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BOOK REVIEWS

FORNBERG, B. *A practical guide to pseudospectral methods* (Cambridge University Press, Cambridge, 1996), x + 231 pp., 0 521 49582 2 (hardback), £37.50 (US$54.95).

Since the late seventies there has been a tremendous growth of interest in spectral and pseudospectral methods. Today they are used to solve large-scale problems in areas such as weather prediction, turbulence and seismic exploration. This book serves as a practical guide to the subject and describes the 'nuts and bolts' of pseudospectral methods. The book is born out of the author's vast experience in the field and we should be indebted to him for making his knowledge available to us in this user-friendly book. This book is not a theoretical guide to the subject – you will not be embedded in Sobolev spaces here! It is more suited for those interested in the implementational aspects of the subject. It serves as a useful successor to the book of Boyd [**1**], which focuses on algorithms for solving differential equations. However, for those interested in all aspects of spectral methods it is a useful partner for the monograph by Canuto *et al.* [**2**].

In his development of the subject matter the author exploits the connection between finite difference and pseudospectral methods. Pseudospectral methods are viewed as limiting cases of increasing-order finite difference methods rather than expansions of smooth global functions. This viewpoint is almost unique and the author is to be congratulated for offering his readership the insights into the behaviour of pseudospectral methods that this stance affords. It is interesting to note that the connection with finite element methods is usually exploited in the analysis of spectral methods.

In Chapter 2 the traditional approach to pseudospectral methods via expansions of smooth global functions is outlined. In Chapter 3 pseudospectral methods are introduced as the limit of finite difference methods. This viewpoint enables the author to highlight similarities between the methods and to show how pseudospectral methods may be generalized using developments of the finite difference technique. For example, ideas such as staggered grids and upwinding may be carried across using this interpretation.

The use of the nodes of Gaussian quadrature rules as collocation points in pseudospectral methods is now part of the established folklore of spectral methods. This is due to the spectacular accuracy of these quadrature rules and also because their use facilitates the development of a convergence theory for the variational formulation of certain types of differential equations. The author is quite fervent in opposing this trend and shows that, although the choice of nodes is crucial to the accuracy of Gaussian quadrature rules, the corresponding choice for pseudospectral methods is not at all as sensitive. Chapter 4 contains an interesting section on time-stepping methods and a discussion on stability conditions. The analysis of the stability of pseudospectral methods is performed using the connection with finite difference methods.

In Chapter 5 some variations to the standard pseudospectral method are described. These are aids which improve the overall performance of the method. Topics covered include using additional information from the governing equations to reduce the size of the problem (for example, building any symmetry present in a problem into the approximation), using different pseudospectral operators for different terms in an equation (the pseudospectral equivalent of upwinding), staggered grids and the use of finite difference preconditioning to reduce the condition numbers of the linear systems and to accelerate convergence of iterative methods. A limited discussion of domain decomposition and spectral element methods is given. This is one of the weaknesses of the book in that only a cursory mention is made of these methods which are essential for many practical engineering problems which are defined in complex geometries.

Chapter 6 investigates problems that are peculiar to spherical and polar coordinates, in particular the impossible task of covering a sphere with a mesh which is dense and uniform. Despite their intrinsic elegance spherical harmonics have had limited numerical use because of the lack of a fast transformation between expansion coefficients and grid point values for all except the case when $m = 0$. For problems defined in polar coordinates the use of a mixed Fourier–Chebyshev representation in the region $-1 \leq r \leq 1$, $-\pi/2 \leq \theta \leq \pi/2$ instead of $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$ circumvents the problem of having to refine the grid unnecessarily in the radial direction. Pseudospectral methods in spherical coordinates are also examined in this chapter for elliptic and hyperbolic problems. The author demonstrates that, while finite difference methods perform well for elliptic problems, they are not so successful for hyperbolic problems. The effectiveness of high order methods is clearly shown for these problems. Smoothing applied in the ϕ direction circumvents restrictive polar CFL stability conditions.

