eta\right)\right)w_+\left(\xi\right)\geq -w_+^{l-1}\left(\eta\right)w_+\left(\xi\right).\tag{3.8}$$

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In fact, when $u(\xi) \geq u(\eta)$, it easy to see that the inequality (3.8) holds. When $u(\xi) < u(\eta)$ and $u(\xi) \leq k$, $w_+(\xi) = 0$, the inequality (3.8) also holds. When $k < u(\xi) < u(\eta)$,

$$J_l(u(\xi) - u(\eta))w_+(\xi) = -|w_+(\xi) - w_+(\eta)|^{l-1} w_+(\xi) \geq -w_+^{l-1}(\eta) w_+(\xi).$$

Thus, we apply (3.8) and (3.4) to get

$$\begin{aligned} J_2 &= 2\int_{\mathbb{H}^n \setminus B_r} \int_{B_r} \left[\frac{J_p(u(\xi) - u(\eta))w_+(\xi)\phi^q(\xi)}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} + a(\xi,\eta) \frac{J_q(u(\xi) - u(\eta))w_+(\xi)\phi^q(\xi)}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+tq}} \right] \\ &\quad \times d\xi d\eta \\ &\geq -c\int_{\mathbb{H}^n \setminus B_r} \int_{B_r} \left[\frac{w_+^{p-1}(\eta)w_+(\xi)\phi^q(\xi)}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{sp}} + a(\xi,\eta) \frac{w_+^{q-1}(\eta)w_+(\xi)\phi^q(\xi)}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{tq}} \right] \frac{d\xi d\eta}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q}} \\ &\geq -c \left(\sup_{\xi \in \operatorname{supp} \phi} \int_{\mathbb{H}^n \setminus B_r} h(\xi,\eta,w_+(\eta)) \frac{d\eta}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q}} \right) \int_{B_r} w_\pm(\xi)\phi^q(\xi)\, d\xi. \end{aligned} \tag{3.9}$$

Combining (3.6), (3.7) with (3.9), we get (3.5). □

The following standard iteration lemma can be found in [25, lemma 7.1].

Lemma 3.4. *Let* $\{y_i\}_{i=0}^{\infty}$ *be a sequence of nonnegative numbers satisfying*

$$y_{i+1} \leq b_1 b_2^i y_i^{1+\beta},\ i = 0, 1, 2, \cdots$$

for some constants $b_1,\ \beta > 0$ *and* $b_2 > 1$. *If*

$$y_0 \leq b_1^{-\frac{1}{\beta}} b_2^{-\frac{1}{\beta^2}},$$

then $y_i \to 0$ *as* $i \to \infty$.

We end this section by providing the proof of boundedness. Lemmas 2.7 and 3.3 play the vital roles in the process.

Proof of theorem 1.2. For convenience, denote

$$H_0(\tau) = \tau^p + \|a\|_{L^\infty} \tau^q, \quad \tau \geq 0.$$

Let $B_r \equiv B_r(\xi_0) \subset\subset \Omega$ be a fixed ball with $r \leq 1$. For $i = 0, 1, 2, \cdots$ and $k_0 > 0$, we write

$$r_i := \frac{r}{2}\left(1 + 2^{-i}\right), \quad \sigma_i := \frac{r_{i-1} + r_i}{2}, \quad k_i := 2k_0\left(1 - 2^{-i-1}\right)$$

and

$$y_i := \int_{A^+(k_i, r_i)} H_0\left((u(\xi) - k_i)_+\right)\, d\xi.$$

In addition, we denote

$$A^+(k_i, r_i) := \left\{\xi \in B_{r_i} : u(\xi) \geq k_i\right\}.$$

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Then via $(u(\xi)-k_i)_+ \leq (u(\xi)-k_{i-1})_+$,

$$A^+(k_i,r_i) \subset A^+(k_{i-1},r_i) \subset A^+(k_{i-1},r_{i-1}). \tag{3.10}$$

Moreover, for $\xi \in A^+(k_i,r_i)$, we have

$$(u(\xi)-k_{i-1})_+ = u(\xi)-k_{i-1} \geq k_i-k_{i-1} = 2^{-i}k_0.$$

Thus, it deduces

$$\left|A^+(k_i,r_i)\right| \leq \int_{A^+(k_i,r_i)} \frac{(u(\xi)-k_{i-1})_+^p}{(k_i-k_{i-1})^p} d\xi \leq k_0^{-p}2^{ip}y_{i-1} \tag{3.11}$$

and

$$\begin{aligned}\int_{Br_{i-1}} (u(\xi)-k_i)_+ \, d\xi &\leq \int_{Br_{i-1}} (u(\xi)-k_{i-1})_+ \left(\frac{(u(\xi)-k_{i-1})_+}{k_i-k_{i-1}}\right)^{p-1} d\xi \\ &\leq k_0^{1-p}2^{i(p-1)} \int_{Br_{i-1}} H_0\left((u(\xi)-k_{i-1})_+\right) \, d\xi \\ &= k_0^{1-p}2^{i(p-1)}y_{i-1}. \end{aligned} \tag{3.12}$$

We use lemma 2.7 with $f := (u-k)_+, a_0 := \|a\|_{L^\infty}$ and (3.11) to get

$$\begin{aligned} y_i &\leq cr_i^Q \fint_{B_{r_i}} H_0\left((u(\xi)-k_i)_+\right) \, d\xi \\ &\leq cr_i^{Q+sp} \fint_{B_{r_i}} \left(\left|\frac{(u(\xi)-k_i)_+}{r_i^s}\right|^p + \|a\|_{L^\infty}\left|\frac{(u(\xi)-k_i)_+}{r_i^t}\right|^q\right) \, d\xi \\ &\leq c\|a\|_{L^\infty} r_i^{Q+sp-tq} D_1^{\frac{q}{p}}(\sigma_i,r_i)\left(\fint_{B_{\sigma_i}}\int_{B_{\sigma_i}} \frac{\left|(u(\xi)-k_i)_+ - (u(\eta)-k_i)_+\right|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta\right)^{\frac{q}{p}} \\ &\quad + cr_i^{Q-sp} D_1(\sigma_i,r_i)\left(A^+(k_i,r_i)\right)^{\frac{sp}{Q}} \fint_{B_{\sigma_i}}\int_{B_{\sigma_i}} \frac{\left|(u(\xi)-k_i)_+ - (u(\eta)-k_i)_+\right|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta \\ &\quad + cr_i^{Q+sp}\left(\frac{A^+(k_i,r_i)}{|B_{r_i}|}\right)^{p-1} \fint_{B_{\sigma_i}} \left(\left|\frac{(u(\xi)-k_i)_+}{r_i^s}\right|^p + \|a\|_{L^\infty}\left|\frac{(u(\xi)-k_i)_+}{r_i^t}\right|^q\right) \, d\xi \\ &\leq c\|a\|_{L^\infty} r_i^{Q+sp-tq} D_1^{\frac{q}{p}}(\sigma_i,r_i)\left(\fint_{B_{\sigma_i}}\int_{B_{\sigma_i}} \frac{\left|(u(\xi)-k_i)_+ - (u(\eta)-k_i)_+\right|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta\right)^{\frac{q}{p}} \\ &\quad + ck_0^{-\frac{sp^2}{Q}} r_i^{Q-sp} 2^{i\frac{sp^2}{Q}} D_1(\sigma_i,r_i) y_{i-1}^{\frac{sp}{Q}} \fint_{B_{\sigma_i}}\int_{B_{\sigma_i}} \frac{\left|(u(\xi)-k_i)_+ - (u(\eta)-k_i)_+\right|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi d\eta \\ &\quad + cr_i^{Q+sp-tq}\left(\frac{k_0^{-p}2^{ip}y_{i-1}}{|B_{r_i}|}\right)^{p-1} \fint_{B_{\sigma_i}} H_0\left((u(\xi)-k_i)_+\right) \, d\xi. \end{aligned} \tag{3.13}$$

