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THE SATURATION OF A PRODUCT OF IDEALS

STANLEY WAGON

In this note we discuss how the saturation of $I \times J$, where I, J are κ-complete ideals on a regular uncountable cardinal κ, depends on the saturation of I and J. We show that if $2^\kappa = \kappa^+$ then the saturation of $I \times J$ is completely determined by the saturation of I and J. A consequence of a negative saturation result is that $NS_\kappa \times NS_\kappa$ is not κ^+-saturated, where NS_κ is the non-stationary ideal on κ (even though it is still open whether NS_κ can be κ^+-saturated). We also discuss the preservation of precipitousness under certain products, obtaining a simple example of an ideal on κ that is precipitous but not κ^+-saturated.

1. Preliminaries. Let κ denote a regular uncountable cardinal. By an *ideal on* κ we mean a κ-complete, non-principal, proper ideal on κ, i.e., a collection $I \subseteq \mathfrak{P}(\kappa)$ such that $\{\alpha\} \in I$ for $\alpha < \kappa$, $\kappa \notin I$, if $A \subseteq B \in I$ then $A \in I$, and if $\beta < \kappa$ and $A_\alpha \in I$ for $\alpha < \beta$ then $\bigcup\{A_\alpha : \alpha < \beta\} \in I$. I^* denotes $\{\kappa - A : A \in I\}$ and I^+ denotes $\{A \subseteq \kappa : A \notin I\}$.

If $A \subseteq \kappa$ then $[A] = \{B \subseteq \kappa$: the symmetric difference of A and B is in $I\}$; $\mathfrak{P}(\kappa)/I$ denotes $\{[A] : A \subseteq \kappa\}$. A collection $\mathfrak{A}$ of sets in I^+ is called an *almost disjoint family for* I (adf for I) if $A \cap B \in I$ whenever $A, B \in \mathfrak{A}$ and $A \neq B$. An ideal I is called λ-*saturated*, λ a cardinal, if whenever $\mathfrak{A}$ is an adf for I then $|\mathfrak{A}| < \lambda$. We let sat I denote the least λ such that I is λ-saturated.

Note that sat $I \leqq (2^\kappa)^+$ if I is an ideal on κ. The saturation of an ideal was first defined by Tarski [**6**], who proved that sat I is always a regular cardinal. The saturation of an ideal I provides a measure of how close I^* is to being an ultrafilter.

If I, J are ideals on κ, we may define the product ideal $I \times J$ on $\kappa \times \kappa$ by setting $A \in I \times J$ if $\{\alpha < \kappa : A_{(\alpha)} \in J^+\} \in I$ where $A_{(\alpha)} = \{\beta < \kappa : (\alpha, \beta) \in A\}$. It is well-known that if I, J are 2-saturated i.e., *prime* (an ideal I is prime if and only if I^* is a measure ultrafilter) then so is $I \times J$. Theorem 1 generalizes this by showing that sat $I \times J$ = max {sat I, sat J} whenever sat $I \leqq \kappa^+$ and sat $J < \kappa$. In the other direction, Theorem 2 shows that sat $I \times J > \kappa^+$ whenever sat $J > \kappa$. These results are motivated, in part, by a paper of Kakuda [**3**], where the preservation of saturation under certain forcing extensions is studied.

2. Products which are saturated. In this section we describe a situation where the saturation of a product is as small as it can be. The following theorem

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is well-known when sat $I < \kappa$, in which case it follows easily from a result of Silver (see [**4**, Theorem 1.45]).

THEOREM 1. *If I, J are ideals on κ,* sat $I = \mu \leqq \kappa^+$, *and* sat $J = \lambda < \kappa$, *then* sat $I \times J = \nu = \max\{\mu, \lambda\}$.

Proof. It is easy to see that sat $I \times J \geqq \nu$. For the reverse inequality we shall use the method of generic ultrapowers introduced by Solovay [**5**] (see also [**2**]). If G is $\mathfrak{P}(\kappa)/I$-generic over the universe V then, in $V[G]$, we may use G to form an ultrapower, Ult (V, G), of V which we denote by V'. (Strictly speaking, such G does not exist. However, for ease of exposition, we use this approach to generic sets rather than considering a Boolean-valued universe, which would be more precise.) V' consists of Scott equivalence classes (induced by G) of functions f in V such that $f:\kappa \rightarrow V$; let $[f]$ denote the equivalence class of f. That V' is well-founded follows from the κ^+-saturation of I. As in the usual ultrapower construction, there is, in $V[G]$, an elementary embedding $i: V \rightarrow V'$ such that κ is the first ordinal moved by i.

