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Dimension estimates for C^1 iterated function systems and repellers. Part I

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Abstract. This is the first paper in a two-part series containing some results on dimension estimates for C^1 iterated function systems and repellers. In this part, we prove that the upper box-counting dimension of the attractor of any C^1 iterated function system (IFS) on $\mathbb{R}^d$ is bounded above by its singularity dimension, and the upper packing dimension of any ergodic invariant measure associated with this IFS is bounded above by its Lyapunov dimension. Similar results are obtained for the repellers for C^1 expanding maps on Riemannian manifolds.

Key words: Hausdorff dimension, packing dimension, iterated function systems, repellers of expanding maps, singularity dimension, Lyapunov dimension
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1. *Introduction*

The dimension theory of iterated function systems (IFSs) and dynamical repellers has developed into an important field of research during the last 40 years. One of the main objectives is to estimate variant notions of dimension of the invariant sets and measures involved. Despite many new and significant developments in recent years, only the cases of conformal repellers and attractors of conformal IFSs under a certain separation condition have been completely understood. In such cases, the topological pressure plays a crucial role in the theory. Indeed, the Hausdorff and box-counting dimensions of the repeller X for a C^1 conformal expanding map f are given by the unique number s satisfying the Bowen–Ruelle formula $P(X, f, -s \log \|D_x f\|) = 0$, where the functional P is the topological pressure; see [8, 23, 38]. A similar formula is obtained for the Hausdorff and box-counting dimensions of the attractor of a conformal IFS satisfying the open set condition (see, for example, [34]).

The study of dimension in the non-conformal cases has proved to be much more difficult. In his seminal paper [13], Falconer established a general upper bound on the Hausdorff and box-counting dimensions of a self-affine set (which is the attractor of an IFS consisting of contracting affine maps) in terms of the so-called affinity dimension, and proved that for typical self-affine sets under a mild assumption this upper bound is equal to the dimension. So far substantial progress has been made towards understanding when the Hausdorff and box-counting dimensions of a concrete planar self-affine set are equal to its affinity dimension; see [3, 24] and the references therein. However, very little has been known in the higher-dimensional case.

In [15, Theorem 5.3], by developing a subadditive version of the thermodynamical formalism, Falconer showed that the upper box-counting dimension of a mixing repeller Λ of a non-conformal C^2 mapping ψ, under the distortion condition

$$\|(D_x\psi)^{-1}\|^2\|D_x\psi\| < 1 \quad \text{for } x \in \Lambda, \tag{1.1}$$

is bounded above by the zero point of the (subadditive) topological pressure associated with the singular value functions of the derivatives of iterates of ψ. We write this zero point as $\dim_{S^*} \Lambda$ and call it the *singularity dimension of* Λ (see Definition 6.1 for the details). Condition (1.1) is used to prove the bounded distortion property of the singular value functions, which enables one to control the distortion of balls after many iterations (see [15, Lemma 5.2]). Examples involving 'triangular maps' were constructed in [32] to show that the condition (1.1) is necessary for this bounded distortion property.

Using a quite different approach, Zhang [40] proved that the Hausdorff dimension of the repeller of an arbitrary C^1 expanding map ψ is also bounded above by the singularity dimension. (We remark that the upper bound given by Zhang was defined in a slightly different way, but it is equal to the singularity dimension; see [2, Corollary 2] for a proof.) The basic technique used in Zhang's proof is to estimate the Hausdorff measure of $\psi(A)$ for small sets A (see [40, Lemma 3]), which is applied to only one iteration so it avoids assuming any distortion condition. However, his method does not apply to the box-counting dimension.

Thanks to the results of Falconer and Zhang, a natural question arises as to whether the upper box-counting dimension of a C^1 repeller is always bounded above by its singularity dimension. The challenge here is the lack of valid tools to analyze the geometry of the images of balls under a large number of iterations of C^1 maps. In [4, Theorem 3], Barreira made a positive claim concerning this question, but his proof contains a crucial mistake†, as found by Manning and Simon [32]. In a recent paper [9, Theorem 3.2], Cao, Pesin and Zhao obtained an upper bound for the upper box-counting dimension of the repellers

† This result was cited or applied in several papers (e.g., [2, Corollary 4], [11, Theorems 4.4–4.7]) without noticing the mistake in [4].

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of $C^{1+\alpha}$ maps satisfying a certain dominated splitting property. However, that upper bound depends on the splitting involved and is usually strictly larger than the singularity dimension.

In the present paper, we give an affirmative answer to the above question. We also establish an analogous result for the attractors of C^1 non-conformal IFSs. Meanwhile we prove that the upper packing dimension of an ergodic invariant measure supported on a C^1 repeller (respectively, the projection of an ergodic measure on the attractor of a C^1 IFS) is bounded above by its Lyapunov dimension.

In the continuation of this paper [22] we verify that upper bound estimates of the dimensions of the attractors and ergodic measures in the previous paragraph give the exact values of the dimensions for some families of C^1 non-conformal IFSs in $\mathbb{R}^d$ at least typically. 'Typically' means that the assertions hold for almost all translations of the system. These families include the C^1 non-conformal IFSs in $\mathbb{R}^d$ for which all differentials are either diagonal matrices, or all differentials are lower triangular matrices and satisfy a certain domination condition.

We first state our results for C^1 IFSs. To this end, let us introduce some notation and definitions. Let Z be a compact subset of $\mathbb{R}^d$. A finite family $\{f_i\}_{i=1}^{\ell}$ of contracting self-maps on Z is called a C^1 *iterated function system*, if there exists an open set $U \supset Z$ such that each f_i extends to a contracting C^1-diffeomorphism $f_i : U \to f_i(U) \subset U$. Let K be the attractor of the IFS, that is, K is the unique non-empty compact subset of $\mathbb{R}^d$ such that

$$K = \bigcup_{i=1}^{\ell} f_i(K) \tag{1.2}$$

(cf. [16]).

Let (Σ, σ) be the one-sided full shift over the alphabet $\{1, \ldots, \ell\}$ (cf. [7]). Let $\Pi : \Sigma \to K$ denote the canonical coding map associated with the IFS $\{f_i\}_{i=1}^{\ell}$. That is,

$$\Pi(x) = \lim_{n\to\infty} f_{x_1} \circ \cdots \circ f_{x_n}(z), \quad x = (x_n)_{n=1}^{\infty}, \tag{1.3}$$

with $z \in U$. The definition of Π is independent of the choice of z.

For any compact subset X of Σ with $\sigma X \subset X$, we call (X, σ) a *one-sided subshift* or simply a *subshift* over $\{1, \ldots, \ell\}$ and let $\dim_S X$ denote the singularity dimension of X with respect to the IFS $\{f_i\}_{i=1}^{\ell}$ (cf. Definition 2.5).

For a set $E \subset \mathbb{R}^d$, let $\overline{\dim}_B E$ denote the upper box-counting dimension of E (cf. [16]). The first result in this paper is the following theorem, stating that the upper box-counting dimension of $\Pi(X)$ is bounded above by the singularity dimension of X.

THEOREM 1.1. *Let $X \subset \Sigma$ be compact and $\sigma X \subset X$. Then $\overline{\dim}_B \Pi(X) \le \dim_S X$. In particular,*

$$\overline{\dim}_B K \le \dim_S \Sigma.$$

For an ergodic σ-invariant measure m on Σ, let $\dim_L m$ denote the Lyapunov dimension of m with respect to $\{f_i\}_{i=1}^{\ell}$ (cf. Definition 2.6). For a Borel probability measure η on $\mathbb{R}^d$

(or a manifold), let $\overline{\dim}_P\eta$ denote the upper packing dimension of η. That is,

$$\overline{\dim}_P\eta = \operatorname{esssup}_{x\in \operatorname{supp}(\eta)}\overline{d}(\eta, x), \quad \text{with } \overline{d}(\eta, x) := \limsup_{r\to 0} \frac{\log \eta(B(x, r))}{\log r},$$

where $B(x, r)$ denotes the closed ball centered at x of radius r. Equivalently,

$$\overline{\dim}_P\eta = \inf\{\dim_P F : F \text{ is a Borel set with } \eta(F) = 1\},$$

where $\dim_P F$ stands for the packing dimension of F (cf. [16]). See, for example, [17] for a proof.

Our second result can be viewed as a measure analogue of Theorem 1.1.

THEOREM 1.2. *Let m be an ergodic σ-invariant measure on Σ. Then*

$$\overline{\dim}_P(m \circ \Pi^{-1}) \le \dim_L m,$$

where $m \circ \Pi^{-1}$ stands for the push-forward of m by Π.

The above theorem improves a result of Jordan and Pollicott [26, Theorem 1] which states that

$$\overline{\dim}_H(m \circ \Pi^{-1}) \le \dim_L m$$

under a slightly more general setting, where $\overline{\dim}_H(m \circ \Pi^{-1})$ stands for the upper Hausdorff dimension of $m \circ \Pi^{-1}$. Recall that the upper Hausdorff dimension of a measure is the infimum of the Hausdorff dimension of Borel sets of full measure, which is always less than or equal to the upper packing dimension of the measure. It is worth pointing out that Theorem 1.2 was proved previously by Jordan [25] and Rossi [36] in the special case when $\{f_i\}_{i=1}^{\ell}$ is an affine IFS.

Next, we turn to the case of repellers. Let $\boldsymbol{M}$ be a smooth Riemannian manifold of dimension d and $\psi : \boldsymbol{M} \to \boldsymbol{M}$ a C^1-map. Let Λ be a compact subset of $\boldsymbol{M}$ such that $\psi(\Lambda) = \Lambda$. We say that ψ is *expanding* on the *repeller* Λ if:

(a) there exists $\lambda > 1$ such that $\|(D_z\psi)v\| \ge \lambda\|v\|$ for all $z \in \Lambda$, $v \in T_z\boldsymbol{M}$ (with respect to a Riemannian metric on $\boldsymbol{M}$);
(b) there exists an open neighborhood V of Λ such that

$$\Lambda = \{z \in V : \psi^n(z) \in V \text{ for all } n \ge 0\}.$$

We refer the reader to [35, §20] for more details. In what follows we always assume that Λ is a repeller of ψ. Let $\dim_{S^*} \Lambda$ denote the singular dimension of Λ with respect to ψ (see Definition 6.1). For an ergodic ψ-invariant measure μ on Λ, let $\dim_{L^*} \mu$ be the Lyapunov dimension of μ with respect to ψ (see Definition 6.2). Analogously to Theorems 1.1–1.2, we have the following results.

THEOREM 1.3. *Let Λ be the repeller of ψ. Then*

$$\overline{\dim}_B\Lambda \le \dim_{S^*} \Lambda.$$

THEOREM 1.4. *Let μ be an ergodic ψ-invariant measure supported on Λ. Then*

$$\overline{\dim}_P \mu \le \dim_{L^*} \mu.$$

For the estimates of the box-counting dimension of attractors of C^1 non-conformal IFSs (respectively, C^1 repellers), the reader may reasonably ask what difficulties arose in the previous work [15] which the present paper overcomes. Below we give an explanation and roughly illustrate our strategy for the proof.

Let us give an account of the IFS case. The case of repellers is similar. Let K be the attractor of a C^1 IFS $\{f_i\}_{i=1}^{\ell}$. To estimate $\overline{\dim}_B K$, by definition one needs to estimate for given $r > 0$ the smallest number of balls of radius r required to cover K, say, $N_r(K)$. To this end, one may iterate the IFS to get $K = \bigcup_{i_1 \ldots i_n} f_{i_1 \ldots i_n}(K)$ and then estimate $N_r(f_{i_1 \ldots i_n}(K))$ separately, where $f_{i_1 \ldots i_n} := f_{i_1} \circ \cdots \circ f_{i_n}$. For this purpose, one needs to estimate $N_r(f_{i_1 \ldots i_n}(B))$, where B is a fixed ball covering K.

Under the strong assumption of distortion property, Falconer was able to show that $f_{i_1 \ldots i_n}(B)$ is roughly comparable to the ellipsoid $(D_x f_{i_1 \ldots i_n})(B)$ for each $x \in B$ (see [15, Lemma 5.2]); then, by cutting the ellipsoid into roughly round pieces, he could use a certain singular value function to give an upper bound of $N_r(f_{i_1 \ldots i_n}(B))$, and then apply the subadditive thermodynamic formalism to estimate the growth rate of $\sum_{i_1 \ldots i_n} N_r(f_{i_1 \ldots i_n}(B))$. However, in the general C^1 non-conformal case, this approach is no longer feasible, since it seems hopeless to analyze the geometric shape of $f_{i_1 \ldots i_n}(B)$ when n is large.

The strategy of our approach is quite different. We use an observation going back to Douady and Oesterlé [12] (see also [40]) that, for a given C^1 map f, when B_0 is a small enough ball in a fixed bounded region, $f(B_0)$ is close to being an ellipsoid and so can be covered by a certain number of balls controlled by the singular values of the differentials of f (see Lemma 4.1). Since the maps f_i in the IFS are contracting, we may apply this fact to the maps $f_{i_n}, f_{i_{n-1}}, \ldots, f_{i_1}$ recursively. Roughly speaking, suppose that B_0 is a ball of small radius r_0. Then $f_{i_n}(B_0)$ can be covered by N_1 balls of radius r_1, and the image of each of them under $f_{i_{n-1}}$ can be covered by N_2 balls of radius of r_2, and so on, where $N_j, r_j/r_{j-1}$ $(j = 1, \ldots, n)$ can be controlled by the singular values of the differentials of $f_{i_{n-j+1}}$. In this way we get an estimate that $N_{r_n}(f_{i_1 \ldots i_n}(B_0)) \le N_1 \ldots N_n$ (see Proposition 4.2 for a more precise statement), which is in spirit analogous to the corresponding estimate for the Hausdorff measure by Zhang [40]. In this process, we do not need to consider the differentials of $f_{i_1 \ldots i_n}$ and so no distortion property is required. By developing a key technique from the thermodynamic formalism (see Proposition 3.4), we can get an upper bound for $\overline{\dim}_B K$, say s_1. Replacing the IFS $\{f_i\}_{i=1}^{\ell}$ by its nth iteration $\{f_{i_1 \ldots i_n}\}$, we get other upper bounds s_n. Again using a technique in the thermodynamic formalism (see Proposition 3.1), we manage to show that $\liminf s_n$ is bounded above by the singularity dimension.

