8
Group integration

Wilson's use of the invariant measure in his definition of lattice gauge theory lends a flair of mathematical elegance to the subject. This measure is essential to the simplicity of the gauge symmetries in the cutoff theory. In this chapter we review some general properties of invariant integrals over compact Lie groups. We will explicitly display the measure for some simple cases and then discuss integrals over polynomials of $SU(n)$ matrices.

To begin, we must have the basic properties of any integral

$$\int \mathrm{d}g\,(af(g)+bh(g)) = a\int \mathrm{d}g\,f(g)+b\int \mathrm{d}g\,h(g), \tag{8.1}$$

$$\int \mathrm{d}g\,f(g) > 0 \quad \text{whenever} \quad f(g) > 0 \quad \text{for all} \quad g. \tag{8.2}$$

Here f and h are arbitrary functions over the group and a and b are arbitrary complex numbers. We now impose the additional constraint that the measure be left-invariant

$$\int \mathrm{d}g\,f(g) = \int \mathrm{d}g\,f(g'g), \tag{8.3}$$

where g' is an arbitrary fixed element of the group. In an ordinary integral, this corresponds to a shift of the integration variable. As we will only be considering compact groups, we can normalize the measure such that

$$\int \mathrm{d}g\,1 = 1. \tag{8.4}$$

We will now show that this measure exists and is unique. We do this by first finding an expression for it under the assumption of its existence, and then we will show that this expression works.

To begin, we consider an arbitrary parametrization of the group elements in terms of a set of parameters α_i where the index i runs from one to n, the dimension of the group manifold. We assume that as the parameters α run over some domain D of R^n, the corresponding group element runs once over the group

$$G = \{g(\alpha)\,|\,\alpha \in D\}. \tag{8.5}$$

The group multiplication is represented by a function $\alpha(\beta, \gamma)$ satisfying

$$g(\alpha(\beta, \gamma)) = g(\beta) g(\gamma), \tag{8.6}$$

where α, β, and γ all reside in D. We now wish to find a weight $J(\alpha)$ such that the group integral is an ordinary n-dimensional integral

$$\int \mathrm{d}g f(g) = \int \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n \, J(\alpha) f(g(\alpha)). \tag{8.7}$$

The integral on the right hand side of this equation is over the domain D. Writing the group invariance property in this notation gives

$$\int \mathrm{d}\beta J(\beta) f(g(\beta)) = \int \mathrm{d}\beta J(\beta) f(g(\alpha(\gamma, \beta))), \tag{8.8}$$

where γ parametrizes the factor g' in eq. (8.3). We now change variables to $\alpha(\gamma, \beta)$ with the result

$$\int \mathrm{d}\beta J(\beta) f(g(\beta)) = \int \mathrm{d}\alpha \left\| \frac{\partial \alpha}{\partial \beta} \right\|^{-1} J(\beta) f(g(\alpha)), \tag{8.9}$$

where $\| \partial\alpha / \partial\beta \|$ represents the Jacobian determinant for the change of variables. Since this is true for arbitrary f, we conclude

$$J(\alpha) = \|\partial\alpha / \partial\beta\|^{-1} J(\beta). \tag{8.10}$$

Taking β to the identity, denoted by e, we find

$$J(\gamma) = K \| \partial(\alpha(\beta, \gamma)) / \partial\beta \|^{-1} |_{\beta = e}, \tag{8.11}$$

where $K = J(e)$ is a normalization factor, determined in magnitude with eq. (8.4). Thus the group measure is simply a Jacobian factor. It represents the shift of a small standard volume from near the identity to any point in the group.