When we apply lemma 3.3, we choose a cut-off function $\phi \in C_0^\infty\left(B_{\frac{\sigma_i+r_{i-1}}{2}}\right)$ satisfying $0 \leq \phi \leq 1$, $\phi \equiv 1$ in B_{σ_i} and $|\nabla_H \phi| \leq \frac{c}{r_{i-1}-\sigma_i} = \frac{c}{r}2^i$. Then we have that, from (3.12),

$$
\begin{aligned}
&\fint_{B_{\sigma_i}} \int_{B_{\sigma_i}} \frac{\left|(u(\xi)-k_i)_+ - (u(\eta)-k_i)_+\right|^p}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\, d\xi d\eta \\
&\leq \fint_{B_{\sigma_i}} \int_{B_{\sigma_i}} H\left(\xi,\eta,\left|(u(\xi)-k_i)_+ - (u(\eta)-k_i)_+\right|\right) \frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}} \\
&\leq c r^{-p} 2^{ip} \fint_{B_{r_{i-1}}} (u(\xi)-k_i)_+^p \int_{B_{r_{i-1}}} \frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+(s-1)p}} \\
&\quad + c\|a\|_{L^\infty} r^{-q} 2^{iq} \fint_{B_{r_{i-1}}} (u(\xi)-k_i)_+^q \int_{B_{r_{i-1}}} \frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+(t-1)q}} \\
&\quad + c \sup_{\xi\in \operatorname{supp}\phi} \int_{\mathbb{H}^n\setminus B_{r_{i-1}}} \left(\frac{(u(\eta)-k_i)_+^{p-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} + \|a\|_{L^\infty} \frac{(u(\eta)-k_i)_+^{q-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}} \right) d\eta \\
&\qquad \cdot \fint_{B_{r_{i-1}}} (u(\xi)-k_i)_+ \, d\xi \\
&\leq c r^{-p} 2^{ip} r_{i-1}^{(1-s)p} \fint_{B_{r_{i-1}}} (u(\xi)-k_i)_+^p \, d\xi \\
&\quad + c\|a\|_{L^\infty} r^{-q} 2^{iq} r_{i-1}^{(1-t)q} \fint_{B_{r_{i-1}}} (u(\xi)-k_i)_+^q \, d\xi + c\left(\frac{r_{i-1}+\sigma_i}{r_{i-1}-\sigma_i}\right)^{Q+tq} \\
&\qquad \cdot \int_{\mathbb{H}^n\setminus B_{r_{i-1}}} \left(\frac{(u(\eta)-k_i)_+^{p-1}}{\|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n}^{Q+sp}} + \|a\|_{L^\infty} \frac{(u(\eta)-k_i)_+^{q-1}}{\|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n}^{Q+tq}} \right) d\eta \fint_{B_{r_{i-1}}} (u(\xi)-k_i)_+ \, d\xi \\
&\leq c r^{-q} 2^{iq} r_{i-1}^{(1-t)p} \fint_{B_{r_{i-1}}} H_0\left((u(\xi)-k_i)_+\right) \, d\xi \\
&\quad + c 2^{i(Q+tq)} T\left((u-k_i)_+;\xi_0,r_{i-1}\right) \fint_{B_{r_{i-1}}} (u(\xi)-k_i)_+ \, d\xi \\
&\leq c 2^{i(Q+q+p-1)} y_{i-1},
\end{aligned}
$$

where we used the fact that

$$
T\left((u-k_i)_+;\xi_0,r_{i-1}\right) \leq T\left(u;\xi_0,\frac{r}{2}\right) < \infty,
$$

and

$$
\frac{\|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}} \leq 1 + \frac{\|\xi_0^{-1}\circ\xi\|_{\mathbb{H}^n}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}} \leq 1 + \frac{r_{i-1}+\sigma_i}{r_{i-1}-\sigma_i} \leq 2\frac{r_{i-1}+\sigma_i}{r_{i-1}-\sigma_i} \leq c2^i
$$

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for $\xi \in \operatorname{supp} \phi$ and $\eta \in \mathbb{H}^n \backslash B_{r_{i-1}}$. Noting that $D_1(\sigma_i, r_i) \leq c2^{i(Q+p)}$, it follows from (3.13) that

$$y_i \leq c2^{i\left[\frac{q(Q+p)}{p}+\frac{q(Q+q+p)}{p}\right]} y_{i-1}^{\frac{q}{p}} + c2^{i\left(\frac{p^2}{Q}+Q+p+\frac{q(Q+q+p)}{p}\right)} y_{i-1}^{\frac{sp}{Q}+1} + c2^{ip(p-1)} y_{i-1}^{p}. \tag{3.14}$$

Since $H_0(u) \in L^1(\Omega)$ from the assumption (1.8), we get that

$$y_0 = \int_{A^+(k_0,r)} H_0\left((u(\xi) - k_0)_+\right) \, d\xi \to 0 \quad \text{as } k_0 \to \infty.$$

First, we consider $k_0 > 1$ so large that

$$y_i \leq y_{i-1} \leq \cdots \leq y_0 \leq 1, \quad i = 1, 2, \cdots.$$

Then, we have from (3.14) that

$$y_i \leq c2^{\theta i} y_{i-1}^{\beta},$$

where

$$\theta = 2\left(\frac{(Q+p+q)\,q}{p} + p^2\right), \qquad \beta = \min\left\{\frac{q}{p} - 1, \frac{sp}{Q}, p - 1\right\}.$$

Finally, we can choose k_0 so large that

$$y_0 \leq \tilde{c}^{-\frac{1}{\beta}} 2^{-\frac{\theta}{\beta^2}}$$

holds. Then lemma 3.4 implies

$$y_\infty = \int_{A^+\left(2k_0, \frac{r}{2}\right)} H_0\left((u(\xi) - 2k_0)_+\right) \, d\xi = 0,$$

which means that $u \leq 2k_0$ a.e. in $B_{\frac{r}{2}}$.

Applying the same argument to $-u$, we consequently obtain $u \in L^\infty(B_{\frac{r}{2}})$.

4. Hölder continuity

We are going to demonstrate the Hölder regularity of weak solutions to equation (1.1) in the last section. First, the second important tool, logarithmic estimate, is established as follows. Throughout this part, we fix any subdomain $\Omega' \subset\subset \Omega$.

LEMMA 4.1. Logarithmic inequality *Let s, t, p, q satisfy (1.4) and $a(\cdot,\cdot)$ fulfil (1.5), (1.6) with (1.9). Let also $u \in \mathcal{A}(\Omega)$ be a weak solution of (1.1) such that $u \in L^\infty(\Omega')$*

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and $u \geq 0$ *in* $B_R := B_R(\xi_0) \subset \Omega'$ *with* $R \leq 1$. *Then for any* $0 < r \leq \frac{R}{2}$ *and* $d > 0$,

$$\int_{B_r}\int_{B_r}\left|\log\frac{u(\xi)+d}{u(\eta)+d}\right|\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}$$
$$\leq cK^2\left(r^Q+\frac{r^{Q+sp}}{d^{p-1}}\int_{\mathbb{H}^n\setminus B_R}\frac{u_-^{p-1}(\eta)+u_-^{q-1}(\eta)}{\|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n}^{Q+sp}}d\eta+\frac{r^{Q+tq}}{d^{q-1}}\int_{\mathbb{H}^n\setminus B_R}\frac{u_-^{q-1}(\eta)}{\|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n}^{Q+tq}}d\eta\right),$$

holds true. Here $K := 1 + d^{q-p} + \|u\|_{L^\infty(\Omega')}^{q-p}$ *and the constant* $c \geq 1$ *depends on* ***data***.

Proof. Let us give some notations as below,

$$H_\rho(\xi,\eta,\tau)=\frac{\tau^p}{\rho^{sp}}+a(\xi,\eta)\frac{\tau^q}{\rho^{tq}},\quad h_\rho(\xi,\eta,\tau)=\frac{\tau^{p-1}}{\rho^{sp}}+a(\xi,\eta)\frac{\tau^{q-1}}{\rho^{tq}}$$

and

$$G_\rho(\tau)=\frac{\tau^p}{\rho^{sp}}+a_\rho^+\frac{\tau^q}{\rho^{tq}},\quad g_\rho(\tau)=\frac{\tau^{p-1}}{\rho^{sp}}+a_\rho^+\frac{\tau^{q-1}}{\rho^{tq}},$$

with $a_\rho^+ := \sup_{B_\rho\times B_\rho} a(\cdot,\cdot)$ and $\tau \geq 0$.

Consider a cut-off function $\phi \in C_0^\infty\left(B_{\frac{3r}{2}}(\xi_0)\right)$ satisfying

$$0\leq\phi\leq 1,\quad \phi\equiv 1 \text{in} B_r \quad\text{and}\quad |\nabla_H\phi|\leq\frac{c}{r}\text{in} B_{\frac{3r}{2}}.$$

Taking the test function $\varphi(\xi) := \frac{\phi^q(\xi)}{g_{2r}(u(\xi)+d)}$, we have from the weak formulation that

$$\begin{aligned}
0=&\int_{B_{2r}}\int_{B_{2r}}\left[\frac{J_p(u(\xi)-u(\eta))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\left(\frac{\phi^q(\xi)}{g_{2r}(\overline{u}(\xi))}-\frac{\phi^q(\eta)}{g_{2r}(\overline{u}(\eta))}\right)\right.\\
&\left.+a(\xi,\eta)\frac{J_q(u(\xi)-u(\eta))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\left(\frac{\phi^q(\xi)}{g_{2r}(\bar{u}(\xi))}-\frac{\phi^q(\eta)}{g_{2r}(\bar{u}(\eta))}\right)\right]d\xi d\eta\\
&+2\int_{\mathbb{H}^n\setminus B_{2r}}\int_{B_{2r}}\left[\frac{J_p(u(\xi)-u(\eta))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}+a(\xi,\eta)\frac{J_q(u(\xi)-u(\eta))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\right]\frac{\phi^q(\xi)}{g_{2r}(\overline{u}(\xi))}d\xi d\eta\\
=:&\, I_1+I_2,
\end{aligned}\tag{4.1}$$

with $\bar{u} := u + d$.