Now, suppose, to get a contradiction, that $\{A_\alpha : \alpha < \nu\}$ is an adf for $I \times J$. Define $f_\alpha : \kappa \rightarrow \mathfrak{P}(\kappa)$, for $\alpha < \nu$, by $f_\alpha(\gamma) = A_{\alpha(\gamma)}$. Then $\| \!\! -_P [f_\alpha] \subseteq i(\kappa)$, where $P = \mathfrak{P}(\kappa)/I$. We claim that $\| \!\! -_P V' \models \exists \beta < \nu \forall \alpha > \beta\ [f_\alpha] \in i(J)$. For if not then, since ν is regular, there is some p in P, $p \neq 0$, such that

$$p \,\|\, \quad V' \models \text{there is a } \nu\text{-sized adf for } i(J).$$

Since $\lambda \leqq \nu$, this contradicts the fact that $\| \!\! -_P V' \models i(J)$ is $i(\lambda) = \lambda$-saturated.

Now, use the claim to choose $\{Y_\delta : \delta < \mu'\}$, $\{\beta_\delta : \delta < \mu'\}$ such that $\{Y_\delta : \delta < \mu'\}$ is a maximal adf for I and $[Y_\delta] \| \!\! - \beta_\delta$ is the least $\beta < \nu$ such that $[f_\alpha] \in i(J)$ for all $\beta \leqq \alpha < \nu$. Then $\mu' < \mu$ and so, since ν is regular and $\nu \geqq \mu$, $\beta = \sup\{\beta_\delta : \delta < \mu'\} < \nu$. But then $\| \!\! -_P [f_\beta] \in i(J)$, which implies (by the fundamental theorem on ultrapowers in this context) that $\{\gamma < \kappa : f_\beta(\gamma) \in J\} \in I^*$. Since this means that $\{\gamma < \kappa : A_{\beta(\gamma)} \in J^+\} \in I$, we have that $A_\beta \in I \times J$, a contradiction.

In the case sat $I = \kappa$, the following version of Silver's lemma referred to earlier is true (proved independently by A. Taylor and the referee), and this gives a simpler proof of the theorem above. Namely, if I is a κ-saturated ideal on κ, $\lambda < \kappa$, and $\{A_\alpha : \alpha < \kappa\} \subseteq I^+$ then there is some $Y \subseteq \kappa$ with $|Y| = \lambda$ and $\bigcap\{A_\alpha : \alpha \in Y\} \neq 0$. It is not as clear how to proceed in case sat $I = \kappa^+$, but one can obtain a combinatorial proof based on the above metamathematical proof.

It is easy to see that if J is prime then sat $I \times J =$ sat I, with no restriction on I. However, it will follow from Theorem 2 that it is not necessarily true that sat $I \times J =$ sat J when I is prime.

3. Products which are not saturated. We now show that if J is mildly unsaturated then $I \times J$ is badly unsaturated. We need the following lemma, most of which is well-known.

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LEMMA. (a) *If κ is regular there are $g_\alpha:\kappa \to \kappa$ for $\alpha < \kappa^+$ such that if $\alpha < \beta < \kappa^+$ then $|\{\xi < \kappa : g_\alpha(\xi) = g_\beta(\xi)\}| < \kappa$.*

(b) *If κ is a regular limit (i.e., weakly inaccessible) cardinal then there are g_α as in* (a) *such that, in addition, $g_\alpha(\xi) < |\xi|^+$.*

(c) *If κ is strongly inaccessible there are $g_\alpha:\kappa \to \kappa$ for $\alpha < 2^\kappa$ such that if $\alpha < \beta < \kappa^+$ then $|\{\xi < \kappa : g_\alpha(\xi) = g_\beta(\xi)\}| < \kappa$ and $g_\alpha(\xi) < 2^{|\xi|}$.*

Proof. We first prove (b); the proof of (a) is similar. Define g_α by induction on α, letting g_0 be identically 0. Suppose g_α has been defined for $\alpha < \beta$, and $g_\alpha(\xi) < |\xi|^+$. Let $h:\beta \to \kappa$ be one-one and, for $\xi < \kappa$, let $g_\beta(\xi)$ be such that $g_\beta(\xi) < |\xi|^+$ and $g_\beta(\xi) > g_\alpha(\xi)$ for each α such that $h(\alpha) < \xi$. This is possible since $\{g_\alpha(\xi) : h(\alpha) < \xi\}$ has size at most $|\xi|$ and so is not cofinal in $|\xi|^+$. It is easy to see that this construction produces the sequence of functions as required.

