Proceedings of the Edinburgh Mathematical Society (2012) **55**, 731–769
DOI:10.1017/S0013091512000053

MINIMAL SPECTRAL FUNCTIONS OF AN ORDINARY DIFFERENTIAL OPERATOR

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(Received 7 October 2010)

Abstract Let $l[y]$ be a formally self-adjoint differential expression of an even order on the interval $[0,b\rangle (b \leqslant \infty)$ and let L_0 be the corresponding minimal operator. By using the concept of a decomposing boundary triplet, we consider the boundary problem formed by the equation $l[y] - \lambda y = f$, $f \in L_2[0,b\rangle$, and the Nevanlinna λ-dependent boundary conditions with constant values at the regular endpoint 0. For such a problem we introduce the concept of the m-function, which in the case of self-adjoint separated boundary conditions coincides with the classical characteristic (Titchmarsh–Weyl) function. Our method allows one to describe all minimal spectral functions of the boundary problem, i.e. all spectral functions of the minimally possible dimension. We also improve (in the case of intermediate deficiency indices $n_\pm(L_0)$ and non-separated boundary conditions) the known estimate of the spectral multiplicity of the (exit space) self-adjoint extension $\tilde{A} \supset L_0$. Results are obtained for expressions $l[y]$ with operator-valued coefficients and arbitrary (equal or unequal) deficiency indices $n_\pm(L_0)$.

Keywords: differential operator; decomposing D-boundary triplet; boundary conditions; minimal spectral function; spectral multiplicity

2010 *Mathematics subject classification:* Primary 34B05; 34B20; 34B40
Secondary 47E05

1. Introduction

The paper deals with differential operators generated by a formally self-adjoint differential expression $l[y]$ of an even order $2n$ on an interval $\Delta = [0,b\rangle$, $b \leqslant \infty$, with a regular endpoint 0 and either regular or singular endpoint b. We consider the expression $l[y]$ with operator-valued coefficients and arbitrary (possibly unequal) deficiency indices, but in order to simplify presentation of the main results let us assume that

$$l[y] = \sum_{k=1}^{n} (-1)^k (p_{n-k} y^{(k)})^{(k)} + p_n y \tag{1.1}$$

is a scalar expression with real-valued coefficients $p_k(t)$, $t \in \Delta$, such that $p_k \in L_1(0,\beta)$ for each $\beta \in (0,b)$ [**30**]. Denote by L_0 and L $(= L_0^*)$ minimal and maximal operators, respectively, generated by the expression (1.1) in the Hilbert space $\mathfrak{H} := L_2(\Delta)$ and let $\mathcal{D}$ be the domain of L. As is known, L_0 is a symmetric operator with equal deficiency indices $m = n_\pm(L_0)$ and $n \leqslant m \leqslant 2n$.

We develop an approach based on the concept of a decomposing boundary triplet for a differential operator (see [**27**–**29**]). First recall the following definitions.

Definition 1.1 (Gorbachuk and Gorbachuk [14]). Let $\mathfrak{H}$ be a Hilbert space, let A be a densely defined symmetric operator in $\mathfrak{H}$ with equal deficiency indices and let $\mathcal{D}(A^*)$ be the domain of its adjoint A^*. A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 \colon \mathcal{D}(A^*) \to \mathcal{H}$ are linear maps, is called a boundary triplet for A^* if the map $\Gamma := (\Gamma_0, \Gamma_1)^{\mathrm{T}}$ is surjective and the following 'abstract Green's identity' holds:

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \quad f, g \in \mathcal{D}(A^*).$$

Definition 1.2 (Derkach and Malamud [4]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The operator-valued (Nevanlinna) function $M(\lambda)$ ($\in [\mathcal{H}]$) defined by

$$\Gamma_1 f_\lambda = M(\lambda) \Gamma_0 f_\lambda, \quad f_\lambda \in \mathfrak{N}_\lambda := \operatorname{Ker}(A^* - \lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$$

is called the Weyl function corresponding to Π.

By choosing a suitable boundary triplet for a concrete problem, one can parametrize various classes of extensions $\tilde{A} \supset A$ in the most convenient form. In particular, for the minimal operator L_0 generated by the expression (1.1) we suggest the use of the special boundary triplet called decomposing in [**27**]. According to [**27**] such a triplet is of the form $\Pi = \{\mathbb{C}^n \oplus \mathbb{C}^{n_b}, \Gamma_0, \Gamma_1\}$, where $n_b = m - n$ and $\Gamma_j \colon \mathcal{D} \to \mathbb{C}^n \oplus \mathbb{C}^{n_b}$, $j \in \{0, 1\}$, are the linear maps given by

$$\Gamma_0 y = \{y^{(2)}(0), \Gamma_0' y\} \ (\in \mathbb{C}^n \oplus \mathbb{C}^{n_b}), \qquad \Gamma_1 y = \{-y^{(1)}(0), \Gamma_1' y\} \ (\in \mathbb{C}^n \oplus \mathbb{C}^{n_b}). \tag{1.2}$$

Here $y^{(j)}(0)$ are vectors of quasi-derivatives (2.18) at the point 0 and $\Gamma_0' y, \Gamma_1' y (\in \mathbb{C}^{n_b})$ are boundary values of a function $y \in \mathcal{D}$ at the singular endpoint b, which satisfy the identity (2.24) (for more details see Definition 2.5).

Next assume that $\mathcal{P} = \{C_0(\lambda), C_1(\lambda)\} (\lambda \in \mathbb{C} \setminus \mathbb{R})$ is a pair of holomorphic operator functions $C_0(\lambda)$ and $C_1(\lambda)$ defined by the block representations

$$C_0(\lambda) = (\hat{C}_0, C_0'(\lambda)) \colon \mathbb{C}^n \oplus \mathbb{C}^{n_b} \to \mathbb{C}^m, \qquad C_1(\lambda) = (\hat{C}_1, C_1'(\lambda)) \colon \mathbb{C}^n \oplus \mathbb{C}^{n_b} \to \mathbb{C}^m \tag{1.3}$$

with the constant entries $\hat{C}_0$, $\hat{C}_1$. Moreover, assume that such a pair belongs to the Nevanlinna class (in short, the Nevanlinna pair), which means that the corresponding family of linear relations $\tau = \tau(\lambda) := \{\{h, h'\} \colon C_0(\lambda) h + C_1(\lambda) h' = 0\}$ is a Nevanlinna family (see, for example, [**7**]).

Denote by $\hat{\mathcal{K}}$ the range of the operator $\hat{C} = (\hat{C}_0, \hat{C}_1)$ and let

$$\hat{n} = \dim \hat{\mathcal{K}} = \operatorname{rank}(\hat{C}_0, \hat{C}_1), \qquad n' = m - \hat{n}.$$

Then $n \leqslant \hat{n} \leqslant m$ and the operator pair (1.3) admits the block representation

$$C_0(\lambda) = \begin{pmatrix} N_0 & C_{01}'(\lambda) \\ 0 & C_{02}'(\lambda) \end{pmatrix} : \mathbb{C}^n \oplus \mathbb{C}^{n_b} \to \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp, \tag{1.4}$$

$$C_1(\lambda) = \begin{pmatrix} N_1 & C_{11}'(\lambda) \\ 0 & C_{12}'(\lambda) \end{pmatrix} : \mathbb{C}^n \oplus \mathbb{C}^{n_b} \to \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^\perp, \tag{1.5}$$

where N_j are $(\hat{n}\times n)$-matrices with $\operatorname{rank}(N_0,N_1)=\hat{n}$, and $C'_{j1}(\lambda)$, $C'_{j2}(\lambda)$, $j\in\{0,1\}$, are, respectively, $(\hat{n}\times n_b)$- and $(n'\times n_b)$-matrix functions. By using the boundary operators (1.2), consider the boundary problem

$$l[y]-\lambda y=f \tag{1.6}$$

$$C_0(\lambda)\Gamma_0 y - C_1(\lambda)\Gamma_1 y = 0. \tag{$*$}$$

It follows from (1.4), (1.5) that the boundary condition $(*)$ can be written as two equalities

$$N_0 y^{(2)}(0)+N_1 y^{(1)}(0)+C'_{01}(\lambda)\Gamma'_0 y - C'_{11}(\lambda)\Gamma'_1 y=0, \tag{1.7}$$

$$C'_{02}(\lambda)\Gamma'_0 y - C'_{12}(\lambda)\Gamma'_1 y = 0, \tag{1.8}$$

which in fact define m linearly independent boundary conditions in the sense of [**8**].

The problem (1.6)–(1.8) is a particular case of a general Nevanlinna-type boundary problem and hence it generates a generalized resolvent $R(\lambda)=R_\tau(\lambda)$ and the corresponding spectral function $F(t)=F_\tau(t)$ of the operator L_0 [**29**]. Moreover, each self-adjoint boundary problem is given by the boundary condition $(*)$ with a constant-valued Nevanlinna pair $\mathcal{P}=\{C_0,C_1\}$, which implies that each canonical resolvent of the operator L_0 is generated by the boundary problem (1.6)–(1.8) with $C'_{j1}(\lambda)\equiv C'_{j1}$ and $C'_{j2}(\lambda)\equiv C'_{j2}$, $j\in\{0,1\}$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Observe also that the problem (1.6)–(1.8) contains as a particular case a decomposing boundary problem. Namely, if (and only if) the equality $\hat{n}=n$ is satisfied, then $C'_{01}(\lambda)=C'_{11}(\lambda)=0$ and the boundary conditions (1.7), (1.8) become decomposing.

Next assume that $M(\cdot)$ is the Weyl function of the decomposing boundary triplet (1.2) in the sense of Definition 1.2 and let

$$M(\lambda)=\begin{pmatrix} m(\lambda) & M_2(\lambda)\\ M_3(\lambda) & M_4(\lambda)\end{pmatrix}:\mathbb{C}^n\oplus\mathbb{C}^{n_b}\to\mathbb{C}^n\oplus\mathbb{C}^{n_b},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}, \tag{1.9}$$

be the block representation of $M(\lambda)$. Moreover, let $\Omega(\lambda)=\Omega_\tau(\lambda)$ be the Štraus characteristic matrix of the generalized resolvent $R(\lambda)=R_\tau(\lambda)$ [**35**] (for more details see (2.35)). Then, according to [**29**], $\Omega_\tau(\lambda)$ is defined immediately in terms of a Nevanlinna boundary parameter τ by the equalities

$$\tilde{\Omega}_\tau(\lambda)=\begin{pmatrix} M(\lambda)-M(\lambda)(\tau(\lambda)+M(\lambda))^{-1}M(\lambda) & -\tfrac{1}{2}I+M(\lambda)(\tau(\lambda)+M(\lambda))^{-1}\\ -\tfrac{1}{2}I+(\tau(\lambda)+M(\lambda))^{-1}M(\lambda) & -(\tau(\lambda)+M(\lambda))^{-1}\end{pmatrix}, \tag{1.10}$$

$$\Omega_\tau(\lambda)=P_{\mathbb{C}^n\oplus\mathbb{C}^n}\tilde{\Omega}_\tau(\lambda)\restriction\mathbb{C}^n\oplus\mathbb{C}^n,\quad \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

For a given operator pair (1.4), (1.5), consider also the operator function

$$\Omega_{\tau,W'}(\lambda)=(W')^{-1}\Omega_\tau(\lambda)(W')^{-1*},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}, \tag{1.11}$$

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where

$$W' = \begin{pmatrix} -N_0^* & * \\ N_1^* & * \end{pmatrix}$$

is an invertible $(2n \times 2n)$-matrix (the form of the entries $*$ does not matter). We shall show that the operator function (1.11) is of the form

$$\Omega_{\tau,W'}(\lambda) = \begin{pmatrix} m_{\mathcal{P}}(\lambda) & C^* \\ C & 0 \end{pmatrix} : \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^{\perp} \to \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^{\perp}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{1.12}$$

where C is a constant operator. The equality (1.12) generates the uniformly strict Nevanlinna operator function $m_{\mathcal{P}}(\cdot)$, which we call an m-function of the boundary problem (1.6)–(1.8). This function can also be defined explicitly in terms of the boundary conditions (1.7), (1.8) (see Theorem 3.11 (3)). Moreover, in the case of self-adjoint decomposing boundary conditions, the function $m_{\mathcal{P}}(\cdot)$ coincides with the classical characteristic (Titchmarsh–Weyl) function [**30**].

It turns out that the characteristic matrix $\Omega_\tau(\cdot)$ and the m-function $m_{\mathcal{P}}(\cdot)$ are connected by

$$m_{\mathcal{P}}(\lambda) = \hat{N}^* \Omega_\tau(\lambda) \hat{N} + \hat{C}, \quad \hat{C} = \hat{C}^*,$$

where $\hat{N}$ is the right inverse operator for $N' = (-N_0, N_1)$. This implies that $m_{\mathcal{P}}(\cdot)$ is the uniformly strict part of the Nevanlinna function $\Omega_\tau(\cdot)$ and the function $\Omega_\tau(\cdot)$ is uniformly strict if and only if $m = \hat{n} = 2n$ and the $(2n \times 2n)$-matrix (N_0, N_1) is invertible.

In the final part of the paper we consider some questions of the eigenfunction expansion. Namely, let $\varphi(t,\lambda) = (\varphi_1(t,\lambda), \varphi_2(t,\lambda), \dots, \varphi_d(t,\lambda))$ be a system of d linearly independent solutions of the equation $l[y] - \lambda y = 0$ with the constant initial data $\varphi^{(j)}(0,\lambda) \equiv \varphi_j$, $j \in \{0,1\}$. Recall that a $(d \times d)$-matrix distribution function $\Sigma(s) = \Sigma_{\tau,\varphi}(s)$, $s \in \mathbb{R}$, is called a spectral function of the boundary problem (1.6)–(1.8) corresponding to the solution $\varphi(\cdot,\lambda)$ if, for each function $f \in \mathfrak{H}$ with compact support, the Fourier transform

$$g_f(s) = \int_0^b \varphi^{\mathrm{T}}(t,s) f(t)\, \mathrm{d}t$$

satisfies the equality

$$((F_\tau(\beta) - F_\tau(\alpha))f, f)_{\mathfrak{H}} = \int_{[\alpha,\beta)} (\mathrm{d}\Sigma_{\tau,\varphi}(s) g_f(s), g_f(s)), \quad [\alpha,\beta) \subset \mathbb{R}$$

(here $F_\tau(\cdot)$ is the spectral function of L_0). As is known [**8**, **30**, **35**], in the case $d = 2n$ there exists a unique spectral function $\Sigma_{\tau,\varphi}(\cdot)$ of the problem (1.6)–(1.8). At the same time, for simplification of calculations, it is important to make d as small as possible [**8**, Chapter 13.5]. Therefore, the natural problem seems to be a description of all spectral functions $\Sigma_{\tau,\varphi}(\cdot)$ with the minimally possible value of d (we denote this value by $d_{\min}$ and we call the corresponding spectral function minimal). It turns out that the complete solution of this problem is based on the concept of the m-function, $m_{\mathcal{P}}(\cdot)$. Namely, the following theorem is immediate from the more general result obtained in the present paper (see Theorem 4.15).

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Theorem 1.3. *Let $\mathcal{P} = \{C_0(\lambda), C_1(\lambda)\}$ be a Nevanlinna pair (1.4), (1.5) and let $\varphi_N(t,\lambda) = (\varphi_1(t,\lambda), \varphi_2(t,\lambda), \dots, \varphi_{\hat n}(t,\lambda))$ be the $\hat n$-component linearly independent solution of the equation $l[y] - \lambda y = 0$ with the initial data $\varphi_N^{(1)}(0,\lambda) = -N_0^*$, $\varphi_N^{(2)}(0,\lambda) = N_1^*$. Then*

1. *there exists a unique $(\hat n \times \hat n)$-spectral function $\Sigma_{\mathcal{P},N}(s)$ of the problem (1.6)–(1.8) corresponding to $\varphi_N(\cdot,\lambda)$ and this function is calculated by means of the Stieltjes formula (4.18) for the m-function $m_{\mathcal{P}}(\cdot)$,*
2. *$d_{\min} = \hat n$ and the set of all minimal spectral functions $\Sigma_{\min}(\cdot)$ is given by*

$$\Sigma_{\min}(s) = X^* \Sigma_{\mathcal{P},N}(s) X,$$

where X is an invertible $(\hat n \times \hat n)$-matrix.