The proof of Theorem 1.2 is also based on the above strategy. For an ergodic measure m on Σ, we are able, using the above covering arguments and ergodic theorems, to provide sharp estimates on the growth rates of $N_{(u_k)^n}(f_{\mathbf{i}|\mathbf{n}}(K))$ for m-a.e. (almost every) $\mathbf{i} \in \Sigma$, where $u_k = \exp(\lambda_k)$, $k = 1, \ldots, d$, with λ_k being the kth Lyapunov exponent of the

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matrix cocycle $A(\mathbf{i}, n) := D_{\pi\sigma^n\mathbf{i}} f_{\mathbf{i}|n}$ with respect to m. More precisely, we have the following inequality (see Lemma 5.1):

$$\lim_{n\to\infty} \frac{1}{n} \log(N_{(u_k)^n}(f_{\mathbf{i}|\mathbf{n}}(K))) \le (\lambda_1 + \cdots + \lambda_{k-1}) - (k-1)\lambda_k$$

for m-a.e. $\mathbf{i} \in \Sigma$, with the convention that $\lambda_1 + \cdots + \lambda_{k-1} = 0$ if $k = 1$. The proof of Lemma 5.1 is delicate. In particular, we need to apply a special version of Kingman's subadditive ergodic theorem which is stated and proved in Lemma 2.8. Theorem 1.2 is then derived from Lemma 5.1 by using an idea employed in [**25**, **36**].

This paper is organized as follows. In §2 we give some preliminaries about the subadditive thermodynamic formalism and give the definitions of the singularity and Lyapunov dimensions with respect to a C^1 IFS. In §3 we prove two auxiliary results (Propositions 3.1 and 3.4) which play a key role in the proof of Theorem 1.1 (and of Theorem 1.3). The proofs of Theorems 1.1–1.2 are given in §§4–5, respectively. In §6 we give the definitions of the singularity and Lyapunov dimensions in the repeller case and prove Theorems 1.3–1.4. For the convenience of the reader, in the Appendix we summarize the main notation and typographical conventions used in this paper.

2. *Preliminaries*

2.1. *Variational principle for subadditive pressure.* In order to define the singularity and Lyapunov dimensions and prove our main results, we require some elements from the subadditive thermodynamic formalism.

Let (X, d) be a compact metric space and $T : X \to X$ a continuous mapping. We call (X, T) a *topological dynamical system*. For $x, y \in X$ and $n \in \mathbb{N}$, we define

$$d_n(x, y) := \max_{0\le i\le n-1} d(T^i(x), T^i(y)). \tag{2.1}$$

A set $E \subset X$ is called (n, ε)-*separated* if for every distinct $x, y \in E$ we have $d_n(x, y) > \varepsilon$.

Let $C(X)$ denote the set of real-valued continuous functions on X. Let $\mathcal{G} = \{g_n\}_{n=1}^\infty$ be a *subadditive potential* on X, that is, $g_n \in C(X)$ for all $n \ge 1$ such that

$$g_{m+n}(x) \le g_n(x) + g_m(T^n x) \quad \text{for all } x \in X \text{ and } n, m \in \mathbb{N}. \tag{2.2}$$

Following [**10**], below we define the topological pressure of $\mathcal{G}$.

Definition 2.1. For given $n \in \mathbb{N}$ and $\varepsilon > 0$, we define

$$P_n(X, T, \mathcal{G}, \varepsilon) := \sup\left\{ \sum_{x\in E} \exp(g_n(x)) : E \text{ is an } (n, \varepsilon)\text{-separated set} \right\}. \tag{2.3}$$

Then the *topological pressure of* $\mathcal{G}$ *with respect to* T is defined by

$$P(X, T, \mathcal{G}) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_n(X, T, \mathcal{G}, \varepsilon). \tag{2.4}$$

If the potential $\mathcal{G}$ is additive, that is, $g_n = S_n g := \sum_{k=0}^{n-1} g \circ T^k$ for some $g \in C(X)$, then $P(X, T, \mathcal{G})$ recovers the classical topological pressure $P(X, T, g)$ of g (see, for example, [**39**]).

Let $\mathcal{M}(X)$ denote the set of Borel probability measures on X, and $\mathcal{M}(X, T)$ the set of T-invariant Borel probability measures on X. For $\mu \in \mathcal{M}(X, T)$, let $h_\mu(T)$ denote the measure-theoretic entropy of μ with respect to T (cf. [39]). Moreover, for $\mu \in \mathcal{M}(X, T)$, by subadditivity we have

$$\mathcal{G}_*(\mu) := \lim_{n\to\infty} \frac{1}{n} \int g_n \, d\mu = \inf_n \frac{1}{n} \int g_n \, d\mu \in [-\infty, \infty). \tag{2.5}$$

See, for example, [39, Theorem 10.1]. We call $\mathcal{G}_*(\mu)$ the *Lyapunov exponent* of $\mathcal{G}$ with respect to μ.

The proofs of our main results rely on the following general variational principle for the topological pressure of subadditive potentials.

THEOREM 2.2. [10, Theorem 1.1] *Let $\mathcal{G} = \{g_n\}_{n=1}^\infty$ be a subadditive potential on a topological dynamical system (X, T). Suppose that the topological entropy of T is finite. Then*

$$P(X, T, \mathcal{G}) = \sup\{h_\mu(T) + \mathcal{G}_*(\mu) : \mu \in \mathcal{M}(X, T)\}. \tag{2.6}$$

Particular cases of the above result, under stronger assumptions on the dynamical systems and the potentials, were previously obtained by many authors; see, for example, [5, 14, 18, 21, 28, 33] and references therein.

Measures that achieve the supremum in (2.6) are called *equilibrium measures* for the potential $\mathcal{G}$. There exists at least one ergodic equilibrium measure when the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous; this is the case when (X, T) is a subshift (see, for example, [19, Proposition 3.5] and the remark there).

The following well-known result is also needed in our proofs.

LEMMA 2.3. *Let $X_i, i = 1, 2$, be compact metric spaces and let $T_i : X_i \to X_i$ be continuous. Suppose $\pi : X_1 \to X_2$ is a continuous surjection such that the following diagram commutes:*

$$\begin{tikzcd} X_1 \arrow[r, "T_1"] \arrow[d, "\pi"'] & X_1 \arrow[d, "\pi"] \\ X_2 \arrow[r, "T_2"'] & X_2 \end{tikzcd}$$

Then $\pi_* : \mathcal{M}(X_1, T_1) \to \mathcal{M}(X_2, T_2)$ (defined by $\mu \mapsto \mu \circ \pi^{-1}$) is surjective.

If, furthermore, there is an integer $q > 0$ so that $\pi^{-1}(y)$ has at most q elements for each $y \in X_2$, then

$$h_\mu(T_1) = h_{\mu\circ\pi^{-1}}(T_2)$$

for each $\mu \in \mathcal{M}(X_1, T_1)$.

Proof. The first part of the result is the same as [31, Ch. IV, Lemma 8.3]. The second part follows from the Abramov–Rokhlin formula (see [6]). □

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2.2. *Subshifts.* In this subsection we introduce some basic notation and definitions about subshifts.

Let (Σ, σ) be the one-sided full shift over the alphabet $\mathcal{A} = \{1, \ldots, \ell\}$. That is, $\Sigma = \mathcal{A}^{\mathbb{N}}$ endowed with the product topology, and $\sigma : \Sigma \to \Sigma$ is the left shift defined by $(x_i)_{i=1}^{\infty} \mapsto (x_{i+1})_{i=1}^{\infty}$. The topology of Σ is compatible with the following metric on Σ:

$$d(x, y) = 2^{-\inf\{k : x_{k+1} \neq y_{k+1}\}} \quad \text{for } x = (x_i)_{i=1}^{\infty},\ y = (y_i)_{i=1}^{\infty}.$$

For $x = (x_i)_{i=1}^{\infty} \in \Sigma$ and $n \in \mathbb{N}$, write $x|n = x_1 \ldots x_n$.

Let X be a non-empty compact subset of Σ satisfying $\sigma X \subset X$. We call (X, σ) a *one-sided subshift* or simply a *subshift* over $\mathcal{A}$. We denote the collection of finite words allowed in X by X^*, and the subset of X^* of words of length n by X_n^*. In particular, define, for $n \in \mathbb{N}$,

$$X^{(n)} := \{(x_i)_{i=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} : x_{kn+1}x_{kn+2} \ldots x_{(k+1)n} \in X_n^* \text{ for all } k \geq 0\}. \tag{2.7}$$

It is clear that $X \subset X^{(n)}$ for every $n \in \mathbb{N}$.

Let $\mathcal{G} = \{g_n\}_{n=1}^{\infty}$ be a subadditive potential on a subshift (X, σ). It is known that in such a case, the topological pressure of $\mathcal{G}$ can alternatively be defined by

$$P(X, \sigma, \mathcal{G}) = \lim_{n\to\infty} \frac{1}{n} \log \bigg(\sum_{\mathbf{i} \in X_n^*} \sup_{x \in [\mathbf{i}] \cap X} \exp(g_n(x)) \bigg), \tag{2.8}$$

where $[\mathbf{i}] := \{x \in \Sigma : x|n = \mathbf{i}\}$ for $\mathbf{i} \in \mathcal{A}^n$; see [**10**, p. 649]. The limit can be seen to exist by using a standard subadditivity argument. We remark that (2.8) was first introduced by Falconer in [**14**] for the definition of the topological pressure of subadditive potentials on a mixing repeller.

Below we provide a useful lemma.

LEMMA 2.4. *Let $(\mathcal{A}^{\mathbb{N}}, \sigma)$ be the one-sided full shift space over a finite alphabet $\mathcal{A}$. Let $\nu \in \mathcal{M}(\mathcal{A}^{\mathbb{N}}, \sigma^m)$ for some $m \in \mathbb{N}$. Set $\mu = (1/m) \sum_{k=0}^{m-1} \nu \circ \sigma^{-k}$. Then $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{N}}, \sigma)$ and $h_\mu(\sigma) = (1/m) h_\nu(\sigma^m)$.*

Proof. The lemma might be well known. However, we are not able to find a reference, so for the convenience of the reader we provide a self-contained proof. The σ-invariance of μ follows directly from its definition, and we only need to prove that $h_\mu(\sigma) = (1/m)h_\nu(\sigma^m)$.

Clearly, $\nu \circ \sigma^{-k} \in \mathcal{M}(\mathcal{A}^{\mathbb{N}}, \sigma^m)$ for $k = 0, \ldots, m-1$. We claim that $h_{\nu\circ\sigma^{-k}}(\sigma^m) = h_\nu(\sigma^m)$ for $k = 1, \ldots, m-1$. Without loss of generality we prove this in the case when $k = 1$. For $n \in \mathbb{N}$, let $\mathcal{P}_n$ denote the partition of $\mathcal{A}^{\mathbb{N}}$ consisting of the nth cylinders of $\mathcal{A}^{\mathbb{N}}$, that is, $\mathcal{P}_n = \{[I] : I \in \mathcal{A}^n\}$, and set $\sigma^{-1}\mathcal{P}_n = \{\sigma^{-1}([I]) : I \in \mathcal{A}^n\}$. Then it is direct to see that any element in $\mathcal{P}_n$ intersects at most $\#\mathcal{A}$ elements in $\sigma^{-1}\mathcal{P}_n$, and vice versa. Hence,

$$|H_\nu(\mathcal{P}_n) - H_\nu(\sigma^{-1}\mathcal{P}_n)| \leq \log \#\mathcal{A};$$

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see, for example, [20, Lemma 4.6]. It follows that

$$h_{\nu\circ\sigma^{-1}}(\sigma^m) = \lim_{n\to\infty}\frac{1}{n}H_{\nu\circ\sigma^{-1}}(\mathcal{P}_{nm}) = \lim_{n\to\infty}\frac{1}{n}H_\nu(\sigma^{-1}\mathcal{P}_{nm})$$
$$= \lim_{n\to\infty}\frac{1}{n}H_\nu(\mathcal{P}_{nm}) = h_\nu(\sigma^m).$$

This proves the claim.

By the affinity of the measure-theoretic entropy $h_{(\cdot)}(\sigma^m)$ (see [39, Theorem 8.1]), we have

$$h_\mu(\sigma^m) = \frac{1}{m}\sum_{k=0}^{m-1} h_{\nu\circ\sigma^{-k}}(\sigma^m) = h_\nu(\sigma^m),$$

where the second equality follows from the above claim. Hence, $h_\mu(\sigma) = (1/m)h_\mu(\sigma^m) = (1/m)h_\nu(\sigma^m)$. □

2.3. *Singularity dimension and Lyapunov dimension with respect to C^1 IFSs.* In this subsection, we define the singularity and Lyapunov dimensions with respect to C^1 IFSs. The corresponding definitions with respect to C^1 repellers will be given in §5.

Let $\{f_i\}_{i=1}^{\ell}$ be a C^1 IFS on $\mathbb{R}^d$ with attractor K. Let (Σ, σ) be the one-sided full shift over the alphabet $\{1, \ldots, \ell\}$ and let $\Pi : \Sigma \to K$ denote the corresponding coding map defined as in (1.3). For a differentiable function $f : U \subset \mathbb{R}^d \to \mathbb{R}^d$, let $D_z f$ denote the differential of f at $z \in U$.

For $T \in \mathbb{R}^{d\times d}$, let $\alpha_1(T) \geq \cdots \geq \alpha_d(T)$ denote the singular values of T. Following [13], for $s \geq 0$ we define the *singular value function* $\phi^s : \mathbb{R}^{d\times d} \to [0, \infty)$ as

$$\phi^s(T) = \begin{cases} \alpha_1(T)\cdots\alpha_k(T)\alpha_{k+1}^{s-k}(T) & \text{if } 0 \leq s \leq d, \\ \det(T)^{s/d} & \text{if } s > d, \end{cases} \tag{2.9}$$

where $k = [s]$ is the integral part of s.

Definition 2.5. For a compact subset X of Σ with $\sigma(X) \subset X$, the *singularity dimension of X with respect to $\{f_i\}_{i=1}^{\ell}$*, written as $\dim_S X$, is the unique non-negative value s for which

$$P(X, \sigma, \mathcal{G}^s) = 0,$$

where $\mathcal{G}^s = \{g_n^s\}_{n=1}^{\infty}$ is the subadditive potential on Σ defined by

$$g_n^s(x) = \log \phi^s(D_{\Pi\sigma^n x} f_{x|n}), \quad x \in \Sigma, \tag{2.10}$$

with $f_{x|n} := f_{x_1} \circ \cdots \circ f_{x_n}$ for $x = (x_n)_{n=1}^{\infty}$.