The proof of (c) is essentially the standard proof that, for a strongly inaccessible κ, there are 2^κ κ-sized subsets of κ with pairwise intersections having size less than κ. Let T be the full binary tree with κ many levels. Since κ is strongly inaccessible this tree has κ nodes and, in fact, one can label these nodes with ordinals less than κ so that if a node has level γ, then its label is less than $2^{|\gamma|}$. Now, each of the 2^κ paths through T of length κ induces a function from κ to κ and the collection of such functions satisfies the conditions of the lemma.

THEOREM 2. *If I, J are ideals on κ such that* sat $J \geqq \kappa$ *then* sat $I \times J > \kappa^+$. *If, in addition, κ is strongly inaccessible, then* sat $I \times J = (2^\kappa)^+$.

Proof. First, suppose κ is a successor cardinal. Then, by a theorem of Ulam [**8**], sat $J > \kappa$ and so there is $\{B(\gamma):\gamma < \kappa\}$ which is an adf for J. Now, for $\alpha < \kappa^+$ let

$$A_\alpha = \{(\xi, \delta) : \delta \in B(g_\alpha(\xi))\},$$

where the g_α are as in (a) of the lemma. Then $\{A_\alpha : \alpha < \kappa^+\}$ is an adf for $I \times J$. If κ is a limit cardinal then let $\{g_\alpha : \alpha < \kappa^+\}$ be as in (b) of the lemma. For each $\beta < \kappa$, let $\{B(\beta, \delta) : \delta < |\beta|^+\}$ be an adf for J; such exists because J is not $|\beta|^+$-saturated. Define A_α for $\alpha < \kappa^+$ by

$$A_\alpha = \{(\xi, \delta) : \delta \in B(\xi, g_\alpha(\xi))\}.$$

Then $\{A_\alpha : \alpha < \kappa^+\}$ is an adf for $I \times J$. The strongly inaccessible case is similar to the previous case, using the functions $\{g_\alpha : \alpha < 2^\kappa\}$ of part (c) of the lemma.

It is easy to modify this proof slightly and obtain that if J is nowhere λ-saturated for any $\lambda < \kappa$ then $I \times J$ is nowhere κ^+-saturated. (I is nowhere λ-saturated if sat $I|A > \lambda$ for each $A \in I^*$, where $I|A = \{X \subseteq \kappa : X \cap A \in I\}$.) If NS_κ denotes the ideal of nonstationary subsets of κ then NS_κ is nowhere κ-saturated (Solovay [**5**]) but it is not known whether NS_κ can be κ^+-saturated (although recent results of van Wesep show, under some very strong assump-

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tions, that NS_{ω_1} can be ω_2-saturated). Theorem 2 yields some information about the saturation of $NS_\kappa \times NS_\kappa$.

COROLLARY. *$NS_\kappa \times NS_\kappa$ is nowhere κ^+-saturated.*

If $2^\kappa = \kappa^+$ then Theorems 1 and 2 handle all possible cases and the saturation of $I \times J$ is completely determined by the saturation of I and J.

COROLLARY. *If $2^\kappa = \kappa^+$ then* $\text{sat } I \times J = \max \{\text{sat } I, \text{sat } J\}$ *unless* $\text{sat } J \geqq \kappa$ *in which case* $\text{sat } I \times J = \kappa^{++}$.

Proof. If $\text{sat } J \geqq \kappa$ then Theorem 2 implies that $\text{sat } I \times J = \kappa^{++}$. If $\text{sat } J < \kappa$ and $\text{sat } I \leqq \kappa^+$ then Theorem 1 implies that $\text{sat } I \times J$ equals $\max \{\text{sat } I, \text{sat } J\}$. Lastly, if $\text{sat } I > \kappa^+$ then $\text{sat } I = \kappa^{++}$ so $\text{sat } I \times J = \kappa^{++}$.