Moreover, we show that for a fixed pair $N = (N_0, N_1)$ the set of all spectral functions $\Sigma_{\mathcal{P},N}(s)$ is parametrized by the Stieltjes formula (4.18) and the following equality:

$$m_{\mathcal{P}}(\lambda) = T_{N,0}(\lambda) + T_N(\lambda)(C_0(\lambda) - C_1(\lambda)M(\lambda))^{-1} C_1(\lambda) T_N^*(\bar\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{1.13}$$

which is similar to the known Krein formula for resolvents (see, for example, [**4**]). In (1.13) $M(\lambda)$ is the Weyl function (1.9) and $T_{N,0}(\lambda), T_N(\lambda)$ are the matrix functions defined by means of $M(\lambda)$ and the pair N. The role of a parameter in (1.13) is played by a Nevanlinna pair $\mathcal{P} = \{C_0(\lambda), C_1(\lambda)\}$ given by (1.4), (1.5) with fixed N_0, N_1 and all possible $C'_{ij}(\lambda)$. Note in this connection that, for a decomposing boundary problem, formula (1.13) leads to a similar formula given in [**26**]. Moreover, (1.13) implies the known description of all Titchmarsh–Weyl functions $m(\cdot)$ obtained for quasi-regular expressions $l[y]$ by Fulton [**9**] and Khol′kin [**18**, **33**] (we shall touch upon these problems elsewhere).

Next, by using Theorem 1.3, we prove the inequality $\operatorname{sm}(\tilde A) \leqslant \hat n$, where $\operatorname{sm}(\tilde A)$ is the spectral multiplicity of the (exit space) self-adjoint extension $\tilde A \supset L_0$ given by the boundary conditions (1.7), (1.8). This result improves the known estimate $\operatorname{sm}(\tilde A) \leqslant m$ implied by simplicity of the operator L_0. In this connection note that, in the case $\Delta = [0,b]$, one can let $\Gamma'_0 y = y^{(2)}(b)$, $\Gamma'_1 y = y^{(1)}(b)$ in (1.7), (1.8), which implies that the multiplicity of each eigenvalue of the canonical extension $\tilde A = \tilde A^*$ does not exceed $\hat n$ (= $\operatorname{rank}(N_0, N_1)$). Hence, in the case $\Delta = [0,b]$, the estimate $\operatorname{sm}(\tilde A) \leqslant \hat n$ (for the canonical extension $\tilde A$) is immediate from (1.7), (1.8) and the discreteness of the spectrum of $\tilde A$. Meanwhile, such an estimate does not seem to be so obvious in the case of intermediate deficiency indices $n < m < 2n$ and non-separated boundary conditions.

In conclusion, note that spectral multiplicity of differential operators and spectral matrix-functions of minimal dimension have been studied by many authors. Thus, Kac [**16**] and Gilbert [**13**] examined the spectral multiplicity of the self-adjoint operator $\tilde A$ generated by the Sturm–Liouville expression

$$l[y] = -y'' + V(t)y, \qquad \overline{V(t)} = V(t) \tag{1.14}$$

on $\mathbb{R}$, which is supposed to be in the limit point case at $+\infty$ and $-\infty$. In [**13**, **16**] under the above assumptions the spectral multiplicity of $\tilde A$ was described in terms of

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properties of the classical Titchmarsh–Weyl functions $m_+(\cdot)$ and $m_-(\cdot)$ associated with the restrictions $l[y] \upharpoonright \mathbb{R}_+$ and $l[y] \upharpoonright \mathbb{R}_-$, respectively.

In the recent papers by Gesztesy and Zinchenko [**12**] and Fulton and Langer [**10**, **11**], some special classes of the expression (1.14) on $(0,\infty)$ were examined. These authors modified the well-known Titchmarsh–Weyl method by considering certain singular and regular solutions of the equation $l[y] = \lambda y$ and thus obtained a generalized Titchmarsh–Weyl function $m(\cdot)$, which is a scalar function but no longer belongs to the Nevanlinna class. Such an approach enables one to obtain a scalar spectral function and as a result to show that the corresponding self-adjoint operator $\tilde{A}$ has a simple spectrum. We also refer the reader to [**31**], in which scalar differential expressions $l[y]$ of a higher order with minimal equal deficiency indices are considered and the upper bound for the multiplicity of the continuous spectrum of an arbitrary self-adjoint realization $\tilde{A}$ of $l[y]$ is found. Moreover, the method for calculation of the absolutely continuous spectrum's multiplicity in terms of the abstract Weyl function is provided in [**23**].

Note, in connection with the above, that our approach is applicable to differential operators of a (higher) order $2n$ with arbitrary (possibly unequal) deficiency indices. Moreover, our main results are formulated immediately in terms of the operators N_0 and N_1 or, equivalently, in terms of boundary conditions at the regular endpoint 0 (see (1.7)). Therefore, our approach seems to be convenient for applications, especially in the case of intermediate deficiency indices $n < m < 2n$ (we emphasize that, strictly, this case does not hold for the Sturm–Liouville operator (1.14)).

2. Preliminaries

2.1. Notation

The following notation will be used throughout the paper.

- $\mathfrak{H}$, $\mathcal{H}$ denote Hilbert spaces.
- $[\mathcal{H}_1, \mathcal{H}_2]$ is the set of all bounded linear operators defined on $\mathcal{H}_1$ with values in $\mathcal{H}_2$.
- $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$.
- $P_{\mathcal{L}}$ is the orthogonal projector in $\mathfrak{H}$ onto the subspace $\mathcal{L} \subset \mathfrak{H}$.
- $\mathbb{C}_+$ ($\mathbb{C}_-$) is the upper (lower) half-plane of the complex plane.

Recall that a closed linear relation from $\mathcal{H}_0$ to $\mathcal{H}_1$ is a closed subspace in $\mathcal{H}_0 \oplus \mathcal{H}_1$. The set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ (from $\mathcal{H}$ to $\mathcal{H}$) will be denoted by $\tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ ($\tilde{\mathcal{C}}(\mathcal{H})$). A closed linear operator T from $\mathcal{H}_0$ to $\mathcal{H}_1$ is identified with its graph $\operatorname{gr} T \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$.

For a relation $T \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$, we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\operatorname{Ker} T$ the domain, range and the kernel, respectively. Moreover, T^{-1} ($\in \tilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$) and T^* ($\in \tilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$) denote the inverse and adjoint relations.

In the case $T \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we write

- $0 \in \rho(T)$ if $\operatorname{Ker} T = \{0\}$ and $\mathcal{R}(T) = \mathcal{H}_1$, or equivalently if $T^{-1} \in [\mathcal{H}_1, \mathcal{H}_0]$,
- $0 \in \hat{\rho}(T)$ if $\operatorname{Ker} T = \{0\}$ and $\mathcal{R}(T)$ is closed.

For a linear relation $T \in \tilde{\mathcal{C}}(\mathcal{H})$ we denote by $\rho(T) = \{\lambda \in \mathbb{C} \colon 0 \in \rho(T - \lambda)\}$ and $\hat{\rho}(T) = \{\lambda \in \mathbb{C} \colon 0 \in \hat{\rho}(T - \lambda)\}$ the resolvent set and the set of regular-type points of T, respectively.

2.2. Holomorphic operator pairs

Recall that a holomorphic operator function $\Phi(\cdot) \colon \mathbb{C} \setminus \mathbb{R} \to [\mathcal{H}]$ is called a Nevanlinna function if $\operatorname{Im} \lambda \cdot \operatorname{Im} \Phi(\lambda) \geqslant 0$ and $\Phi^*(\lambda) = \Phi(\bar{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, the Nevanlinna function $\Phi(\cdot)$ is said to be uniformly strict if $0 \in \rho(\operatorname{Im} \Phi(\lambda))$ when $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Next assume that Λ is an open set in $\mathbb{C}$, that $\mathcal{K}$, $\mathcal{H}_0$ and $\mathcal{H}_1$ are Hilbert spaces and that $C_j(\cdot) \colon \Lambda \to [\mathcal{H}_j, \mathcal{K}]$, $j \in \{0, 1\}$, is a pair of holomorphic operator functions (hereafter 'a holomorphic pair'). In what follows, we identify such a pair with a holomorphic operator function

$$C(\lambda) = (C_0(\lambda), C_1(\lambda)) \colon \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{K}, \quad \lambda \in \Lambda. \tag{2.1}$$

A pair (2.1) will be called admissible if $\mathcal{R}(C(\lambda)) = \mathcal{K}$ for all $\lambda \in \Lambda$. In the following all pairs (2.1) are admissible unless otherwise stated.

Definition 2.1. Two holomorphic pairs $C(\cdot) \colon \Lambda \to [\mathcal{H}_0 \oplus \mathcal{H}_1, \mathcal{K}]$ and $C'(\cdot) \colon \Lambda \to [\mathcal{H}_0 \oplus \mathcal{H}_1, \mathcal{K}']$ are said to be equivalent if $C'(\lambda) = \varphi(\lambda) C(\lambda)$, $\lambda \in \Lambda$, with a holomorphic isomorphism $\varphi(\cdot) \colon \Lambda \to [\mathcal{K}, \mathcal{K}']$.

Clearly, the set of all holomorphic pairs (2.1) falls into non-intersecting classes of equivalent pairs. Moreover, such a class can be identified with a function $\tau(\cdot) \colon \Lambda \to \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$, given for all $\lambda \in \Lambda$ by

$$\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{K}\} := \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 \colon C_0(\lambda) h_0 + C_1(\lambda) h_1 = 0\}. \tag{2.2}$$

In what follows, we suppose that $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$, $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$ and P_j is the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j \in \{1, 2\}$. With each linear relation $\theta \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we associate a $\times$-adjoint linear relation $\theta^\times \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$, defined as the set of all $\{k_0, k_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1$ such that

$$(k_1, h_0) - (k_0, h_1) + \mathrm{i}(P_2 k_0, P_2 h_0) = 0, \quad \{h_0, h_1\} \in \theta.$$

Clearly, in the case $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$, the equality $\theta^\times = \theta^*$ is valid.

Next assume that $\mathcal{K}_0$ is an auxiliary Hilbert space, $\mathcal{K}_1$ is a subspace in $\mathcal{K}_0$ and

$$C(\lambda) = (C_0(\lambda), C_1(\lambda)) \colon \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{K}_0, \quad \lambda \in \mathbb{C}_+, \tag{2.3}$$
$$D(\lambda) = (D_0(\lambda), D_1(\lambda)) \colon \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{K}_1, \quad \lambda \in \mathbb{C}_-, \tag{2.4}$$

are holomorphic operator pairs with the block-matrix representations

$$C_0(\lambda) = (C_{01}(\lambda), C_{02}(\lambda)) \colon \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{K}_0, \tag{2.5}$$
$$D_0(\lambda) = (D_{01}(\lambda), D_{02}(\lambda)) \colon \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{K}_1. \tag{2.6}$$

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Definition 2.2. A Nevanlinna collection of holomorphic operator pairs (hereafter 'a Nevanlinna collection') is a totality $\{C(\cdot), D(\cdot)\}$ of holomorphic pairs (2.3), (2.4) satisfying

$$2\,\mathrm{Im}(C_1(\lambda)C_{01}^*(\lambda)) + C_{02}(\lambda), C_{02}^*(\lambda) \geqslant 0, \quad 0 \in \rho(C_0(\lambda) - \mathrm{i}C_1(\lambda)P_1), \quad \lambda \in \mathbb{C}_+, \quad (2.7)$$
$$2\,\mathrm{Im}(D_1(\lambda)D_{01}^*(\lambda)) + D_{02}, (\lambda)D_{02}^*(\lambda) \leqslant 0, \quad 0 \in \rho(D_{01}(\lambda) + \mathrm{i}D_1(\lambda)), \quad \lambda \in \mathbb{C}_-, \quad (2.8)$$

and

$$C_1(\lambda)D_{01}^*(\bar\lambda) - C_{01}(\lambda)D_1^*(\bar\lambda) + \mathrm{i}C_{02}(\lambda)D_{02}^*(\bar\lambda) = 0, \quad \lambda \in \mathbb{C}_+. \quad (2.9)$$

A Nevanlinna collection (2.3), (2.4) is said to be constant if $\mathcal{K}_0 = \mathcal{K}_1 =: \mathcal{K}$ and $C_j(\lambda) = D_j(z) \equiv C_j$, $j \in \{0,1\}$, for all $\lambda \in \mathbb{C}_+$, $z \in \mathbb{C}_-$.

Clearly, a constant Nevanlinna collection can be regarded as an operator pair

$$C = (C_0, C_1)\colon \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{K} \quad (2.10)$$

with the block-matrix representation $C_0 = (C_{01}, C_{02})\colon \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{K}$ satisfying

$$2\,\mathrm{Im}(C_1C_{01}^*) + C_{02}C_{02}^* = 0, \quad 0 \in \rho(C_0 - \mathrm{i}C_1P_1), \quad 0 \in \rho(C_{01} + \mathrm{i}C_1). \quad (2.11)$$

Equation (2.11) and [**24**, Proposition 3.4] imply that the equality

$$\theta = \{(C_0, C_1); \mathcal{K}\} := \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : C_0h_0 + C_1h_1 = 0\} \quad (2.12)$$

define a linear relation $\theta \in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ such that $(-\theta)^\times = -\theta$. Moreover, a constant Nevanlinna collection exists if and only if $\dim \mathcal{H}_1 = \dim \mathcal{H}_0$ $(= \dim \mathcal{K})$.

We emphasize that in Definition 2.2 we just mean a collection that consists of the two holomorphic operator pairs $\{C_0(\cdot), C_1(\cdot)\}$ and $\{D_0(\cdot), D_1(\cdot)\}$.

Definition 2.3. A collection $\tau = \{\tau_+, \tau_-\}$ of two functions $\tau_+(\cdot)\colon \mathbb{C}_+ \to \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ and $\tau_-(\cdot)\colon \mathbb{C}_- \to \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ is said to be of the class $\tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ if, for all $\lambda \in \mathbb{C}_+$ and $z \in \mathbb{C}_-$, it admits the representation

$$\tau_+(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{K}_0\}, \qquad \tau_-(z) = \{(D_0(z), D_1(z)); \mathcal{K}_1\} \quad (2.13)$$

with a Nevanlinna collection $\{C(\cdot), D(\cdot)\}$ (see (2.2)).

A collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to the class $\tilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$ if it admits the representation $\tau_\pm(\lambda) = \{(C_0, C_1); \mathcal{K}\} = \theta$, $\lambda \in \mathbb{C}_\pm$, with a constant Nevanlinna collection (operator pair) (2.10).

It follows from Definition 2.3 that a collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ can be regarded as a collection of two equivalence classes of holomorphic pairs (2.3) and (2.4) satisfying (2.7)–(2.9). Moreover, according to [**24**],

$$\mathrm{Im}\,\lambda \cdot (2\,\mathrm{Im}(h_1, h_0) - \|P_2h_0\|^2) \geqslant 0, \quad \{h_0, h_1\} \in \tau_\pm(\lambda), \quad (2.14)$$

and $-\tau_+(\lambda) = (-\tau_-(\bar\lambda))^\times$, $\lambda \in \mathbb{C}_+$, for any collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$.

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Remark 2.4.

1. Clearly, a Nevanlinna collection (2.3), (2.4) satisfies the equalities

$$\dim \mathcal{H}_0 = \dim \mathcal{K}_0, \qquad \dim \mathcal{H}_1 = \dim \mathcal{K}_1. \tag{2.15}$$

Therefore, the representation (2.13) with $\mathcal{K}_0 = \mathcal{K}_1 =: \mathcal{K}$ is possible if and only if $\dim \mathcal{H}_1 = \dim \mathcal{H}_0$, in which case the corresponding Nevanlinna collection (2.3), (2.4) can be regarded as the unique holomorphic operator pair defined on $\mathbb{C}_+ \cup \mathbb{C}_-$.

2. The above concepts of a Nevanlinna collection and the class $\tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ can be regarded as a natural generalization of the well-known concepts of a Nevanlinna pair and the class $\tilde{R}(\mathcal{H})$ of $\tilde{\mathcal{C}}(\mathcal{H})$-valued functions. To explain this assertion, recall (see, for example, [**7**]) that a function $\tau(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, whose values are linear relations in the Hilbert space $\mathcal{H}$ belongs to the class $\tilde{R}(\mathcal{H})$ if for every $\lambda \in \mathbb{C}_+$ the relation $\tau(\lambda)$ is maximal dissipative, $\tau(\lambda)^* = \tau(\bar{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $(\tau(\lambda) + \mathrm{i})^{-1}$, $\lambda \in \mathbb{C}_+$, is a holomorphic operator function. A holomorphic operator pair

$$C(\lambda) = (C_0(\lambda), C_1(\lambda)) \colon \mathcal{H} \oplus \mathcal{H} \to \mathcal{K}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

is said to be a Nevanlinna pair if (cf. (2.7)–(2.9))

$$\operatorname{Im} \lambda \cdot \operatorname{Im}(C_1(\lambda) C_0^*(\lambda)) \geqslant 0, \quad 0 \in \rho(C_0(\lambda) - \lambda C_1(\lambda))$$

and

$$C_1(\lambda) C_0^*(\bar{\lambda}) - C_0(\lambda) C_1^*(\bar{\lambda}) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

or, equivalently, if the equality

$$\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{K}\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.16}$$

defines a function $\tau(\cdot) \in \tilde{R}(\mathcal{H})$ [**7**]. Moreover, a constant Nevanlinna pair can be identified by means of (2.10) and (2.12) with a self-adjoint linear relation (operator pair) $\theta = \theta^* \in \tilde{\mathcal{C}}(\mathcal{H})$.