Chapter 7 attempts to compare the computational costs of finite difference and pseudospectral methods. This is a very important exercise, for in many situations particularly in the engineering world it is important to estimate a priori the computational expense required to solve a problem to a desired accuracy. The only rule which the author can offer here is that for simple model problems in smooth domains pseudospectral and high-order finite difference methods perform extremely well. However, for problems with singularities either in the solution or the geometry low-order finite difference methods may be more economical and robust. The author considers model 1-D periodic and nonperiodic problems, *viz.* the hyperbolic equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \tag{1}$$

and the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{9\pi^2} \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

It is difficult to predict how these methods will behave on more difficult problems and as the author says 'the design of a comprehensive, convenient and sharp a priori estimation technique for cost versus accuracy remains an elusive goal.'

It is fitting that after a discussion of pseudospectral methods for model problems in the previous chapter the attention to applications is drawn in Chapter 8. This is a very interesting chapter for a number of reasons. First, it shows that the recent interest in spectral methods arose out of a necessity to develop accurate and efficient numerical methods for large-scale computations in meteorology and turbulence modelling. Secondly, it gives an historical background to turbulence modelling, nonlinear wave equations, weather prediction and seismic exploration, including the development of numerical methods to solve problems in these important areas.

The appendices in the book are comprehensive and essential for all who work on the implementation of pseudospectral and high-order finite difference methods. I thoroughly enjoyed reading this book and the opportunity to look at pseudospectral methods from a different perspective. I would recommend this book to those interested in pseudospectral methods and those whose research involves solving differential equations using high-order discretization methods.

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REFERENCES

1. J. P. Boyd, *Chebyshev and Fourier spectral methods* (Lecture Notes in Engineering No. 49, Springer-Verlag, Berlin, 1989).

2. C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral methods in fluid dynamics* (Springer-Verlag, Berlin, 1988).

Falconer, K. J. *Techniques in fractal geometry* (Wiley, Chichester–New York–Weinheim–Brisbane–Singapore–Toronto, 1997), xvii+256 pp., 0 471 95724 0, £24.95.

This book has proved surprisingly hard to write a review for. This is not due to problems with the book (indeed it is excellent) but more due to the reviewer's concerns with his own bias: he was involved in the proof-reading.

This book is a sequel to the author's earlier book, *Fractal Geometry* [1] (recently published by Wiley in paperback). It follows closely the style of its ancestor and details in a clear and elegant manner developments in fractal geometry since the earlier book was published. The author concentrates on presenting the key ideas in the subject rather than obtaining the most general results. This makes the book a useful primer before reading more technical papers/works.

The book begins by reviewing some of the more useful results from *Fractal Geometry*, obviating the need for the reader to possess this book as well (although it would not do any harm).

The first part of the book discusses general methods for calculating the dimension (be it Hausdorff, packing or Box) of a set and introduces the family of examples which the book is primarily concerned with: cookie-cutter sets. These are essentially non-linear versions of the more familiar self-similar sets and many of the results for self-similar sets hold for this more general class. The thermodynamic formalism is introduced and it is shown how this enables dimensions of these non-linear sets to be (theoretically) found. The author explains clearly the analogies with thermodynamics and makes the methods used appear very natural.

The middle part of the book describes how probabilistic ideas may be used in the study of fractals. In particular, the author states and proves versions of the Ergodic theorem, Renewal theorem and the Martingale Convergence theorem. He gives a nice application of the Martingale Convergence theorem to the study of Random cut-out sets (a set which is formed by randomly cutting out pieces of decreasing size from some initial set, such as a square). If you know how the size of the cut-out pieces varies, then with positive probability you know the dimension of the cut-out set.

The latter part of the book is a survey of several topics from geometric measure theory and fractal geometry, discussing some of the developments of recent years. There is an introduction to the theory of tangent measures and an explanation of their rôle in studying the geometry of measures (and sets) in Euclidean spaces. This chapter serves as a useful primer to the more technical account given by Mattila in [2].

There is a chapter discussing dimensions of measures; this involves, essentially, looking for "typical" values of

$$\underline{\alpha}(\mu, x) = \liminf_{r \searrow 0} \frac{\mu(B(x,r))}{\log r}, \quad \overline{\alpha}(\mu, x) = \limsup_{r \searrow 0} \frac{\mu(B(x,r))}{\log r}$$

for a measure μ (here $B(x, r)$ denotes the closed ball of centre x and radius r). When $\underline{\alpha}$ (or $\overline{\alpha}$) may take many different values we are led naturally to the idea of decomposing measures into pieces where they do behave similarly. This in turn leads, ultimately, to investigating the