In what follows, we deal with I_1 in the case $\bar{u}(\xi) \geq \bar{u}(\eta)$ that is divided into two subcases:

$$\bar{u}(\xi)\geq\bar{u}(\eta)\geq\frac{1}{2}\bar{u}(\xi),\tag{4.2}$$

and

$$\bar{u}(\xi)\geq 2\bar{u}(\eta).\tag{4.3}$$

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If (4.2) occurs, we first observe that

$$
\begin{aligned}
&\frac{\phi^q(\xi)}{g_{2r}(\bar{u}(\xi))}-\frac{\phi^q(\eta)}{g_{2r}(\bar{u}(\eta))}\\
&\quad\leq \frac{c\phi^{q-1}(\xi)\sup_{B_{3R/2}}|\nabla_H\phi|\,\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}}{g_{2r}(\bar{u}(\eta))}\\
&\quad+\phi^q(\xi)\int_0^1\frac{d}{d\sigma}\left(g_{2r}^{-1}(\sigma\bar{u}(\xi)+(1-\sigma)\bar{u}(\eta))\right)d\sigma\\
&\quad\leq \frac{c\phi^{q-1}(\xi)\,r^{-1}\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}}{g_{2r}(\bar{u}(\eta))}-\frac{(p-1)\,\phi^q(\xi)\,(\bar{u}(\xi)-\bar{u}(\eta))}{2^qG_{2r}(\bar{u}(\eta))},
\end{aligned}\tag{4.4}
$$

where the first inequality holds naturally when $\phi(\xi)\leq\phi(\eta)$. Here, we have used (4.2) and

$$
\begin{aligned}
\int_0^1\frac{d}{d\sigma}\left(g_{2r}^{-1}(\sigma\bar{u}(\xi)+(1-\sigma)\bar{u}(\eta))\right)d\sigma &\geq \frac{(p-1)(\bar{u}(\xi)-\bar{u}(\eta))}{G_{2r}(\bar{u}(\xi))}\\
&\geq \frac{(p-1)(\bar{u}(\xi)-\bar{u}(\eta))}{2^qG_{2r}(\bar{u}(\eta))},
\end{aligned}
$$

the details of which can be found in [4]. Then, combining (4.4) and Young's inequality yields

$$
\begin{aligned}
F(\xi,\eta) &:= \left(\frac{J_p(u(\xi)-u(\eta))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{sp}}+a(\xi,\eta)\frac{J_q(u(\xi)-u(\eta))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{tq}}\right)\left(\frac{\phi^q(\xi)}{g_{2r}(\bar{u}(\xi))}-\frac{\phi^q(\eta)}{g_{2r}(\bar{u}(\eta))}\right)\\
&\leq \frac{c\phi^{q-1}(\xi)\,r^{-1}\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}\bar{u}(\eta)}{G_{2r}(\bar{u}(\eta))}\\
&\left(\frac{|\bar{u}(\xi)-\bar{u}(\eta)|^{p-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{sp}}+a(\xi,\eta)\frac{|\bar{u}(\xi)-\bar{u}(\eta)|^{q-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{tq}}\right)\\
&\quad-\frac{(p-1)\,\phi^q(\xi)\,H(\xi,\eta,\bar{u}(\xi)-\bar{u}(\eta))}{2^qG_{2r}(\bar{u}(\eta))}\\
&\leq \frac{\varepsilon\phi^{\frac{(q-1)p}{p-1}}(\xi)\,|\bar{u}(\xi)-\bar{u}(\eta)|^p}{G_{2r}(\bar{u}(\eta))\,\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{sp}}+a(\xi,\eta)\frac{\varepsilon\phi^q(\xi)\,|\bar{u}(\xi)-\bar{u}(\eta)|^q}{G_{2r}(\bar{u}(\eta))\,\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{tq}}\\
&\quad-\frac{(p-1)\,\phi^q(\xi)\,H(\xi,\eta,\bar{u}(\xi)-\bar{u}(\eta))}{2^qG_{2r}(\bar{u}(\eta))}\\
&\quad+c(\varepsilon)\frac{r^{-p}\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{p}|\bar{u}(\eta)|^p}{G_{2r}(\bar{u}(\eta))\,\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{sp}}+c(\varepsilon)\,a_{2r}^{+}\frac{r^{-q}\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{q}|\bar{u}(\eta)|^q}{G_{2r}(\bar{u}(\eta))\,\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{tq}}\\
&\leq -\frac{(p-1)\,\phi^q(\xi)\,H(\xi,\eta,\bar{u}(\xi)-\bar{u}(\eta))}{2^{q+1}G_{2r}(\bar{u}(\eta))}+c\frac{r^{p(s-1)}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{p(s-1)}}+c\frac{r^{q(t-1)}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{q(t-1)}},
\end{aligned}\tag{4.5}
$$

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where ε was chosen as $\frac{p-1}{2^{q+1}}$, $\frac{(q-1)p}{p-1} > q$ and $c > 0$ is independent of a. We proceed to evaluate $G_{2r}\left(\bar{u}\left(\eta\right)\right)$. For $\xi, \eta \in B_{2r}$, recalling the Hölder continuity of a, we get

$$a_{2r}^{+} = a_{2r}^{+} - a\left(\xi,\eta\right) + a\left(\xi,\eta\right) \leq 2[a]_{\alpha}(4r)^{\alpha} + a\left(\xi,\eta\right).$$

Thus this implies by the facts that $r \leq 1$ and $tq \leq sp + \alpha$ that

$$\begin{aligned} G_{2r}\left(\bar{u}\left(\eta\right)\right) &\leq \frac{\bar{u}^{p}\left(\eta\right)}{(2r)^{sp}} + 2[a]_{\alpha}(4r)^{\alpha}\frac{\bar{u}^{q}\left(\eta\right)}{(2r)^{tq}} + a\left(\xi,\eta\right)\frac{\bar{u}^{q}\left(\eta\right)}{(2r)^{tq}} \\ &\leq \left(1 + 8[a]_{\alpha}r^{\alpha+sp-tq}\|u\|_{L^{\infty}(\Omega')}^{q-p}\right)\frac{\bar{u}^{p}\left(\eta\right)}{(2r)^{sp}} + a\left(\xi,\eta\right)\frac{\bar{u}^{q}\left(\eta\right)}{(2r)^{tq}} \\ &\leq c\left(1 + \|u\|_{L^{\infty}(\Omega')}^{q-p}\right)H_{2r}\left(\xi,\eta,\bar{u}\left(\eta\right)\right). \end{aligned} \tag{4.6}$$

Next, we will obtain an estimate on $\log \bar{u}$. It is easy to find

$$\log\frac{\bar{u}\left(\xi\right)}{\bar{u}\left(\eta\right)} = \int_{0}^{1}\frac{\bar{u}\left(\xi\right)-\bar{u}\left(\eta\right)}{\bar{u}\left(\eta\right)+\sigma\left(\bar{u}\left(\xi\right)-\bar{u}\left(\eta\right)\right)}\,d\sigma \leq \frac{(\bar{u}(\xi)-\bar{u}(\eta))/\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{s}}{\bar{u}(\eta)/(2r)^{s}}\frac{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{s}}{(2r)^{s}},$$

so, by the monotonicity of the function $f(\tau) = (\tau^{p} + a(\xi,\eta)\tau^{q}\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{-(t-s)q})/\tau$ with $\tau \geq 0$,

$$\begin{aligned} \log\frac{\bar{u}(\xi)}{u(\eta)} &\leq \frac{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{s}}{(2r)^{s}}\left[\frac{\left(\frac{\bar{u}(\xi)-\bar{u}(\eta)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{s}}\right)^{p} + a(\xi,\eta)\left(\frac{\bar{u}(\xi)-\bar{u}(\eta)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{s}}\right)^{q}\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{-(t-s)q}}{\left(\frac{\bar{u}(\eta)}{(2r)^{s}}\right)^{p} + a(\xi,\eta)\left(\frac{\bar{u}(\eta)}{(2r)^{s}}\right)^{q}\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{-(t-s)q}} + 1\right] \\ &\leq \frac{cH\left(\xi,\eta,\bar{u}\left(\xi\right)-\bar{u}\left(\eta\right)\right)}{H_{2r}\left(\xi,\eta,\bar{u}\left(\eta\right)\right)} + \frac{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{s}}{(2r)^{s}}, \end{aligned} \tag{4.7}$$

where we need to note $\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}} \leq 4r$. It follows from (4.5)–(4.7) that

$$F\left(\xi,\eta\right) \leq -\frac{\phi^{q}\left(\xi\right)}{cK}\log\frac{\bar{u}\left(\xi\right)}{\bar{u}\left(\eta\right)} + \frac{c\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{s}}{(2r)^{s}} + \frac{c\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{p(1-s)}}{(2r)^{p(1-s)}} + \frac{c\|\eta^{-1}\circ\xi\|_{\mathbb{H}^{n}}^{q(1-t)}}{(2r)^{q(1-t)}}.$$