The class $\tilde{R}(\mathcal{H})$ plays an important role in the parametrization of all (exit space) self-adjoint extensions of a symmetric operator A with equal deficiency indices and related problems (see [**4**, **7**, **20**, **21**, **36**] and the references therein). At the same time, in the case of unequal deficiency indices of the operator A, it is necessary to consider unequal spaces $\mathcal{H}_1 \neq \mathcal{H}_0$ and two functions $\tau_+(\lambda)$, $\lambda \in \mathbb{C}_+$, and $\tau_-(z)$, $z \in \mathbb{C}_-$, which form a collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ [**24**, **25**]. Of course, in the case $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$, one has $\tilde{R}(\mathcal{H}, \mathcal{H}) = \tilde{R}(\mathcal{H})$ and the representations (2.13) turn into (2.16).

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2.3. Differential operators

Let $\varDelta = [0, b\rangle$, $b \leqslant \infty$, be an interval on the real axis (in the case $b < \infty$, the point b may or may not belong to $\varDelta$), let H be a separable Hilbert space and let

$$l[y] = \sum_{k=1}^{n} (-1)^k ((p_{n-k} y^{(k)})^{(k)} - \tfrac{1}{2}\mathrm{i}[(q_{n-k}^* y^{(k)})^{(k-1)} + (q_{n-k} y^{(k-1)})^{(k)}]) + p_n y \tag{2.17}$$

be a differential expression of an even order $2n$ with sufficiently smooth operator-valued coefficients $p_k(\cdot), q_k(\cdot)\colon \varDelta \to [H]$ such that $p_k(t) = p_k^*(t)$ and $0 \in \rho(p_0(t))$. Denote by $y^{[k]}(\cdot)$, $k = 0, 1, \ldots, 2n$, the quasi-derivatives of a vector-function $y(\cdot)\colon \varDelta \to H$, corresponding to the expression (2.17) [**30, 32, 33**] and let $\mathcal{D}(l)$ be the set of functions $y(\cdot)$ for which this expression makes sense. With every function $y \in \mathcal{D}(l)$, we associate the functions $y^{(j)}(\cdot)\colon \varDelta \to H^n$, $j \in \{1, 2\}$, and $\tilde{y}(\cdot)\colon \varDelta \to H^n \oplus H^n$ by setting

$$\left.\begin{aligned} y^{(1)}(t) &:= \{y^{[k-1]}(t)\}_{k=1}^n (\in H^n), \\ y^{(2)}(t) &:= \{y^{[2n-k]}(t)\}_{k=1}^n (\in H^n), \end{aligned}\right\} \tag{2.18}$$

$$\tilde{y}(t) = \{y^{(1)}(t), y^{(2)}(t)\}\ (\in H^n \oplus H^n), \quad t \in \varDelta. \tag{2.19}$$

Let $\mathcal{K}$ be a Hilbert space and let $Y(\cdot)\colon \varDelta \to [\mathcal{K}, H]$ be an operator solution of the differential equation

$$l[y] - \lambda y = 0. \tag{2.20}$$

With each such solution we associate the operator functions $Y^{(j)}(\cdot)\colon \varDelta \to [\mathcal{K}, H^n]$, $j \in \{1, 2\}$, and $\tilde{Y}(\cdot)\colon \varDelta \to [\mathcal{K}, H^n \oplus H^n]$:

$$\begin{aligned} Y^{(1)}(t) &= (Y(t), Y^{[1]}(t), \ldots, Y^{[n-1]}(t))^{\mathrm{T}}, \\ Y^{(2)}(t) &= (Y^{[2n-1]}(t), Y^{[2n-2]}(t), \ldots, Y^{[n]}(t))^{\mathrm{T}}, \\ \tilde{Y}(t) &= (Y^{(1)}(t), Y^{(2)}(t))^{\mathrm{T}}\colon \mathcal{K} \to H^n \oplus H^n, \quad t \in \varDelta, \end{aligned}$$

where $Y^{[k]}(\cdot)$, $k = 0, 1, \ldots, 2n-1$ are quasi-derivatives of $Y(\cdot)$.

In what follows, $\mathfrak{H}$ $(= L_2(\varDelta; H))$ is the Hilbert space of all measurable functions $f(\cdot)\colon \varDelta \to H$ such that

$$\int_0^b \|f(t)\|^2 \,\mathrm{d}t < \infty.$$

Moreover, $L_2'[\mathcal{K}, H]$ denotes the set of all operator functions $Y(\cdot)\colon \varDelta \to [\mathcal{K}, H]$ such that $Y(t)h \in \mathfrak{H}$ for all $h \in \mathcal{K}$.

It is known [**30, 32, 33**] that the expression (2.17) generates the maximal operator L in $\mathfrak{H}$, defined on the domain $\mathcal{D} = \mathcal{D}(L) := \{y \in \mathcal{D}(l) \cap \mathfrak{H}\colon l[y] \in \mathfrak{H}\}$ by $Ly = l[y]$, $y \in \mathcal{D}$. Moreover, for all $y, z \in \mathcal{D}$ there exists the limit

$$[y, z](b) := \lim_{t \uparrow b} (y^{(1)}(t), z^{(2)}(t))_{H^n} - (y^{(2)}(t), z^{(1)}(t))_{H^n}. \tag{2.21}$$

Let

$$\mathcal{D}_0 = \{y \in \mathcal{D} \colon \tilde{y}(0) = 0 \text{ and } [y, z](b) = 0,\ z \in \mathcal{D}\}$$

and let $L_0 = L \restriction \mathcal{D}_0$ be the minimal operator generated by the expression (2.17). Then L_0 is a closed densely defined symmetric operator in $\mathfrak{H}$ and $L_0^* = L$ [**30**, **32**, **33**].

Next denote by $\mathfrak{N}_\lambda(L_0) := \operatorname{Ker}(L - \lambda)(\bar{\lambda} \in \hat{\rho}(L_0))$ the defect subspace of the operator L_0 and let $n_\pm(L_0) := \dim \mathfrak{N}_\lambda(L_0)$, $\lambda \in \mathbb{C}_\pm$, be its deficiency indices. As is known, these indices are not necessarily equal [**19**].

Let $\theta = \theta^* \in \tilde{\mathcal{C}}(H^n)$ and let L_θ be a symmetric extension of L_0 with the domain $\mathcal{D}(L_\theta) = \{y \in \mathcal{D} \colon \tilde{y}(0) \in \theta,\ [y, z](b) = 0, \text{ for all } z \in \mathcal{D}\}$. According to [**27**] deficiency indices $n_\pm(L_\theta)$ of an operator L_θ do not depend on θ $(= \theta^*)$, which enables us to introduce the deficiency indices at the right endpoint b as $n_{b\pm} := n_\pm(L_\theta)$.

2.4. Decomposing boundary triplets and Weyl functions

Assume that $\mathcal{H}_1'$ is a subspace in a Hilbert space $\mathcal{H}_0'$, $\mathcal{H}_2' := \mathcal{H}_0' \ominus \mathcal{H}_1'$, $\Gamma_0' \colon \mathcal{D} \to \mathcal{H}_0'$ and $\Gamma_1' \colon \mathcal{D} \to \mathcal{H}_1'$ are linear maps and P_j' is the orthoprojector in $\mathcal{H}_0'$ onto $\mathcal{H}_j'$, $j \in \{1, 2\}$. Moreover, let $\mathcal{H}_0 = H^n \oplus \mathcal{H}_0'$, $\mathcal{H}_1 = H^n \oplus \mathcal{H}_1'$ and let $\Gamma_j \colon \mathcal{D} \to \mathcal{H}_j$, $j \in \{0, 1\}$, be linear maps given for all $y \in \mathcal{D}$ by

$$\Gamma_0 y = \{y^{(2)}(0), \Gamma_0' y\}\ (\in H^n \oplus \mathcal{H}_0'), \qquad \Gamma_1 y = \{-y^{(1)}(0), \Gamma_1' y\}\ (\in H^n \oplus \mathcal{H}_1'). \tag{2.22}$$

Definition 2.5 (Mogilevskii [27]). A collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where Γ_0 and Γ_1 are linear maps (2.22), is said to be a decomposing D-boundary triplet (briefly a decomposing D-triplet) for L if the map $\Gamma' = (\Gamma_0', \Gamma_1')^{\mathrm{T}} \colon \mathcal{D} \to \mathcal{H}_0' \oplus \mathcal{H}_1'$ is surjective and the following identity holds:

$$[y, z](b) = (\Gamma_1' y, \Gamma_0' z) - (\Gamma_0' y, \Gamma_1' z) + \mathrm{i}(P_2' \Gamma_0' y, P_2' \Gamma_0' z), \quad y, z \in \mathcal{D}. \tag{2.23}$$

In the case $\mathcal{H}_0' = \mathcal{H}_1' =: \mathcal{H}' (\Longleftrightarrow \mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H})$ a decomposing D-triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a decomposing boundary triplet for L. For such a triplet, the identity (2.23) takes the form

$$[y, z](b) = (\Gamma_1' y, \Gamma_0' z) - (\Gamma_0' y, \Gamma_1' z), \quad y, z \in \mathcal{D}. \tag{2.24}$$

As was shown in [**27**, Lemma 3.4], a decomposing D-triplet (boundary triplet) for L exists if and only if $n_{b-} \leqslant n_{b+}$ (respectively, $n_{b-} = n_{b+}$), in which case

$$\dim \mathcal{H}_1' = n_{b-} \leqslant n_{b+} = \dim \mathcal{H}_0', \qquad \dim \mathcal{H}_1 = n_-(L_0) \leqslant n_+(L_0) = \dim \mathcal{H}_0 \tag{2.25}$$

(respectively, $n_{b-} = n_{b+} = \dim \mathcal{H}'$ and $n_-(L_0) = n_+(L_0) = \dim \mathcal{H}$). Therefore, in the following we suppose (without loss of generality) that $n_{b-} \leqslant n_{b+}$ and, consequently, $n_-(L_0) \leqslant n_+(L_0)$.

Remark 2.6.

1. It follows from (2.25) that, in the case $\dim H < \infty$, the last term in (2.23) appears if and only if $n_+(L_0) > n_-(L_0)$.

2. To illustrate the concept of a decomposing D-triplet consider the following example. Assume that $\dim H = 1$ (the scalar case) and $n_+(L_0) = n+1$, $n_-(L_0) = n$ (such a situation is possible in view of [**19**]). Then, according to [**27**, Proposition 3.10], the following statements hold:

   (i) the linear space $\tilde{\mathfrak{N}}_i = \{y \in L_2(\Delta) \colon l[y] = \mathrm{i}y,\ y'(0) = 0\}$ is one dimensional;

   (ii) if $v \in \tilde{\mathfrak{N}}_i$ and $\|v\|_{L_2(\Delta)} = 1$, then the operators

$$\Gamma_0(y) = \{y^{(2)}(0), \tfrac{1}{\sqrt{2}}[y,v](b)\}\ (\in \mathbb{C}^n \oplus \mathbb{C}), \quad \Gamma_1 y = -y^{(1)}(0)\ (\in \mathbb{C}^n), \quad y \in \mathcal{D},$$

   form the decomposing D-boundary triplet $\Pi = \{(\mathbb{C}^n \oplus \mathbb{C}) \oplus \mathbb{C}^n, \Gamma_0, \Gamma_1\}$ for L.

Proposition 2.7 (Mogilevskii [27]). *Let $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet (2.22) for L. Then the following hold.*

1. *For each $\lambda \in \mathbb{C}_+ (z \in \mathbb{C}_-)$ there exists a unique operator function $Z_+(\cdot,\lambda) \in L_2'[\mathcal{H}_0, H] (Z_-(\cdot,z) \in L_2'[\mathcal{H}_1, H])$ satisfying (2.20) and the boundary condition $\Gamma_0(Z_+(t,\lambda)h_0) = h_0$, $h_0 \in \mathcal{H}_0$ (respectively, $P_1\Gamma_0(Z_-(t,z)h_1) = h_1$, $h_1 \in \mathcal{H}_1$). If*

$$Z_+(t,\lambda) = (v_0(t,\lambda), u_+(t,\lambda)) \colon H^n \oplus \mathcal{H}_0' \to H, \quad \lambda \in \mathbb{C}_+, \tag{2.26}$$
$$Z_-(t,z) = (v_0(t,z), u_-(t,z)) \colon H^n \oplus \mathcal{H}_1' \to H, \quad z \in \mathbb{C}_-, \tag{2.27}$$

   are the block representations of $Z_+(\cdot,\lambda)$ and $Z_-(\cdot,z)$, then the above boundary condition can be represented as

$$v_0^{(2)}(0,\mu) = I_{H^n} (\mu \in \mathbb{C}\setminus\mathbb{R}), \quad \Gamma_0'(v_0(t,\lambda)\hat{h}) = 0, \quad P_1'\Gamma_0'(v_0(t,z)\hat{h}) = 0, \quad \hat{h} \in H^n,$$
$$u_+^{(2)}(0,\lambda) = 0, \qquad \Gamma_0'(u_+(t,\lambda)h_0') = h_0', \quad \lambda \in \mathbb{C}_+, \quad h_0' \in \mathcal{H}_0',$$
$$u_-^{(2)}(0,z) = 0, \quad P_1'\Gamma_0'(u_-(t,z)h_1') = h_1', \quad z \in \mathbb{C}_-, \quad h_1' \in \mathcal{H}_1'.$$

2. *The equalities $(\gamma_+(\lambda)h_0)(t) = Z_+(t,\lambda)h_0$, $h_0 \in \mathcal{H}_0$, and $(\gamma_-(z)h_1)(t) = Z_-(t,z)h_1$, $h_1 \in \mathcal{H}_1$, define the isomorphisms $\gamma_+(\lambda) \in [\mathcal{H}_0, \mathfrak{N}_\lambda(L_0)]$ and $\gamma_-(z) \in [\mathcal{H}_1, \mathfrak{N}_z(L_0)]$.*

Definition 2.8. The operator functions $M_+(\cdot) \colon \mathbb{C}_+ \to [\mathcal{H}_0, \mathcal{H}_1]$ and $M_-(\cdot) \colon \mathbb{C}_- \to [\mathcal{H}_1, \mathcal{H}_0]$ given for all $\lambda \in \mathbb{C}_+$ and $z \in \mathbb{C}_-$ by

$$\left.\begin{aligned} M_+(\lambda)h_0 &= \Gamma_1(Z_+(t,\lambda)h_0), & h_0 \in \mathcal{H}_0,\\ M_-(z)h_1 &= (\Gamma_1 + \mathrm{i}P_2\Gamma_0)(Z_-(t,z)h_1), & h_1 \in \mathcal{H}_1,\end{aligned}\right\} \tag{2.28}$$

are called the Weyl functions for the decomposing D-triplet $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$.

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As was proved in [**27**, Theorem 3.12] all the entries of the block representations

$$M_+(\lambda) = \begin{pmatrix} m(\lambda) & M_{2+}(\lambda) \\ M_{3+}(\lambda) & M_{4+}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}_0' \to H^n \oplus \mathcal{H}_1', \quad \lambda \in \mathbb{C}_+, \tag{2.29}$$

$$M_-(z) = \begin{pmatrix} m(z) & M_{2-}(z) \\ M_{3-}(z) & M_{4-}(z) \end{pmatrix} : H^n \oplus \mathcal{H}_1' \to H^n \oplus \mathcal{H}_0', \quad z \in \mathbb{C}_-, \tag{2.30}$$

can be defined immediately in terms of boundary values of the functions $v_0(\cdot,\lambda)$ and $u_\pm(\cdot,\lambda)$. In particular, (2.29) and (2.30) generate the uniformly strict Nevanlinna function $m(\lambda) = -v_0^{(1)}(0,\lambda)$, which in [**27**] we called the m-function.

Observe also that, in the case of a decomposing boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ (i.e. if $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H} = H^n \oplus \mathcal{H}'$), we can let $Z(t,\lambda) = Z_\pm(t,\lambda)$, $\lambda \in \mathbb{C}_\pm$ (cf. (2.26) and (2.27)), in which case the relations (2.28) can be written as

$$M(\lambda)h = \Gamma_1(Z(t,\lambda)h), \quad h \in \mathcal{H},\ \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.31}$$

Formula (2.31) defines the Weyl function $M(\lambda)$ $(\in [\mathcal{H}])$ of the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with the block representation

$$M(\lambda) = \begin{pmatrix} m(\lambda) & M_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}' \to H^n \oplus \mathcal{H}', \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.32}$$

Moreover, $M(\cdot)$ is a Nevanlinna function.