If an invariant measure exists, eq. (8.11) is an expression for it. We must now show that this formula works. In particular, eq. (8.10) must be true for all β. We need to show that

$$J(\alpha(\beta, \gamma)) = K \| \partial(\alpha(\delta, \alpha(\beta, \gamma))) / \partial\delta \|_{\delta = e}^{-1} \tag{8.12}$$

is equal to

$$\| \partial\alpha(\beta, \gamma) / \partial\beta \| \, J(\beta) = K \left\| \frac{\partial\alpha(\beta, \gamma)}{\partial\beta} \right\|^{-1} \left\| \frac{\partial\alpha(\delta, \beta)}{\partial\delta} \right\|_{\delta = e}^{-1} \tag{8.13}$$

For this we need associativity, which implies

$$\alpha(\delta, \alpha(\beta, \gamma)) = \alpha(\alpha(\delta, \beta), \gamma). \tag{8.14}$$

Differentiating with respect to δ gives

$$\left\| \frac{\partial\alpha(\delta, \alpha(\beta, \gamma))}{\partial\delta} \right\| = \left\| \frac{\partial\alpha(\rho, \gamma)}{\partial\rho} \right\|_{\rho = \alpha(\delta, \beta)} \left\| \frac{\partial\alpha(\delta, \beta)}{\partial\delta} \right\|. \tag{8.15}$$

Setting δ to the identity gives the desired result.

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Barring a singular parametrization of the group, this analysis proves existence and uniqueness of the measure and provides a formal expression for it. We now show that the right- and left-invariant measures are the same. Clearly a modification of the above arguments will produce a measure which is right-invariant

$$\int (\mathrm{d}g)_r f(g) = \int (\mathrm{d}g)_r f(gg'). \tag{8.16}$$

Suppose we now define

$$\int (\mathrm{d}g)' f(g) = \int (\mathrm{d}g)_r f(g_0 g g_0^{-1}), \tag{8.17}$$

where g_0 is some arbitrary fixed element of the group. This new measure satisfies

$$\begin{aligned}\int (\mathrm{d}g)' f(gg_1) &= \int (\mathrm{d}g)_r f(g_0 g g_0^{-1} g_1) \\ &= \int (\mathrm{d}g)_r f(g_0 g g_0^{-1}) = \int (\mathrm{d}g)' f(g),\end{aligned} \tag{8.18}$$

where we have used the right-invariance of $(\mathrm{d}g)_r$. Thus $(\mathrm{d}g)'$ is also right-invariant. Uniqueness implies $(\mathrm{d}g)' = (\mathrm{d}g)_r$. But now we can use right-invariance again in eq. (8.17) to obtain

$$\int (\mathrm{d}g)_r f(g) = \int (\mathrm{d}g)_r f(g_0 g g_0^{-1}) = \int (\mathrm{d}g)_r f(g_0 g). \tag{8.19}$$

We conclude that the right measure is also left-invariant and, by uniqueness, the measures must be equal. Note that we have used compactness in a rather subtle way. If the integration measures cannot be normalized as in eq. (8.3), the various measures discussed here may differ by constant factors.

We note in passing that

$$\int \mathrm{d}g\, f(g^{-1}) = \int \mathrm{d}g\, f(g). \tag{8.20}$$

This follows because the left hand side defines another invariant measure which, by uniqueness, must equal the right hand side. In lattice gauge theory, the directions of the bonds do not enter in the measure.

Knowing of its existence may not be useful if the group combination law is complicated. A somewhat more explicit formula for the measure for groups of matrices follows from the definition of a metric tensor on the group

$$M_{ij} = \operatorname{Tr}(g^{-1}(\partial_i g)\, g^{-1}(\partial_j g)), \tag{8.21}$$

where the derivatives are with respect to the parameters α_i

$$\partial_i g = (\partial/\partial\alpha_i)\, g(\alpha). \tag{8.22}$$

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In terms of this metric, the invariant measure is

$$\int \mathrm{d}g f(g) = K \int \mathrm{d}\alpha \,|\det(M)|^{\frac{1}{2}} f(g(\alpha)), \tag{8.23}$$

where the factor of K is again a normalization. This is a standard formula of differential geometry.

We now give some simple examples. For a discrete group the measure is an ordinary sum over the elements. For the group $U(1)$ of relevance to electrodynamics

$$U(1) = \{\mathrm{e}^{\mathrm{i}\theta} \,|\, -\pi < \theta \leqslant \pi\} \tag{8.24}$$

the measure is

$$\int \mathrm{d}g f(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta f(\mathrm{e}^{\mathrm{i}\theta}). \tag{8.25}$$

Functions over the group are periodic functions of the angle θ. Group-invariance is under shifts of phase.