Second, we in the case (4.3) tackle the integral I_1. Applying lemma 3.2 and the relation $\bar{u}\left(\xi\right) \geq 2\bar{u}\left(\eta\right)$, we could derive

$$\begin{aligned} \frac{\phi^{q}\left(\xi\right)}{g_{2r}\left(\bar{u}\left(\xi\right)\right)} - \frac{\phi^{q}(\eta)}{g_{2r}\left(\bar{u}\left(\eta\right)\right)} &\leq \frac{\phi^{q}\left(\xi\right)-\phi^{q}\left(\eta\right)}{g_{2r}\left(\bar{u}\left(\xi\right)\right)} + \phi^{q}\left(\eta\right)\left(\frac{1}{g_{2r}\left(2\bar{u}\left(\eta\right)\right)} - \frac{1}{g_{2r}\left(\bar{u}\left(\eta\right)\right)}\right) \\ &\leq \frac{\varepsilon\phi^{q}\left(\eta\right) + c\left(\varepsilon\right)\left|\phi\left(\xi\right)-\phi\left(\eta\right)\right|^{q}}{g_{2r}\left(\bar{u}\left(\xi\right)\right)} - \frac{2^{p-1}-1}{2^{p-1}}\frac{\phi^{q}(\eta)}{g_{2r}\left(\bar{u}\left(\eta\right)\right)} \\ &\leq \frac{c|\phi(\xi)-\phi(\eta)|^{q}}{g_{2r}\left(\bar{u}\left(\xi\right)\right)} - \frac{\left(2^{p-1}-1\right)\phi^{q}\left(\eta\right)}{2^{p}g_{2r}\left(\bar{u}\left(\eta\right)\right)}, \end{aligned}$$

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with $\varepsilon = \frac{2^{p-1}-1}{2^p}$. Thereby, it holds that

$$
\begin{aligned}
F(\xi,\eta) &\le \frac{ch(\xi,\eta,\bar u(\xi)-\bar u(\eta))\,|\phi(\xi)-\phi(\eta)|^q}{g_{2r}(\bar u(\xi))} - \frac{h(\xi,\eta,\bar u(\xi)-\bar u(\eta))\,\phi^q(\eta)}{cg_{2r}(\bar u(\eta))} \\
&\le \frac{c(2r)^{-q}\|\eta^{-1}\circ\xi\|^q_{\mathbb{H}^n} h(\xi,\eta,\bar u(\xi)-\bar u(\eta))}{g_{2r}(\bar u(\xi))} - \frac{h(\xi,\eta,\bar u(\xi)-\bar u(\eta))\,\phi^q(\eta)}{cKh_{2r}(\xi,\eta,\bar u(\eta))}.
\end{aligned}
$$

Here $F(\xi,\eta)$ is the same as that in (4.5) and the estimate for $g_{2r}(\bar u(\eta))$ is similar to (4.6). Moreover, via $\bar u(\xi) \ge 2\bar u(\eta) \ge 0$ in B_{2r},

$$
\begin{aligned}
\frac{h(\xi,\eta,\bar u(\xi)-\bar u(\eta))}{g_{2r}(\bar u(\xi))} &\le \frac{\frac{|\bar u(\xi)-\bar u(\eta)|^{p-1}}{\|\eta^{-1}\circ\xi\|^{sp}_{\mathbb{H}^n}} + a(\xi,\eta)\frac{|\bar u(\xi)-\bar u(\eta)|^{q-1}}{\|\eta^{-1}\circ\xi\|^{tq}_{\mathbb{H}^n}}}{\frac{|\bar u(\xi)-\bar u(\eta)|^{p-1}}{(2r)^{sp}} + a^+_{2r}\frac{|\bar u(\xi)-\bar u(\eta)|^{q-1}}{(2r)^{tq}}} \\
&\le \frac{(2r)^{sp}}{\|\eta^{-1}\circ\xi\|^{sp}_{\mathbb{H}^n}} + \frac{(2r)^{tq}}{\|\eta^{-1}\circ\xi\|^{tq}_{\mathbb{H}^n}},
\end{aligned}
$$

and further

$$
F(\xi,\eta) \le \frac{c\|\eta^{-1}\circ\xi\|^{q-sp}_{\mathbb{H}^n}}{(2r)^{q-sp}} + \frac{\|\eta^{-1}\circ\xi\|^{q(1-t)}_{\mathbb{H}^n}}{(2r)^{q(1-t)}} - \frac{h(\xi,\eta,\bar u(\xi)-\bar u(\eta))\,\phi^q(\eta)}{cKh_{2r}(\xi,\eta,\bar u(\eta))}.
$$

Now we obtain an estimate on $\log\frac{\bar u(\xi)}{\bar u(\eta)}$ under (4.3). Notice $\bar u(\xi) \le 2(\bar u(\xi)-\bar u(\eta))$. we get

$$
\begin{aligned}
&\log\frac{\bar u(\xi)}{\bar u(\eta)} \\
&\le \frac{c\left((\bar u(\xi)-\bar u(\eta))/\|\eta^{-1}\circ\xi\|^s_{\mathbb{H}^n}\right)^{p-1}}{(\bar u(\eta)/(2r)^s)^{p-1}}\frac{\|\eta^{-1}\circ\xi\|^{s(p-1)}_{\mathbb{H}^n}}{(2r)^{s(p-1)}} \\
&\le c\frac{\|\eta^{-1}\circ\xi\|^{s(p-1)}_{\mathbb{H}^n}}{(2r)^{s(p-1)}}\left[\frac{\left(\frac{\bar u(\xi)-\bar u(\eta)}{\|\eta^{-1}\circ\xi\|^s_{\mathbb{H}^n}}\right)^{p-1} + a(\xi,\eta)\left(\frac{\bar u(\xi)-\bar u(\eta)}{\|\eta^{-1}\circ\xi\|^s_{\mathbb{H}^n}}\right)^{q-1}\|\eta^{-1}\circ\xi\|^{-(t-s)q}_{\mathbb{H}^n}}{\left(\frac{\bar u(\eta)}{(2r)^s}\right)^{p-1} + a(\xi,\eta)\left(\frac{\bar u(\eta)}{(2r)^s}\right)^{q-1}\|\eta^{-1}\circ\xi\|^{-(t-s)q}_{\mathbb{H}^n}} + 1\right] \\
&\le \frac{ch(\xi,\eta,\bar u(\xi)-\bar u(\eta))}{h_{2r}(\xi,\eta,\bar u(\eta))} + \frac{c\|\eta^{-1}\circ\xi\|^{s(p-1)}_{\mathbb{H}^n}}{(2r)^{s(p-1)}},
\end{aligned}
$$

where the fact $\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n} \le 4r$ was utilized. Noting $q \ge p$ and $\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n} \le 4r$ again,

$$
F(\xi,\eta) \le -\frac{\phi^q(\xi)}{cK}\log\frac{\bar u(\xi)}{\bar u(\eta)} + \frac{c\|\eta^{-1}\circ\xi\|^{p(1-s)}_{\mathbb{H}^n}}{(2r)^{p(1-s)}} + \frac{c\|\eta^{-1}\circ\xi\|^{q(1-t)}_{\mathbb{H}^n}}{(2r)^{q(1-t)}} + \frac{c\|\eta^{-1}\circ\xi\|^{s(p-1)}_{\mathbb{H}^n}}{(2r)^{s(p-1)}}.
$$

At this moment, for $\bar{u}(\xi) \geq \bar{u}(\eta)$, the integral I_1 is evaluated as

$$\begin{aligned} I_1 \leq & -\frac{1}{cK}\int_{B_{2r}}\int_{B_{2r}} \min\left\{\phi^q(\xi), \phi^q(\eta)\right\}\left|\log\frac{\bar{u}(\xi)}{\bar{u}(\eta)}\right| \frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^Q} \\ & + c\int_{B_{2r}}\int_{B_{2r}}\left[\frac{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{p-sp}}{r^{p(1-s)}}\right. \\ & \left. + \frac{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{q(1-t)}}{r^{q(1-t)}} + \frac{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{s(p-1)}}{r^{s(p-1)}} + \frac{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{s}}{r^{s}}\right]\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^Q} \\ \leq & -\frac{1}{cK}\int_{B_{2r}}\int_{B_{2r}}\left|\log\frac{\bar{u}(\xi)}{\bar{u}(\eta)}\right|\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^Q} + cr^Q, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} \int_{B_{2r}}\int_{B_{2r}} \frac{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{l}}{r^l}\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^Q} &\leq \int_{B_{2r}}\int_{B_{4r}(\eta)} \frac{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{l}}{r^l}\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^Q} \\ &\leq \frac{c}{r^l}\int_{B_{2r}}\int_0^{4r}\rho^{l-1}\,d\rho d\eta \leq cr^Q. \end{aligned}$$

Furthermore, if $\bar{u}(\xi) < \bar{u}(\eta)$, the same estimate still holds true through exchanging the roles of ξ and η.