Suppose I is a prime ideal on a measurable cardinal and J is defined as follows. Choose $\{A_\alpha : \alpha < \kappa\}$, a partition of κ into sets of size κ, and $\{f_\alpha : \alpha < \kappa\}$ such that $f_\alpha : \kappa \to A_\alpha$ is a bijection, and then let $X \in J$ if $f_\alpha^{-1}(X) \in I$ for each $\alpha < \kappa$. Then $\text{sat } J = \kappa^+$ and so, by Theorem 2, $\text{sat } I \times J > \kappa^+$. Thus a product with a prime left factor need not preserve saturation, while a product with a prime right factor does (see remark at end of Section 2). Note that, by Theorem 3 below, it follows that this ideal, $I \times J$, is a precipitous ideal on κ which is not κ^+-saturated.

4. Products which are precipitous. In the proof of Theorem 1 we made use of the fact that if I is a κ^+-saturated ideal on κ then, for any $\mathfrak{P}(\kappa)/I$-generic set G, the ultrapower Ult (V, G) is well-founded. An ideal satisfying this latter property is called *precipitous;* this is a weaker condition than being a κ^+-saturated ideal on κ (see [**1**]). In this section we show that in some ways this notion is more well-behaved under products than saturation is. The following theorem should be compared with the result of the previous section which showed that saturation need not be preserved under formation of a product with a prime ideal.

THEOREM 3. *If I is a prime ideal on the measurable cardinal κ and J is a precipitous ideal on κ then $I \times J$ and $J \times I$ are both precipitous.*

Proof. We first consider $I \times J$. Suppose G is $\mathfrak{P}(\kappa \times \kappa)/I \times J$-generic over V. Let $V' = \text{Ult}(V, I)$, the standard ultrapower with respect to a measure ultrafilter (prime ideal), and let $i : V \to V'$ be the canonical elementary embedding. Then, in V', $i(J)$ is a precipitous ideal on $i(\kappa)$. We shall show that Ult (V, G) is well-founded by defining a set G' which is $\mathfrak{P}(i\kappa) \cap V'/i(J)$-generic over V' such that Ult $(V, G) \cong$ Ult (V', G'); since $i(J)$ is precipitous, this suffices.

Define G' as follows. If $[f]_I \in \mathfrak{P}(i\kappa) \cap V'$, put $[[f]_I]_{iJ}$ in G' if and only if $[A]_{I \times J} \in G$ where $A = \{(\alpha, \beta) : \beta \in f(\alpha)\}$. It is easy to check that G' is well-defined. To prove that G' is appropriately generic it suffices to show that if, in V', $\{[g_\alpha]_I : \alpha < \theta\}$ is a maximal adf for $i(J)$, then for some α, $[[g_\alpha]] \in G'$. Let A_α be defined from g_α as in the definition of G'. Then $\{A_\alpha : \alpha < \theta\}$ is in V and is an adf for $I \times J$. In fact, it is a maximal adf for suppose $A \in (I \times J)^+$. Define $h : \kappa \to \mathfrak{P}(\kappa)$ by $h(\gamma) = A_{(\gamma)}$. Then $[h]_I \in (iJ)^+$ since $\{\gamma < \kappa : A_{(\gamma)} \in J^+\} \in I^+ = I^*$. So for some $\alpha < \theta$, $[h]_I \cap [g_\alpha]_I \in (iJ)^+$ which implies that $A \cap A_\alpha \in (I \times J)^+$. So, since G is generic, some $[A_\alpha]_{I \times J} \in G$ and so some $[[g_\alpha]] \in G'$.

$$\begin{array}{ccc} V[G] & \supseteq & V'[G'] \\ \cup\!\!\mid & & \cup\!\!\mid \\ \mathrm{Ult}\,(V, G) & \xrightarrow{\ \Psi\ } & \mathrm{Ult}\,(V', G') \\ \uparrow & & \uparrow \\ V & \xrightarrow{\ i\ } & V' \end{array}$$