For $SU(2)$ we can parametrize the elements as the surface of a four-dimensional sphere (S_3)

$$SU(2) = \{a_0 + \mathrm{i}\mathbf{a}\cdot\boldsymbol{\sigma} \,|\, a_0^2 + \mathbf{a}^2 = 1\}. \tag{8.26}$$

The matrices $\boldsymbol{\sigma}$ are the Pauli matrices used in chapter 5 when we discussed fermions. With this parametrization the group measure assumes a particularly simple form

$$\int \mathrm{d}g f(g) = \pi^{-2} \int \mathrm{d}^4 a \,\delta(a^2 - 1) f(g). \tag{8.27}$$

Here we use the shorthand notation

$$a^2 = a_0^2 + \mathbf{a}\cdot\mathbf{a}. \tag{8.28}$$

For $SU(3)$ we refer the reader to the discussion by Beg and Ruegg (1965).

For many purposes an explicit form for the measure is unnecessary. In Monte Carlo simulations, to be discussed later, certain algorithms move randomly around in the group in a uniform manner and automatically generate the correct measure. For analytic work, many integrals can often be done using symmetry arguments. For example, the expression

$$\int \mathrm{d}g \, R_{\alpha\beta}(g) \tag{8.29}$$

will vanish if $R_{\alpha\beta}$ is a non-trivial irreducible matrix representation of the group. A group integral selects the singlet part of any function over the group. In particular, we have the relation

$$\int \mathrm{d}g \, \chi_{R_1}(g) \ldots \chi_{R_k}(g) = n_\mathrm{s}(R_1 \otimes \ldots \otimes R_k), \tag{8.30}$$

where the character $\chi_R(g)$ denotes the trace of the matrix corresponding to g in representation R, and $n_\mathrm{s}(R_1 \ldots R_k)$ is the number of times the singlet representation occurs in the direct product of the representations R_1 to R_k. If R and R' are irreducible, we have the orthogonality of the characters

$$\int \mathrm{d}g\, \chi_R^*(g)\, \chi_{R'}(g) = \delta_{R,R'}. \tag{8.31}$$

For $SU(3)$ we have the integral

$$\int \mathrm{d}g\, (\chi_3(g))^3 = 1. \tag{8.32}$$

For the strong coupling expansion we will need integrals of polynomials of the group elements in the fundamental representation. We now turn to a set of graphical rules for the evaluation of such integrals with the groups $SU(n)$ (Creutz, 1978*b*). We are interested in expressions of the form

$$I = \int \mathrm{d}g\, g_{i_1 j_1} \cdots g_{i_n j_n} g^{-1}_{k_1 l_1} \cdots g^{-1}_{k_m l_m}, \tag{8.33}$$

where we explicitly indicate the matrix indices on the group elements. It is useful to introduce a generating function for these integrals

$$W(J,K) = \int \mathrm{d}g \exp(\mathrm{Tr}\,(Jg+Kg^{-1})). \tag{8.34}$$

Here J and K are arbitrary n-by-n matrices. To obtain the integral in eq. (8.33), we take derivatives of this generating function

$$I = \left(\frac{\partial}{\partial J_{j_1 i_1}} \cdots \frac{\partial}{\partial K_{l_m k_m}}\right) W(J,K)|_{J=K=0}. \tag{8.35}$$

Invariance of the group measure gives W the symmetry properties

$$W(J,K) = W(K,J) = W(g_0^{-1} J g_1, g_1^{-1} K g_0), \tag{8.36}$$

where g_0 and g_1 are arbitrary $SU(n)$ matrices.

The generating function satisfies an interesting system of differential equations. Since $gg^{-1} = 1$, we have

$$(\partial/\partial K_{ik})(\partial/\partial J_{kj})\, W(J,K) = \delta_{ij}. \tag{8.37}$$

And since the determinant of an $SU(n)$ matrix is unity, we have

$$\det(\partial/\partial J)\, W(J,K) = 1. \tag{8.38}$$

Along with the initial condition

$$W(0,0) = 1, \tag{8.39}$$

these differential equations are sufficient to determine W. Several authors have studied these equations in the large n limit (Brower and Nauenberg, 1980; Bars, 1981). We will solve them iteratively in powers of J and K and give a graphical algorithm for evaluating the coefficients in this expansion.