Definition 2.6. Let m be an ergodic σ-invariant Borel probability measure on Σ. For any $i \in \{1, \ldots, d\}$, the ith Lyapunov exponent of m is

$$\lambda_i(m) := \lim_{n\to\infty}\frac{1}{n}\int \log(\alpha_i(D_{\Pi\sigma^n x} f_{x|n}))\, dm(x). \tag{2.11}$$

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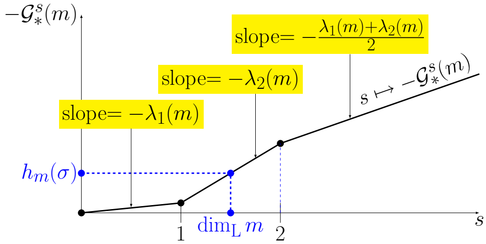

FIGURE 1. The connection between Lyapunov dimension, entropy and the function $s \mapsto -\mathcal{G}_*^s(m)$ when $d = 2$.

The *Lyapunov dimension of m with respect to* $\{f_i\}_{i=1}^{\ell}$, written as $\dim_L m$, is the unique non-negative value s for which

$$h_m(\sigma) + \mathcal{G}_*^s(m) = 0,$$

where $\mathcal{G}^s = \{g_n^s\}_{n=1}^{\infty}$ is defined as in (2.10) and $\mathcal{G}_*^s(m) := \lim_{n\to\infty}(1/n) \int g_n^s \, dm$. See Figure 1 for the mapping $s \mapsto -\mathcal{G}_*^s(m)$ in the case when $d = 2$.

It follows from the definition of the singular value function ϕ^s that, for an ergodic measure m, we have

$$\mathcal{G}_*^s(m) = \begin{cases} \lambda_1(m) + \cdots + \lambda_{[s]}(m) + (s - [s])\lambda_{[s]+1}(m) & \text{if } s < d, \\ \frac{s}{d}(\lambda_1(m) + \cdots + \lambda_d(m)) & \text{if } s \geq d. \end{cases}$$

Observe that, in the special case when all the Lyapunov exponents are equal to the same λ, we have $\dim_L m = h_m(\sigma)/-\lambda$.

Remark 2.7

(i) The concept of singularity dimension was first introduced by Falconer [13, 15]; see also [29]. It is also called *affinity dimension* when the IFS $\{f_i\}_{i=1}^{\ell}$ is affine, that is, each map f_i is affine.

(ii) The definition of Lyapunov dimension of ergodic measures with respect to an IFS presented above was taken from [26]. It is a generalization of that given in [27] for affine IFSs.

2.4. *A special consequence of Kingman's subadditive ergodic theorem.* Here we state a special consequence of Kingman's subadditive ergodic theorem which will be needed in the proof of Lemma 5.1.

LEMMA 2.8. *Let T be a measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$, and let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of L^1 functions satisfying the following subadditivity relation:*

$$g_{n+m}(x) \leq g_n(x) + g_m(T^n x) \quad \textit{for all } x \in X.$$

Suppose that there exists $C > 0$ such that

$$|g_n(x)| \leq Cn \quad \textit{for all } x \in X \textit{ and } n \in \mathbb{N}. \tag{2.12}$$

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Then

$$\lim_{n\to\infty} E\bigg(\frac{g_n}{n}\bigg|C_n\bigg)(x) = g(x) := \lim_{n\to\infty} \frac{g_n(x)}{n}$$

for m-a.e. x, where

$$C_n := \{B \in \mathcal{B} : T^{-n}B = B \text{ a.e.}\},$$

$E(\cdot|\cdot)$ denotes the conditional expectation and $g(x)$ is T-invariant.

Proof. By Kingman's subadditive ergodic theorem, g_n/n converges pointwise to a T-invariant function g-almost everywhere. Meanwhile, by Birkhoff's ergodic theorem, for each $n \in \mathbb{N}$,

$$E(g_n|C_n)(x) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} g_n(T^{jn}x) \text{ almost everywhere.}$$

Since $\sum_{j=0}^{k-1} g_n(T^{jn}x) \geq g_{kn}(x)$ by subadditivity, it follows that

$$E\bigg(\frac{g_n}{n}\bigg|C_n\bigg)(x) \geq \lim_{k\to\infty} \frac{g_{nk}(x)}{nk} = g(x) \text{ almost everywhere.} \tag{2.13}$$

Since the sequence $\{g_n\}$ is subadditive and satisfies (2.12), by [28, Lemma 2.2], for any $0 < k < n$,

$$\frac{g_n(x)}{n} \leq \frac{1}{kn} \sum_{j=0}^{n-1} g_k(T^j x) + \frac{3kC}{n}, \quad x \in X.$$

As a consequence,

$$E\bigg(\frac{g_n}{n}\bigg|C_n\bigg)(x) \leq \frac{1}{kn} E\bigg(\sum_{j=0}^{n-1} g_k \circ T^j \bigg| C_n\bigg)(x) + \frac{3kC}{n}. \tag{2.14}$$

Notice that, for each $f \in L^1$ and $n \in \mathbb{N}$,

$$E\bigg(\sum_{j=0}^{n-1} f \circ T^j \bigg| C_n\bigg) = nE(f|C_1) \text{ almost everywhere.} \tag{2.15}$$

To see the above identity, one simply applies Birkhoff's ergodic theorem (with respect to the transformations T^n and T, respectively) to the following limits:

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} (f + f\circ T + \cdots + f \circ T^{n-1})(T^{nj}x) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{nk-1} f(T^j x).$$

Now applying the identity (2.15) (with $f = g_k$) to (2.14) yields

$$E\bigg(\frac{g_n}{n}\bigg|C_n\bigg) \leq E\bigg(\frac{g_k}{k}\bigg|C_1\bigg) + \frac{3kC}{n} \text{ almost everywhere} \quad \text{for } 0 < k < n.$$

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It follows that

$$\limsup_{n\to\infty} E\left(\frac{g_n}{n}\middle| C_n\right) \le E\left(\frac{g_k}{k}\middle| C_1\right) \text{ almost everywhere} \quad \text{for each } k,$$

so by the dominated convergence theorem,

$$\limsup_{n\to\infty} E\left(\frac{g_n}{n}\middle| C_n\right) \le \lim_{k\to\infty} E\left(\frac{g_k}{k}\middle| C_1\right) = E(g|C_1) = g \text{ almost everywhere.}$$

Combining it with (2.13) yields the desired result $\lim_{n\to\infty} E((g_n/n)|C_n) = g$ almost everywhere. □

3. *Some auxiliary results*

In this section we give two auxiliary results (Propositions 3.1 and 3.4) which are needed in the proof of Theorem 1.1.

PROPOSITION 3.1. *Let (X, σ) be a one-sided subshift over a finite alphabet $\mathcal{A}$ and $\mathcal{G} = \{g_n\}_{n=1}^{\infty}$ a subadditive potential on $\mathcal{A}^{\mathbb{N}}$. Then*

$$P(X, \sigma, \mathcal{G}) = \lim_{n\to\infty} \frac{1}{n} P(X^{(n)}, \sigma^n, g_n) = \inf_{n\ge 1} \frac{1}{n} P(X^{(n)}, \sigma^n, g_n), \tag{3.1}$$

where $X^{(n)}$ is defined as in (2.7), and $P(X^{(n)}, \sigma^n, g_n)$ denotes the classical topological pressure of g_n over the full shift space $(X^{(n)}, \sigma^n)$.

Remark 3.2. Instead of (3.1), it was proved in [2, Proposition 2.2] that

$$P(X, \sigma, \mathcal{G}) = \lim_{n\to\infty} \frac{1}{n} P(X, \sigma^n, g_n)$$

under a more general setting. We remark that the proof of (3.1) is more subtle.

To prove Proposition 3.1, we need the following lemma.

LEMMA 3.3. [10, Lemma 2.3] *Under the assumptions of Proposition 3.1, suppose that $\{\nu_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{M}(\mathcal{A}^{\mathbb{N}})$, where $\mathcal{M}(\mathcal{A}^{\mathbb{N}})$ denotes the space of all Borel probability measures on $\mathcal{A}^{\mathbb{N}}$ with the weak* topology. We form the new sequence $\{\mu_n\}_{n=1}^{\infty}$ by $\mu_n = (1/n)\sum_{i=0}^{n-1} \nu_n \circ \sigma^{-i}$. Assume that μ_{n_i} converges to μ in $\mathcal{M}(\mathcal{A}^{\mathbb{N}})$ for some subsequence $\{n_i\}$ of natural numbers. Then $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{N}}, \sigma)$ and, moreover,*

$$\limsup_{i\to\infty} \frac{1}{n_i} \int g_{n_i}(x)\, d\nu_{n_i}(x) \le \mathcal{G}_*(\mu) := \lim_{n\to\infty} \frac{1}{n} \int g_n \, d\mu.$$

Proof of Proposition 3.1. We first prove that, for each $n \in \mathbb{N}$,

$$P(X, \sigma, \mathcal{G}) \le \frac{1}{n} P(X^{(n)}, \sigma^n, g_n). \tag{3.2}$$

To see this, fix $n \in \mathbb{N}$ and let μ be an equilibrium measure for the potential $\mathcal{G}$. Then

$$\begin{aligned} P(X, \sigma, \mathcal{G}) &= h_\mu(\sigma) + \mathcal{G}_*(\mu) \\ &\le h_\mu(\sigma) + \frac{1}{n}\int g_n \, d\mu \quad \text{(by (2.5))} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\left(h_\mu(\sigma^n) + \int g_n \, d\mu\right)\\
&\leq \frac{1}{n} P(X^{(n)}, \sigma^n, g_n),
\end{aligned}
$$

where in the last inequality, we use the fact that $\mu \in \mathcal{M}(X^{(n)}, \sigma^n)$ and the classical variational principle for the topological pressure of additive potentials. This proves (3.2).

In what follows we prove that

$$
P(X, \sigma, \mathcal{G}) \geq \limsup_{n\to\infty} \frac{1}{n} P(X^{(n)}, \sigma^n, g_n). \tag{3.3}
$$

Clearly (3.2) and (3.3) imply (3.1). To prove (3.3), by the classical variational principle we can take a subsequence $\{n_i\}$ of natural numbers and $\nu_{n_i} \in \mathcal{M}(X^{(n_i)}, \sigma^{n_i})$ such that

$$
\limsup_{n\to\infty} \frac{1}{n} P(X^{(n)}, \sigma^n, g_n) = \lim_{i\to\infty} \frac{1}{n_i}\left(h_{\nu_{n_i}}(\sigma^{n_i}) + \int g_{n_i} \, d\nu_{n_i}\right). \tag{3.4}
$$

Set $\mu^{(i)} = (1/n_i)\sum_{k=0}^{n_i-1} \nu_{n_i} \circ \sigma^{-k}$ for each i. Taking a subsequence if necessary, we may assume that $\mu^{(i)}$ converges to an element $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{N}})$ in the weak* topology. By Lemma 3.3, $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{N}}, \sigma)$ and, moreover,

$$
\limsup_{i\to\infty} \frac{1}{n_i} \int g_{n_i}(x) \, d\nu_{n_i}(x) \leq \mathcal{G}_*(\mu). \tag{3.5}
$$

Next, we show that μ is supported on X. For this we adopt some arguments from the proof of [30, Theorem 1.1]. Notice that, for each i, $\mu^{(i)}$ is σ-invariant supported on

$$
\bigcup_{k=0}^{n_i-1} \sigma^k X^{(n_i)} = \bigcup_{k=1}^{n_i} \sigma^{n_i-k} X^{(n_i)}.
$$

Hence, μ is supported on

$$
\bigcap_{N=1}^{\infty} \overline{\bigcup_{i=N}^{\infty} \bigcup_{k=1}^{n_i} \sigma^{n_i-k} X^{(n_i)}}.
$$

If x is in this set, then, for each $N \geq 1$, there exist integers $i(N) \geq N$ and $k(N) \in [1, n_{i(N)}]$ for which $d(x, \sigma^{n_{i(N)}-k(N)} X^{(n_{i(N)})}) < 1/N$ (i.e., $d(x, z) < 1/N$ for some $z \in \sigma^{n_{i(N)}-k(N)} X^{(n_{i(N)})}$), hence

$$
d(x, X) \leq d(x, z) + d(z, X) < \frac{1}{N} + \sup_{y \in \sigma^{n_{i(N)}-k(N)} X^{(n_{i(N)})}} d(y, X) \leq \frac{1}{N} + 2^{-k(N)} \tag{3.6}
$$

and

$$
\begin{aligned}
d(\sigma^{k(N)} x, X) &\leq d(\sigma^{k(N)} x, \sigma^{k(N)} z) + d(\sigma^{k(N)} z, X)\\
&\leq 2^{k(N)} d(x, z) + d(\sigma^{k(N)} z, X)
\end{aligned}
$$

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$$
\begin{aligned}
&< \frac{2^{k(N)}}{N} + \sup_{y\in\sigma^{n_{i(N)}-k(N)}X^{(n_{i(N)})}} d(\sigma^{k(N)}y, X) \\
&\le \frac{2^{k(N)}}{N} + 2^{-n_{i(N)}} \quad (\text{since } \sigma^{k(N)}y \in \sigma^{n_{i(N)}}X^{(n_{i(N)})} = X^{(n_{i(N)})}).
\end{aligned}
\tag{3.7}
$$

If the values $k(N)$ are unbounded as $N \to \infty$, then (3.6) yields $x \in X$, while if they are bounded then some value of k recurs infinitely often as $k(N)$, which implies that $\sigma^k x \in X$ by (3.7). Thus μ is supported on

$$\bigcup_{k=0}^{\infty} \sigma^{-k}X.$$

Since $\sigma X \subset X$, the set $(\sigma^{-1}X)\backslash X$ is wandering under σ^{-1} (i.e., its preimages under powers of σ are disjoint), so it must have zero μ-measure. Consequently, $\mu \in \mathcal{M}(X, \sigma)$.

Notice that $h_{\mu^{(i)}}(\sigma) = (1/n_i)h_{\nu_{n_i}}(\sigma^{n_i})$ (see Lemma 2.4). By the upper semi-continuity of the entropy map,

$$h_\mu(\sigma) \ge \limsup_{i\to\infty} h_{\mu^{(i)}}(\sigma) = \limsup_{i\to\infty} \frac{1}{n_i} h_{\nu_{n_i}}(\sigma^{n_i}),$$

which, together with (3.5), yields that

$$
\begin{aligned}
h_\mu(\sigma) + \mathcal{G}_*(\mu) &\ge \limsup_{i\to\infty} \frac{1}{n_i}\bigg(h_{\nu_{n_i}}(\sigma^{n_i}) + \int g_{n_i}\, d\nu_{n_i}\bigg) \\
&= \limsup_{n\to\infty} \frac{1}{n} P(X^{(n)}, \sigma^n, g_n).
\end{aligned}
$$

Applying Theorem 2.2, we obtain (3.3). This completes the proof of the proposition. □

Next, we present another auxiliary result.