Remark 2.9. Recall that the concept of a D-boundary triplet and the corresponding Weyl function for an abstract symmetric operator with arbitrary (possibly unequal) deficiency indices was introduced in [**25**]. As proved in [**27**], a decomposing D-triplet (2.22) for L is a D-boundary triplet, while $\gamma_\pm(\cdot)$ and $M_\pm(\cdot)$ are the corresponding γ-fields and Weyl functions in the sense of [**25**]. Moreover, if Π is a decomposing boundary triplet (that is, $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ and $n_-(L_0) = n_+(L_0)$), then the operator functions $\gamma(\lambda) = \gamma_\pm(\lambda)$ and $M(\lambda) = M_\pm(\lambda)$, $\lambda \in \mathbb{C}_\pm$, are respectively the γ-field and the abstract Weyl function in the sense of Derkach and Malamud [**4**].

2.5. Generalized resolvents and characteristic matrices

Let $\tilde{A} \supset L_0$ be an exit-space self-adjoint extension of the operator L_0 acting in the Hilbert space $\tilde{\mathfrak{H}} \supset \mathfrak{H}$ and let $\tilde{E}(t)$ be the orthogonal spectral function of the operator $\tilde{A}$. Recall that the operator functions $R(\lambda) = P_{\mathfrak{H}}(\tilde{A} - \lambda)^{-1} \restriction \mathfrak{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $F(t) = P_{\mathfrak{H}}\tilde{E}(t) \restriction \mathfrak{H}$ are called the generalized resolvent and the spectral function, respectively, of the operator L_0. In the following we suppose that the spectral function $\tilde{E}(t)$ (or equivalently the extension $\tilde{A}$) is minimal, which means that $\operatorname{span}\{\mathfrak{H}, \tilde{E}(t)\mathfrak{H} \colon t \in \mathbb{R}\} = \tilde{\mathfrak{H}}$.

Let $Y_0(\cdot,\lambda) \colon \Delta \to [H^n \oplus H^n, H]$ be the 'canonical' operator solution of (2.20) with the initial data $\tilde{Y}_0(0,\lambda) = I_{H^n \oplus H^n}$ and let

$$J_{H^n} := \begin{pmatrix} 0 & -I_{H^n} \\ I_{H^n} & 0 \end{pmatrix} : H^n \oplus H^n \to H^n \oplus H^n. \tag{2.33}$$

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According to [**3, 35**] the generalized resolvent $R(\lambda)$ admits the representation

$$(R(\lambda)f)(x) = \int_0^b G(x,t,\lambda)f(t)\,\mathrm{d}t := \lim_{\eta\uparrow b}\int_0^\eta G(x,t,\lambda)f(t)\,\mathrm{d}t, \quad f = f(\cdot)\in\mathfrak{H}, \tag{2.34}$$

with the Green function $G(\cdot,\cdot,\lambda)\colon \Delta\times\Delta\to[H]$ given by

$$G(x,t,\lambda) = Y_0(x,\lambda)(\Omega(\lambda)+\tfrac{1}{2}\operatorname{sgn}(t-x)J_{H^n})Y_0^*(t,\bar{\lambda}), \quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{2.35}$$

Here $\Omega(\lambda)$ ($\in[H^n\oplus H^n]$) is a Nevanlinna operator function, which is called a characteristic matrix of the generalized resolvent $R(\lambda)$ [**35**].

Next assume that $\Pi = \{\mathcal{H}_0\oplus\mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a decomposing D-triplet (2.22) for L and $\tau = \{\tau_+,\tau_-\}\in\tilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ is a collection of holomorphic pairs (2.13) with the block representations

$$\left.\begin{aligned} C_0(\lambda) &= (\hat{C}_0(\lambda), C_0'(\lambda))\colon H^n\oplus\mathcal{H}_0'\to\mathcal{K}_0, \\ C_1(\lambda) &= (\hat{C}_1(\lambda), C_1'(\lambda))\colon H^n\oplus\mathcal{H}_1'\to\mathcal{K}_0, \quad \lambda\in\mathbb{C}_+, \end{aligned}\right\} \tag{2.36}$$

$$\left.\begin{aligned} D_0(\lambda) &= (\hat{D}_0(\lambda), D_0'(\lambda))\colon H^n\oplus\mathcal{H}_0'\to\mathcal{K}_1, \\ D_1(\lambda) &= (\hat{D}_1(\lambda), D_1'(\lambda))\colon H^n\oplus\mathcal{H}_1'\to\mathcal{K}_1, \quad \lambda\in\mathbb{C}_-. \end{aligned}\right\} \tag{2.37}$$

For a given function $f\in\mathfrak{H}$ consider the boundary-value problem

$$l[y]-\lambda y = f, \tag{2.38}$$

$$\hat{C}_0(\lambda)y^{(2)}(0)+\hat{C}_1(\lambda)y^{(1)}(0)+C_0'(\lambda)\Gamma_0'y - C_1'(\lambda)\Gamma_1'y = 0, \quad \lambda\in\mathbb{C}_+, \tag{2.39}$$

$$\hat{D}_0(\lambda)y^{(2)}(0)+\hat{D}_1(\lambda)y^{(1)}(0)+D_0'(\lambda)\Gamma_0'y - D_1'(\lambda)\Gamma_1'y = 0, \quad \lambda\in\mathbb{C}_-. \tag{2.40}$$

In view of (2.36) and (2.37) the conditions (2.39) and (2.40) can be written as

$$C_0(\lambda)\Gamma_0 y - C_1(\lambda)\Gamma_1 y = 0, \quad \lambda\in\mathbb{C}_+; \qquad D_0(\lambda)\Gamma_0 y - D_1(\lambda)\Gamma_1 y = 0, \quad \lambda\in\mathbb{C}_-. \tag{2.41}$$

A function $y(\cdot,\cdot)\colon \Delta\times(\mathbb{C}\setminus\mathbb{R})\to H$ is called a solution of the boundary problem (2.38)–(2.40) if, for each $\lambda\in\mathbb{C}\setminus\mathbb{R}$, the function $y(\cdot,\lambda)$ belongs to $\mathcal{D}$ and satisfies (2.38) and the boundary conditions (2.39), (2.40).

Theorem 2.10 (Mogilevskii [29]). *Let $\tau = \{\tau_+,\tau_-\}\in\tilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ be a collection given by (2.13) and (2.36), (2.37) and let $\tilde{\Omega}_{\tau+}(\lambda)$ be the operator function defined for all $\lambda\in\mathbb{C}_+$ by (1.10) (with τ_+ and M_+ in place of τ and M). Then we have the following.*

1. *For each $f\in\mathfrak{H}$ the boundary problem (2.38)–(2.40) has the unique solution $y(t,\lambda) = y_f(t,\lambda)$ and the equality $(R(\lambda)f)(t) = y_f(t,\lambda)$, $f\in\mathfrak{H}$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, defines a generalized resolvent $R(\lambda) := R_\tau(\lambda)$ of the minimal operator L_0.*

2. *The characteristic matrix of the generalized resolvent $R_\tau(\lambda)$ is*

$$\Omega(\lambda) = \Omega_\tau(\lambda) := P_{H^n\oplus H^n}\tilde{\Omega}_{\tau+}(\lambda)\restriction H^n\oplus H^n, \quad \lambda\in\mathbb{C}_+. \tag{2.42}$$

Conversely, for each generalized resolvent $R(\lambda)$ there exists a unique $\tau\in\tilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ such that $R(\lambda) = R_\tau(\lambda)$. Moreover, $R_\tau(\lambda)$ is a canonical resolvent if and only if $\tau\in\tilde{R}^0(\mathcal{H}_0,\mathcal{H}_1)$.

Proposition 2.11. *The characteristic matrix $\Omega_\tau(\cdot)$ satisfies the equality*

$$s - \lim_{y\to\infty} \Omega_\tau(\mathrm{i}y)/y = 0. \tag{2.43}$$

Proof. For simplicity, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a decomposing boundary triplet for L. Since the operator L_0 is densely defined, it follows from [**6**, **7**] that

$$s - \lim_{y\to\infty} (M(\mathrm{i}y) - M(\mathrm{i}y)(\tau(\mathrm{i}y) + M(\mathrm{i}y))^{-1} M(\mathrm{i}y))/y = 0,$$
$$s - \lim_{y\to\infty} (\tau(\mathrm{i}y) + M(\mathrm{i}y))^{-1}/y = 0.$$

Therefore, by (1.10), $s - \lim_{y\to\infty} \tilde{\Omega}_\tau(\mathrm{i}y)/y = 0$, which in view of (2.42) gives (2.43).

In the case of a decomposing D-triplet the proof is similar. □

Remark 2.12. It follows from Theorem 2.10 that the boundary problem (2.38)–(2.40) gives a parametrization of all generalized resolvents $R(\lambda) = R_\tau(\lambda)$ and characteristic matrices $\Omega(\lambda) = \Omega_\tau(\lambda)$ by means of the Nevanlinna boundary parameter τ. Moreover, since a spectral function $F(t)$ is uniquely defined by the corresponding generalized resolvent $R(\lambda)$, one obtains the parametrization $F(t) = F_\tau(t)$ of all spectral functions of the operator L_0 by means of the same boundary parameter τ.

3. m-functions and characteristic matrices

3.1. Quasi-constant and N-triangular Nevanlinna collections

Let

$$\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$$

be a decomposing D-triplet (2.22) for L (with $\mathcal{H}_j = H^n \oplus \mathcal{H}'_j$, $j \in \{0, 1\}$). A Nevanlinna collection $\{C(\cdot), D(\cdot)\}$ defined by (2.3), (2.4) and the block representations (2.36), (2.37) will be called quasi-constant if $\hat{C}_j(\lambda) = \hat{D}_j(z) \equiv \hat{C}_j (\in [H^n, \mathcal{K}_1])$, $j \in \{0, 1\}$, for all $\lambda \in \mathbb{C}_+$ and $z \in \mathbb{C}_-$ (such a definition is correct, since $\mathcal{K}_1 \subset \mathcal{K}_0$). Clearly, each constant pair $\theta (= \theta^*) = \{(C_0, C_1); \mathcal{K}\}$ is quasi-constant.

Next assume that

$$N = (N_0, N_1) \colon H^n \oplus H^n \to \hat{\mathcal{K}} \tag{3.1}$$

is an admissible operator pair (i.e. $\mathcal{R}(N) = \hat{\mathcal{K}}$) and let $\theta_N \in \tilde{\mathcal{C}}(H^n)$ be a linear relation given by $\theta_N = \{(N_0, N_1); \hat{\mathcal{K}}\}$. The operator pair (3.1) will be called symmetric (self-adjoint) if the linear relation θ_N is symmetric (self-adjoint).

Definition 3.1. A Nevanlinna collection $\{C(\cdot), D(\cdot)\}$ defined by (2.3), (2.4) will be called N-triangular if there exist a Hilbert space $\mathcal{K}'_0$ and a subspace $\mathcal{K}'_1 \subset \mathcal{K}'_0$ such that $\mathcal{K}_j = \hat{\mathcal{K}} \oplus \mathcal{K}'_j$, $j \in \{0, 1\}$, and the following block representations hold:

$$C_0(\lambda) = \begin{pmatrix} N_0 & C'_{01}(\lambda) \\ 0 & C'_{02}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}'_0 \to \hat{\mathcal{K}} \oplus \mathcal{K}'_0, \quad \lambda \in \mathbb{C}_+, \tag{3.2}$$

$$C_1(\lambda) = \begin{pmatrix} N_1 & C'_{11}(\lambda) \\ 0 & C'_{12}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}'_1 \to \hat{\mathcal{K}} \oplus \mathcal{K}'_0, \quad \lambda \in \mathbb{C}_+, \tag{3.3}$$

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$$D_0(\lambda) = \begin{pmatrix} N_0 & D'_{01}(\lambda) \\ 0 & D'_{02}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}'_0 \to \hat{\mathcal{K}} \oplus \mathcal{K}'_1, \quad \lambda \in \mathbb{C}_-, \tag{3.4}$$

$$D_1(\lambda) = \begin{pmatrix} N_1 & D'_{11}(\lambda) \\ 0 & D'_{12}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}'_1 \to \hat{\mathcal{K}} \oplus \mathcal{K}'_1, \quad \lambda \in \mathbb{C}_-. \tag{3.5}$$

A constant N-triangular collection can be regarded as an operator pair

$$C = (C_0, C_1) \colon (H^n \oplus \mathcal{H}'_0) \oplus (H^n \oplus \mathcal{H}'_1) \to \hat{\mathcal{K}} \oplus \mathcal{K}' \tag{3.6}$$

defined by the block-matrix representations

$$C_0 = \begin{pmatrix} N_0 & C'_{01} \\ 0 & C'_{02} \end{pmatrix} : H^n \oplus \mathcal{H}'_0 \to \underbrace{\hat{\mathcal{K}} \oplus \mathcal{K}'}_{\mathcal{K}}, \qquad C_1 = \begin{pmatrix} N_1 & C'_{11} \\ 0 & C'_{12} \end{pmatrix} : H^n \oplus \mathcal{H}'_1 \to \underbrace{\hat{\mathcal{K}} \oplus \mathcal{K}'}_{\mathcal{K}} \tag{3.7}$$

and satisfying the relations (2.11).

Assume now that $\{C(\cdot), D(\cdot)\}$ is a quasi-constant Nevanlinna collection defined by (2.3), (2.4) and (2.36), (2.37) and let $\hat{\mathcal{K}} (\subset \mathcal{K}_1)$ be the range of the operator

$$\hat{C} := C(\lambda) \restriction H^n \oplus H^n = (\hat{C}_0, \hat{C}_1) \colon H^n \oplus H^n \to \mathcal{K}_1. \tag{3.8}$$

It is clear that the collection $\{C(\cdot), D(\cdot)\}$ is N-triangular with some N if and only if $\hat{\mathcal{K}}$ is closed, in which case $N_j = \hat{C}_j (\in [H^n, \hat{\mathcal{K}}])$, $j \in \{0, 1\}$ (here $\hat{C}_j$ is considered as acting from H^n to $\hat{\mathcal{K}}$). In this connection the following proposition holds.

Proposition 3.2. *If $n_{b+} < \infty$ (in particular, if $\dim H < \infty$), then each quasi-constant Nevanlinna collection is N-triangular.*

Proof. Since the operator pair (2.3) is admissible, it follows that $\mathcal{R}(C(\lambda)) = \mathcal{K}_0$ and, therefore, the range of the operator $C^*(\lambda)$ is a closed subspace in $\mathcal{H}_0 \oplus \mathcal{H}_1$. Moreover, by (3.8), $\hat{C}^* = P_{H^n \oplus H^n} C^*(\lambda)$ and, consequently,

$$\mathcal{R}(\hat{C}^*) = P_{H^n \oplus H^n} \mathcal{R}(C^*(\lambda)). \tag{3.9}$$

Since in view of (2.25) $\operatorname{codim}(H^n \oplus H^n) = \dim(\mathcal{H}'_0 \oplus \mathcal{H}'_1) < \infty$, it follows from (3.9) that $\mathcal{R}(\hat{C}^*)$ is a closed subspace in $H^n \oplus H^n$. This implies that $\hat{\mathcal{K}}$ $(= \mathcal{R}(\hat{C}))$ is also closed. □

Remark 3.3. In the case $n_{b+} = \infty$ ($\Longleftrightarrow \dim \mathcal{H}'_0 = \infty$) one can easily construct a quasi-constant (and even constant) Nevanlinna collection $\{C(\cdot), D(\cdot)\}$ with non-closed subspace $\hat{\mathcal{K}}$, which implies that this collection is not N-triangular with any N. Hence, the condition $n_{b+} < \infty$ in Proposition 3.2 is essential.

Two N-triangular Nevanlinna collections $\{C(\cdot), D(\cdot)\}$ and $\{\tilde{C}(\cdot), \tilde{D}(\cdot)\}$ (with the same N) are said to be equivalent if the operator pairs $C(\cdot)$ and $\tilde{C}(\cdot)$ as well as $D(\cdot)$ and $\tilde{D}(\cdot)$ are equivalent in the sense of Definition 2.1. It is clear that for a given operator pair N (see (3.1)) the set of all N-triangular Nevanlinna collections falls into nonintersecting equivalence classes. In what follows, the set of all such classes will be denoted by $\mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$. Moreover, we shall denote by $\mathcal{P} = \{C(\cdot), D(\cdot)\}$ both an N-triangular Nevanlinna collection and the corresponding equivalence class.

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Definition 3.4. A collection (the corresponding equivalence class) $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$ is said to belong to the class $\mathrm{TR}^0(\mathcal{H}_0, \mathcal{H}_1)$ if it admits the representation (3.6), (3.7) as a constant N-triangular collection.

In the following, we write $\mathcal{P} = \{C_0, C_1\} \in \mathrm{TR}^0(\mathcal{H}_0, \mathcal{H}_1)$ identifying the collection $\mathcal{P} \in \mathrm{TR}^0(\mathcal{H}_0, \mathcal{H}_1)$ and the corresponding operator pair (3.6), (3.7).