We first eliminate the K dependence in W using the expression for g^{-1} in terms of the cofactors of g

$$\begin{aligned}(g^{-1})_{ij} &= (\operatorname{cof}(g))_{ij} \\ &= (1/(n-1)!)\epsilon_{j,i_1\dots,i_{n-1}}\epsilon_{i,j_1\dots,j_{n-1}}g_{i_1j_1}\cdots g_{i_{n-1}j_{n-1}},\end{aligned} \tag{8.40}$$

where ϵ denotes the totally antisymmetric tensor with $\epsilon_{1\dots n} = 1$. This allows us to solve eq. (8.37), replacing derivatives with respect to K by derivatives with respect to J

$$W(J,K) = \exp(\operatorname{Tr}(K\operatorname{cof}(\partial/\partial J)))\, W(J), \tag{8.41}$$

where
$$W(J) = W(J, K=0) = \int \mathrm{d}g \exp(\operatorname{Tr}(Jg)). \tag{8.42}$$

To evaluate $W(J)$ we use the invariance of eq. (8.36), which now reads

$$W(J) = W(g_0^{-1}Jg_1). \tag{8.43}$$

In an appendix of Creutz (1978*a*) it is proven that any analytic function of J satisfying this symmetry property is a function only of the determinant of J. Thus we expand

$$W(J) = \sum_{i=0}^{\infty} a_i (\det J)^i. \tag{8.44}$$

Normalization of the integration measure implies

$$a_0 = 1. \tag{8.45}$$

A recursion relation determining further a_i follows from the second differential equation, eq. (8.38). A tedious combinatoric exercise (Creutz, 1978*b*) shows

$$(\det(\partial/\partial J))(\det J)^i = \frac{(i+n-1)!}{(i-1)!}(\det J)^{i-1}. \tag{8.46}$$

From eqs (8.38), (8.44) and (8.46) we find

$$a_i = \frac{(i-1)!}{(i+n-1)!}a_{i-1}. \tag{8.47}$$

With eq. (8.45), this is solved to give

$$a_i = \frac{2!3!\dots(n-1)!}{i!(i+1)!\dots(i+n-1)!}. \tag{8.48}$$

Our final power series expression for $W(J)$ is

$$W(J) = \sum_{i=0}^{\infty} \frac{2!\dots(n-1)!}{i!\dots(i+n-1)!}(\det J)^i. \tag{8.49}$$

Note that the determinant of a matrix is simply expressed in terms of the antisymmetric tensor ϵ

$$\det J = (1/n!)\epsilon_{i_1\dots i_n}\epsilon_{j_1\dots j_n}J_{i_1j_1}\cdots J_{i_nj_n}. \tag{8.50}$$

A graphical notation is useful for carrying out the derivatives in eq.

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(8.35). We use directed line segments to denote group elements. In figure 8.1 we illustrate the convention of upward directed lines representing factors of g while downward lines represent g^{-1}. The ends of these line segments carry as labels the matrix indices of the respective elements. The line direction runs from the first to the second index, as shown in the figure. In figure 8.2 we show how the generic integral from eq. (8.33) appears in this notation.

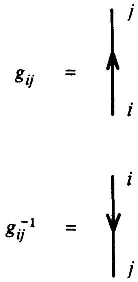

Fig. 8.1. Graphical representation of g and g^{-1} (Creutz, 1978*b*).

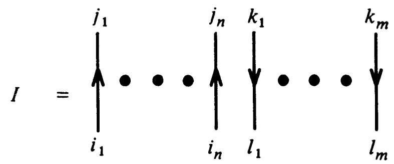

Fig. 8.2. The generic integral under consideration (Creutz, 1978*b*).