PROPOSITION 3.4. *Let (X, σ) be a one-sided subshift over a finite alphabet $\mathcal{A}$ and $g, h \in C(X)$. Assume, in addition, that $h(x) < 0$ for all $x \in X$. Let*

$$r_0 = \sup_{x\in X} \exp(h(x)).$$

Set, for $0 < r < r_0$,

$$\mathcal{A}_r := \Big\{i_1 \ldots i_n \in X^* : \sup_{x\in[i_1\ldots i_n]\cap X} \exp(S_n h(x)) < r \le \sup_{y\in[i_1\ldots i_{n-1}]\cap X} \exp(S_{n-1}h(y))\Big\},$$

where X^* is the collection of finite words allowed in X and $S_n h(x) := \sum_{k=0}^{n-1} h(\sigma^k x)$. Then

$$\lim_{r\to 0} \frac{\log(\sum_{I\in\mathcal{A}_r} \sup_{x\in[I]\cap X} \exp(S_{|I|}g(x)))}{\log r} = -t, \tag{3.8}$$

where t is the unique real number such that $P(X, \sigma, g + th) = 0$, and $|I|$ stands for the length of I.

To prove the above result, we need the following lemma.

LEMMA 3.5. *Let (X, σ) be a one-sided subshift over a finite alphabet $\mathcal{A}$ and $f \in C(X)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sup\{|S_n f(x) - S_n f(y)| : x_i = y_i \textit{ for all } 1 \le i \le n\} = 0.$$

Proof. The result is well known. For the reader's convenience, we include a proof.

Define, for $n \in \mathbb{N}$,

$$\mathrm{var}_n f = \sup\{|f(x) - f(y)| : x_i = y_i \text{ for all } 1 \le i \le n\}.$$

Since f is uniformly continuous, $\mathrm{var}_n f \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathrm{var}_i f = 0.$$

This concludes the result of the lemma since

$$\sup\{|S_n f(x) - S_n f(y)| : x_i = y_i \text{ for all } 1 \le i \le n\}$$

is bounded above by $\sum_{i=1}^{n} \mathrm{var}_i f$. □

Proof of Proposition 3.4. Set

$$\Theta_r = \sum_{I\in\mathcal{A}_r} \sup_{x\in[I]\cap X} \exp(S_{|I|} g(x)), \quad r \in (0, r_0).$$

Let $\epsilon > 0$. It is enough to show that

$$r^{-t+\epsilon} \le \Theta_r \le r^{-t-\epsilon} \tag{3.9}$$

for sufficiently small r.

To this end, set, for $0 < r < r_0$,

$$m(r) = \min\{|I| : I \in \mathcal{A}_r\}, \quad M(r) = \max\{|I| : I \in \mathcal{A}_r\}.$$

From the definition of $\mathcal{A}_r$ and the negativity of h, it follows that there exist two positive constants a, b such that

$$a \log(1/r) \le m(r) \le M(r) \le b \log(1/r) \quad \text{for all } r \in (0, r_0). \tag{3.10}$$

Define

$$\Gamma_r = \sum_{I\in\mathcal{A}_r} \sup_{x\in[I]\cap X} \exp(S_{|I|}(g + th)(x)), \quad r \in (0, r_0).$$

By Lemma 3.5 and (3.10), it is readily checked that

$$r^{t+\epsilon/2}\Theta_r \le \Gamma_r \le r^{t-\epsilon/2}\Theta_r \quad \text{for sufficiently small } r, \tag{3.11}$$

Hence, to prove (3.9), it suffices to prove that $r^{\epsilon/2} \le \Gamma_r \le r^{-\epsilon/2}$ for small r.

We first prove $\Gamma_r > r^{\epsilon/2}$ when r is small. Suppose to the contrary that this is not true. Then by (3.10) we can find some $r \in (0, r_0)$ and $\lambda > 0$ such that $Z(r, \lambda) < 1$, where

$$Z(r, \lambda) := \sum_{I\in\mathcal{A}_r} \exp(\lambda|I|) \sup_{x\in[I]\cap X} \exp(S_{|I|}(g + th)(x)). \tag{3.12}$$

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Observe that $\{[I] : I \in \mathcal{A}_r\}$ is a cover of X. From [7] it follows that $P(X, \sigma, g + th) \leq -\lambda$, contradicting the fact that $P(X, \sigma, g + th) = 0$. Hence, we have $\Gamma_r > r^{\epsilon/2}$ when r is sufficiently small.

Next, we prove the inequality $\Gamma_r \leq r^{-\epsilon/2}$ for small r. To do this, fix $\lambda \in (0, \epsilon/(2b))$, where b is the constant in (3.10). We claim that there exists $0 < r_1 < r_0$ such that

$$Z(r, -\lambda) < 1 \quad \text{for all } r \in (0, r_1), \tag{3.13}$$

where Z is defined as in (3.12). Since $\lambda \in (0, \epsilon/(2b))$, it follows from (3.10) that, for any $I \in \mathcal{A}_r$,

$$\exp(-\lambda|I|) \geq \exp(-\lambda b \log(1/r)) = r^{b\lambda} \geq r^{\epsilon/2}.$$

Hence, (3.13) implies that $\Gamma_r \leq r^{-\epsilon/2}$ for $0 < r < r_1$.

Now it remains to prove (3.13). Since $P(X, \sigma, g + th) = 0$, by definition we have

$$\lim_{n\to\infty} \frac{1}{n} \log\bigg(\sum_{I\in X^*:\, |I|=n} \sup_{x\in[I]\cap X} \exp(S_{|I|}(g + th)(x))\bigg) = 0.$$

Hence, there exists a large N such that $e^{-\lambda N/2} < 1 - e^{-\lambda/2}$ and, for any $n > N$,

$$\gamma_n := \sum_{I\in X^*:\, |I|=n} \exp(-\lambda|I|) \sup_{x\in[I]\cap X} \exp(S_{|I|}(g + th)(x)) \leq \exp(-\lambda n/2).$$

Take a small $r_1 \in (0, r_0)$ so that $m(r) \geq N$ for any $0 < r < r_1$. By the definition of $m(r)$, for any $0 < r < r_1$ we have $\mathcal{A}_r \subset \{I \in X^* : |I| \geq N\}$ and so

$$Z(r, -\lambda) \leq \sum_{n=N}^{\infty} \gamma_n \leq \sum_{n=N}^{\infty} \exp(-\lambda n/2) = \frac{e^{-\lambda N/2}}{1 - e^{-\lambda/2}} < 1.$$

This proves (3.13). □

4. *The proof of Theorem 1.1*

Recall that, for $T \in \mathbb{R}^{d\times d}$, $\alpha_1(T) \geq \cdots \geq \alpha_d(T)$ are the singular values of T, and $\phi^s(T)$ ($s \geq 0$) is defined as in (2.9). We begin with an elementary but important lemma.

LEMMA 4.1. *Let $E \subset U \subset \mathbb{R}^d$, where E is compact and U is open. Let $k \in \{0, 1, \ldots, d - 1\}$. Then, for any non-degenerate C^1 map $f : U \to \mathbb{R}^d$, there exists $r_0 > 0$ such that, for any $y \in E$, $z \in B(y, r_0)$ and $0 < r < r_0$, the set $f(B(z, r))$ can be covered by*

$$(4d)^d \cdot \frac{\phi^k(D_y f)}{(\alpha_{k+1}(D_y f))^k}$$

balls of radius $\alpha_{k+1}(D_y f)r$.

Proof. The result was implicitly proved in [40, Lemma 3] by using an idea of [12]. For the convenience of the reader, we provide a detailed proof.

Set

$$\gamma = \min_{y\in E} \alpha_d(D_y f).$$

Then

$$B(0,\gamma) \subset D_y f(B(0,1)) \quad \text{for all } y \in E. \tag{4.1}$$

Since f is C^1, non-degenerate on U and E is compact, it follows that $\gamma > 0$. Take $\epsilon = (2\sqrt{d}-1)/2$. Then there exists a small $r_0 > 0$ such that, for $u, v, w \in V_{2r_0}(E) := \{x : d(x, E) < 2r_0\}$,

$$|f(u) - f(v) - D_v f(u - v)| \le \epsilon\gamma|u - v| \quad \text{if } |u - v| \le r_0, \tag{4.2}$$

and

$$D_v f(B(0,1)) \subset ((1+\epsilon)D_w f)(B(0,1)) \quad \text{if } |v - w| \le r_0. \tag{4.3}$$

Now let $y \in E$ and $z \in B(y, r_0)$. For any $0 < r < r_0$ and $x \in B(z, r)$, taking $u = x$ and $v = z$ in (4.2) gives

$$f(x) - f(z) - D_z f(x - z) \in B(0, \epsilon\gamma r),$$

so by (4.3) and (4.1),

$$\begin{aligned}
f(x) - f(z) &\in D_z f(B(0,r)) + B(0,\epsilon\gamma r)\\
&\subset ((1+\epsilon)D_y f)(B(0,r)) + B(0,\epsilon\gamma r) \quad (\text{by } (4.3))\\
&\subset D_y f(B(0,(1+\epsilon)r) + B(0,\epsilon r)) \quad (\text{by } (4.1))\\
&\subset D_y f(B(0,(1+2\epsilon)r))\\
&= D_y f(B(0, 2\sqrt{d}r)),
\end{aligned}$$

where $A + A' := \{u + v : u \in A,\ v \in A'\}$. Therefore,

$$f(B(z,r)) \subset f(z) + D_y f(B(0, 2\sqrt{d}r)).$$

That is, $f(B(z,r))$ is contained in an ellipsoid which has principle axes of lengths $4\sqrt{d}\alpha_i(D_y f)r$, $i = 1, \ldots, d$. Hence, $f(B(z,r))$ is contained in a rectangular parallelepiped of side lengths $2\sqrt{d}\alpha_i(D_y f)r$, $i = 1, \ldots, d$. Now we can divide such a parallelepiped into at most

$$\left(\prod_{i=1}^{k+1} \frac{4d\alpha_i(D_y f)}{\alpha_{k+1}(D_y f)}\right) \cdot (4d)^{d-k-1} \le (4d)^d \cdot \frac{\phi^k(D_y f)}{(\alpha_{k+1}(D_y f))^k}$$

cubes of side $(2/\sqrt{d}) \cdot \alpha_{k+1}(D_y f)r$. Therefore, this parallelepiped (and $f(B(z,r))$ as well) can be covered by

$$(4d)^d \cdot \frac{\phi^k(D_y f)}{(\alpha_{k+1}(D_y f))^k}$$

balls of radius $\alpha_{k+1}(D_y f)r$. □

In the remainder of this section let $\{f_i\}_{i=1}^{\ell}$ be a C^1 IFS on $\mathbb{R}^d$ with attractor K. Let (Σ, σ) be the one-sided full shift over the alphabet $\{1, \ldots, \ell\}$ and $\Pi : \Sigma \to K$ the canonical coding map associated with the IFS (cf. (1.3)). As a consequence of Lemma 4.1, we obtain the following proposition.

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PROPOSITION 4.2. *Let* $k \in \{0, 1, \ldots, d-1\}$. *Set* $C = (4d)^d$. *Then there exists* $C_1 > 0$ *such that, for* $\mathbf{i} = (i_p)_{p=1}^{\infty} \in \Sigma$ *and* $n \in \mathbb{N}$, *the set* $f_{\mathbf{i}|n}(K)$ *can be covered by* $C_1 \prod_{p=0}^{n-1} G(\sigma^p \mathbf{i})$ *balls of radius* $\prod_{p=0}^{n-1} H(\sigma^p \mathbf{i})$, *where*

$$G(\mathbf{i}) := \frac{C\phi^k(D_{\Pi\sigma\mathbf{i}} f_{i_1})}{\alpha_{k+1}(D_{\Pi\sigma\mathbf{i}} f_{i_1})^k}, \quad H(\mathbf{i}) := \alpha_{k+1}(D_{\Pi\sigma\mathbf{i}} f_{i_1}). \tag{4.4}$$

Proof. Since $\{f_i\}_{i=1}^{\ell}$ is a C^1 IFS, there exists an open set $U \supset K$ such that each f_i extends to a C^1 diffeomorphism $f_i : U \to f_i(U)$. Applying Lemma 4.1 to the mappings f_i, we see that there exists $r_0 > 0$ such that, for any $y \in K$, $z \in B(y, r_0)$, $0 < r < r_0$ and $i \in \{1, \ldots, \ell\}$, the set $f_i(B(z, r))$ can be covered by

$$\theta(y, i) := \frac{C\phi^k(D_y f_i)}{\alpha_{k+1}(D_y f_i)^k}$$

balls of radius $\alpha_{k+1}(D_y f_i) r$.

Since $f_1, \ldots, f_\ell$ are contracting on U, there exists $\gamma \in (0, 1)$ such that

$$|f_i(x) - f_i(y)| \le \gamma |x - y| \quad \text{for all } x, y \in U, \ i \in \{1, \ldots, \ell\}.$$

This implies that $\alpha_1(D_y f_i) \le \gamma$ for any $y \in K$ and $i \in \{1, \ldots, \ell\}$. Take a large integer n_0 such that

$$\gamma^{n_0} \max\{1, \operatorname{diam}(K)\} < r_0/2.$$

Clearly there exists a large number C_1 so that the conclusion of the proposition holds for any positive integer $n \le n_0$ and $\mathbf{i} \in \Sigma$, that is, the set $f_{\mathbf{i}|n}(K)$ can be covered by $C_1 \prod_{p=0}^{n-1} G(\sigma^p \mathbf{i})$ balls of radius $\prod_{p=0}^{n-1} H(\sigma^p \mathbf{i})$. Below we show by induction that this holds for all $n \in \mathbb{N}$ and $\mathbf{i} \in \Sigma$.