If $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a decomposing boundary triplet (2.22) for L (i.e. if $\mathcal{H}_0' = \mathcal{H}_1' =: \mathcal{H}'$), then the above definition of the N-triangular Nevanlinna collection can be rather simplified. Namely, in this case one can let $\mathcal{K}_0' = \mathcal{K}_1' =: \mathcal{K}'$ in (3.2)–(3.5), so that the Nevanlinna collection $\{C(\cdot), D(\cdot)\}$ given by (3.2)–(3.5) can be considered as a unique Nevanlinna operator pair $(C_0(\lambda), C_1(\lambda)) \colon \mathcal{H} \oplus \mathcal{H} \to \mathcal{K}$ defined for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ by the block representations

$$C_0(\lambda) = \begin{pmatrix} N_0 & C_{01}'(\lambda) \\ 0 & C_{02}'(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}' \to \hat{\mathcal{K}} \oplus \mathcal{K}', \tag{3.10}$$

$$C_1(\lambda) = \begin{pmatrix} N_1 & C_{11}'(\lambda) \\ 0 & C_{12}'(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}' \to \hat{\mathcal{K}} \oplus \mathcal{K}'. \tag{3.11}$$

As before, we identify equivalent N-triangular Nevanlinna pairs (3.10), (3.11) (with the same N) and denote by $\mathcal{P} = \{C_0(\cdot), C_1(\cdot)\}$ both a pair (3.10), (3.11) and the corresponding equivalence class. Observe also that in this case the constant operator pair (3.6), (3.7) is self-adjoint, i.e. it defines by means of (2.12) a self-adjoint relation $\theta = \theta^*$ in $H^n \oplus \mathcal{H}'$ (cf. Remark 2.4 (2)).

In the following we denote by $\mathrm{TR}(\mathcal{H})$ the set of all equivalence classes of N-triangular Nevanlinna pairs (3.10), (3.11) and by $\mathrm{TR}^0(\mathcal{H})$ the set of all equivalence classes $\mathcal{P} \in \mathrm{TR}(\mathcal{H})$ containing a constant (self-adjoint) pair (3.7).

Proposition 3.5. *Assume that $N = (N_0, N_1)$ is an operator pair (3.1) and that $\{C(\cdot), D(\cdot)\} \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$ is a collection (3.2)–(3.5). Then the pair N is symmetric and*

$$n \dim H \leqslant \dim \hat{\mathcal{K}} \leqslant n_-(L_0). \tag{3.12}$$

Proof. Let $\tau_\pm(\lambda)$ be linear relations (2.13). Then in view of (3.2)–(3.5) $\theta_N = \tau_\pm(\lambda) \cap (H^n \oplus H^n)$, $\lambda \in \mathbb{C}_\pm$, and (2.14) shows that θ_N is a symmetric linear relation. Therefore, $\dim H^n \leqslant \operatorname{codim} \theta_N = \dim \hat{\mathcal{K}}$, which, together with (2.15) and the second relation in (2.25), gives (3.12). □

3.2. Generalized resolvents and the Green function

Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet (2.22) for L and let $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$ be a collection (3.2)–(3.5). Then the corresponding boundary

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problem (2.38)–(2.40) can be written as

$$l[y]-\lambda y=f, \tag{3.13}$$

$$N_0y^{(2)}(0)+N_1y^{(1)}(0)+C_{01}'(\lambda)\Gamma_0'y-C_{11}'(\lambda)\Gamma_1'y=0, \tag{3.14}$$

$$C_{02}'(\lambda)\Gamma_0'y-C_{12}'(\lambda)\Gamma_1'y=0,\quad \lambda\in\mathbb{C}_+, \tag{3.15}$$

$$N_0y^{(2)}(0)+N_1y^{(1)}(0)+D_{01}'(\lambda)\Gamma_0'y-D_{11}'(\lambda)\Gamma_1'y=0, \tag{3.16}$$

$$D_{02}'(\lambda)\Gamma_0'y-D_{12}'(\lambda)\Gamma_1'y=0,\quad \lambda\in\mathbb{C}_-. \tag{3.17}$$

Moreover, in the case $\mathcal{P}=\{C_0,C_1\}\in \mathrm{TR}^0(\mathcal{H}_0,\mathcal{H}_1)$ (see (3.6) and (3.7)) the boundary conditions (3.14)–(3.17) take the form

$$N_0y^{(2)}(0)+N_1y^{(1)}(0)+C_{01}'\Gamma_0'y-C_{11}'\Gamma_1'y=0, \tag{3.18}$$

$$C_{02}'\Gamma_0'y-C_{12}'\Gamma_1'y=0. \tag{3.19}$$

Observe also that, in the particular case of a decomposing boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ and a pair $\mathcal{P}=\{C_0(\cdot),C_1(\cdot)\}\in\mathrm{TR}(\mathcal{H})$ given by (3.10), (3.11), the boundary conditions (3.14)–(3.17) can be written in the simpler form

$$N_0y^{(2)}(0)+N_1y^{(1)}(0)+C_{01}'(\lambda)\Gamma_0'y-C_{11}'(\lambda)\Gamma_1'y=0, \tag{3.20}$$

$$C_{02}'(\lambda)\Gamma_0'y-C_{12}'(\lambda)\Gamma_1'y=0,\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{3.21}$$

The following corollary is immediate from Theorem 2.10.

Corollary 3.6. *Let $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ be a decomposing D-triplet (2.22) for L and let $\mathcal{P}=\{C(\cdot),D(\cdot)\}\in\mathrm{TR}(\mathcal{H}_0,\mathcal{H}_1)$ be a collection given by (3.2)–(3.5). Then the boundary problem (3.13)–(3.17) generates the generalized resolvent $R(\lambda)=R_{\mathcal{P}}(\lambda)$ of the operator L_0 (in the same way as in Theorem 2.10). Moreover, $R(\lambda)$ is a canonical resolvent if and only if $\mathcal{P}\in\mathrm{TR}^0(\mathcal{H}_0,\mathcal{H}_1)$, in which case the corresponding boundary conditions can be defined by (3.18) and (3.19).*

If, in addition, $\mathcal{H}_0=\mathcal{H}_1=:\mathcal{H}$ (that is, Π is a decomposing boundary triplet), then the above statements hold for the boundary problem (3.13), (3.20) and (3.21).

Remark 3.7. Note that, in view of Corollary 3.6, the generalized resolvent $R(\lambda)=R_{\mathcal{P}}(\lambda)$ can also be defined by $R_{\mathcal{P}}(\lambda)=(\tilde{A}(\lambda)-\lambda)^{-1}$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, where $\tilde{A}(\lambda)=L\restriction\mathcal{D}(\tilde{A}(\lambda))$, and $\mathcal{D}(\tilde{A}(\lambda))$ is the set of all functions $y\in\mathcal{D}$ satisfying the boundary conditions (3.14)–(3.17) or, equivalently, (2.41).

Assume that $\mathcal{P}=\{C(\cdot),D(\cdot)\}\in\mathrm{TR}(\mathcal{H}_0,\mathcal{H}_1)$ is a collection (3.2)–(3.5) and let $\tilde{D}_1(\lambda)$ ($\in[\mathcal{H}_1,\mathcal{K}_1]$) and $\tilde{D}_0(\lambda)$ ($\in[\mathcal{H}_0,\mathcal{K}_1]$) be defined by

$$\tilde{D}_1(\lambda):=D_0(\lambda)\restriction\mathcal{H}_1,\quad \tilde{D}_0(\lambda)=D_1(\lambda)P_1+\mathrm{i}D_0(\lambda)P_2,\quad \lambda\in\mathbb{C}_-.$$

It follows from (3.4) and (3.5) that the following block representations hold:

$$\tilde{D}_1(\lambda)=\begin{pmatrix}N_0 & \tilde{D}_{01}'(\lambda)\\ 0 & \tilde{D}_{02}'(\lambda)\end{pmatrix}:H^n\oplus\mathcal{H}_1'\to\hat{\mathcal{K}}\oplus\mathcal{K}_1',\quad \lambda\in\mathbb{C}_-,$$

$$\tilde{D}_0(\lambda)=\begin{pmatrix}N_1 & \tilde{D}_{11}'(\lambda)\\ 0 & \tilde{D}_{12}'(\lambda)\end{pmatrix}:H^n\oplus\mathcal{H}_0'\to\hat{\mathcal{K}}\oplus\mathcal{K}_1',\quad \lambda\in\mathbb{C}_-.$$

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Proposition 3.8. *Let the conditions of Corollary 3.6 be satisfied. Then, we have the following.*

1. *For each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists the unique operator function $v(\cdot, \lambda) \in L_2'[\hat{\mathcal{K}}, H]$ satisfying the equation $l[y] - \lambda y = 0$ and the boundary conditions*

$$(N_0 v^{(2)}(0,\lambda) + N_1 v^{(1)}(0,\lambda))\hat{h} + (C_{01}'(\lambda)\Gamma_0' - C_{11}'\Gamma_1')(v(t,\lambda)\hat{h}) = \hat{h}, \tag{3.22}$$

$$(C_{02}'(\lambda)\Gamma_0' - C_{12}'(\lambda)\Gamma_1')(v(t,\lambda)\hat{h}) = 0, \quad \hat{h} \in \hat{\mathcal{K}},\ \lambda \in \mathbb{C}_+, \tag{3.23}$$

$$(N_0 v^{(2)}(0,\lambda) + N_1 v^{(1)}(0,\lambda))\hat{h} + (D_{01}'(\lambda)\Gamma_0' - D_{11}'(\lambda)\Gamma_1')(v(t,\lambda)\hat{h}) = \hat{h}, \tag{3.24}$$

$$(D_{02}'(\lambda)\Gamma_0' - D_{12}'(\lambda)\Gamma_1')(v(t,\lambda)\hat{h}) = 0, \quad \hat{h} \in \hat{\mathcal{K}},\ \lambda \in \mathbb{C}_-. \tag{3.25}$$

2. *The functions $v(\cdot, \lambda)$ and $Z_\pm(\cdot, \lambda)$ (see (2.26) and (2.27)) are connected by*

$$v(t,\lambda) = \begin{cases} Z_+(t,\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1} \restriction \hat{\mathcal{K}}, & \lambda \in \mathbb{C}_+, \\ Z_-(t,\lambda)(\tilde{D}_1(\lambda) - \tilde{D}_0(\lambda)M_-(\lambda))^{-1} \restriction \hat{\mathcal{K}}, & \lambda \in \mathbb{C}_-. \end{cases} \tag{3.26}$$

where $M_\pm(\cdot)$ are the Weyl functions (2.29) and (2.30) for Π.

Proof. It follows from (2.22) and (3.2)–(3.5) that the conditions (3.22)–(3.25) are equivalent to

$$(C_0(\lambda)\Gamma_0 - C_1(\lambda)\Gamma_1)(v(t,\lambda)\hat{h}) = \hat{h}, \quad \hat{h} \in \hat{\mathcal{K}},\ \lambda \in \mathbb{C}_+, \tag{3.27}$$

$$(D_0(\lambda)\Gamma_0 - D_1(\lambda)\Gamma_1)(v(t,\lambda)\hat{h}) = \hat{h}, \quad \hat{h} \in \hat{\mathcal{K}},\ \lambda \in \mathbb{C}_-. \tag{3.28}$$

As was shown in [**29**], $0 \in \rho(C_0(\lambda) - C_1(\lambda)M_+(\lambda))$, $0 \in \rho(\tilde{D}_1(\lambda) - \tilde{D}_0(\lambda)M_-(\lambda))$ and

$$(C_0(\lambda)\Gamma_0 - C_1(\lambda)\Gamma_1)(Z_+(t,\lambda)h) = (C_0(\lambda) - C_1(\lambda)M_+(\lambda))h, \quad h \in \mathcal{H}_0,\ \lambda \in \mathbb{C}_+,$$

$$(D_0(\lambda)\Gamma_0 - D_1(\lambda)\Gamma_1)(Z_-(t,\lambda)h) = (\tilde{D}_1(\lambda) - \tilde{D}_0(\lambda)M_-(\lambda))h, \quad h \in \mathcal{H}_1,\ \lambda \in \mathbb{C}_-.$$

Hence, the equality (3.26) correctly defines the function $v(\cdot, \lambda) \in L_2'[\hat{\mathcal{K}}, H]$ satisfying (3.27), (3.28) and consequently (3.22)–(3.25). The uniqueness of such a function follows from the inclusion $\lambda \in \rho(\tilde{A}(\lambda))$, where $\tilde{A}(\lambda)$ is defined in Remark 3.7. □

Remark 3.9. One can easily verify that for a given operator pair $N = (N_0, N_1)$ the operator function $v(\cdot, \lambda)$ is uniquely defined by the equivalence class $\mathcal{P} \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$, i.e. $v(\cdot, \lambda)$ does not depend on the choice of an N-triangular Nevanlinna collection (3.2)–(3.5) inside the equivalence class. To emphasize this fact we shall write $v(\cdot, \lambda) = v_{\mathcal{P}}(\cdot, \lambda)$.

Moreover, it is easy to prove that, for each $\lambda \in \mathbb{C}_+$ (respectively, $\lambda \in \mathbb{C}_-$), the equality $y(t) = v(t,\lambda)\hat{h}$ gives a bijective correspondence between all $\hat{h} \in \hat{\mathcal{K}}$ and all solutions $y(\cdot)$ of (2.20) that belong to $\mathfrak{H}$ and satisfy the boundary condition (3.15) (respectively, (3.17)). Therefore, the operator function $v_{\mathcal{P}}(\cdot, \lambda)$ is a fundamental solution of the boundary problems (2.20), (3.15) for $\lambda \in \mathbb{C}_+$ and (2.20), (3.17) for $\lambda \in \mathbb{C}_-$ (see [**28, 33**]).

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Theorem 3.10. *Assume that $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a decomposing D-triplet (2.22) for L, $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$ is a collection (3.2)–(3.5) and $\varphi_N(\cdot,\lambda)\colon \Delta \to [\hat{\mathcal{K}}, H]$, $\lambda \in \mathbb{C}$, is the operator solution of (2.20) with the initial data*

$$\varphi_N^{(1)}(0,\lambda) = -N_0^*, \quad \varphi_N^{(2)}(0,\lambda) = N_1^*, \quad \lambda \in \mathbb{C}. \tag{3.29}$$

Then the generalized resolvent $R(\lambda) = R_{\mathcal{P}}(\lambda)$ generated by the boundary problem (3.13)–(3.17) admits the representation (2.34) with the Green function $G(x,t,\lambda) = G_{\mathcal{P}}(x,t,\lambda)$ given by

$$G_{\mathcal{P}}(x,t,\lambda) = \begin{cases} v_{\mathcal{P}}(x,\lambda)\varphi_N^*(t,\bar{\lambda}), & x > t, \\ \varphi_N(x,\lambda)v_{\mathcal{P}}^*(t,\bar{\lambda}), & x < t, \end{cases} \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.30}$$

Proof. Let $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ be a collection given by (2.13) and (2.36), (2.37), and let $Y_+(\cdot,\lambda)\colon \Delta \to [\mathcal{K}_1, H]$, $\lambda \in \mathbb{C}_+$, and $Y_-(\cdot,z)\colon \Delta \to [\mathcal{K}_0, H]$, $z \in \mathbb{C}_-$, be the operator solutions of (2.20) with the initial data

$$\tilde{Y}_+(0,\lambda) = (-\hat{D}_0^*(\bar{\lambda}), \hat{D}_1^*(\bar{\lambda}))^{\mathrm{T}}, \qquad \tilde{Y}_-(0,z) = (-\hat{C}_0^*(\bar{z}), \hat{C}_1^*(\bar{z}))^{\mathrm{T}}. \tag{3.31}$$

Assume also that $\mathcal{Z}_+(\cdot,\lambda) \in L_2'[\mathcal{K}_0, H]$ and $\mathcal{Z}_-(\cdot,z) \in L_2'[\mathcal{K}_1, H]$ are given by

$$\mathcal{Z}_+(t,\lambda) = Z_+(t,\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_+, \tag{3.32}$$

$$\mathcal{Z}_-(t,z) = Z_-(t,z)(\tilde{D}_1(z) - \tilde{D}_0(z)M_-(z))^{-1}, \quad z \in \mathbb{C}_-, \tag{3.33}$$

and let

$$Y(t,\lambda) = \begin{cases} Y_+(t,\lambda), & \lambda \in \mathbb{C}_+, \\ Y_-(t,\lambda), & \lambda \in \mathbb{C}_-, \end{cases} \qquad \mathcal{Z}(t,\lambda) = \begin{cases} \mathcal{Z}_+(t,\lambda), & \lambda \in \mathbb{C}_+, \\ \mathcal{Z}_-(t,\lambda), & \lambda \in \mathbb{C}_-. \end{cases}$$