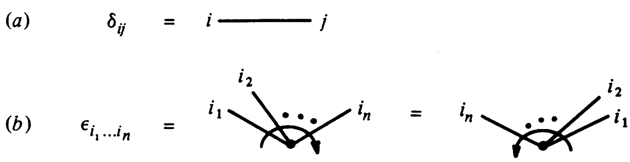

Fig. 8.3. Representation of (*a*) the Kronecker symbol and (*b*) the antisymmetric tensor (Creutz, 1978*b*).

We represent the Kronecker delta symbol δ_{ij} with an undirected line connecting the indices i and j, as shown in figure 8.3*a*. The antisymmetric epsilon symbol $\epsilon_{i_1 \ldots i_n}$ appears as a vertex joining n lines from the indices i_1 to i_n. As the order of these lines is important, we attach to the vertex an arrow running from the first to the last index, as shown in figure 8.3*b*. Finally, whenever two line segments are connected, a matrix product is understood; i.e., the indices associated with the connected ends are summed over.

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In the evaluation of group integrals, products of ϵ symbols often occur. Some useful identities involving such products are:

$$\epsilon_{i_1 \dots i_n} \epsilon_{i_1 \dots i_n} = n!, \tag{8.51}$$

$$\epsilon_{i, i_1 \dots i_{n-1}} \epsilon_{j, i_1 \dots i_{n-1}} = (n-1)!\, \delta_{ij}, \tag{8.52}$$

$$\epsilon_{i, j, i_1 \dots i_{n-2}} \epsilon_{k, l, i_1 \dots i_{n-2}} = (n-2)!\, (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}). \tag{8.53}$$

In our graphical notation these relations appear in figure 8.4.

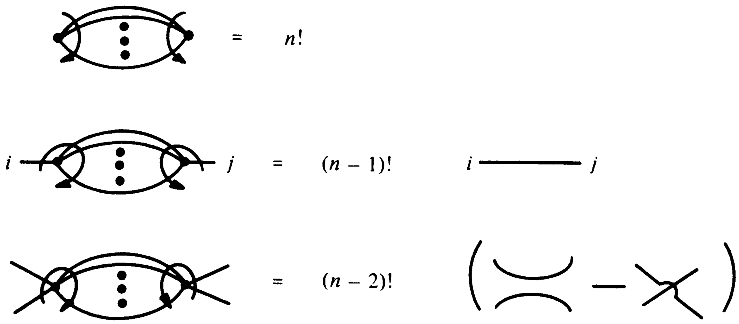

Fig. 8.4. Some combinatoric identities (Creutz, 1978*b*).

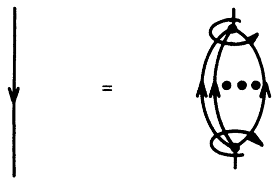

Fig. 8.5. Replacing g^{-1} with the cofactors of g (Creutz, 1978*b*).

Evaluation of a group integral consists of a replacement of the directed lines in figure 8.2 with vertices and undirected lines, thus expressing the result in terms of antisymmetric ϵ and Kronecker δ symbols. The first step in this procedure is to convert all directed lines into a set of lines directed only upward. This is accomplished using eq. (8.40), which is shown graphically in figure 8.5. If there were initially more downward than upward lines, it would be simplest to first use eq. (8.20), which says that the arrows on all lines can be simultaneously reversed. Once all lines have the same orientation, we use eqs (8.49) and (8.50) to reduce the integral to a sum of terms involving antisymmetric tensors. Note that the integral automatically vanishes unless the number of group lines is a multiple of n. Supposing we have np lines, where p is an integer, we display eq. (8.49) graphically in figure 8.6. The indicated sum over permutations is over all topologically distinct ways of connecting the group indices to pairs of vertices. The factor in the figure already includes permutations of indices coupled to the same vertex pair and permutations of the vertex pairs. The resulting sum has $(np)!/(p!(n!)^p)$ terms.