Suppose, for some $m \ge n_0$, that the conclusion of the proposition holds for any positive integer $n \le m$ and $\mathbf{i} \in \Sigma$. Then, for $\mathbf{i} \in \Sigma$, $f|_{(\sigma\mathbf{i})|m}(K)$ can be covered by $C_1 \prod_{p=0}^{m-1} G(\sigma^{p+1} \mathbf{i})$ balls of radius $\prod_{p=0}^{m-1} H(\sigma^{p+1} \mathbf{i})$. Let $B_1, \ldots B_N$ denote these balls. We may assume that $B_j \cap f|_{(\sigma\mathbf{i})|m}(K) \ne \emptyset$ for each j. Since

$$\prod_{p=0}^{m-1} H(\sigma^{p+1} \mathbf{i}) \le \gamma^m \le \gamma^{n_0} < r_0/2$$

and

$$d(\Pi\sigma\mathbf{i}, B_j \cap f_{(\sigma\mathbf{i})|m}(K)) \le \operatorname{diam}(f_{(\sigma\mathbf{i})|m}(K)) \le \gamma^{n_0} \operatorname{diam}(K) < r_0/2,$$

the center of B_j is in $B(\Pi\sigma\mathbf{i}, r_0)$. Therefore, $f_{i_1}(B_j)$ can be covered by $\theta(\Pi\sigma\mathbf{i}, i_1) = G(\mathbf{i})$ balls of radius

$$H(\mathbf{i}) \cdot \prod_{p=0}^{m-1} H(\sigma^{p+1} \mathbf{i}) = \prod_{p=0}^{m} H(\sigma^p \mathbf{i}).$$

Since $f_{\mathbf{i}|(m+1)}(K) \subset \bigcup_{j=1}^{N} f_{i_1}(B_j)$, it follows that $f_{\mathbf{i}|(m+1)}(K)$ can be covered by

$$G(\mathbf{i})N \leq G(\mathbf{i}) \cdot C_1 \prod_{p=0}^{m-1} G(\sigma^{p+1}\mathbf{i}) = C_1 \prod_{p=0}^{m} G(\sigma^{p}\mathbf{i})$$

balls of radius $\prod_{p=0}^{m} H(\sigma^p \mathbf{i})$. Thus the proposition also holds for $n = m+1$ and all $\mathbf{i} \in \Sigma$, as desired. □

Next, we provide an upper bound on the upper box-counting dimension of the attractor K of the IFS $\{f_i\}_{i=1}^{\ell}$.

PROPOSITION 4.3. *Let $k \in \{0, 1, \ldots, d-1\}$. Let $G, H : \Sigma \to \mathbb{R}$ be defined as in (4.4). Let t be the unique real number so that*

$$P(\Sigma, \sigma, (\log G) + t(\log H)) = 0.$$

Then $\overline{\dim}_B K \leq t$.

Proof. Write $g = \log G$ and $h = \log H$ for short. Define

$$r_{\min} = \min_{x\in\Sigma} \alpha_{k+1}(D_{\Pi\sigma x} f_{x_1}), \qquad r_{\max} = \max_{x\in\Sigma} \alpha_{k+1}(D_{\Pi\sigma x} f_{x_1}).$$

Then $0 < r_{\min} \leq r_{\max} < 1$. For $0 < r < r_{\min}$, define

$$\mathcal{A}_r = \Big\{ i_1 \ldots i_n \in \Sigma^* : \sup_{x\in[i_1\ldots i_n]} S_n h(x) < \log r \leq \sup_{y\in[i_1\ldots i_{n-1}]} S_{n-1}h(y) \Big\}; \tag{4.5}$$

clearly $\{[I] : I \in \mathcal{A}_r\}$ is a partition of Σ. By Proposition 4.2, there exists a constant $C_1 > 0$ such that, for each $0 < r < r_{\min}$, every $I \in \mathcal{A}_r$ and $x \in [I]$, $f_I(K)$ can be covered by

$$C_1 \exp(S_{|I|}g(x)) \leq C_1 \exp\Big(\sup_{y\in[I]} S_{|I|}g(y) \Big)$$

balls of radius

$$\exp(S_{|I|}h(x)) \leq \exp\Big(\sup_{y\in[I]} S_{|I|}h(y) \Big) < r.$$

It follows that K can be covered by

$$C_1 \sum_{I\in\mathcal{A}_r} \exp\Big(\sup_{y\in[I]} S_{|I|}g(y) \Big)$$

balls of radius r. Hence, by Proposition 3.4,

$$\overline{\dim}_B K \leq \limsup_{r\to 0} \frac{\log\big(\sum_{I\in\mathcal{A}_r} \exp\big(\sup_{y\in[I]} S_{|I|}g(y) \big)\big)}{\log(1/r)} = t.$$

This completes the proof of the proposition. □

As an application of Proposition 4.3, we may estimate the upper box-counting dimension of the projections of a class of σ-invariant sets under the coding map.

PROPOSITION 4.4. *Let X be a compact subset of Σ satisfying $\sigma X \subset X$, and $k \in \{0, \ldots, d-1\}$. Then, for each $n \in \mathbb{N}$,*

$$\overline{\dim}_B \Pi(X^{(n)}) \le t_n,$$

where $X^{(n)}$ is defined as in (2.7), t_n is the unique number for which

$$P(X^{(n)}, \sigma^n, (\log G_n) + t_n(\log H_n)) = 0,$$

and G_n, H_n are continuous functions on Σ defined by

$$G_n(y) := \frac{C\phi^k(D_{\Pi\sigma^n y} f_{y|n})}{\alpha_{k+1}(D_{\Pi\sigma^n y} f_{y|n})^k}, \quad H_n(y) := \alpha_{k+1}(D_{\Pi\sigma^n y} f_{y|n}), \tag{4.6}$$

with $C = (4d)^d$.

Proof. The result is obtained by applying Proposition 4.3 to the IFS $\{f_I : I \in X_n^*\}$ instead of $\{f_i\}_{i=1}^{\ell}$, where X_n^* stands for the collection of words of length n allowed in X. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Write $s = \dim_S X$. We may assume that $s < d$; otherwise we have nothing left to prove. Set $k = [s]$, that is, k is the largest integer less than or equal to s. Let $\mathcal{U} = \{u_n\}_{n=1}^{\infty}$ be the subadditive potential on Σ defined by

$$u_n(x) = \log \phi^s(D_{\Pi\sigma^n x} f_{x|n}).$$

Then $P(X, \sigma, \mathcal{U}) = 0$ by the definition of $\dim_S X$. Hence, by Proposition 3.1,

$$\lim_{n\to\infty} \frac{1}{n} P(X^{(n)}, \sigma^n, u_n) = \inf_{n\ge 1} \frac{1}{n} P(X^{(n)}, \sigma^n, u_n) = 0.$$

It follows that, for each $\epsilon > 0$, there exists $N_\epsilon > 0$ such that

$$0 \le P(X^{(n)}, \sigma^n, u_n) \le n\epsilon \quad \text{for } n > N_\epsilon. \tag{4.7}$$

For $n \in \mathbb{N}$, let s_n be the unique real number such that $P(X^{(n)}, \sigma^n, v_n) = 0$, where v_n is a continuous function on Σ defined by

$$\begin{aligned} v_n(x) &= \log G_n(x) + s_n \log H_n(x) \\ &= \log(C\phi^k(D_{\Pi\sigma^n x} f_{x|n})\alpha_{k+1}(D_{\Pi\sigma^n x} f_{x|n})^{s_n - k}), \end{aligned}$$

where G_n, H_n are defined in (4.6) and $C = (4d)^d$. By Proposition 4.4,

$$\overline{\dim}_B \Pi(X) \le \overline{\dim}_B \Pi(X^{(n)}) \le s_n.$$

If $s \ge s_n$ for some n, then $\overline{\dim}_B \Pi(X) \le s_n \le s$ and we are done. In what follows, we assume that $s < s_n$ for each n. Then, for each $x \in \Sigma$,

$$\begin{aligned} u_n(x) - v_n(x) &= -\log C + (s - s_n) \log \alpha_{k+1}(D_{\Pi\sigma^n x} f_{x|n}) \\ &\ge -\log C + (s - s_n) \log \alpha_1(D_{\Pi\sigma^n x} f_{x|n}) \\ &\ge -\log C + n(s - s_n) \log \theta, \end{aligned}$$

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where

$$\theta := \max_{y\in\Sigma} \alpha_1(D_{\Pi\sigma y} f_{y_1}) < 1.$$

Hence,

$$\begin{aligned} P(X^{(n)}, \sigma^n, u_n) &= P(X^{(n)}, \sigma^n, u_n) - P(X^{(n)}, \sigma^n, v_n) \\ &\geq \inf_{x\in\Sigma}(u_n(x) - v_n(x)) \\ &\geq -\log C + n(s - s_n)\log\theta, \end{aligned}$$

where in the second inequality, we used [39, Theorem 9.7(iv)]. Combining this with (4.7) yields that, for $n \geq N_\epsilon$,

$$n\epsilon \geq -\log C + n(s - s_n)\log\theta,$$

so

$$s \geq s_n + \frac{\epsilon + n^{-1}\log C}{\log\theta} \geq \overline{\dim}_B \Pi(X) + \frac{\epsilon + n^{-1}\log C}{\log\theta}.$$

Letting $n \to \infty$ and then $\epsilon \to 0$, we obtain $s \geq \overline{\dim}_B \Pi(X)$, as desired. □

5. *The proof of Theorem 1.2*

Let $\Pi : \Sigma \to \mathbb{R}^d$ be the coding map associated with a C^1 IFS $\{f_i\}_{i=1}^{\ell}$ on $\mathbb{R}^d$ (cf. (1.3)). For $E \subset \mathbb{R}^d$ and $\delta > 0$, let $N_\delta(E)$ denote the smallest integer N for which E can be covered by N closed balls of radius δ. For $T \in \mathbb{R}^{d\times d}$, let $\alpha_1(T) \geq \cdots \geq \alpha_d(T)$ denote the singular values of T, and let $\phi^s(T)$ be the singular value function defined as in (2.9).

The following geometric counting lemma plays an important role in the proof of Theorem 1.2. It is of independent interest as well.

LEMMA 5.1. *Let m be an ergodic σ-invariant Borel probability measure on Σ. Set*

$$\lambda_i := \lim_{n\to\infty} \frac{1}{n}\int \log(\alpha_i(D_{\Pi\sigma^n x} f_{x|n}))\, dm(x), \quad i = 1, \ldots, d. \tag{5.1}$$

Let $k \in \{0, \ldots, d-1\}$. Write $u := \exp(\lambda_{k+1})$. Then, for m-a.e. $x \in \Sigma$,

$$\limsup_{n\to\infty} \frac{1}{n}\log N_{u^n}(\Pi([x|n])) \leq (\lambda_1 + \cdots + \lambda_k) - k\lambda_{k+1}. \tag{5.2}$$

Proof. It is known (see, for example, [1, Theorem 3.3.3]) that, for m-a.e. x,

$$\lim_{p\to\infty} \frac{1}{p}\log(\alpha_i(D_{\Pi\sigma^p x} f_{x|p})) = \lambda_i, \quad i = 1, \ldots, d,$$

and

$$\lim_{p\to\infty} \frac{1}{p}\log(\phi^i(D_{\Pi\sigma^p x} f_{x|p})) = \lambda_1 + \cdots + \lambda_i, \quad i = 1, \ldots, d.$$

For $i \in \{1, \ldots, d\}$, $x \in \Sigma$ and $p \in \mathbb{N}$, set

$$w_p^{(i)}(x) = \log \phi^i(D_{\Pi\sigma^p x} f_{x|p}),$$
$$v_p^{(i)}(x) = \log \alpha_i(D_{\Pi\sigma^p x} f_{x|p}).$$

Then, by the definition of ϕ^i,

$$w_p^{(i)} = v_p^{(1)} + \cdots + v_p^{(i)}. \tag{5.3}$$

Since ϕ^i is submultiplicative on $\mathbb{R}^{d\times d}$ (cf. [13, Lemma 2.1]), $\{w_p^{(i)}\}_{p=1}^{\infty}$ is a subadditive potential satisfying

$$|w_p^{(i)}(x)| \le pC, \quad p \in \mathbb{N}, x \in \Sigma,$$

for some constant $C > 0$. Set $C_p := \{B \in \mathcal{B}(\Sigma) : \sigma^{-p}B = B$ almost everywhere$\}$. Then, by Lemma 2.8, for m-a.e. x,

$$\lim_{p\to\infty} E\bigg(\frac{w_p^{(i)}}{p}\bigg|C_p\bigg)(x) = \lim_{p\to\infty} \frac{1}{p} w_p^{(i)}(x) = \lambda_1 + \cdots + \lambda_i, \quad i = 1, \ldots, d, \tag{5.4}$$

and so by (5.3),

$$\lim_{p\to\infty} E\bigg(\frac{v_p^{(i)}}{p}\bigg|C_p\bigg)(x) = \lambda_i, \quad i = 1, \ldots, d. \tag{5.5}$$

Let $p \in \mathbb{N}$. Applying Proposition 4.2 to the IFS $\{f_I : I \in \mathcal{A}^p\}$, we see that there exists a positive number $C_1(p)$ such that, for any $x \in \Sigma$ and $n \in \mathbb{N}$, the set $\Pi([x|np]) = f_{x|np}(K)$ can be covered by $C_1(p)\prod_{i=0}^{n-1} G_p(\sigma^{pi}x)$ balls of radius $\prod_{i=0}^{n-1} H_p(\sigma^{ip}x)$, where

$$G_p(x) := \frac{C\phi^k(D_{\Pi\sigma^p x} f_{x|p})}{\alpha_{k+1}(D_{\Pi\sigma^p x} f_{x|p})^k}, \quad H_p(x) := \alpha_{k+1}(D_{\Pi\sigma^p x} f_{x|p}), \tag{5.6}$$

with $C = (4d)^d$. By Birkhoff's ergodic theorem, for m-a.e. $x \in \Sigma$,

$$\lim_{n\to\infty} \frac{1}{np} \log\bigg(C_1(p)\prod_{i=0}^{n-1} G_p(\sigma^{pi}x)\bigg) = \frac{\log C}{p} + E\bigg(\frac{w_p^{(k)}}{p}\bigg|C_p\bigg)(x) - kE\bigg(\frac{v_p^{(k+1)}}{p}\bigg|C_p\bigg)(x) \tag{5.7}$$

and

$$\lim_{n\to\infty} \frac{1}{np} \log\bigg(\prod_{i=0}^{n-1} H_p(\sigma^{ip}x)\bigg) = E\bigg(\frac{v_p^{(k+1)}}{p}\bigg|C_p\bigg)(x). \tag{5.8}$$

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Let $\epsilon > 0$. By (5.4), (5.5), (5.7), (5.8), for m-a.e. x there exists a positive integer $p_0(x)$ such that, for any $p \geq p_0(x)$,

$$\prod_{i=0}^{n-1} H_p(\sigma^{ip}x) \leq (u+\epsilon)^{np} \quad \text{for large enough } n, \tag{5.9}$$

and

$$\frac{1}{np}\log\bigg(C_1(p)\prod_{i=0}^{n-1} G_p(\sigma^{pi}x)\bigg) \leq \lambda_1 + \cdots + \lambda_k - k\lambda_{k+1} + \epsilon \quad \text{for large enough } n. \tag{5.10}$$

Fix such an x and let $p \geq p_0(x)$. By (5.9),

$$N_{(u+\epsilon)^{np}}(\Pi([x|pn])) \leq C_1(p)\prod_{i=0}^{n-1} G_p(\sigma^{pi}x) \quad \text{for large enough } n.$$

Notice that there exists a constant $C_2 = C_2(d) > 0$ such that a ball of radius $(u+\epsilon)^{np}$ in $\mathbb{R}^d$ can be covered by $C_2(1+\epsilon/u)^{dnp}$ balls of radius u^{np}. It follows that, for large enough n,

$$\begin{aligned} N_{u^{np}}(\Pi([x|pn])) &\leq C_2(1+\epsilon/u)^{dnp} N_{(u+\epsilon)^{np}}(\Pi([x|pn])) \\ &\leq C_1(p)C_2(1+\epsilon/u)^{dnp}\prod_{i=0}^{n-1} G_p(\sigma^{pi}x). \end{aligned}$$

Hence, by (5.10),

$$\begin{aligned} \limsup_{n\to\infty}\frac{1}{n}\log N_{u^n}(\Pi([x|n])) &= \limsup_{n\to\infty}\frac{1}{np}\log N_{u^{np}}(\Pi([x|pn])) \\ &\leq d\log(1+\epsilon/u) + \limsup_{n\to\infty}\frac{1}{np}\log\bigg(\prod_{i=0}^{n-1} G_p(\sigma^{pi}x)\bigg) \\ &\leq d\log(1+\epsilon/u) + \epsilon + \lambda_1 + \cdots + \lambda_k - k\lambda_{k+1}, \end{aligned}$$

where the first equality follows from the fact that, for $pn \leq m < p(n+1)$,

$$\begin{aligned} N_{u^m}(\Pi([x|m])) \leq N_{u^m}(\Pi([x|pn])) &\leq 4^d(u^{pn-m})^d N_{u^{np}}(\Pi([x|pn])) \\ &\leq 4^d u^{-pd} N_{u^{np}}(\Pi([x|pn])), \end{aligned}$$

using the fact that, for $R > r > 0$, a ball of radius R in $\mathbb{R}^d$ can be covered by $(4R/r)^d$ balls of radius r. Letting $\epsilon \to 0$ yields the desired inequality (5.2). □

The following result is also needed in the proof of Theorem 1.2.