Then according to [**29**, Theorem 16] the Green function in (2.34) is

$$G(x,t,\lambda) = \begin{cases} \mathcal{Z}(x,\lambda)Y^*(t,\bar{\lambda}), & x > t, \\ Y(x,\lambda)\mathcal{Z}^*(t,\bar{\lambda}), & x < t, \end{cases} \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.34}$$

Next, in the case of the block representations (3.2)–(3.5) one has

$$\hat{C}_j(\lambda) = (N_j, 0)^{\mathrm{T}} \in [H^n, \hat{\mathcal{K}} \oplus \mathcal{K}_0'], \quad \hat{D}_j(\lambda) = (N_j, 0)^{\mathrm{T}} \in [H^n, \hat{\mathcal{K}} \oplus \mathcal{K}_1'], \quad j \in \{0,1\}.$$

Therefore, the initial data (3.31) can be written in the form

$$\tilde{Y}_+(0,\lambda) = \begin{pmatrix} -N_0^* & 0 \\ N_1^* & 0 \end{pmatrix} \in [\hat{\mathcal{K}} \oplus \mathcal{K}_1', H^n \oplus H^n],$$

$$\tilde{Y}_-(0,z) = \begin{pmatrix} -N_0^* & 0 \\ N_1^* & 0 \end{pmatrix} \in [\hat{\mathcal{K}} \oplus \mathcal{K}_0', H^n \oplus H^n],$$

which, in view of (3.29), gives the block representations

$$Y_+(t,\lambda) = (\varphi_N(t,\lambda),0)\colon \hat{\mathcal{K}}\oplus\mathcal{K}_1' \to H, \qquad Y_-(t,z) = (\varphi_N(t,z),0)\colon \hat{\mathcal{K}}\oplus\mathcal{K}_0' \to H. \quad (3.35)$$

Moreover, by (3.26), the operator functions (3.32) have the block representations

$$\mathcal{Z}_+(t,\lambda) = (v_{\mathcal{P}}(t,\lambda), u_+(t,\lambda)), \qquad \mathcal{Z}_-(t,z) = (v_{\mathcal{P}}(t,z), u_+(t,z)) \quad (3.36)$$

with some operator functions $u_+(t,\lambda)$ and $u_-(t,z)$. Now, combining (3.35) and (3.36) with (3.34), we arrive at the equality (3.30). □

3.3. m-functions

Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet (2.22) for L and let $N = (N_0, N_1)$ be an admissible operator pair (3.1). Since $\mathcal{R}(N) = \hat{\mathcal{K}}$, it follows that $\operatorname{Ker} N^* = \{0\}$ and $\mathcal{R}(N^*)$ is a closed subspace in $H^n \oplus H^n$. Therefore, there exist a Hilbert space $\hat{\mathcal{K}}^{\perp}$ and operators $T_j \in [H^n, \hat{\mathcal{K}}^{\perp}]$, $j \in \{0,1\}$, such that the operator

$$W' = \begin{pmatrix} -N_0^* & -T_0^* \\ N_1^* & T_1^* \end{pmatrix} : \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^{\perp} \to H^n \oplus H^n \quad (3.37)$$

is an isomorphism.

Next assume that W' is an isomorphism (3.37) and let $Y_{W'}(\cdot,\lambda)$ $(\in [\hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^{\perp}, H])$ be the operator solution of (2.20) such that $\tilde{Y}_{W'}(0,\lambda) = W'$. Then

$$Y_{W'}(t,\lambda) = (\varphi_N(t,\lambda), \varphi_T(t,\lambda))\colon \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^{\perp} \to H, \quad \lambda \in \mathbb{C}, \quad (3.38)$$

where $\varphi_T(\cdot,\lambda)\colon \Delta \to [\hat{\mathcal{K}}^{\perp}, H]$ is the operator solution of (2.20) given by (3.29) with T in place of N. We also introduce the operator $\mathcal{J}_{W'} = (W')^{-1} J_{H^n} (W')^{-1*}$ $(\in [\hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^{\perp}])$, where J_{H^n} is the operator (2.33). Since $\mathcal{J}_{W'}^* = -\mathcal{J}_{W'}$, the operator $\mathcal{J}_{W'}$ has the block representation

$$\mathcal{J}_{W'} = \begin{pmatrix} \mathcal{J}_1 & -\mathcal{J}_2^* \\ \mathcal{J}_2 & \mathcal{J}_4 \end{pmatrix} : \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^{\perp} \to \hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^{\perp}, \quad (3.39)$$

with $\mathcal{J}_1 = -\mathcal{J}_1^*$ and $\mathcal{J}_4 = -\mathcal{J}_4^*$.

Theorem 3.11. *Assume that the following assumptions are satisfied:*

(i) $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *is a decomposing D-triplet (2.22) for L;*

(ii) $N = (N_0, N_1)$ *is an operator pair (3.1);*

(iii) $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$ *is a collection of holomorphic pairs (3.2)–(3.5);*

(iv) $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ *is the corresponding collection (2.13);*

(v) $\Omega_\tau(\cdot)$ *is the characteristic matrix (2.42).*

Moreover, let W' be an isomorphism (3.37) and let $\varOmega_{\tau,W'}(\cdot)\colon \mathbb{C}\setminus\mathbb{R}\to[\hat{\mathcal{K}}\oplus\hat{\mathcal{K}}^{\perp}]$ be the operator function given by

$$\varOmega_{\tau,W'}(\lambda)=(W')^{-1}\varOmega_{\tau}(\lambda)(W')^{-1*},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{3.40}$$

Then the following hold.

1. *The Green function (3.30) admits the representation*

$$G_{\mathcal{P}}(x,t,\lambda)=Y_{W'}(x,\lambda)(\varOmega_{\tau,W'}(\lambda)+\tfrac{1}{2}\operatorname{sgn}(t-x)\mathcal{J}_{W'})Y^{*}_{W'}(t,\bar{\lambda}). \tag{3.41}$$

2. *The operator function (3.40) has the block representation*

$$\varOmega_{\tau,W'}(\lambda)=\begin{pmatrix} m_{\mathcal{P}}(\lambda) & -\frac{1}{2}\mathcal{J}_2^{*}\\ -\frac{1}{2}\mathcal{J}_2 & 0\end{pmatrix}\colon \hat{\mathcal{K}}\oplus\hat{\mathcal{K}}^{\perp}\to\hat{\mathcal{K}}\oplus\hat{\mathcal{K}}^{\perp},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{3.42}$$

3. *The equality (3.42) generates the holomorphic operator function $m_{\mathcal{P}}(\cdot)\colon\mathbb{C}\setminus\mathbb{R}\to[\hat{\mathcal{K}}]$, which can also be defined by the following statement.*

   There exists a unique operator function $m_{\mathcal{P}}(\cdot)\colon\mathbb{C}\setminus\mathbb{R}\to[\hat{\mathcal{K}}]$ such that, for every $\lambda\in\mathbb{C}\setminus\mathbb{R}$, the operator function

$$v(t,\lambda):=\varphi_N(t,\lambda)(m_{\mathcal{P}}(\lambda)-\tfrac{1}{2}\mathcal{J}_1)-\varphi_T(t,\lambda)\mathcal{J}_2 \tag{3.43}$$

   belongs to $L_2'[\hat{\mathcal{K}},H]$ and satisfies the boundary conditions (3.22)–(3.25).

Proof. 1. The representation (3.41) is immediate from (2.35) and the obvious equality $Y_0(t,\lambda)=Y_{W'}(t,\lambda)(W')^{-1}$, $\lambda\in\mathbb{C}$.

2. Let $v_{\mathcal{P}}(\cdot,\lambda)$ be the operator function defined in Proposition 3.8 and let

$$u(x,\lambda)=(v_{\mathcal{P}}(x,\lambda),0)\colon\hat{\mathcal{K}}\oplus\hat{\mathcal{K}}^{\perp}\to H,\quad \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

Comparing (3.30) with (3.41), one obtains

$$u(x,\lambda)Y^{*}_{W'}(t,\bar{\lambda})=Y_{W'}(x,\lambda)(\varOmega_{\tau,W'}(\lambda)-\tfrac{1}{2}\mathcal{J}_{W'})Y^{*}_{W'}(t,\bar{\lambda}),\quad x>t, \tag{3.44}$$

for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Since $0\in\rho(\tilde{Y}_{W'}(t,\bar{\lambda}))$, it follows from (3.44) that

$$u(x,\lambda)=Y_{W'}(x,\lambda)(\varOmega_{\tau,W'}(\lambda)-\tfrac{1}{2}\mathcal{J}_{W'}),\quad x\in\varDelta,\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{3.45}$$

Next assume that the block representation of the operator function $\varOmega_{\tau,W'}(\lambda)$ is

$$\varOmega_{\tau,W'}(\lambda)=\begin{pmatrix} m_{\mathcal{P}}(\lambda) & \varOmega_3(\lambda)\\ \varOmega_2(\lambda) & \varOmega_4(\lambda)\end{pmatrix}\colon \hat{\mathcal{K}}\oplus\hat{\mathcal{K}}^{\perp}\to\hat{\mathcal{K}}\oplus\hat{\mathcal{K}}^{\perp},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{3.46}$$

Then the equality (3.45) can be written as

$$(v_{\mathcal{P}}(x,\lambda),0)=(\varphi_N(x,\lambda),\varphi_T(x,\lambda))\begin{pmatrix} m_{\mathcal{P}}(\lambda)-\frac{1}{2}\mathcal{J}_1 & \varOmega_3(\lambda)+\frac{1}{2}\mathcal{J}_2^{*}\\ \varOmega_2(\lambda)-\frac{1}{2}\mathcal{J}_2 & \varOmega_4(\lambda)-\frac{1}{2}\mathcal{J}_4\end{pmatrix},$$

which implies the relations

$$v_{\mathcal{P}}(x,\lambda) = \varphi_N(x,\lambda)(m_{\mathcal{P}}(\lambda) - \tfrac{1}{2}\mathcal{J}_1) + \varphi_T(x,\lambda)(\Omega_2(\lambda) - \tfrac{1}{2}\mathcal{J}_2), \tag{3.47}$$

$$\Omega_3(\lambda) + \tfrac{1}{2}\mathcal{J}_2^* = 0, \quad \Omega_4(\lambda) - \tfrac{1}{2}\mathcal{J}_4 = 0, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{3.48}$$

Since $\Omega_\tau(\lambda) = \Omega_\tau^*(\bar\lambda)$, it follows from (3.40) that $\Omega_{\tau,W'}(\lambda) = \Omega_{\tau,W'}^*(\bar\lambda)$ and by (3.46) one has $\Omega_2(\lambda) = \Omega_3^*(\bar\lambda)$, $\Omega_4(\lambda) = \Omega_4^*(\bar\lambda)$. Combining these relations with (3.48) and taking the equality $\mathcal{J}_4 = -\mathcal{J}_4^*$ into account, one obtains

$$\Omega_3(\lambda) = -\tfrac{1}{2}\mathcal{J}_2^*, \quad \Omega_2(\lambda) = -\tfrac{1}{2}\mathcal{J}_2, \quad \Omega_4(\lambda) = 0, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{3.49}$$

Therefore, the block-matrix representation (3.46) takes the form (3.42).

3. In view of (3.47) and the second equality in (3.49), the function $v(\cdot,\lambda) = v_{\mathcal{P}}(\cdot,\lambda)$ admits the representation (3.43). This and Proposition 3.8 give statement 3. □

Definition 3.12. The operator function $m_{\mathcal{P}}(\cdot)$ introduced in Theorem 3.11 will be called an m-function corresponding to the collection $\mathcal{P} \in \mathrm{TR}(\mathcal{H}_0,\mathcal{H}_1)$ or, equivalently, to the boundary-value problem (3.13)–(3.17).

The m-function $m_{\mathcal{P}}(\cdot)$ will be called canonical if $\mathcal{P} \in \mathrm{TR}^0(\mathcal{H}_0,\mathcal{H}_1)$ or, equivalently, if it corresponds to the canonical boundary problem (3.13), (3.18), (3.19).

Remark 3.13. Under the conditions of Theorem 3.11, let W' and $\tilde W'$ be different isomorphisms (3.37) (with the same first column), let $\Omega_{\tau,W'}(\cdot)$ and $\Omega_{\tau,\tilde W'}(\cdot)$ be the corresponding functions (3.40) and let $m_{\mathcal{P}}(\lambda)$ and $\tilde m_{\mathcal{P}}(\lambda)$ be upper left entries in the representations (3.42). One can easily verify that $\tilde m_{\mathcal{P}}(\lambda) = m_{\mathcal{P}}(\lambda) + C$, $C = C^*$, which implies that the m-function $m_{\mathcal{P}}(\cdot)$ is defined by a collection $\mathcal{P} \in \mathrm{TR}(\mathcal{H}_0,\mathcal{H}_1)$ up to the self-adjoint constant.

For a given operator pair (3.1), introduce the operator $N' \in [H^n \oplus H^n, \hat{\mathcal{K}}]$ and the subspaces θ and $\theta^\perp$ in $H^n \oplus H^n$ by

$$N' = (-N_0, N_1)\colon H^n \oplus H^n \to \hat{\mathcal{K}}, \quad \theta^\perp = \operatorname{Ker} N', \quad \theta = (H^n \oplus H^n) \ominus \theta^\perp. \tag{3.50}$$

Clearly, the operator $N_0' := N' \restriction \theta$ isomorphically maps θ onto $\hat{\mathcal{K}}$ and the operator

$$\hat N := (N_0')^{-1}, \quad \hat N \in [\hat{\mathcal{K}}, H^n \oplus H^n], \tag{3.51}$$

is the right inverse for N', i.e. $N'\hat N = I_{\hat{\mathcal{K}}}$.

Proposition 3.14. *Let assumptions (i)–(v) of Theorem 3.11 be satisfied. Then the following hold.*

1. *The m-function $m_{\mathcal{P}}(\cdot)$ is a uniformly strict Nevanlinna function such that*

$$(\operatorname{Im}\lambda)^{-1}\operatorname{Im}(m_{\mathcal{P}}(\lambda)) \geqslant \int_0^b v_{\mathcal{P}}^*(t,\lambda)v_{\mathcal{P}}(t,\lambda)\,\mathrm{d}t, \quad \lambda \in \mathbb{C}_+. \tag{3.52}$$

https://doi.org/10.1017/S0013091512000053 Published online by Cambridge University Press

Moreover, the canonical m-function $m_{\mathcal{P}}(\cdot)$ satisfies the identity

$$m_{\mathcal{P}}(\mu) - m_{\mathcal{P}}^*(\lambda) = (\mu - \bar{\lambda}) \int_0^b v_{\mathcal{P}}^*(t,\lambda) v_{\mathcal{P}}(t,\mu)\,\mathrm{d}t, \quad \mu,\lambda \in \mathbb{C}_+, \tag{3.53}$$

where

$$\int_0^b v_{\mathcal{P}}^*(t,\lambda) v_{\mathcal{P}}(t,\mu)\,\mathrm{d}t = s - \lim_{\eta \uparrow b} \int_0^\eta v_{\mathcal{P}}^*(t,\lambda) v_{\mathcal{P}}(t,\mu)\,\mathrm{d}t.$$

Formula (3.53) implies that, for the canonical m-function $m_{\mathcal{P}}(\cdot)$, the inequality (3.52) turns into the equality.

2. *The characteristic matrix $\Omega_\tau(\cdot)$ admits the representation*

$$\Omega_\tau(\lambda) = \begin{pmatrix} \Omega_0(\lambda) & \Omega_1^* \\ \Omega_1 & \Omega_2 \end{pmatrix} : \theta \oplus \theta^\perp \to \theta \oplus \theta^\perp, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{3.54}$$

where $\Omega_2 = \Omega_2^* \in [\theta^\perp]$ and $\Omega_0(\cdot)\colon \mathbb{C}\setminus\mathbb{R} \to [\theta]$ is a uniformly strict Nevanlinna function associated with $m_{\mathcal{P}}(\cdot)$ by

$$\Omega_0(\lambda) = N_0'^* m_{\mathcal{P}}(\lambda) N_0' + C, \quad C = C^* \in [\theta]. \tag{3.55}$$

Moreover, the following equality holds

$$m_{\mathcal{P}}(\lambda) = \hat{N}^* \Omega_\tau(\lambda) \hat{N} + \hat{C}, \qquad \hat{C} = \hat{C}^* \in [\hat{\mathcal{K}}],\ \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{3.56}$$

Proof. Let W' be an isomorphism (3.37) and let $\Omega_{\tau,W'}(\lambda)$ be the operator function (3.40). Then $\Omega_\tau(\lambda) = W'\Omega_{\tau,W'}(\lambda)W'^*$ and the immediate calculation with taking (3.42) into account shows that

$$\Omega_\tau(\lambda) = N'^* m_{\mathcal{P}}(\lambda) N' + \tilde{C} \tag{3.57}$$

with some $\tilde{C} = \tilde{C}^* \in [H^n \oplus H^n]$. Multiplying the equality (3.57) by $\hat{N}^*$ from the left and by $\hat{N}$ from the right one obtains (3.56). Therefore, $m_{\mathcal{P}}(\cdot)$ is a Nevanlinna function.