For the second contribution I_2 in (4.1), we first observe that if $\eta \in B_R$, then $(u(\xi) - u(\eta))_+ \leq u(\xi) + d$ by $u(\eta) \geq 0$, and that if $\eta \in \mathbb{H}^n \backslash B_R$, then $(u(\xi) - u(\eta))_+ \leq u(\xi) + u_-(\eta) \leq \bar{u}(\xi) + u_-(\eta)$. From this and $\operatorname{supp}\phi \subset B_{\frac{3r}{2}}$, we can evaluate I_2 as

$$\begin{aligned} I_2 \leq & \, 2\int_{B_R\backslash B_{2r}}\int_{B_{\frac{3r}{2}}}\left[\frac{(u(\xi)-u(\eta))_+^{p-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} + a(\xi,\eta)\frac{(u(\xi)-u(\eta))_+^{q-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\right]\frac{d\xi d\eta}{g_{2r}(\bar{u}(\xi))} \\ & + 2\int_{\mathbb{H}^n\backslash B_R}\int_{B_{\frac{3r}{2}}}\left[\frac{(u(\xi)-u(\eta))_+^{p-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} + a(\xi,\eta)\frac{(u(\xi)-u(\eta))_+^{q-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\right]\frac{d\xi d\eta}{g_{2r}(\bar{u}(\xi))} \\ \leq & \int_{\mathbb{H}^n\backslash B_{2r}}\int_{B_{\frac{3r}{2}}}\frac{ch(\xi,\eta,\bar{u}(\xi))}{g_{2r}(\bar{u}(\xi))\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^Q}\,d\xi d\eta + \int_{\mathbb{H}^n\backslash B_R}\int_{B_{\frac{3r}{2}}} \\ & \frac{ch(\xi,\eta,u(\eta))}{g_{2r}(\bar{u}(\xi))\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^Q}\,d\xi d\eta \\ =: & \, I_{21} + I_{22}. \end{aligned} \tag{4.9}$$

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We now intend to control precisely the term $\frac{h(\xi,\eta,\bar{u}(\xi))}{g_{2r}(\bar{u}(\xi))}$ by some constants. In view of the condition (1.6), there holds that, for $\xi \in B_{2r}$ and $\eta \in \mathbb{H}^n$,

$$\begin{aligned} a(\xi,\eta) &\leq a(\xi,\eta) - a(\xi,\xi) + a_{2r}^+ \\ &\leq (2\|a\|_{L^\infty})^{1-\frac{tq-sp}{\alpha}} |a(\xi,\eta) - a(\xi,\xi)|^{\frac{tq-sp}{\alpha}} + a_{2r}^+ \\ &\leq c\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{tq-sp} + a_{2r}^+. \end{aligned} \tag{4.10}$$

This indicates

$$\begin{aligned} I_{21} &\leq c\int_{\mathbb{H}^n\setminus B_{2r}}\int_{B_{\frac{3r}{2}}} \frac{\frac{\bar{u}^{p-1}(\xi)+\bar{u}^{q-1}(\xi)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{sp}} + a_{2r}^+\frac{\bar{u}^{q-1}(\xi)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{tq}}}{\frac{\bar{u}^{p-1}(\xi)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{sp}}\frac{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{sp}}{(2r)^{sp}} + a_{2r}^+\frac{\bar{u}^{q-1}(\xi)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{tq}}\frac{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{tq}}{(2r)^{tq}}}\frac{d\xi d\eta}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}} \\ &\leq cK\int_{\mathbb{H}^n\setminus B_{2r}}\int_{B_{\frac{3r}{2}}} \frac{(r/2)^{sp}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}}\,d\xi d\eta, \end{aligned}$$

by virtue of $\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n} > \frac{r}{2}$. For $\xi \in B_{\frac{3r}{2}}$ and $\eta \in \mathbb{H}^n\setminus B_{2r}$, via the triangle inequality,

$$\begin{aligned} \|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n} &\leq \left(1 + \frac{\|\xi^{-1}\circ\xi_0\|_{\mathbb{H}^n}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}}\right)\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n} \\ &\leq \left(1+\frac{3r/2}{r/2}\right)\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n} = 4\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}, \end{aligned} \tag{4.11}$$

Thus by [31, Lemma 2.6],

$$I_{21} \leq cK\left|B_{\frac{3r}{2}}\right|\int_{\mathbb{H}^n\setminus B_{2r}}\frac{r^{sp}}{\|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n}^{Q+sp}}\,d\eta \leq cKr^Q. \tag{4.12}$$

Let us proceed to examine I_{22}. With the aid of (4.10), (4.11) and $u(\xi) \geq 0$ in $B_{\frac{3r}{2}}$,

$$\begin{aligned} I_{22} &\leq c\int_{\mathbb{H}^n\setminus B_R}\int_{B_{\frac{3r}{2}}}\left(\frac{u_-^{p-1}(\eta)+u_-^{q-1}(\eta)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} + a_{2r}^+\frac{u_-^{q-1}(\eta)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\right)g^{-1}(d)\,d\xi d\eta \\ &\leq cr^Q g^{-1}(d)\int_{\mathbb{H}^n\setminus B_R}\left(\frac{u_-^{p-1}(\eta)+u_-^{q-1}(\eta)}{\|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n}^{Q+sp}} + a_{2r}^+\frac{u_-^{q-1}(\eta)}{\|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n}^{Q+tq}}\right)d\eta \\ &\leq cr^{Q+sp}d^{1-p}\int_{\mathbb{H}^n\setminus B_R}\frac{u_-^{p-1}(\eta)+u_-^{q-1}(\eta)}{\|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n}^{Q+sp}}d\eta \\ &\quad + cr^{Q+tq}d^{1-q}\int_{\mathbb{H}^n\setminus B_R}\frac{u_-^{q-1}(\eta)}{\|\eta^{-1}\circ\xi_0\|_{\mathbb{H}^n}^{Q+tq}}\,d\eta, \end{aligned} \tag{4.13}$$

where we notice $\eta \in \mathbb{H}^n\setminus B_R \subset \mathbb{H}^n\setminus B_{2r}$.

Merging (4.8), (4.9), (4.12), (4.13) with (4.1) arrives eventually at the desired estimate with the positive constant c depending upon $n, p, q, s, t, \alpha, [a]_\alpha$ and $\|a\|_{L^\infty}$. $\square$

Corollary 4.2. *Let the assumptions of lemma 4.1 be in force. Define*

$$w := \min\left\{(\log(\tau + d) - \log(u + d))_+, \log b\right\}$$

with $\tau, d > 0$ and $b > 1$. Then for the weak solution u of (1.1) it holds that

$$\begin{aligned}
&\fint_{B_r} |w - (w)_r| \, d\eta \\
&\le cK^2 \left(1 + \frac{r^{sp}}{d^{p-1}} \int_{\mathbb{H}^n \setminus B_R} \frac{u_-^{p-1}(\eta) + u_-^{q-1}(\eta)}{\|\eta^{-1} \circ \xi_0\|_{\mathbb{H}^n}^{Q+sp}} \, d\eta + \frac{r^{tq}}{d^{q-1}} \int_{\mathbb{H}^n \setminus B_R} \frac{u_-^{q-1}(\eta)}{\|\eta^{-1} \circ \xi_0\|_{\mathbb{H}^n}^{Q+tq}} \, d\eta\right),
\end{aligned}$$

*where $c > 1$ depends on **data**, and K is defined as in lemma 4.1.*

Proof. Notice that, since w is a truncation of $\log(u + d)$,

$$\begin{aligned}
\fint_{B_r} |w - (w)_r| \, d\eta &\le \fint_{B_r} \left| \fint_{B_r} (w(\eta) - w(\xi)) \, d\xi \right| \, d\eta \\
&\le \fint_{B_r} \fint_{B_r} |w(\xi) - w(\eta)| \, d\xi d\eta \\
&\le \fint_{B_r} \fint_{B_r} \frac{|\log(u(\xi) + d) - \log(u(\eta) + d)|}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^Q / (2r)^Q} \, d\xi d\eta \\
&\le \fint_{B_r} \int_{B_r} \left| \log \frac{u(\xi) + d}{u(\eta) + d} \right| \frac{d\xi d\eta}{\|\eta^{-1} \circ \xi\|_{\mathbb{H}^n}^Q}.
\end{aligned}$$

Then the desired result is a plain consequence of lemma 4.1. $\square$

In the end, we will focus on establishing Hölder regularity of weak solutions. For this aim, it is sufficient to show an oscillation improvement result, theorem 4.3. Before proceeding, let us introduce some notations. For $j \in \mathbb{N} \cup \{0\}$, set

$$r_j := \sigma^j r, \quad \sigma \in (0, 1/4], \quad B_j := B_{r_j}(\xi_0) \quad \text{and} \quad 2B_j := B_{2r_j},$$

where we fix any ball $B_{2r}(\xi_0) \subset \Omega' \subset\subset \Omega$. Furthermore, define

$$\begin{aligned}
\omega(r_0) := 2 \sup_{B_r} |u| &+ \left(r^{sp} \int_{\mathbb{H}^n \setminus B_r} \frac{|u|^{p-1} + |u|^{q-1}}{\|\xi_0^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi\right)^{\frac{1}{p-1}} \\
&+ \left(r^{tq} \int_{\mathbb{H}^n \setminus B_r} \frac{|u|^{q-1}}{\|\xi_0^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+tq}} \, d\xi\right)^{\frac{1}{q-1}},
\end{aligned}$$

and

$$\omega(r_j) := \left(\frac{r_j}{r_0}\right)^\beta \omega(r_0) = \sigma^{j\beta} \omega(r) \quad \text{for some } 0 < \beta < \frac{sp}{q-1}.$$

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Let us point out that σ and β are to be determined later.