Certain identities on the group elements have a simple graphical representation. For example, invariance of the Kronecker symbol

$$g_{ij}\delta_{jk}(g^{-1})_{kl} = \delta_{il}, \tag{8.54}$$

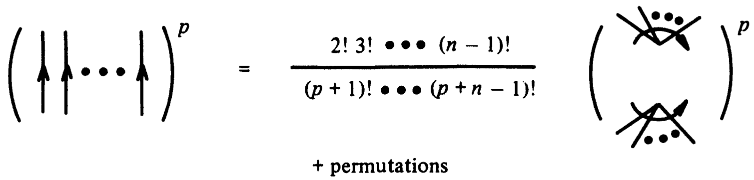

Fig. 8.6. Evaluation of the integral (Creutz, 1978*b*).

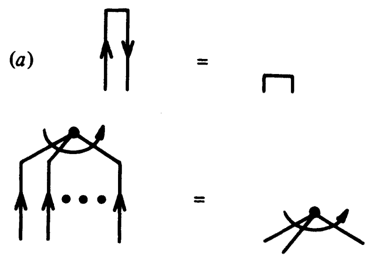

Fig. 8.7. Invariance of the (*a*) Kronecker symbol and (*b*) antisymmetric tensor (Creutz, 1978*b*).

is shown in figure 8.7*a*. In terms of the sources J and K, this figure corresponds to eq. (8.37). Invariance of the antisymmetric symbol

$$g_{i_1 j_1} \cdots g_{i_n j_n} \epsilon_{j_1 \ldots j_n} = \epsilon_{i_1 \ldots i_n}, \tag{8.55}$$

is shown in figure 8.7*b*. Contracting the indices with an additional ϵ symbol gives the graphical representation of eq. (8.38). Both the identities represented in figure 8.7 are valid regardless of any other lines present in the diagram.

We conclude this chapter with some simple examples to illustrate these rules. First consider $p = 1$ in figure 8.6. This immediately gives

$$\int \mathrm{d}g\, g_{i_1 j_1} \cdots g_{i_n j_n} = (1/n!)\, \epsilon_{i_1 \ldots i_n} \epsilon_{j_1 \ldots j_n}. \tag{8.56}$$

In low-order strong coupling expansions a useful integral is

$$I_{ijkl} = \int \mathrm{d}g\, g_{ij}\, (g^{-1})_{kl}. \tag{8.57}$$

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This is evaluated graphically in figure 8.8 Here we first use figure 8.5 to direct all lines upwards, then we use figure 8.6 to eliminate these lines, and finally we use the identity from figure 8.4 to obtain the result

$$I_{ijkl} = (1/n)\,\delta_{jk}\,\delta_{il}. \tag{8.58}$$

As a final example consider

$$I = \int \mathrm{d}g\, g_{ij}\,(g^{-1})_{kl}\, g_{mn}(g^{-1})_{pq}. \tag{8.59}$$

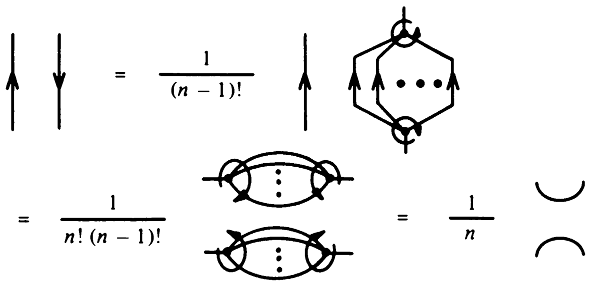

Fig. 8.8. Evaluation of the integral $\int \mathrm{d}g\, g_{ij}\, g^{-1}_{kl}$ (Creutz, 1978*b*).

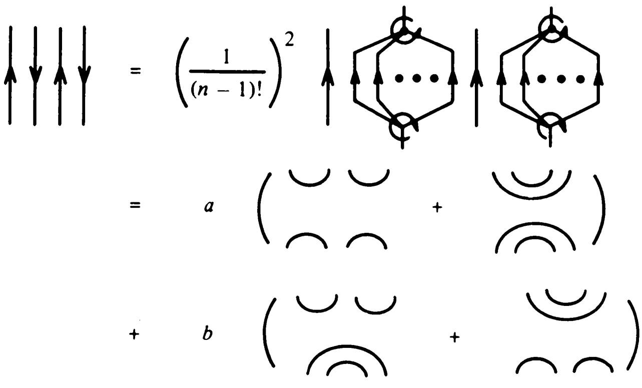

Fig. 8.9. The integral $\int \mathrm{d}g\, g_{ij}\, g^{-1}_{kl}\, g_{mn}\, g^{-1}_{pq}$ (Creutz, 1978*b*).