Next assume that $\gamma_+(\lambda)$ is the isomorphism from Proposition 2.7 (2) and let $\gamma_\tau(\lambda) \in [H^n \oplus H^n, \mathfrak{N}_\lambda(L_0)]$ be the operator given by

$$\gamma_\tau(\lambda) = \gamma_+(\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}(-\hat{C}_0(\lambda)\colon \hat{C}_1(\lambda)), \quad \lambda \in \mathbb{C}_+. \tag{3.58}$$

Then by (3.56) and [**29**, Proposition 23]

$$(\operatorname{Im}\lambda)^{-1} \cdot \operatorname{Im}(m_{\mathcal{P}}(\lambda)) \geqslant \hat{N}^* \gamma_\tau^*(\lambda)\gamma_\tau(\lambda)\hat{N} = \gamma_c^*(\lambda)\gamma_c(\lambda), \quad \lambda \in \mathbb{C}_+, \tag{3.59}$$

where $\gamma_c(\lambda) = \gamma_\tau(\lambda)\hat{N} (\in [\hat{\mathcal{K}}, \mathfrak{N}_\lambda(L_0)])$. Moreover, the canonical m-function $m_{\mathcal{P}}(\cdot)$ satisfies the identity

$$m_{\mathcal{P}}(\mu) - m_{\mathcal{P}}^*(\lambda) = (\mu - \bar{\lambda})\gamma_c^*(\lambda)\gamma_c(\mu), \quad \mu,\lambda \in \mathbb{C}_+. \tag{3.60}$$

It follows from (3.2) and (3.3) that the operator $(-\hat{C}_0(\lambda)\colon \hat{C}_1(\lambda))$ in (3.58) coincides with N'. This and the equality $N'\hat{N} = I_{\hat{\mathcal{K}}}$ imply that

$$\gamma_c(\lambda) = \gamma_+(\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1} \restriction \hat{\mathcal{K}}, \tag{3.61}$$

and, consequently, $0 \in \rho(\gamma_c^*(\lambda)\gamma_c(\lambda))$. Therefore, by (3.59) the Nevanlinna function $m_{\mathcal{P}}(\cdot)$ is uniformly strict. Moreover, in view of (3.61) and (3.26) one has $(\gamma_c(\lambda)\hat{h})(t) = v_{\mathcal{P}}(t,\lambda)\hat{h}(\hat{h} \in \hat{\mathcal{K}})$. Now, applying [**28**, Lemma 4.1 (3)] to the operator function $v_{\mathcal{P}}(t,\lambda)$, one obtains

$$\gamma_c^*(\lambda)\gamma_c(\mu) = \int_0^b v_{\mathcal{P}}^*(t,\lambda)v_{\mathcal{P}}(t,\mu)\,\mathrm{d}t := s - \lim_{\eta\uparrow b}\int_0^\eta v_{\mathcal{P}}^*(t,\lambda)v_{\mathcal{P}}(t,\mu)\,\mathrm{d}t.$$

Combining this relation with (3.59) and (3.60), we arrive at (3.52) and (3.53).

Finally, the equality (3.54) is immediate from (3.57) and the block representation $N' = (N_0', 0)\colon \theta \oplus \theta^\perp \to \hat{\mathcal{K}}$. □

Corollary 3.15. *Let assumptions (i)–(v) of Theorem 3.11 be satisfied. Then the following statements are equivalent:*

(i) *the characteristic matrix $\Omega_\tau(\cdot)$ is a uniformly strict Nevanlinna function;*

(ii) *the operator $N = (N_0, N_1)$ in (3.1) isomorphically maps $H^n \oplus H^n$ onto $\hat{\mathcal{K}}$.*

If in addition $\dim H < \infty$, then statement (i) is equivalent to the following:

(iii) *the operator L_0 has maximal deficiency indices $n_+(L_0) = n_-(L_0) = 2n\dim H$, $\mathcal{H}_0' = \mathcal{H}_1' =: \mathcal{H}'$ (i.e. $\Pi = \{H^n \oplus \mathcal{H}', \Gamma_0, \Gamma_1\}$ is a decomposing boundary triplet for L), $\dim \mathcal{H}' = n\dim H$ and the collection $\mathcal{P}$ can be represented as the holomorphic Nevanlinna pair (see Remark 2.4 (2)) $C(\lambda) = (C_0(\lambda), C_1(\lambda))$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$,*

$$C_0(\lambda) = (N_0, C_0'(\lambda))\colon H^n \oplus \mathcal{H}' \to \hat{\mathcal{K}}, \qquad C_1(\lambda) = (N_1, C_1'(\lambda))\colon H^n \oplus \mathcal{H}' \to \hat{\mathcal{K}}, \tag{3.62}$$

where $\dim \hat{\mathcal{K}} = 2n\dim H$ and the operator $N = (N_0, N_1)\colon H^n \oplus H^n \to \hat{\mathcal{K}}$ is an isomorphism.

Proof. It follows from (3.54) that the Nevanlinna function $\Omega_\tau(\cdot)$ is uniformly strict if and only if $\theta^\perp = \{0\}$. Moreover, by (3.50), one has $\theta^\perp = \{0\} \Longleftrightarrow \operatorname{Ker} N\ (= \operatorname{Ker} N') = \{0\}$. This yields the equivalence (i) $\Longleftrightarrow$ (ii).

Next assume that $\dim H < \infty$ and prove the equivalence (ii) $\Longleftrightarrow$ (iii). If $0 \in \rho(N)$, then $\dim \hat{\mathcal{K}} = \dim(H^n \oplus H^n) = 2n\dim H$ and by (3.12) $n_-(L_0) = n_+(L_0) = 2n\dim H$. This and the second relation in (2.25) imply that $\dim \mathcal{H}_1 = \dim \mathcal{H}_0 = 2n\dim H$, and hence $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$. Therefore, $\mathcal{H}_0' = \mathcal{H}_1'$ and $\Pi = \{H^n \oplus \mathcal{H}', \Gamma_0, \Gamma_1\}$ is a decomposing boundary triplet for L. Moreover, by (2.15), the Hilbert spaces $\hat{\mathcal{K}} \oplus \mathcal{K}_j'$ in (3.2)–(3.5) satisfy the equalities $\dim(\hat{\mathcal{K}} \oplus \mathcal{K}_j') = \dim \mathcal{H} = 2n\dim H = \dim \hat{\mathcal{K}}$. Hence, $\mathcal{K}_j' = \{0\}$, $j \in \{0, 1\}$, and the equalities (3.2)–(3.5) take the form (3.62), which yields the implication (ii) $\Longrightarrow$ (iii). The inverse implication (iii) $\Longrightarrow$ (ii) is obvious. Thus, in the case $\dim H < \infty$, the equivalences (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) hold. □

https://doi.org/10.1017/S0013091512000053 Published online by Cambridge University Press

Remark 3.16.

1. It follows from Corollary 3.15 that in the case $\dim H < \infty$ and $n_-(L_0) < 2n \dim H$ the characteristic matrix $\Omega_\tau(\cdot)$ corresponding to the boundary operators (3.2)–(3.5) is not a uniformly strict Nevanlinna function. In particular, by Proposition 3.2, this statement holds for each canonical characteristic matrix corresponding to the constant Nevanlinna collection (2.10).

2. Let $\mathcal{P}_0 \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$ be the collection (3.2)–(3.5) with $\hat{\mathcal{K}} = H^n$, $\mathcal{K}'_j = \mathcal{H}'_j$, $j \in \{0, 1\}$, and $C_0(\lambda) = I_{\mathcal{H}_0}$, $C_1(\lambda) = 0_{\mathcal{H}_1, \mathcal{H}_0}$, $D_0 = P_1$ and $D_1 = 0_{\mathcal{H}_1}$. Then the corresponding m-function $m_{\mathcal{P}_0}(\cdot)$ coincides with the operator function $m(\cdot)$ defined by (2.29) and (2.30). Note in this connection that the statements of Theorem 3.11 and Proposition 3.14 for $m(\lambda)$ $(= m_{\mathcal{P}_0}(\lambda))$ were obtained in [**27**].

3. One can prove that, in the case of a decomposing boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, each canonical m-function $m_{\mathcal{P}}(\cdot)$ is the Weyl function of some symmetric extension $\tilde{A} \supset L_0$, while a uniformly strict canonical characteristic matrix $\Omega_\tau(\cdot)$ is the Weyl function of the minimal operator L_0. More precisely, this means that one can construct the boundary triplet for $\tilde{A}^*$ (respectively, for L) such that $m_{\mathcal{P}}(\cdot)$ (respectively, $\Omega_\tau(\cdot)$) is the Weyl function for this triplet in the sense of [**4**].

4. Spectral functions of differential operators

4.1. The space $L_2(\Sigma; \mathcal{H})$

Let $\mathcal{H}$ be a separable Hilbert space.

Definition 4.1. A non-decreasing operator function $\Sigma\colon \mathbb{R} \to [\mathcal{H}]$ is called a distribution function if it is strongly left continuous and satisfies the equality $\Sigma(0) = 0$.

Let $\Sigma\colon \mathbb{R} \to [\mathcal{H}]$ be a distribution function and let $f(\cdot)$, $g(\cdot)$ be vector functions defined on the segment $[\alpha, \beta]$ with values in $\mathcal{H}$. Consider the Riemann–Stieltjes integral [**1**]

$$\int_\alpha^\beta (\mathrm{d}\Sigma(t) f(t), g(t)) = \lim_{d_\pi \to 0} \sum_{k=1}^{n} ((\Sigma(t_k) - \Sigma(t_{k-1})) f(\xi_k), g(\xi_k)), \tag{4.1}$$

where $\pi = \{\alpha = t_0 < t_1 < \cdots < t_n = \beta\}$ is a partition of $[\alpha, \beta]$, $\xi_k \in [t_{k-1}, t_k]$ and d_π is the diameter of π. As is known (see, for example, [**22**]), in the case $\dim \mathcal{H} = \infty$ there exist a distribution function $\Sigma(\cdot)$ and continuous functions $f(\cdot)$ and $g(\cdot)$ for which the integral (4.1) does not exist. At the same time, the holomorphy of $f(\cdot)$ and $g(\cdot)$ on $[\alpha, \beta]$ is a sufficient condition for the existence of such an integral [**34**].

Definition 4.2. A function $f\colon [\alpha, \beta) \to \mathcal{H}$ will be called piecewise holomorphic if there is a partition $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ such that each restriction $f \restriction [t_{k-1}, t_k)$ admits a holomorphic continuation $\tilde{f}_k(\cdot)$ on some interval $(\tilde{t}_{k-1}, \tilde{t}_k) \supset [t_{k-1}, t_k]$.

A function $f\colon \mathbb{R} \to \mathcal{H}$ will be called piecewise holomorphic if it is piecewise holomorphic on each finite half-interval $[\alpha, \beta)$.

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It follows from Definition 4.2 that a piecewise holomorphic function is strongly right continuous.

Let $\Sigma\colon \mathbb{R} \to [\mathcal{H}]$ be a distribution function and let $f, g\colon [\alpha,\beta) \to \mathcal{H}$ be a pair of piecewise holomorphic functions. It is clear that there exists a partition of $[\alpha,\beta]$ satisfying the conditions of Definition 4.2 for both functions $f(\cdot)$ and $g(\cdot)$. By using such a partition, we introduce the integral

$$\int_{[\alpha,\beta)} (\mathrm{d}\Sigma(t)f(t), g(t)) = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (\mathrm{d}\Sigma(t)\tilde{f}_k(t), \tilde{g}_k(t)). \tag{4.2}$$

Note that for a pair of continuous functions $f, g\colon [\alpha,\beta] \to \mathcal{H}$ piecewise holomorphic on $[\alpha,\beta)$ there exists the integral (4.1) which coincides with that of (4.2).

For a given distribution function $\Sigma\colon \mathbb{R} \to [\mathcal{H}]$ denote by $\mathrm{Hol}(\Sigma,\mathcal{H})$ the set of all piecewise holomorphic functions $f\colon \mathbb{R} \to \mathcal{H}$ such that

$$\int_{\mathbb{R}} (\mathrm{d}\Sigma(t)f(t), f(t)) := \lim_{[\alpha,\beta)\to\mathbb{R}} \int_{[\alpha,\beta)} (\mathrm{d}\Sigma(t)f(t), f(t)) < \infty.$$

One can easily prove that for each pair $f, g \in \mathrm{Hol}(\Sigma,\mathcal{H})$ there exists the integral

$$(f,g)_{\mathrm{Hol}(\Sigma,\mathcal{H})} = \int_{\mathbb{R}} (\mathrm{d}\Sigma(t)f(t), g(t)) := \lim_{[\alpha,\beta)\to\mathbb{R}} \int_{[\alpha,\beta)} (\mathrm{d}\Sigma(t)f(t), g(t)). \tag{4.3}$$

This implies that $\mathrm{Hol}(\Sigma,\mathcal{H})$ is a linear space with the semi-definite scalar product (4.3).

Next recall the definition of the space $L_2(\Sigma; H)$ as given in [**1**].

A function $f\colon \mathbb{R} \to \mathcal{H}$ is called finite dimensional if there is a subspace $\mathcal{H}_f \subset \mathcal{H}$ such that $\dim \mathcal{H}_f < \infty$ and $f(t) \in \mathcal{H}_f$, $t \in \mathbb{R}$. For a given distribution function $\Sigma\colon \mathbb{R} \to [\mathcal{H}]$ denote by $C_{00}(\mathcal{H})$ the linear space of all strongly continuous finite-dimensional functions $f\colon \mathbb{R} \to \mathcal{H}$ with compact support $\operatorname{supp} f$. Clearly, the equality

$$(f,g)_{L_2(\Sigma;\mathcal{H})} = \int_{\mathbb{R}} (\mathrm{d}\Sigma(t)f(t), g(t)) := \int_{\alpha}^{\beta} (\mathrm{d}\Sigma(t)f(t), g(t)), \quad f, g \in C_{00}(\mathcal{H}), \tag{4.4}$$

with $[\alpha,\beta] \supset (\operatorname{supp} f \cup \operatorname{supp} g)$ defines the semi-definite scalar product on $C_{00}(\mathcal{H})$. The completion of $C_{00}(\mathcal{H})$ with respect to this product is a semi-Hilbert space $\tilde{L}_2(\Sigma;\mathcal{H})$. The quotient of $\tilde{L}_2(\Sigma;\mathcal{H})$ over the kernel $\{f \in \tilde{L}_2(\Sigma;\mathcal{H})\colon (f,f)_{L_2(\Sigma;\mathcal{H})} = 0\}$ is the Hilbert space $L_2(\Sigma; H)$.

Denote by $\mathrm{Hol}_0(\Sigma,\mathcal{H})$ the set of all strongly continuous, piecewise holomorphic and finite-dimensional functions $f\colon \mathbb{R} \to \mathcal{H}$ with a compact support. It is clear that $\mathrm{Hol}_0(\Sigma,\mathcal{H}) = \mathrm{Hol}(\Sigma,\mathcal{H}) \cap C_{00}(\mathcal{H})$ and, consequently, $\mathrm{Hol}_0(\Sigma,\mathcal{H})$ is a linear manifold both in $\mathrm{Hol}(\Sigma,\mathcal{H})$ and $C_{00}(\mathcal{H})$. Moreover, the semi-scalar products (4.3) and (4.4) coincide on $\mathrm{Hol}_0(\Sigma,\mathcal{H})$.

By using the Taylor expansions of the function $f \in \mathrm{Hol}(\Sigma,\mathcal{H})$ one can prove the following proposition.

Proposition 4.3. *The set* $\mathrm{Hol}_0(\Sigma,\mathcal{H})$ *is a dense linear manifold both in* $\mathrm{Hol}(\Sigma,\mathcal{H})$ *and* $C_{00}(\mathcal{H})$*, which implies that the closure of* $\mathrm{Hol}(\Sigma,\mathcal{H})$ *coincides with* $\tilde{L}_2(\Sigma;\mathcal{H})$*. In other words, the semi-Hilbert space* $\tilde{L}_2(\Sigma;\mathcal{H})$ *can be considered as the completion of* $\mathrm{Hol}(\Sigma,\mathcal{H})$*.*

Remark 4.4. In connection with Proposition 4.3, note that the intrinsic functional description of the spaces $\tilde{L}_2(\Sigma;\mathcal{H})$ and $L_2(\Sigma;\mathcal{H})$ in the case $\dim\mathcal{H}<\infty$ was obtained in [**15**]. Moreover, in the case $\dim\mathcal{H}=\infty$, the description of these spaces in terms of the direct integrals of Hilbert spaces can be found in [**22**].

4.2. Spectral functions

Let $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ be a decomposing D-triplet (2.22) for L and let $\tau=\{\tau_+,\tau_-\}\in\tilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ be a Nevanlinna collection defined by (2.13) and (2.36), (2.37). For this collection consider the boundary problem (2.38)–(2.40). According to Remark 2.12, this problem defines the spectral function $F_\tau(t)$ of the operator L_0.