LEMMA 5.2. *Let m be a Borel probability measure on Σ. Let $\rho, \epsilon \in (0, 1)$. Then, for m-a.e. $x = (x_n)_{n=1}^{\infty} \in \Sigma$,*

$$m \circ \Pi^{-1}(B(\Pi x, 2\rho^n)) \geq (1-\epsilon)^n \frac{m([x_1 \ldots x_n])}{N_{\rho^n}(\Pi([x_1 \ldots x_n]))} \quad \textit{for large enough } n. \tag{5.11}$$

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Proof. The formulation and the proof of the above lemma are adapted from an argument given by Jordan [25]. A similar idea was also employed in the proof of [36, Theorem 2.2].

For $n \in \mathbb{N}$, let Λ_n denote the set of the points $x = (x_n)_{n=1}^{\infty} \in \Sigma$ such that

$$m \circ \Pi^{-1}(B(\Pi x, 2\rho^n)) < (1-\epsilon)^n \frac{m([x_1 \ldots x_n])}{N_{\rho^n}(\Pi([x_1 \ldots x_n]))}.$$

To prove that (5.11) holds almost everywhere, by the Borel–Cantelli lemma it suffices to show that

$$\sum_{n=1}^{\infty} m(\Lambda_n) < \infty. \tag{5.12}$$

For this purpose, let us estimate $m(\Lambda_n)$. Fix $n \in \mathbb{N}$ and $I \in \mathcal{A}^n$. Notice that $\Pi([I])$ can be covered by $N_{\rho^n}(\Pi([I]))$ balls of radius ρ^n. As a consequence, there exists $L \leq N_{\rho^n}(\Pi([I]))$ such that $\Pi(\Lambda_n \cap [I])$ can be covered by L balls of radius ρ^n, say, $B_1, \ldots, B_L$. We may assume that $\Pi(\Lambda_n \cap [I]) \cap B_i \neq \emptyset$ for each $1 \leq i \leq L$. Hence, for each i, we may pick $x^{(i)} \in \Lambda_n \cap [I]$ such that $\Pi x^{(i)} \in B_i$. Clearly $B_i \subset B(\Pi x^{(i)}, 2\rho^n)$. Since $x^{(i)} \in \Lambda_n \cap [I]$, by the definition of Λ_n we obtain

$$m \circ \Pi^{-1}(B(\Pi x^{(i)}, 2\rho^n)) < (1-\epsilon)^n \frac{m([I])}{N_{\rho^n}(\Pi([I]))}.$$

It follows that

$$\begin{aligned} m(\Lambda_n \cap [I]) &\leq m \circ \Pi^{-1}(\Pi(\Lambda_n \cap [I])) \\ &\leq m \circ \Pi^{-1}\bigg(\bigcup_{i=1}^{L} B_i\bigg) \\ &\leq m \circ \Pi^{-1}\bigg(\bigcup_{i=1}^{L} B(\Pi x^{(i)}, 2\rho^n)\bigg) \\ &\leq L(1-\epsilon)^n \frac{m([I])}{N_{\rho^n}(\Pi([I]))} \\ &\leq (1-\epsilon)^n m([I]). \end{aligned}$$

Summing over $I \in \mathcal{A}^n$ yields that $m(\Lambda_n) \leq (1-\epsilon)^n$, which implies (5.12). □

Remark 5.3. Lemma 5.2 remains valid when the coding map $\Pi : \Sigma \to \mathbb{R}^d$ is replaced by any Borel measurable map from Σ to $\mathbb{R}^d$.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We may assume that $s := \dim_L m < d$; otherwise there is nothing left to prove. Set $k = [s]$. Let λ_i, $i = 1, \ldots, d$, be defined as in (5.1). Then, by Definition 2.6,

$$h_m(\sigma) + \lambda_1 + \cdots + \lambda_k + (s-k)\lambda_{k+1} = 0.$$

Let $u = \exp(\lambda_{k+1})$ and $\epsilon \in (0, 1)$. Applying Lemma 5.2 yields that, for m-a.e. $x = (x_n)_{n=1}^{\infty} \in \Sigma$,

$$m \circ \Pi^{-1}(B(\Pi x, 2u^n)) \geq (1-\epsilon)^n \frac{m([x_1 \ldots x_n])}{N_{u^n}(\Pi([x_1 \ldots x_n]))} \quad \text{for large enough } n. \tag{5.13}$$

It follows that, for m-a.e. $x = (x_n)_{n=1}^{\infty} \in \Sigma$,

$$\begin{aligned}
&\overline{d}(m \circ \Pi^{-1}, \Pi x)\\
&\quad = \limsup_{n\to\infty} \frac{\log(m \circ \Pi^{-1}(B(\Pi x, 2u^n)))}{n \log u}\\
&\quad \leq \frac{\log(1-\epsilon)}{\log u} + \limsup_{n\to\infty} \bigg(\frac{\log m([x_1 \ldots x_n])}{n \log u} - \frac{\log N_{u^n}(\Pi([x_1 \ldots x_n]))}{n \log u}\bigg)\\
&\quad \leq \frac{\log(1-\epsilon) - h_m(\sigma) - (\lambda_1 + \cdots + \lambda_k) + k\lambda_{k+1}}{\log u}\\
&\quad = \frac{\log(1-\epsilon) + s\lambda_{k+1}}{\lambda_{k+1}},
\end{aligned}$$

where in the third inequality, we used the Shannon–McMillan–Breiman theorem (cf. [39, p. 93]) and Lemma 5.1 (keeping in mind that $\log u = \lambda_{k+1} < 0$). Letting $\epsilon \to 0$ yields the desired result. □

6. *Upper bound for the box-counting dimension of C^1-repellers and the Lyapunov dimensions of ergodic invariant measures*

Throughout this section let $\boldsymbol{M}$ be a smooth Riemannian manifold of dimension d and $\psi : \boldsymbol{M} \to \boldsymbol{M}$ a C^1-map. Let Λ be a compact subset of $\boldsymbol{M}$ such that $\psi(\Lambda) = \Lambda$, and assume that Λ is a repeller of ψ (cf. §1). Below we first introduce the definitions of singularity and Lyapunov dimensions for the case of repellers, which are sightly different from that for the case of IFSs.

Definition 6.1. The *singularity dimension* of Λ with respect to ψ, written as $\dim_{S^*} \Lambda$, is the unique real value s for which

$$P(\Lambda, \psi, \mathcal{G}^s) = 0,$$

where $\mathcal{G}^s = \{g_n^s\}_{n=1}^{\infty}$ is the subadditive potential on Λ defined by

$$g_n^s(z) = \log \phi^s((D_z\psi^n)^{-1}), \quad z \in \Lambda. \tag{6.1}$$

Definition 6.2. For an ergodic ψ-invariant measure μ supported on Λ, the *Lyapunov dimension* of μ with respect to ψ, written as $\dim_{L^*} \mu$, is the unique real value s for which

$$h_\mu(\psi) + \mathcal{G}_*^s(\mu) = 0,$$

where $\mathcal{G}^s = \{g_n^s\}_{n=1}^{\infty}$ is defined as in (6.1) and $\mathcal{G}_*^s(\mu) = \lim_{n\to\infty}(1/n) \int g_n^s \, d\mu$.

Before proving Theorems 1.3–1.4, we recall some definitions and necessary facts about C^1 repellers.

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A finite closed cover $\{R_1, \ldots, R_\ell\}$ of Λ is called a *Markov partition* of Λ with respect to ψ if:

(i) $\overline{\mathrm{int} R_i} = R_i$ for each $i = 1, \ldots, \ell$;

(ii) $\mathrm{int} R_i \cap \mathrm{int} R_j = \emptyset$ for $i \neq j$; and

(iii) each $\psi(R_i)$ is the union of a subfamily of $\{R_j\}_{j=1}^{\ell}$.

It is well known that any repeller of an expanding map has Markov partitions of arbitrary small diameter (see [37, p. 146]). Let $\{R_1, \ldots, R_\ell\}$ be a Markov partition of Λ with respect to ψ. It is known that this dynamical system induces a subshift space of finite type (Σ_A, σ) over the alphabet $\{1, \ldots, \ell\}$, where $A = (a_{ij})$ is the transfer matrix of the Markov partition, namely, $a_{ij} = 1$ if $\mathrm{int} R_i \cap \psi^{-1}(\mathrm{int} R_j) \neq \emptyset$ and $a_{ij} = 0$ otherwise [37], and

$$\Sigma_A = \{(i_n)_{n=1}^{\infty} \in \{1, \ldots, \ell\}^{\mathbb{N}} : a_{i_n i_{n+1}} = 1 \text{ for all } n \geq 1\}.$$

This gives the coding map $\Pi : \Sigma_A \to \Lambda$ such that

$$\Pi(\mathbf{i}) = \bigcap_{n \geq 1} \psi^{-(n-1)}(R_{i_n}) \quad \text{for all } \mathbf{i} = (i_n)_{n=1}^{\infty} \in \Sigma_A, \tag{6.2}$$

and the following diagram commutes:

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ \Pi \downarrow & & \downarrow \Pi \\ \Lambda & \xrightarrow[\psi]{} & \Lambda \end{array} \tag{6.3}$$

(Keep in mind that throughout this section, Π denotes the coding map for the repeller Λ and no longer for the coding map for an IFS as used in the previous sections.)

The coding map Π is a Hölder continuous surjection. Moreover, there is a positive integer q such that $\Pi^{-1}(z)$ has at most q elements for each $z \in \Lambda$ (see [37, p. 147]).

For $n \geq 1$, define

$$\Sigma_{A,n} := \{i_1 \ldots i_n \in \{1, \ldots, \ell\}^n : a_{i_k i_{k+1}} = 1 \text{ for } 1 \leq k \leq n-1\}.$$

For any word $I = i_1 \ldots i_n \in \Sigma_{A,n}$, the set $\bigcap_{k=1}^{n} \psi^{-(k-1)}(R_{i_k})$ is called a *basic set* and is denoted by R_I.

The proof of Theorem 1.3 is similar to that of Theorem 1.1. We begin with the following lemma, which is a slight variant of Lemma 4.1.

LEMMA 6.3. *Let $E \subset U \subset \boldsymbol{M}$, where E is compact and U is open. Let $k \in \{0, 1, \ldots, d-1\}$. Then, for any non-degenerate C^1 map $f : U \to \boldsymbol{M}$, there exists $r_0 > 0$ so that, for any $y \in E$, $z \in B(y, r_0)$ and $0 < r < r_0$, the set $f(B(z, r))$ can be covered by*

$$C_{\boldsymbol{M}} \cdot \frac{\phi^k(D_y f)}{(\alpha_{k+1}(D_y f))^k}$$

balls of radius $\alpha_{k+1}(D_y f) r$, where $C_{\boldsymbol{M}}$ is a positive constant depending on $\boldsymbol{M}$.

Proof. This can be done by routinely modifying the proof of Lemma 4.1 and using similar arguments to the proof of [40, Corollary 1]. □

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Let $\delta > 0$ be small enough so that $\psi : B(z, \delta) \to \psi(B(z, \delta))$ is a diffeomorphism for each z in the δ-neighborhood of Λ. Suppose that $\{R_1, \ldots, R_\ell\}$ is a Markov partition of Λ with diameter less than δ.

The following result is an analogue of Proposition 4.2.

PROPOSITION 6.4. *Let $k \in \{0, 1, \ldots, d-1\}$. Set $C_{\boldsymbol{M}}$ be the constant in Lemma 6.3. Then there exists $C_1 > 0$ such that, for all $\mathbf{i} = (i_p)_{p=1}^{\infty} \in \Sigma_A$ and $n \in \mathbb{N}$, the basic set $R_{\mathbf{i}|n}$ can be covered by $C_1 \prod_{p=0}^{n-1} G(\sigma^p \mathbf{i})$ balls of radius $\prod_{p=0}^{n-1} H(\sigma^p \mathbf{i})$, where*

$$G(\mathbf{i}) := \frac{C_{\boldsymbol{M}} \phi^k((D_{\Pi\mathbf{i}}\psi)^{-1})}{\alpha_{k+1}((D_{\Pi\mathbf{i}}\psi)^{-1})^k}, \quad H(\mathbf{i}) := \alpha_{k+1}((D_{\Pi\mathbf{i}}\psi)^{-1}). \tag{6.4}$$

Remark 6.5. The definitions of the functions G and H in the above proposition are slightly different from that in Proposition 4.2.

Proof of Proposition 6.4. The proof is adapted from that of Proposition 4.2. For the reader's convenience, we provide the full details.