Now we are in a position to prove the following iteration lemma, which suggests $u \in C^{0,\beta}(B_r)$.

THEOREM 4.3 *Let $u \in \mathcal{A}(\Omega) \cap L^{q-1}_{sp}(\mathbb{H}^n)$ be a weak solution to (1.1). Under the conditions (1.4), (1.5) and (1.6) with $tq \le sp + \alpha$, there holds that*

$$\operatorname*{osc}_{B_j} u \le \omega(r_j) \quad \textit{for any } j \in \mathbb{N} \cup \{0\},$$

where these notations are fixed as above.

Proof. Argue by induction. The conclusion is obvious for $j = 0$ and then assume it holds true for $i \le j$. Now we show this claim for $j+1$. Let us notice the simple fact that either

$$\left| 2B_{j+1} \cap \left\{ u \ge \inf_{B_j} u + \omega(r_j)/2 \right\} \right| \ge \frac{1}{2} |2B_{j+1}|, \tag{4.14}$$

or

$$\left| 2B_{j+1} \cap \left\{ u < \inf_{B_j} u + \omega(r_j)/2 \right\} \right| \ge \frac{1}{2} |2B_{j+1}|. \tag{4.15}$$

Define

$$u_j = \begin{cases} u - \inf_{B_j} u, & \text{if (4.14) occurs,} \\ \sup_{B_j} u - u, & \text{if (4.15) occurs.} \end{cases}$$

Obviously, $u_j \ge 0$ in B_j and

$$|2B_{j+1} \cap \{u_j \ge \omega(r_j)/2\}| \ge \frac{1}{2} |2B_{j+1}|. \tag{4.16}$$

Moreover, u_j is a weak solution to (1.1) such that

$$\sup_{B_i} |u_j| \le \omega(r_i) \quad \text{for any } i \in \{0, 1, 2, \cdots, j\}. \tag{4.17}$$

Now we set an auxiliary function

$$w := \min \left\{ \left[\log \left(\frac{\omega(r_j)/2 + d}{u_j + d} \right) \right]_+, k \right\} \quad \text{with } k > 0.$$

Applying corollary 4.2 derives

$$\begin{aligned} &\fint_{2B_{j+1}} |w - (w)_{2B_{j+1}}| \, d\xi \\ &\le CK^2 \left(1 + d^{1-p} r_{j+1}^{sp} \int_{\mathbb{H}^n \setminus B_j} \frac{|u_j|^{p-1} + |u_j|^{q-1}}{\|\xi_0^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi + d^{1-q} r_{j+1}^{tq} \int_{\mathbb{H}^n \setminus B_j} \frac{|u_j|^{q-1}}{\|\xi_0^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+tq}} \, d\xi \right), \end{aligned} \tag{4.18}$$

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with K defined as in lemma 4.1. We evaluate the second integral at the right-hand side. By means of (4.17) and the definition of $\omega(r_0)$,

$$\begin{aligned} & r_{j+1}^{tq} \int_{\mathbb{H}^n \setminus B_j} \frac{|u_j|^{q-1}}{\|\xi_0^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+tq}} \, d\xi \\ & = r_j^{tq} \sum_{i=1}^{j} \int_{B_{i-1} \setminus B_i} \frac{|u_j|^{q-1}}{\|\xi_0^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+tq}} \, d\xi + r_j^{tq} \int_{\mathbb{H}^n \setminus B_0} \frac{|u_j|^{q-1}}{\|\xi_0^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+tq}} \, d\xi \\ & \leq \sum_{i=1}^{j} \omega(r_{i-1})^{q-1} \left(\frac{r_j}{r_i}\right)^{tq} + C r_j^{tq} \int_{\mathbb{H}^n \setminus B_0} \frac{|u|^{q-1} + (\sup_{B_0} |u|)^{q-1}}{\|\xi_0^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+tq}} \, d\xi \\ & \leq C \sum_{i=1}^{j} \left(\frac{r_j}{r_i}\right)^{tq} \omega(r_{i-1})^{q-1} \\ & \leq C \frac{4^{tq-\beta(q-1)}}{(tq-\beta(q-1)) \log 4} \sigma^{-\beta(q-1)} \omega(r_j)^{q-1}, \end{aligned} \tag{4.19}$$

where we used the fact that $\beta < \frac{sp}{q-1} \left(\leq \frac{tq}{q-1}\right)$. Analogously,

$$\begin{aligned} r_j^{sp} \int_{\mathbb{H}^n \setminus B_j} \frac{|u_j|^{p-1} + |u_j|^{q-1}}{\|\xi_0^{-1} \circ \xi\|_{\mathbb{H}^n}^{Q+sp}} \, d\xi & \leq C(1 + \|u\|_{L^\infty(\Omega')}^{q-p}) \sum_{i=1}^{j} \left(\frac{r_j}{r_i}\right)^{sp} \omega(r_{i-1})^{p-1} \\ & \leq C N \sigma^{-\beta(p-1)} \omega(r_j)^{p-1}, \end{aligned} \tag{4.20}$$

with $\beta < \frac{sp}{q-1} \left(\leq \frac{sp}{p-1}\right)$, where $N := 1 + \|u\|_{L^\infty(\Omega')}^{q-p}$ and the derivation of $\|u\|_{L^\infty(\Omega')}^{q-p}$ is from the term $|u_j|^{q-1}$, and $C > 0$ depends on n, p, s and the difference of $\frac{sp}{p-1}$ and β. Combining (4.19), (4.20) with (4.18) and remembering $\frac{r_{j+1}}{r_j} = \sigma$, we get

$$\begin{aligned} & ⨍_{2B_{j+1}} |w - (w)_{2B_{j+1}}| \, d\xi \\ & \leq C K^2 \left(1 + N d^{1-p} \sigma^{sp-\beta(p-1)} \omega(r_j)^{p-1} + d^{1-q} \sigma^{tq-\beta(q-1)} \omega(r_j)^{q-1}\right), \end{aligned}$$

where C depends on n, p, q, s, t and the difference of β and $\frac{tq}{q-1}$, and $\frac{sp}{p-1}$.

In what follows, picking

$$d := \sigma^{\frac{sp}{q-1} - \beta} \omega(r_j),$$

and recalling $\omega(r_j) = \sigma^{j\beta} \omega(r_0)$, we find

$$\begin{aligned} & ⨍_{2B_{j+1}} |w - (w)_{2B_{j+1}}| \, d\xi \\ & \leq C K^2 \left[1 + N \sigma^{\left(\frac{sp}{q-1} - \beta\right)(1-p) + \left(\frac{sp}{p-1} - \beta\right)(p-1)} + \sigma^{\left(\frac{sp}{q-1} - \beta\right)(1-q) + \left(\frac{tq}{q-1} - \beta\right)(q-1)}\right] \leq C N^3, \end{aligned}$$

where C depends on $n, p, q, s, t, \alpha, [a]_\alpha, \|a\|_{L^\infty}$ and the difference of β and $\frac{tq}{q-1}$, and $\frac{sp}{p-1}$. Here we need to utilize the definition of K as in lemma 4.1, and $\omega(r_j) \leq$

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$2\|u\|_{L^\infty(\Omega')}$. From the last inequality,

$$\frac{|2B_{j+1}\cap\{w=k\}|}{|2B_{j+1}|}\le\frac{CN^3}{k}.$$

We refer to [14, page 1296] for the details. By taking

$$k=\log\left(\frac{\omega(r_j)/2+\varepsilon\omega(r_j)}{3\varepsilon\omega(r_j)}\right)=\log\left(\frac{1/2+\varepsilon}{3\varepsilon}\right)\approx\log\frac{1}{\varepsilon},$$

with $\varepsilon:=\sigma^{\frac{sp}{q-1}-\beta}$, it holds that

$$\frac{|2B_{j+1}\cap\{u_j\le 2\varepsilon\omega(r_j)\}|}{|2B_{j+1}|}\le\frac{CN^3}{k}\le\frac{C_{\log}N^3}{\log\frac{1}{\sigma}}\tag{4.21}$$

for the constant $C_{\log}>0$ depending on $n,p,q,s,t,\alpha,[a]_\alpha,\|a\|_{L^\infty}$ and β.