In figure 8.9 we use figure 8.5 to express I in terms of $2n$ upward lines. Use of figure 8.6 at this point would give an expression with $(2n)!/(2!n!^2)$ terms. Some simple tricks allow us to simplify this expression for general n. All terms in this sum have four, an even number, of ϵ vertices both at the top and at the bottom of the diagram. These can all be eliminated using identities similar to those in figure 8.4. Thus the result must finally appear in terms of sets of Kronecker δ symbols connecting separately indices at the top and bottom of the diagram. Furthermore, note that a Kronecker

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δ cannot connect the indices i and m or j and n because they can be initially coupled only through an odd number of ϵ vertices. Thus the final expression for the integral must take the form

$$I = a(\delta_{il}\,\delta_{mq}\,\delta_{jk}\,\delta_{np}+\delta_{iq}\,\delta_{ml}\,\delta_{jp}\,\delta_{nk}) + b(\delta_{il}\,\delta_{mq}\,\delta_{jp}\,\delta_{nk}+\delta_{iq}\,\delta_{ml}\,\delta_{jk}\,\delta_{np}), \quad (8.60)$$

where only two independent coefficients are needed because of the $kl \leftrightarrow pq$ symmetry of the integrand. The coefficients a and b can now be determined by multiplying by δ_{jk} and using figure 8.7a to reduce the integral to that

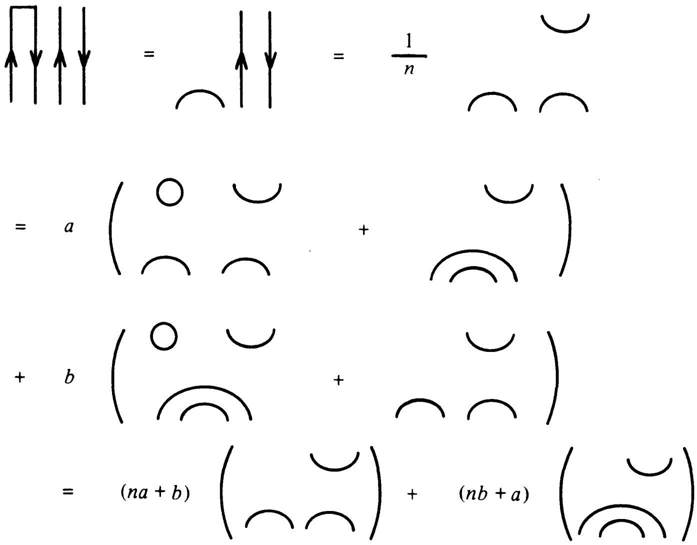

Fig. 8.10. Evaluation of the coefficients a and b. The closed circles represent $\Sigma_i\,\delta_{ii} = n$ (Creutz, 1978b).

already evaluated in figure 8.8. This sequence of steps appears in figure 8.10 and leads to the conclusion

$$a = 1/(n^2-1),$$
$$b = -1/(n(n^2-1)). \quad (8.61)$$

Inserted into eq. (8.60), this gives the desired integral.

Problems

1. Show that for 2-by-2 matrices $\det(A) = \frac{1}{2}((\mathrm{Tr}\,A)^2 - \mathrm{Tr}\,(A^2))$. What is the corresponding formula for 3-by-3 matrices?

2. For $SU(n)$ evaluate $\int \mathrm{d}g\,\mathrm{Tr}\,(g^n)$.

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3. Show that for irreducible representations R and R'

$$\int \mathrm{d}g\, \chi_R^*(g)\, \chi_{R'}(g_1 g) = d_R^{-1} \delta_{RR'}\, \chi_R(g_1),$$

where d_R is the dimension of the matrices in the representation.

4. Prove eq. (8.23).