First, we construct a local inverse $f_{i,j}$ of ψ for each pair (i, j) with $ij \in \Sigma_{A,2}$. To do so, notice that $\psi(R_{ij}) = R_j$ for each $ij \in \Sigma_{A,2}$. Since ψ is a diffeomorphism restricted on a small neighborhood of R_i, we can find open sets $\widetilde{R}_{ij}$ and $\widetilde{R_j}$ such that $\widetilde{R}_{ij} \supset R_{ij}$, $\widetilde{R_j} \supset R_j$, $\psi(\widetilde{R}_{ij}) = \widetilde{R_j}$ and $\psi : \widetilde{R}_{ij} \to \widetilde{R_j}$ is diffeomorphic. Then we take $f_{i,j} : \widetilde{R_j} \to \widetilde{R}_{ij}$ to be the inverse of $\psi : \widetilde{R}_{ij} \to \widetilde{R_j}$, and the construction is done.

For any $\mathbf{i} = (i_n)_{n=1}^{\infty} \in \Sigma_A$, we see that $\Pi\sigma\mathbf{i} \in R_{i_2} \subset \widetilde{R_{i_2}}$ and $(\psi \circ f_{i_1,i_2})|_{\widetilde{R_{i_2}}}$ is the identity restricted on $\widetilde{R_{i_2}}$. Since $\psi(\Pi\mathbf{i}) = \Pi\sigma\mathbf{i}$, it follows that $f_{i_1,i_2}(\Pi\sigma\mathbf{i}) = \Pi\mathbf{i}$. Differentiating $\psi \circ f_{i_1,i_2}$ at $\Pi\sigma\mathbf{i}$ and applying the chain rule, we get

$$(D_{\Pi\mathbf{i}}\psi)(D_{\Pi\sigma\mathbf{i}} f_{i_1,i_2}) = \text{Identity},$$

so

$$D_{\Pi\sigma\mathbf{i}} f_{i_1,i_2} = (D_{\Pi\mathbf{i}}\psi)^{-1}. \tag{6.5}$$

According to Lemma 6.3, there exists $r_0 > 0$ such that, for each $ij \in \Sigma_{A,2}$, $y \in R_j$, $z \in B(y, r_0)$ and $0 < r < r_0$, the set $f_{i,j}(B(z, r))$ can be covered by

$$C_{\boldsymbol{M}} \cdot \frac{\phi^k(D_y f_{i,j})}{(\alpha_{k+1}(D_y f_{i,j}))^k}$$

balls of radius $\alpha_{k+1}(D_y f_{i,j}) r$.

Since ψ is expanding on Λ, there exists $\gamma \in (0, 1)$ such that $\sup_{\mathbf{i}\in\Sigma_A} \alpha_1((D_{\Pi\mathbf{i}}\psi)^{-1}) < \gamma$. Then $\sup_{\mathbf{i}\in\Sigma_A} H(\mathbf{i}) < \gamma$ and

$$\operatorname{diam}(R_{\mathbf{i}|(n+1)}) \le \gamma \operatorname{diam}(R_{(\sigma\mathbf{i})|n}) \tag{6.6}$$

for all $\mathbf{i} \in \Sigma_A$ and $n \in \mathbb{N}$. Take a large integer n_0 such that

$$\gamma^{n_0-1} \max\{1, \operatorname{diam}(\Lambda)\} < r_0/2. \tag{6.7}$$

By (6.6)–(6.7), $\operatorname{diam}(R_{\mathbf{i}|n}) < r_0/2$ for all $\mathbf{i} \in \Sigma_A$ and $n \ge n_0$.

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Clearly there exists a large number C_1 so that the conclusion of the proposition holds for any positive integer $n \le n_0$ and $\mathbf{i} \in \Sigma_A$, that is, the set $R_{\mathbf{i}|n}$ can be covered by $C_1 \prod_{p=0}^{n-1} G(\sigma^p \mathbf{i})$ balls of radius $\prod_{p=0}^{n-1} H(\sigma^p \mathbf{i})$. Below we show by induction that this holds for all $n \in \mathbb{N}$ and $\mathbf{i} \in \Sigma_A$.

Suppose, for some $m \ge n_0$, that the conclusion of the proposition holds for any positive integer $n \le m$ and $\mathbf{i} \in \Sigma_A$. Then, for given $\mathbf{i} = (i_n)_{n=1}^{\infty} \in \Sigma_A$, $R_{(\sigma \mathbf{i})|m}$ can be covered by $C_1 \prod_{p=0}^{m-1} G(\sigma^{p+1}\mathbf{i})$ balls of radius $\prod_{p=0}^{m-1} H(\sigma^{p+1}\mathbf{i})$. Let $B_1, \ldots, B_N$ denote these balls. We may assume that $B_j \cap R_{(\sigma \mathbf{i})|m} \neq \emptyset$ for each j. Since

$$\prod_{p=0}^{m-1} H(\sigma^{p+1}\mathbf{i}) \le \gamma^m \le \gamma^{n_0} < r_0/2$$

and

$$d(\Pi \sigma \mathbf{i}, B_j \cap R_{(\sigma \mathbf{i})|m}) \le \operatorname{diam}(R_{(\sigma \mathbf{i})|m}) < r_0/2,$$

so the center of B_j is in $B(\Pi \sigma \mathbf{i}, r_0)$. Therefore, by Lemma 6.3 and (6.5), $f_{i_1,i_2}(B_j)$ can be covered by

$$C_{\boldsymbol{M}} \cdot \frac{\phi^k(D_{\Pi \sigma \mathbf{i}} f_{i_1,i_2})}{(\alpha_{k+1}(D_{\Pi \sigma \mathbf{i}} f_{i_1,i_2}))^k} = G(\mathbf{i})$$

balls of radius

$$\alpha_{k+1}(D_{\Pi \sigma \mathbf{i}} f_{i_1,i_2}) \cdot \prod_{p=0}^{m-1} H(\sigma^{p+1}\mathbf{i}) = H(\mathbf{i}) \cdot \prod_{p=0}^{m-1} H(\sigma^{p+1}\mathbf{i}) = \prod_{p=0}^{m} H(\sigma^{p}\mathbf{i}).$$

Since $\psi(R_{\mathbf{i}|(m+1)}) \subset R_{(\sigma \mathbf{i})|m}$, it follows that

$$R_{\mathbf{i}|(m+1)} \subset f_{i_1,i_2}(R_{(\sigma \mathbf{i})|m}) \subset \bigcup_{j=1}^{N} f_{i_1,i_2}(B_j),$$

hence $R_{\mathbf{i}|(m+1)}$ can be covered by

$$G(\mathbf{i})N \le G(\mathbf{i}) \cdot C_1 \prod_{p=0}^{m-1} G(\sigma^{p+1}\mathbf{i}) = C_1 \prod_{p=0}^{m} G(\sigma^{p}\mathbf{i})$$

balls of radius $\prod_{p=0}^{m} H(\sigma^p \mathbf{i})$. Thus the proposition also holds for $n = m+1$ and all $\mathbf{i} \in \Sigma_A$, as desired. □

PROPOSITION 6.6. *Let $k \in \{0, 1, \ldots, d-1\}$. Let $G, H : \Sigma_A \to \mathbb{R}$ be defined as in (6.4). Let t be the unique real number so that*

$$P(\Sigma_A, \sigma, (\log G) + t(\log H)) = 0.$$

Then $\overline{\dim}_B \Lambda \le t$.

Proof. Here we use similar arguments to that in the proof of Proposition 4.3. Write $g = \log G$ and $h = \log H$. Define

$$r_{\min} = \min_{\mathbf{i}\in\Sigma_A} h(\mathbf{i}), \qquad r_{\max} = \max_{\mathbf{i}\in\Sigma_A} h(\mathbf{i}).$$

Then $0 < r_{\min} \le r_{\max} < 1$. For $0 < r < r_{\min}$, define

$$\mathcal{A}_r = \Big\{i_1 \ldots i_n \in \Sigma_A^* : \sup_{x\in[i_1\ldots i_n]\cap\Sigma_A} S_n h(x) < \log r \le \sup_{y\in[i_1\ldots i_{n-1}]\cap\Sigma_A} S_{n-1}h(y)\Big\}, \quad (6.8)$$

where Σ_A^* denotes the set of all finite words allowed in Σ_A. Clearly $\{[I] : I \in \mathcal{A}_r\}$ is a partition of Σ_A. By Proposition 6.4, there exists a constant $C_1 > 0$ such that, for each $0 < r < r_{\min}$, every $I \in \mathcal{A}_r$ and $x \in [I]$, R_I can be covered by

$$C_1 \exp(S_{|I|}g(x)) \le C_1 \exp\Big(\sup_{y\in[I]} S_{|I|}g(y)\Big)$$

balls of radius

$$\exp(S_{|I|}h(x)) \le \exp\Big(\sup_{y\in[I]} S_{|I|}h(y)\Big) < r.$$

It follows that Λ can be covered by

$$C_1 \sum_{I\in\mathcal{A}_r} \exp\Big(\sup_{y\in[I]} S_{|I|}g(y)\Big)$$

balls of radius r. Hence, by Proposition 3.4,

$$\overline{\dim}_B \Lambda \le \limsup_{r\to 0} \frac{\sum_{I\in\mathcal{A}_r} \exp\big(\sup_{y\in[I]} S_{|I|}g(y)\big)}{\log(1/r)} = t.$$

This completes the proof of the proposition. □

For $n \in \mathbb{N}$, applying Proposition 6.6 to the mapping ψ^n instead of ψ, we obtain the following result.

PROPOSITION 6.7. *Let $k \in \{0, \ldots, d-1\}$. Then, for each $n \in \mathbb{N}$,*

$$\overline{\dim}_B \Lambda \le t_n,$$

where t_n is the unique number for which $P(\Sigma_A, \sigma^n, (\log G_n) + t_n(\log H_n)) = 0$, and G_n, H_n are continuous functions on Σ_A defined by

$$G_n(y) := \frac{C\phi^k((D_{\Pi y}\psi^n)^{-1})}{\alpha_{k+1}((D_{\Pi y}\psi^n)^{-1}))^k}, \qquad H_n(y) := \alpha_{k+1}((D_{\Pi y}\psi^n)^{-1})), \qquad (6.9)$$

with $C = C_{\boldsymbol{M}}$ being the constant in Lemma 6.3.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. We follow the proof of Theorem 1.1 with slight modifications. Write $s = \dim_{S^*}(\Lambda)$. We may assume that $s < d$; otherwise we have nothing left to prove.

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Set $k=[s]$. Let $\mathcal{G}^s=\{g_n^s\}_{n=1}^\infty$ be the subadditive potential on Λ defined by

$$g_n^s(z)=\log\phi^s((D_z\psi^n)^{-1}).$$

Then $P(\Lambda,\psi,\mathcal{G}^s)=0$ by the definition of $\dim_{S^*}(\Lambda)$. Let $\widehat{\mathcal{G}}^s:=\{\widehat{g}_n^s\}_{n=1}^\infty$, where $\widehat{g}_n^s\in C(\Sigma_A)$ is defined by

$$\widehat{g}_n^s(\mathbf{i})=g_n^s(\Pi\mathbf{i})=\log\phi^s((D_{\Pi\mathbf{i}}\psi^n)^{-1}),\quad \mathbf{i}\in\Sigma_A.$$

Clearly, $\widehat{\mathcal{G}}^s$ is a subadditive potential on Σ_A and

$$(\widehat{\mathcal{G}}^s)_*(m)=(\mathcal{G}^s)_*(m\circ\Pi^{-1}),\quad m\in\mathcal{M}(\Sigma_A,\sigma).$$

Since the factor map $\Pi:\Sigma_A\to\Lambda$ is onto and finite-to-one, by Lemma 2.3, $m\to m\circ\Pi^{-1}$ is a surjective map from $\mathcal{M}(\Sigma_A,\sigma)$ to $\mathcal{M}(\Lambda,\psi)$ and, moreover, $h_m(\sigma)=h_{m\circ\Pi^{-1}}(\psi)$ for $m\in\mathcal{M}(\Sigma_A,\sigma)$. By the variational principle for the subadditive pressure (see Theorem 2.2),

$$\begin{aligned}P(\Lambda,\psi,\mathcal{G}^s)&=\sup\{h_\mu(\psi)+(\mathcal{G}^s)_*(\mu):\mu\in\mathcal{M}(\Lambda,\psi)\}\\&=\sup\{h_{m\circ\Pi^{-1}}(\psi)+(\mathcal{G}^s)_*(m\circ\Pi^{-1}):m\in\mathcal{M}(\Sigma_A,\sigma)\}\\&=\sup\{h_m(\sigma)+(\widehat{\mathcal{G}}^s)_*(m):m\in\mathcal{M}(\Sigma_A,\sigma)\}\\&=P(\Sigma_A,\sigma,\widehat{\mathcal{G}}^s).\end{aligned}$$

It follows that $P(\Sigma_A,\sigma,\widehat{\mathcal{G}}^s)=0$.

By [2],

$$\lim_{n\to\infty}\frac{1}{n}P(\Sigma_A,\sigma^n,\widehat{g}_n^s)=\inf_{n\ge1}\frac{1}{n}P(\Sigma_A,\sigma^n,\widehat{g}_n^s)=P(\Sigma_A,\sigma,\widehat{\mathcal{G}}^s).$$

Since $P(\Sigma_A,\sigma,\widehat{\mathcal{G}}^s)=0$, the above equalities imply that, for each $\epsilon>0$, there exists $N_\epsilon>0$ such that

$$0\le P(\Sigma_A,\sigma^n,\widehat{g}_n^s)\le n\epsilon\quad\text{for } n>N_\epsilon. \tag{6.10}$$

For $n\in\mathbb{N}$, let s_n be the unique real number such that $P(\Sigma_A,\sigma^n,v_n^{s_n})=0$, where $v_n^{s_n}$ is a continuous function on Σ_A defined by

$$\begin{aligned}v_n^{s_n}(x)&=\log G_n(x)+s_n\log H_n(x)\\&=\log(C\phi^k((D_{\Pi x}\psi^n)^{-1})\alpha_{k+1}((D_{\Pi x}\psi^n)^{-1})^{s_n-k}),\end{aligned}$$

where G_n, H_n are defined in (6.9) and $C=C_{\boldsymbol{M}}$. By Proposition 6.7,

$$\overline{\dim}_B\Lambda\le s_n.$$

If $s\ge s_n$ for some n, then $\overline{\dim}_B\Lambda\le s_n\le s$ and we are done. In what follows, we assume that $s<s_n$ for each n. Then, for each $x\in\Sigma_A$,

$$\begin{aligned}\widehat{g}_n^s(x)-v_n^{s_n}(x)&=-\log C+(s-s_n)\log\alpha_{k+1}((D_{\Pi x}\psi^n)^{-1})\\&\ge-\log C+(s-s_n)\log\alpha_1((D_{\Pi x}\psi^n)^{-1})\\&\ge-\log C+n(s-s_n)\log\theta,\end{aligned}$$

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where

$$\theta := \max_{x\in\Sigma_A} \alpha_1((D_{\Pi x}\phi)^{-1}) < 1.$$

Hence,

$$\begin{aligned} P(\Sigma_A, \sigma^n, \widehat{g}_n^s) &= P(\Sigma_A, \sigma^n, \widehat{g}_n^s) - P(\Sigma_A, \sigma^n, v_n^{s_n}) \\ &\geq \inf_{x\in\Sigma_A} (\widehat{g}_n^s(x) - v_n^{s_n}(x)) \\ &\geq -\log C + n(s - s_n)\log\theta. \end{aligned}$$

where in the second inequality, we used [**39**, Theorem 9.7(iv)]. Combining this with (6.10) yields that, for $n \geq N_\epsilon$,

$$n\epsilon \geq -\log C + n(s - s_n)\log\theta,$$

so

$$s \geq s_n + \frac{\epsilon + n^{-1}\log C}{\log\theta} \geq \overline{\dim}_B\Lambda + \frac{\epsilon + n^{-1}\log C}{\log\theta}.$$

Letting $n \to \infty$ and then $\epsilon \to 0$, we obtain $s \geq \overline{\dim}_B\Lambda$, as desired. □

In the remainder of this section we prove Theorem 1.4. To this end, we need the following two lemmas, which are the analogues of Lemmas 5.1–5.2 for C^1 repellers.