At this moment, we are going to perform a suitable iteration. For each $i=0,1,\cdots$, let

$$\rho_i=r_{j+1}+2^{-i}r_{j+1},\quad \hat{\rho}_i=\frac{\rho_i+3\rho_{i+1}}{4},\quad \tilde{\rho}_i=\frac{3\rho_i+\rho_{i+1}}{4},$$

and the corresponding balls

$$B^i=B_{\rho_i},\quad \hat{B}^i=B_{\hat{\rho}_i},\quad \tilde{B}^i=B_{\tilde{\rho}_i}.$$

Then take the cut-off functions $\psi_i\in C_0^\infty(\tilde{B}^i)$ such that

$$0\le\psi_i\le 1,\quad \psi_i\equiv 1\text{in }\hat{B}^i\quad\text{and}\quad |\nabla_H\psi_i|\le 2^{i+2}r_{j+1}^{-1}.$$

Besides, set

$$k_i=(1+2^{-i})\varepsilon\omega(r_j),\quad w_i=(k_i-u_j)_+,$$

and

$$A_i=\frac{|B^i\cap\{u_j\le k_i\}|}{|B^i|}=\frac{|B^i\cap\{w_j\ge 0\}|}{|B^i|}.$$

Observe the apparent facts that

$$r_{j+1}\le\rho_{i+1}<\hat{\rho}_i<\tilde{\rho}_i<\rho_i\le 2r_{j+1},\quad 0\le w_i\le k_i\le 2\varepsilon\omega(r_j),$$

and denote

$$a_{j+1}^+:=\sup_{B_{2r_{j+1}}\times B_{2r_{j+1}}}a(\cdot,\cdot),\quad a_{j+1}^-:=\inf_{B_{2r_{j+1}}\times B_{2r_{j+1}}}a(\cdot,\cdot),\quad \overline{G}(\tau):=\frac{\tau^p}{r_{j+1}^{sp}}+a_{j+1}^+\frac{\tau^q}{r_{j+1}^{tq}}.$$

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With the help of Caccioppoli inequality (lemma 3.3), we derive

$$
\begin{aligned}
&\fint_{\hat{B}^i}\int_{\hat{B}^i}\frac{H(\xi,\eta,|w_i(\xi)-w_i(\eta)|)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\,d\xi d\eta\\
&\le C\fint_{B^i}\int_{B^i}\frac{H(\xi,\eta,(w_i(\xi)+w_i(\eta))|\psi_i(\xi)-\psi_i(\eta)|)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\,d\xi d\eta\\
&\quad+C\fint_{B^i}w_i\psi_i^q\,d\xi\left(\sup_{\eta\in\tilde{B}^i}\int_{\mathbb{H}^n\setminus B^i}\frac{h(\xi,\eta,w_i(\xi))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\,d\xi\right)\\
&=: J_1+J_2.
\end{aligned}
\tag{4.22}
$$

Via the definition of w_i and ψ_i, J_1 is evaluated as

$$
\begin{aligned}
J_1&\le C\frac{2^{ip}k_i^p}{r_{j+1}^p}\int_{B^i\cap\{u_j\le k_i\}}\fint_{B^i}\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{-Q+(1-s)p}\,d\xi d\eta\\
&\quad+Ca_{j+1}^{+}\frac{2^{iq}k_i^q}{r_{j+1}^q}\int_{B^i\cap\{u_j\le k_i\}}\fint_{B^i}\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{-Q+(1-t)q}\,d\xi d\eta\\
&\le C2^{iq}\overline{G}(k_i)A_i,
\end{aligned}
\tag{4.23}
$$

and moreover, we have

$$
\fint_{B^i}w_i\psi_i^q\,d\xi\le Ck_iA_i.
$$

As for the nonlocal integral in J_2, we first note that if $\eta\in\tilde{B}^i$ and $\xi\in\mathbb{H}^n\setminus B^i$, then

$$
\|\xi_0^{-1}\circ\xi\|_{\mathbb{H}^n}\le\left(1+\frac{\|\xi_0^{-1}\circ\eta\|_{\mathbb{H}^n}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}}\right)\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}\le 2^{i+4}\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}.
$$

Furthermore, $w_i\le k_i\le 2\varepsilon\omega(r_j)$ in B_j (by $u_j\ge 0$ in B_j), and $w_i\le k_i+|u|$ in $\mathbb{H}^n\setminus B_j$. In a similar way to treat I_2 in the proof of lemma 4.1, by applying (4.19),

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(4.20), the definition of ε and $B_{j+1} \subset B^i$ we derive

$$\begin{aligned}
&\sup_{\eta\in\tilde{B}^i} \int_{\mathbb{H}^n\setminus B^i} \frac{h(\xi,\eta,w_i(\xi))}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\, d\xi \\
&\le \sup_{\eta\in\tilde{B}^i} \int_{\mathbb{H}^n\setminus B^i} \frac{w_i^{p-1}+w_i^{q-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} + a_{j+1}^{+} \frac{w_i^{q-1}}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\, d\xi \\
&\le C2^{i(Q+sp+tq)} \int_{\mathbb{H}^n\setminus B_{j+1}} \frac{w_i^{p-1}+w_i^{q-1}}{\|\xi_0^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} + a_{j+1}^{+} \frac{w_i^{q-1}}{\|\xi_0^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\, d\xi \\
&\le C2^{i(Q+sp+tq)} \int_{\mathbb{H}^n\setminus B_{j}} \frac{|u_j|^{p-1}+|u_j|^{q-1}}{\|\xi_0^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} + a_{j+1}^{+} \frac{|u_j|^{q-1}}{\|\xi_0^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\, d\xi \\
&\quad + C2^{i(Q+sp+tq)} \int_{\mathbb{H}^n\setminus B_{j+1}} \frac{k_i^{p-1}+k_i^{q-1}}{\|\xi_0^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+sp}} + a_{j+1}^{+} \frac{k_i^{q-1}}{\|\xi_0^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q+tq}}\, d\xi \\
&\le C2^{i(Q+sp+tq)} \left(\frac{N\omega(r_j)^{p-1}}{r_j^{sp}\sigma^{\beta(p-1)}} + a_{j+1}^{+} \frac{\omega(r_j)^{q-1}}{r_j^{tq}\sigma^{\beta(q-1)}} + \frac{k_i^{p-1}+k_i^{q-1}}{r_{j+1}^{sp}} + a_{j+1}^{+}\frac{k_i^{q-1}}{r_{j+1}^{tq}} \right) \\
&\le C2^{i(Q+sp+tq)} \left(\frac{Nk_i^{p-1}}{\varepsilon^{p-1} r_j^{sp}\sigma^{\beta(p-1)}} + a_{j+1}^{+} \frac{k_i^{q-1}}{\varepsilon^{q-1} r_j^{tq}\sigma^{\beta(q-1)}} + \frac{Nk_i^{p-1}}{r_{j+1}^{sp}} + a_{j+1}^{+}\frac{k_i^{q-1}}{r_{j+1}^{tq}} \right) \\
&\le CN2^{i(Q+sp+tq)} \left(\frac{\sigma^{sp-\frac{sp(p-1)}{q-1}} k_i^{p-1}}{r_{j+1}^{sp}} + a_{j+1}^{+} \frac{\sigma^{tq-sp} k_i^{q-1}}{r_{j+1}^{tq}} + \frac{\overline{G}(k_i)}{k_i} \right) \\
&\le CN2^{i(Q+sp+tq)} \frac{\overline{G}(k_i)}{k_i}.
\end{aligned}$$

Therefore,

$$J_2 \le CN2^{i(Q+sp+tq)} \overline{G}(k_i) A_i. \tag{4.24}$$

On the other hand, making use of lemma 2.8 with $u := w_i$ yields that

$$\begin{aligned}
&A_{i+1}^{\frac{1}{\gamma}} \overline{G}(k_i - k_{i+1}) \\
&\le \left(\fint_{B^{i+1}} \left(\left| \frac{w_i}{r_{j+1}^{s}} \right|^p + a_{j+1}^{+} \left| \frac{w_i}{r_{j+1}^{t}} \right|^q \right)^{\gamma} d\xi \right)^{\frac{1}{\gamma}} \\
&\le CN \left(\frac{D_1(\hat{\rho}_i, \rho_{i+1})}{r_{j+1}^{sp}} + \frac{\widetilde{D}_1(\hat{\rho}_i, \rho_{i+1})}{r_{j+1}^{tq}} \right) \fint_{\hat{B}^i} \int_{\hat{B}^i} \frac{H(\xi,\eta,|w_i(\xi)-w_i(\eta)|)}{\|\eta^{-1}\circ\xi\|_{\mathbb{H}^n}^{Q}}\, d\xi d\eta \\
&\quad + CN \fint_{\hat{B}^i} \left| \frac{w_i}{r_{j+1}^{s}} \right|^p + a_{j+1}^{-} \left| \frac{w_i}{r_{j+1}^{t}} \right|^q d\xi.
\end{aligned} \tag{4.25}$$

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Thanks to the definitions of $D_1, \widetilde{D}_1$ and $\hat{\rho}_i, \rho_{i+1}$, we from $\hat{\rho}_i \approx \rho_{i+1} \approx r_{j+1}$ and $\hat{\rho}_i - \rho_{i+1} = 2^{-i-3} r_{j+1}$ calculate

$$\frac{D_1(\hat{\rho}_i, \rho_{i+1})}{r_{j+1}^{sp}} \leq C 2^{i(Q+sp+p)}, \quad \frac{\widetilde{D}_1(\hat{\rho}_i, \rho_{i+1})}{r_{j+1}^{tq}} \leq C 2^{i(Q+tq+q)}.$$

It is easy to obtain

$$⨍_{\hat{B}^i} \left| \frac{w_i}{r_{j+1}^s} \right|^p + a_{j+1}^- \left| \frac{w_i}{r_{j+1}^t} \right|^q d\xi \leq C ⨍_{B^i} \overline{G}(w_i)\, d\xi \leq C\overline{G}(k_i) A_i. \tag{4.26}$$

It follows from (4.22)–(4.26) that

$$\begin{aligned} A_{i+1}^{\frac{1}{\gamma}} \overline{G}(2^{-i-1} \varepsilon \omega(r_j)) &= A_{i+1}^{\frac{1}{\gamma}} \overline{G}(k_i - k_{i+1}) \\ &\leq C N^2 2^{i2(Q+2q)} \overline{G}(k_i) A_i \\ &\leq C N^2 2^{i2(Q+2q)} \overline{G}(\varepsilon \omega(r_j)) A_i, \end{aligned}$$

and further

$$A_{i+1} \leq C N^{2\gamma} 2^{i2(Q+3q)\gamma} A_i^{\gamma},$$

where $\gamma = \min\left\{\frac{p_s^*}{p}, \frac{q_t^*}{q}\right\} > 1$ and C depends on **data** and β.