Form Ult (V', G') and define $\Psi : \mathrm{Ult}\,(V, G) \to \mathrm{Ult}\,(V', G')$ by letting $\Psi([f]_G) = [[h]_I]_{G'}$ where $h : \kappa \to V^\kappa \cap V$ is in V and is defined by setting $h(\gamma)(\delta) = f(\gamma, \delta)$. To see that Ψ is well-defined, suppose that

$$[\{(\gamma, \delta) : f_1(\gamma, \delta) = f_2(\gamma, \delta)\}] \in G.$$

Then $[[H]_I]_{iJ} \in G'$ where $H(\gamma) = \{\delta < \kappa : h_1(\gamma)(\delta) = h_2(\gamma)(\delta)\}$. But $[H]_I = \{\zeta < i(\kappa) : [h_1]_I(\zeta) = [h_2]_I(\zeta)\}$ and so $[[h_1]_I]_{G'} = [[h_2]_I]_{G'}$. This same proof, with $=$ replaced by $\neq$ or $\in$, shows that Ψ is one-one and preserves $\in$. While not strictly necessary for the present theorem, it is worth noting that Ψ is onto, and hence an isomorphism. For if $[[h_I]]_{G'} \in \mathrm{Ult}\,(V', G')$ then, in V', $[h]_I : i(\kappa) \to V'$, and so, for each $\gamma < \kappa$, $h(\gamma) : \kappa \to V$. Let $f(\gamma, \delta) = h(\gamma)(\delta)$. Then $\Psi([f]_G) = [[h]_I]_G$.

The proof that $J \times I$ is precipitous is similar. Suppose G is $\mathfrak{P}(\kappa \times \kappa)/J \times I$-generic over V. We shall show that Ult (V, G) is well-founded by defining a set G' which is $\mathfrak{P}(\kappa)/J$-generic over V such that

$$\mathrm{Ult}\,(V, G) \cong \mathrm{Ult}\,(\mathrm{Ult}\,(V, G'), i(I))$$

where $i : V \to \mathrm{Ult}\,(V, G')$. Since $i(I)$ is a prime ideal on $i(\kappa)$ in the well-founded model Ult (V, G'), this shows that Ult (V, G) is well-founded.

Define G' by setting $[A]_J \in G'$ if and only if $[A \times \kappa]_{J \times I} \in G$. It is easy to check that G' is well-defined and $\mathfrak{P}(\kappa)/J$-generic over V. Thus we may form Ult (V, G') and then Ult (Ult (V, G'), $i(I)$).

Suppose $[f]_G \in \mathrm{Ult}\,(V, G)$. Let $\Psi([f]_G) = [[h]_{G'}]_{i(I)}$ where $h : \kappa \to V^\kappa \cap V$ is defined by letting $h(\gamma)(\delta) = f(\gamma, \delta)$. Then, using the fact that I is prime, one may check that Ψ is well-defined and an isomorphism. This concludes the proof.

The converse to this theorem is valid too (this was pointed out by A. Taylor), in the sense that if $I \times J$ is precipitous then so are I and J.

Mitchell ([**1**]) has shown that if I is a prime ideal on a measurable cardinal κ, P is the Lévy collapse of κ to ω_1, and G is P-generic over V, then, in $V[G]$, $\kappa = \omega_1$ and $\bar{I}$ is a precipitous ideal on ω_1, where $\bar{I}$ is the ideal on κ in $V[G]$ generated by I, i.e., $x \in \bar{I}$ if and only if $x \subseteq y$ for some $y \in I$. This result can be used to point out another difference between precipitous and saturated ideals. By Theorem 2 and the fact that ω_1 bears no ω_1-saturated ideal, if I is an ideal on ω_1 then $I \times I$ is not ω_2-saturated. However, a product can be precipitous. For suppose I, P, G, $\bar{I}$ are as in Mitchell's result. Then $I \times I$ is a prime ideal on κ in V and so, in $V[G]$, $\overline{I \times I}$ is a precipitous ideal on ω_1. It is not difficult to see that $\overline{I \times I} = \bar{I} \times \bar{I}$ (see [**9**, p. 79]).

A. Taylor [**7**] has proved that a κ^+-saturated ideal on a successor cardinal κ is a P-point. The result of the previous paragraph shows that this theorem cannot be improved to hold for precipitous ideals because $\overline{I \times I}$ is precipitous and, since it is a product, it fails to be a P-point.

Remark. Mitchell's result that precipitousness is preserved by a Lévy collapse has been improved recently by Kakuda, who showed that the Lévy collapse could be replaced by any partial ordering with the κ-chain condition.

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Smith College,
Northampton, Massachussets

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