LEMMA 6.8. *Let m be an ergodic σ-invariant Borel probability measure on Σ_A. Set*

$$\lambda_i := \lim_{n\to\infty}\frac{1}{n}\int \log(\alpha_i((D_{\Pi x}\psi^n)^{-1}))\,dm(x), \quad i = 1, \ldots, d. \tag{6.11}$$

Let $k \in \{0, \ldots, d-1\}$. Write $u := \exp(\lambda_{k+1})$. Then, for m-a.e. $x \in \Sigma_A$,

$$\limsup_{n\to\infty}\frac{1}{n}\log N_{u^n}(R_{x|n}) \leq (\lambda_1 + \cdots + \lambda_k) - k\lambda_{k+1}, \tag{6.12}$$

where $N_\delta(E)$ is the smallest integer N for which E can be covered by N closed balls of radius δ.

LEMMA 6.9. *Let m be a Borel probability measure on Σ_A. Let $\rho, \epsilon \in (0, 1)$. Then, for m-a.e. $x = (x_n)_{n=1}^\infty \in \Sigma_A$,*

$$m\circ\Pi^{-1}(B(\Pi x, 2\rho^n)) \geq (1-\epsilon)^n\frac{m([x_1\ldots x_n])}{N_{\rho^n}(R_{x_1\ldots x_n})} \quad \textit{for large enough } n. \tag{6.13}$$

The proofs of these two lemmas are essentially identical to those of Lemmas 5.1–5.2, so we omit them.

Proof of Theorem 1.4. Here we adapt the proof of Theorem 1.2. We may assume that $s := \dim_{L^*}\mu < d$; otherwise there is nothing left to prove. Since $\Pi : \Sigma_A \to \Lambda$ is surjective and finite-to-one, by Lemma 2.3, there exists a σ-invariant ergodic measure m on Σ_A so that $m \circ \Pi^{-1} = \mu$ and $h_m(\sigma) = h_\mu(\psi)$.

Set $k = [s]$. Let λ_i, $i = 1, \ldots, d$, be defined as in (6.11). Then, by Definition 6.2,

$$h_m(\sigma) + \lambda_1 + \cdots + \lambda_k + (s-k)\lambda_{k+1} = 0.$$

Let $u = \exp(\lambda_{k+1})$ and $\epsilon \in (0, 1)$. Applying Lemma 6.9 (in which we take $\rho = u$) yields that, for m-a.e. $x = (x_n)_{n=1}^\infty \in \Sigma_A$,

$$\mu(B(\Pi x, 2u^n)) \geq (1-\epsilon)^n \frac{m([x_1 \ldots x_n])}{N_{u^n}(R_{x_1 \ldots x_n})} \quad \text{for large enough } n. \tag{6.14}$$

It follows that, for m-a.e. $x = (x_n)_{n=1}^\infty \in \Sigma_A$,

$$\begin{aligned}
\overline{d}(\mu, \Pi x) &= \limsup_{n\to\infty} \frac{\log(\mu(B(\Pi x, 2u^n)))}{n \log u} \\
&\leq \frac{\log(1-\epsilon)}{\log u} + \limsup_{n\to\infty}\left(\frac{\log m([x_1 \ldots x_n])}{n \log u} - \frac{\log N_{u^n}(\Pi([x_1 \ldots x_n]))}{n \log u}\right) \\
&\leq \frac{\log(1-\epsilon) - h_m(\sigma) - (\lambda_1 + \cdots + \lambda_k) + k\lambda_{k+1}}{\log u} \\
&= \frac{\log(1-\epsilon) + s\lambda_{k+1}}{\lambda_{k+1}},
\end{aligned}$$

where in the third inequality we used the Shannon–McMillan–Breiman theorem (cf. [**39**, p. 93]) and Lemma 6.8 (keeping in mind that $\log u = \lambda_{k+1} < 0$). Letting $\epsilon \to 0$ yields the desired result. □

Remark 6.10. There is an alternative way to prove Theorem 1.3 in the case when $\boldsymbol{M}$ is an open set of $\mathbb{R}^d$. We can construct a C^1 IFS $\{f_i\}_{i=1}^\ell$ on $\mathbb{R}^d$ so that Λ is the projection of a shift-invariant set under the coding map associated with the IFS. Then we can use Theorem 1.1 to get an upper bound for $\overline{\dim}_B \Lambda$. An additional effort is then required to justify that this upper bound is indeed equal to $\dim_{S^*} \Lambda$. The details of this approach will be given in a forthcoming survey paper.

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A. *Appendix. Main notation and conventions*

For the reader's convenience, we summarize in Table A1 the main notation and typographical conventions used in this paper.

TABLE A1. Main notation and conventions.

$\overline{\dim}_B$	Upper box-counting dimension
$\dim_P$	Packing dimension
$\{f_i\}_{i=1}^{\ell}$	A C^1 IFS (§1)
(Σ, σ)	One-sided full shift over the alphabet $\{1, \ldots, \ell\}$
$\Pi : \Sigma \to K$	Coding map associated with $\{f_i\}_{i=1}^{\ell}$ (§1)
$D_x f$	Differential of f at x
$\dim_S X$	Singular dimension of X with respect to $\{f_i\}_{i=1}^{\ell}$ (cf. Definition 2.5)
$\dim_L m$	Lyapunov dimension of m with respect to $\{f_i\}_{i=1}^{\ell}$ (cf. Definition 2.6)
$P(X, T, \{g_n\}_{n=1}^{\infty})$	Topological pressure of a subadditive potential $\{g_n\}_{n=1}^{\infty}$ on a topological dynamical system (X, T) (cf. Definition 2.1)
$h_{\mu}(T)$	Measure-theoretic entropy of μ with respect to T
$\mathcal{G}_*(\mu)$	Lyapunov exponent of a subadditive potential $\mathcal{G}$ with respect to μ (cf. (2.5))
$\alpha_i(T), i = 1, \ldots, d$	The ith singular value of $T \in \mathbb{R}^{d\times d}$ (§2)
ϕ^s	Singular value function (cf. (2.9))
$S_n g$	$g + g \circ T + \cdots + g \circ T^{n-1}$ for $g \in C(X)$
$\mathcal{G}^s = \{g_n^s\}_{n=1}^{\infty}$	(cf. (2.10))
$\mathcal{G}_*^s(m)$	$\lim_{n\to\infty}(1/n) \int g_n^s \, dm$
$\lambda_i(m), i = 1, \ldots, d$	The ith Lyapunov exponent of m with respect to $\{f_i\}_{i=1}^{\ell}$ (cf. (2.11))
$N_{\delta}(E)$	Smallest number of closed balls of radius δ required to cover E
$\dim_{S^*} \Lambda$	Singularity dimension of Λ with respect to ψ (cf. Definition 6.1)
$\dim_{L^*} \mu$	Lyapunov dimension of μ with respect to ψ (cf. Definition 6.2)

REFERENCES

[1] L. Arnold. *Random Dynamical Systems (Springer Monographs in Mathematics)*. Springer-Verlag, Berlin, 1998.

[2] J. Ban, Y. Cao and H. Hu. The dimensions of a non-conformal repeller and an average conformal repeller. *Trans. Amer. Math. Soc.* **362**(2) (2010), 727–751.

[3] B. Bárány, M. Hochman and A. Rapaport. Hausdorff dimension of planar self-affine sets and measures. *Invent. Math.* **216**(3) (2019), 601–659.

[4] L. Barreira. Dimension estimates in non-conformal hyperbolic dynamics. *Nonlinearity* **16**(5) (2003), 1657–1672.

[5] L. Barreira. Almost additive thermodynamic formalism: some recent developments. *Rev. Math. Phys.* **22**(10) (2010), 1147–1179.

[6] T. Bogenschütz and H. Crauel. The Abramov–Rokhlin formula. *Ergodic Theory and Related Topics, III (Güstrow, 1990) (Lecture Notes in Mathematics, 1514)*. Eds. U. Krengel, K. Richter and V. Warstat. Springer, Berlin, 1992, pp. 32–35.

[7] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics, 470)*. Springer-Verlag, Berlin, 1975.

[8] R. Bowen. Hausdorff dimension of quasicircles. *Publ. Math. Inst. Hautes Études Sci.* **50** (1979), 11–25.

[9] Y. Cao, Y. Pesin and Y. Zhao. Dimension estimates for non-conformal repellers and continuity of sub-additive topological pressure. *Geom. Funct. Anal.* **29**(5) (2019), 1325–1368.

[10] Y.-L. Cao, D.-J. Feng and W. Huang. The thermodynamic formalism for sub-additive potentials. *Discrete Contin. Dyn. Syst.* **20**(3) (2008), 639–657.

[11] J. Chen and Y. Pesin. Dimension of non-conformal repellers: a survey. *Nonlinearity* **23**(4) (2010), R93–R114.

[12] A. Douady and J. Oesterlé. Dimension de Hausdorff des attracteurs. *C. R. Acad. Sci. Paris Sér. A-B* **290**(24) (1980), A1135–A1138.

[13] K. J. Falconer. The Hausdorff dimension of self-affine fractals. *Math. Proc. Cambridge Philos. Soc.* **103**(2) (1988), 339–350.

[14] K. J. Falconer. A subadditive thermodynamic formalism for mixing repellers. *J. Phys. A* **21**(14) (1988), L737–L742.

[15] K. J. Falconer. Bounded distortion and dimension for nonconformal repellers. *Math. Proc. Cambridge Philos. Soc.* **115**(2) (1994), 315–334.
[16] K. J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*, 2nd edn. John Wiley & Sons, Hoboken, NJ, 2003.
[17] A.-H. Fan, K.-S. Lau and H. Rao. Relationships between different dimensions of a measure. *Monatsh. Math.* **135**(3) (2002), 191–201.
[18] D.-J. Feng. The variational principle for products of non-negative matrices. *Nonlinearity* **17**(2) (2004), 447–457.
[19] D.-J. Feng. Equilibrium states for factor maps between subshifts. *Adv. Math.* **226**(3) (2011), 2470–2502.
[20] D.-J. Feng and H. Hu. Dimension theory of iterated function systems. *Comm. Pure Appl. Math.* **62**(11) (2009), 1435–1500.
[21] D.-J. Feng and K.-S. Lau. The pressure function for products of non-negative matrices. *Math. Res. Lett.* **9**(2–3) (2002), 363–378.
[22] D.-J. Feng and K. Simon. Dimension estimates for C^1 iterated function systems and repellers. Part II. *Ergod. Th. & Dynam. Sys.* doi:10.1017/etds.2021.92. Published online 8 September 2021.
[23] D. Gatzouras and Y. Peres. Invariant measures of full dimension for some expanding maps. *Ergod. Th. & Dynam. Sys.* **17**(1) (1997), 147–167.
[24] M. Hochman and A. Rapaport. Hausdorff dimension of planar self-affine sets and measures with overlaps. *J. Eur. Math. Soc. (JEMS)* **24** (2022), 2361–2441.
[25] T. Jordan. Private communication, 2011.
[26] T. Jordan and M. Pollicott. The Hausdorff dimension of measures for iterated function systems which contract on average. *Discrete Contin. Dyn. Syst.* **22**(1–2) (2008), 235–246.
[27] T. Jordan, M. Pollicott and K. Simon. Hausdorff dimension for randomly perturbed self affine attractors. *Comm. Math. Phys.* **270**(2) (2007), 519–544.
[28] A. Käenmäki. On natural invariant measures on generalised iterated function systems. *Ann. Acad. Sci. Fenn. Math.* **29**(2) (2004), 419–458.
[29] A. Käenmäki and M. Vilppolainen. Dimension and measures on sub-self-affine sets. *Monatsh. Math.* **161**(3) (2010), 271–293.
[30] R. Kenyon and Y. Peres. Measures of full dimension on affine-invariant sets. *Ergod. Th. & Dynam. Sys.* **16**(2) (1996), 307–323.
[31] R. Mañé. *Ergodic Theory and Differentiable Dynamics*. Springer-Verlag, Berlin, 1987.
[32] A. Manning and K. Simon. Subadditive pressure for triangular maps. *Nonlinearity* **20**(1) (2007), 133–149.
[33] A. Mummert. The thermodynamic formalism for almost-additive sequences. *Discrete Contin. Dyn. Syst.* **16**(2) (2006), 435–454.
[34] N. Patzschke. Self-conformal multifractal measures. *Adv. in Appl. Math.* **19**(4) (1997), 486–513.
[35] Y. B. Pesin. *Dimension Theory in Dynamical Systems: Contemporary Views and Applications (Chicago Lectures in Mathematics)*. University of Chicago Press, Chicago, 1997.
[36] E. Rossi. Local dimensions of measures on infinitely generated self-affine sets. *J. Math. Anal. Appl.* **413**(2) (2014), 1030–1039.
[37] D. Ruelle. *Thermodynamic Formalism (Encyclopedia of Mathematics and Its Applications, 5)*. Addison-Wesley, Reading, MA, 1978.
[38] D. Ruelle. Repellers for real analytic maps. *Ergod. Th. & Dynam. Sys.* **2**(1) (1982), 99–107.
[39] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer-Verlag, New York, 1982.
[40] Y. Zhang. Dynamical upper bounds for Hausdorff dimension of invariant sets. *Ergod. Th. & Dynam. Sys.* **17**(3) (1997), 739–756.