Now if A_0 fulfils

$$A_0 = \frac{|2B_{j+1} \cap \{u_j \leq 2\varepsilon\omega(r_j)\}|}{|2B_{j+1}|} \leq (CN^{2\gamma})^{-\frac{1}{\gamma-1}} 2^{-\frac{2\gamma(Q+3q)}{(\gamma-1)^2}} =: \mu, \tag{4.27}$$

then by lemma 3.4 we deduce $A_i \to 0$ as $i \to \infty$. This means

$$u_j \geq \varepsilon\omega(r_j) \quad \text{a.e. in } B_{j+1},$$

which together with (4.17) leads to

$$\operatorname*{osc}_{B_{j+1}} u \leq (1-\varepsilon)\omega(r_j) = (1-\varepsilon)\sigma^{-\beta}\omega(r_{j+1}).$$

Finally, choosing $\beta \in \left(0, \frac{sp}{q-1}\right)$ small enough such that

$$\sigma^{\beta} \geq 1 - \varepsilon = 1 - \sigma^{\frac{sp}{q-1} - \beta},$$

then $\operatorname{osc}_{B_{j+1}} u \leq \omega(r_{j+1})$, and β depends on **data** and $\|u\|_{L^\infty(\Omega')}$. Indeed, due to (4.21), it yields that

$$A_0 \leq \frac{C_{\log} N^3}{\log \frac{1}{\sigma}} \leq \mu,$$

by picking $\sigma \leq \exp\left(-\frac{C_{\log} N^3}{\mu}\right)$. Then, we select $\sigma = \min\left\{\frac{1}{4}, \exp\left(-\frac{C_{\log} N^3}{\mu}\right)\right\}$ to ensure the condition (4.27) does hold true. Now we finish the proof. □

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Declarations

Conflict of interest

The authors declare that there is no conflict of interest. We also declare that this manuscript has no associated data.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] A. Adimurthi and A. Mallick. Hardy type inequality on fractional order Sobolev spaces on the Heisenberg group. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **18** (2018), 917–949.

[2] T. P. Branson, L. Fontana and C. Morpurgo. Moser-Trudinger and Beckner-Onofri's Inequalities on the CR sphere. *Ann. of Math.* **177** (2013), 1–52.

[3] S. S. Byun, H. Kim and J. Ok. Local Hölder continuity for fractional nonlocal equations with general growth. *Math. Ann.* **387** (2023), 807–846..

[4] S. S. Byun, J. Ok and K. Song. Hölder regularity for weak solutions to nonlocal double phase problems. *J. Math. Pures Appl.* **168** (2022), 110–142.

[5] J. Chaker, M. Kim and M. Weidner. Harnack inequality for nonlocal problems with non-standard growth. *Math. Ann.* **386** (2022), 533–550.

[6] P. Ciatti, M. G. Cowling and F. Ricci. Hardy and uncertainty inequalities on stratified Lie groups. *Adv. Math.* **277** (2015), 365–387.

[7] E. Cinti and J. Tan. A nonlinear Liouville theorem for fractional equations in the Heisenberg group. *J. Math. Anal. Appl.* **433** (2016), 434–454.

[8] M. Colombo and G. Mingione. Bounded minimisers of double phase variational integrals. *Arch. Ration. Mech. Anal.* **218** (2015), 219–274.

[9] M. Colombo and G. Mingione. Regularity for double phase variational problems. *Arch. Ration. Mech. Anal.* **215** (2015), 443–496.

[10] G. Cupini, P. Marcellini and E. Mascolo. Local boundedness of minimizers with limit growth conditions. *J. Optim. Theory Appl.* **166** (2015), 1–22.

[11] J. Cygan. Subadditicity of homogeneous norms on certain nilpotent Lie groups. *Proc. Am. Math. Soc.* **83** (1981), 69–70.

[12] C. De Filippis and G. Palatucci. Hölder regularity for nonlocal double phase equations. *J. Differential Equations.* **267** (2019), 547–586.

[13] A. Di Castro, T. Kuusi and G. Palatucci. Nonlocal Harnack inequalities. *J. Funct. Anal.* **267** (2014), 1807–1836.

[14] A. Di Castro, T. Kuusi and G. Palatucci. Local behavior of fractional p-minimizers. *Ann. Inst. H. Poincaré Anal. Non LinéAire.* **33** (2016), 1279–1299.

[15] E. Di Nezza, G. Palatucci and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), 521–573.

[16] Y. Fang and C. Zhang. Harnack inequality for the nonlocal equations with general growth. *Proc. Roy. Soc. Edinburgh Sect. A.* **153** (2023), 1479–1502.

[17] Y. Fang and C. Zhang. On weak and viscosity solutions of nonlocal double phase equations. *Int. Math. Res. Not. IMRN.* **5** (2023), 3746–3789.

[18] F. Ferrari and B. Franchi. Harnack inequality for fractional Laplacians in Carnot groups. *Math. Z.* **279** (2015), 435–458.

[19] F. Ferrari, M.Jr. Miranda, D. Pallara, A. Pinamonti and Y. Sire. Fractional Laplacians, perimeters and heat semigroups in Carnot groups. *Discrete Cont. Dyn. Syst. Ser. S.* **11** (2018), 477–491.

[20] R. Frank, M. Gonzalez, D. Monticelli and J. Tan. An extension problem for the CR fractional Laplacian. *Adv. Math.* **270** (2015), 97–137.

[21] N. Garofalo and G. Tralli. Feeling the heat in a group of Heisenberg type. *Adv. Math.* **381** (2021), 107635.

[22] N. Garofalo and G. Tralli. A class of nonlocal hypoelliptic operators and their extensions. *Indiana Univ. Math. J.* **70** (2022), 1717–1744.

[23] J. Giacomoni, D. Kumar and K. Sreenadh. Global regularity results for non-homogeneous growth fractional problems. *J. Geom. Anal.* **32** (2021), 36.

[24] J. Giacomoni, D. Kumar and K. Sreenadh. Hölder regularity results for parabolic nonlocal double phase problems. arXiv:2112.04287v1.

[25] E. Giusti. *Direct Methods in the Calculus of Variations* (World Scientific Publishing Co. Inc., River Edge, NJ, 2003).

[26] A. Iannizzotto, S. Mosconi and M. Squassina. Global Hölder regularity for the fractional p-Laplacian. *Rev. Mat. Iberoam.* **32** (2016), 1353–1392.

[27] A. Kassymov and D. Suragan. Lyapunov-type inequalities for the fractional p-sub-Laplacian. *Adv. Oper. Theory.* **5** (2020), 435–452.

[28] A. Kassymov and D. Surgan. Some functional inequalities for the fractional p-sub-Laplacian. arXiv:1804.01415.

[29] J. Korvenpää, T. Kuusi and G. Palatucci. The obstacle problem for nonlinear integro-differential operators. *Calc. Var. Partial Differential Equations.* **55** (2016), 63.

[30] J. Korvenpää, T. Kuusi and G. Palatucci. Fractional superharmonic functions and the Perron method for nonlinear integro-differential equations. *Math. Ann.* **369** (2017), 1443–1489.

[31] M. Manfredini, G. Palatucci, M. Piccinini and S. Polidoro. Hölder continuity and boundedness estimates for nonlinear fractional equations in the Heisenberg group. *J. Geom. Anal.* **33** (2023), 77.

[32] G. Palatucci and M. Piccinini. Nonlocal Harnack inequalities in the Heisenberg group. *Calc. Var. Partial Differential Equations.* **61** (2022), 185.

[33] G. Palatucci and M. Piccinini. Nonlinear fractional equations in the Heisenberg group. *Bruno Pini Mathematical Analysis Seminar.* **14** (2023), 163–200.

[34] M. Piccinini. The obstacle problem and the Perron Method for nonlinear fractional equations in the Heisenberg group. *Nonlinear Anal.* **222** (2022), 112966.

[35] H. Prasad and V. Tewary. boundedness of variational solutions to nonlocal double phase parabolic equations. *J. Differential Equations.* **351** (2023), 243–276.

[36] L. Roncal and S. Thangavelu. Hardy's Inequality for fractional powers of the sublaplacian on the Heisenberg group. *Adv. Math.* **302** (2016), 106–158.

[37] J. Scott and T. Mengesha. Self-improving inequalities for bounded weak solutions to nonlocal double phase equations. *Commun. Pure Appl. Anal.* **21** (2022), 183–212.

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