Next assume that $\tilde{\mathcal{K}}$ is a separable Hilbert space and $\varphi(\cdot,\lambda)\colon\Delta\to[\tilde{\mathcal{K}},H]$ is an operator solution of the equation (2.20) with the constant initial data $\tilde{\varphi}(0,\lambda)\equiv\tilde{\varphi}_0(\in[\tilde{\mathcal{K}},H^{2n}])$, $\lambda\in\mathbb{C}$, such that $0\in\hat{\rho}(\tilde{\varphi}_0)$. Denote by $\mathfrak{H}_0$ the set of all functions $f\in\mathfrak{H}$ $(=L_2(\Delta;H))$ with $\operatorname{supp} f\subset[0,\beta]$ ($\beta<b$ depends on f) and consider the Fourier transform $g_f\colon\mathbb{R}\to\tilde{\mathcal{K}}$ of a function $f\in\mathfrak{H}_0$ given by

$$g_f(s)=\int_0^b\varphi^*(t,s)f(t)\,\mathrm{d}t. \tag{4.5}$$

Definition 4.5. A distribution function $\Sigma(\cdot)=\Sigma_{\tau,\varphi}(\cdot)\colon\mathbb{R}\to[\tilde{\mathcal{K}}]$ is called a spectral function of the boundary problem (2.38)–(2.40) corresponding to the solution $\varphi(\cdot,\lambda)$ if, for each function $f\in\mathfrak{H}_0$, the Fourier transform (4.5) satisfies the equality

$$((F_\tau(\beta)-F_\tau(\alpha))f,f)_{\mathfrak{H}}=\int_{[\alpha,\beta)}(\mathrm{d}\Sigma_{\tau,\varphi}(s)g_f(s),g_f(s)),\quad [\alpha,\beta)\subset\mathbb{R}, \tag{4.6}$$

where $F_\tau(\cdot)$ is a spectral function of the operator L_0 (see Remark 2.12).

Note that the integral on the right-hand side of (4.6) exists because the function $g_f(\cdot)$ is holomorphic on $\mathbb{R}$. Moreover, by (4.6), $g_f(\cdot)\in\mathrm{Hol}(\Sigma_{\tau,\varphi},\tilde{\mathcal{K}})$ and the following Parseval equality holds:

$$(\|f\|_{\mathfrak{H}}^2=)\ \int_0^b\|f(t)\|_H^2\,\mathrm{d}t=\int_{\mathbb{R}}(\mathrm{d}\Sigma_{\tau,\varphi}(s)g_f(s),g_f(s))\ (=\|g_f\|^2_{L_2(\Sigma_{\tau,\varphi};\tilde{\mathcal{K}})}),\quad f\in\mathfrak{H}_0.$$

This implies that the linear operator $V\colon\mathfrak{H}\to L_2(\Sigma_{\tau,\varphi};\tilde{\mathcal{K}})$ defined on the dense linear manifold $\mathfrak{H}_0\subset\mathfrak{H}$ by $(Vf)(s)=g_f(s)$ is an isometry.

Definition 4.6. A spectral function $\Sigma_{\tau,\varphi}(\cdot)$ is called orthogonal if $V\mathfrak{H}=L_2(\Sigma_{\tau,\varphi};\tilde{\mathcal{K}})$ or, equivalently, if the set of all Fourier transforms $\{g_f(\cdot)\colon f\in\mathfrak{H}_0\}$ is dense in $L_2(\Sigma_{\tau,\varphi};\tilde{\mathcal{K}})$.

Theorem 4.7. *Let $\tilde{H}$ be a Hilbert space with $\dim \tilde{H} = 2n \cdot \dim H$, let $W \in [\tilde{H}, H^{2n}]$ be an isomorphism and let $Y_W(\cdot, \lambda)\colon \Delta \to [\tilde{H}, H]$ be an operator solution of (2.20) with the initial data $\tilde{Y}_W(0, \lambda) = W$, $\lambda \in \mathbb{C}$. Then, for each collection $\tau \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$, there exists a unique spectral function $\Sigma_{\tau,W}\colon \mathbb{R} \to [\tilde{H}]$ of the boundary problem (2.38)–(2.40) corresponding to the solution $Y_W(\cdot, \lambda)$. This function is defined by the equality*

$$\Sigma_{\tau,W}(s) = s - \lim_{\delta \to +0} w - \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \operatorname{Im} \Omega_{\tau,W}(\sigma + \mathrm{i}\varepsilon)\, \mathrm{d}\sigma, \tag{4.7}$$

where $\Omega_{\tau,W}\colon \mathbb{C} \setminus \mathbb{R} \to [\tilde{H}]$ is a Nevanlinna operator function given by

$$\Omega_{\tau,W}(\lambda) = W^{-1} \Omega_\tau(\lambda) W^{-1*}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.8}$$

Moreover, the spectral function $\Sigma_{\tau,W}(\cdot)$ is orthogonal if and only if $\tau \in \tilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$.

One can prove Theorem 4.7 by using the Stieltjes–Livšic formula [**17**, **34**] in the same way as in [**35**] (the scalar case $\dim H = 1$) and [**3**] (the case $\dim H \leqslant \infty$). Moreover, in the scalar case, other methods of proof can be found in [**8**, **30**].

Theorem 4.8. *Assume that, under the conditions of Theorem 4.7, $\Sigma_{\tau,W}(\cdot)$ is a spectral function of the boundary problem (2.38)–(2.40) and $V\colon \mathfrak{H} \to L_2(\Sigma_{\tau,W}; \tilde{H})$ is the corresponding isometry given by the Fourier transform (4.5) with $\varphi(t,s) = Y_W(t,s)$. Moreover, let $\operatorname{Hol}^0(\tilde{H})$ be the linear manifold of all piecewise holomorphic functions $g\colon \mathbb{R} \to \tilde{H}$ with compact support. Then $\operatorname{Hol}^0(\tilde{H})$ is dense in $L_2(\Sigma_{\tau,W}; \tilde{H})$ and*

$$(V^* g)(t) = \int_{\mathbb{R}} Y_W(t,s)\, \mathrm{d}\Sigma_{\tau,W}(s) g(s), \quad g = g(s) \in \operatorname{Hol}^0(\tilde{H}), \tag{4.9}$$

where $V^*\colon L_2(\Sigma_{\tau,W}; \tilde{H}) \to \mathfrak{H}$ is the adjoint operator and, similary to (4.2), the integral is understood as the sum of integrals. In particular, (4.9) implies that the inverse Fourier transform is

$$f(t) = \int_{\mathbb{R}} Y_W(t,s)\, \mathrm{d}\Sigma_{\tau,W}(s) g_f(s). \tag{4.10}$$

In the case $\dim H < \infty$, the proof of Theorem 4.8 can be found in [**8**, **30**, **35**]. In the case $\dim H = \infty$, a somewhat weaker result (only the inverse transform (4.10)) is contained in [**3**] (without the detailed proof). In this connection, note that in the case $\dim H = \infty$ the piecewise holomorphy of a function $g(\cdot)$ is essential, because otherwise the integral in (4.9) may not exist. We omit the proof of Theorem 4.8 for the case $\dim H = \infty$ because it is very technical and tedious.

Our next goal is to obtain a description of all spectral functions $\Sigma_{\tau,W}(\cdot)$ immediately in terms of a boundary parameter τ. Namely, using the block representations (2.29) and (2.30) of the Weyl functions $M_\pm(\cdot)$, we introduce the operator functions

$\Omega_{\tau_0}(\lambda)$ $(\in [H^{2n}])$, $S_+(\lambda)$ $(\in [\mathcal{H}_0, H^{2n}])$ and $S_-(z)$ $(\in [\mathcal{H}_1, H^{2n}])$ by setting

$$\Omega_{\tau_0}(\lambda) = \begin{pmatrix} m(\lambda) & -\frac{1}{2}I_{H^n} \\ -\frac{1}{2}I_{H^n} & 0 \end{pmatrix} : H^n \oplus H^n \to H^n \oplus H^n, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{4.11}$$

$$S_+(\lambda) = \begin{pmatrix} -m(\lambda) & -M_{2+}(\lambda) \\ I_{H^n} & 0 \end{pmatrix} : H^n \oplus \mathcal{H}_0' \to H^n \oplus H^n, \quad \lambda \in \mathbb{C}_+, \tag{4.12}$$

$$S_-(z) = \begin{pmatrix} -m(z) & -M_{2-}(z) \\ I_{H^n} & 0 \end{pmatrix} : H^n \oplus \mathcal{H}_1' \to H^n \oplus H^n, \quad z \in \mathbb{C}_-. \tag{4.13}$$

In the case of a decomposing boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ (i.e. if $\mathcal{H}_0' = \mathcal{H}_1' =: \mathcal{H}'$), in place of (4.12) and (4.13) we let

$$S(\lambda) = \begin{pmatrix} -m(\lambda) & -M_2(\lambda) \\ I_{H^n} & 0 \end{pmatrix} : H^n \oplus \mathcal{H}' \to H^n \oplus H^n, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{4.14}$$

where $m(\lambda)$ and $M_2(\lambda)$ are taken from the block representation (2.32) of $M(\lambda)$.

Note that $\Omega_{\tau_0}(\lambda)$ is a characteristic matrix corresponding to the collection $\tau_0 = \{\tau_{0+}, \tau_{0-}\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ with $\tau_{0+} = \{0\} \oplus \mathcal{H}_1 (\in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1))$.

Theorem 4.9. *Let the assumptions of Theorem 4.7 be satisfied. Let*

$$\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$$

be a decomposing D-triplet (2.22) for L and let $\Omega_{\tau_0,W}(\lambda)$ $(\in [\tilde{H}])$, $S_{W,+}(\lambda)$ $(\in [\mathcal{H}_0, \tilde{H}])$ *and* $\Sigma_{W,-}(z)$ $(\in [\mathcal{H}_1, \tilde{H}])$ *be the operator functions given by*

$$\begin{aligned} \Omega_{\tau_0,W}(\lambda) &= W^{-1}\Omega_{\tau_0}(\lambda)W^{-1*}, & \lambda &\in \mathbb{C} \setminus \mathbb{R}, \\ S_{W,+}(\lambda) &= W^{-1}S_+(\lambda), & \lambda &\in \mathbb{C}_+, \\ S_{W,-}(z) &= W^{-1}S_-(z), & z &\in \mathbb{C}_-. \end{aligned}$$

Then, for each collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$, *the equality*

$$\Omega_{\tau,W}(\lambda) = \Omega_{\tau_0,W}(\lambda) - S_{W,+}(\lambda)(\tau_+(\lambda) + M_+(\lambda))^{-1}S_{W,-}^*(\bar{\lambda}), \quad \lambda \in \mathbb{C}_+, \tag{4.15}$$

together with (4.7), defines a (unique) spectral function $\Sigma_{\tau,W}(\cdot)$ *of the boundary problem (2.38)–(2.40) corresponding to the solution* $Y_W(\cdot, \lambda)$. *Moreover, a spectral function* $\Sigma_{\tau,W}(\cdot)$ *is orthogonal if and only if* $\tau \in \tilde{R}^0(\mathcal{H}_0, \mathcal{H}_1)$.

If in addition $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ *(i.e.* $n_+(L_0) = n_-(L_0)$ *and* Π *is a decomposing boundary triplet), then (4.15) can be written in the simpler form:*

$$\Omega_{\tau,W}(\lambda) = \Omega_{\tau_0,W}(\lambda) - S_W(\lambda)(\tau(\lambda) + M(\lambda))^{-1}S_W^*(\bar{\lambda}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $\tau(\lambda) \in \tilde{R}(\mathcal{H})$ *and* $S_W(\lambda) = W^{-1}S(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. According to [**29**], for each collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$, the corresponding characteristic matrix $\Omega_\tau(\cdot)$ is given by

$$\Omega_\tau(\lambda) = \Omega_{\tau_0}(\lambda) - S_+(\lambda)(\tau_+(\lambda) + M_+(\lambda))^{-1}S_-^*(\bar{\lambda}), \quad \lambda \in \mathbb{C}_+. \tag{4.16}$$

This and Theorem 4.7 yield the desired statement. □

https://doi.org/10.1017/S0013091512000053 Published online by Cambridge University Press

4.3. Minimal spectral functions

We start the subsection with the following lemma, which is immediate from Theorem 4.7.

Lemma 4.10. *Let $\Sigma_{\tau,\varphi}\colon \mathbb{R} \to [\tilde{\mathcal{K}}]$ be a spectral function of the boundary problem (2.38)–(2.40), corresponding to the solution $\varphi(t,\lambda)$ $(\in [\tilde{\mathcal{K}}, H])$ of (2.20) (see Definition 4.5). Assume also that $\tilde{H} \supset \tilde{\mathcal{K}}$, $\tilde{\mathcal{K}}^{\perp} = \tilde{H} \ominus \tilde{\mathcal{K}}$ and $Y_W(\cdot,\lambda)$ $(\in [\tilde{H}, H])$ is a solution of (2.20) satisfying the conditions of Theorem 4.7 and the equality $Y_W(t,\lambda) \upharpoonright \tilde{\mathcal{K}} = \varphi(t,\lambda)$ (such a solution exists because $0 \in \hat{\rho}(\tilde{\varphi}(0,\lambda))$). Then the (unique) spectral function of the boundary problem (2.38)–(2.40) corresponding to $Y_W(\cdot,\lambda)$ is*

$$\Sigma_{\tau,W}(s) = \begin{pmatrix} \Sigma_{\tau,\varphi}(s) & 0 \\ 0 & 0 \end{pmatrix} : \tilde{\mathcal{K}} \oplus \tilde{\mathcal{K}}^{\perp} \to \tilde{\mathcal{K}} \oplus \tilde{\mathcal{K}}^{\perp}, \tag{4.17}$$

which implies that the spectral function $\Sigma_{\tau,\varphi}$ is unique.

Conversely, if a spectral function $\Sigma_{\tau,W}$ is of the form (4.17), then $\Sigma_{\tau,\varphi}(s)$ is a spectral function corresponding to $\varphi(\cdot,\lambda)$.

Now, combining Theorems 4.7 and 4.8 with Lemma 4.10 and taking the equality (3.42) into account, one may derive the following theorem.

Theorem 4.11. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet (2.22) for L, let $N = (N_0, N_1)$ be an admissible operator pair (3.1) and let $\varphi_N(t,\lambda)$ $(\in [\hat{\mathcal{K}}, H])$ be the operator solution of the equation (2.20) with the initial data (3.29). Then the following hold.*

1. *For each collection $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$ of holomorphic pairs (3.2)–(3.5) there exists a unique spectral function $\Sigma_{\mathcal{P},N}\colon \mathbb{R} \to [\hat{\mathcal{K}}]$ of the boundary problem (3.13)–(3.17) corresponding to $\varphi_N(\cdot,\lambda)$. This function is given by*

$$\Sigma_{\mathcal{P},N}(s) = s - \lim_{\delta\to+0} w - \lim_{\varepsilon\to+0} \frac{1}{\pi}\int_{-\delta}^{s-\delta} \operatorname{Im} m_{\mathcal{P}}(\sigma + \mathrm{i}\varepsilon)\,\mathrm{d}\sigma, \tag{4.18}$$

where $m_{\mathcal{P}}(\lambda)$ is the m-function corresponding to the boundary problem (3.13)–(3.17). Moreover, the spectral function $\Sigma_{\mathcal{P},N}$ is orthogonal if and only if $\mathcal{P} \in \mathrm{TR}^0(\mathcal{H}_0, \mathcal{H}_1)$.

2. *Let $\Sigma_{\mathcal{P},N}(\cdot)$ be a spectral function and let $V\colon \mathfrak{H} \to L_2(\Sigma_{\mathcal{P},N}; \hat{\mathcal{K}})$ be an isometry given by the Fourier transform (4.5) with $\varphi(t,s) = \varphi_N(t,s)$. Then*

$$(V^*g)(t) = \int_{\mathbb{R}} \varphi_N(t,s)\,\mathrm{d}\Sigma_{\mathcal{P},N}(s) g(s), \quad g = g(s) \in \mathrm{Hol}^0(\hat{\mathcal{K}}).$$

In particular, the inverse Fourier transform is

$$f(t) = \int_{\mathbb{R}} \varphi_N(t,s)\,\mathrm{d}\Sigma_{\mathcal{P},N}(s) g_f(s).$$

https://doi.org/10.1017/S0013091512000053 Published online by Cambridge University Press

In the next theorem we give a parametrization of all spectral functions $\Sigma_{\mathcal{P},N}(\cdot)$ in terms of a boundary parameter $\mathcal{P} \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$.

Theorem 4.12. *Let the assumptions of Theorem 4.11 be satisfied, let $\hat{N}$ be the operator (3.51) and let $T_{N,0}\colon \mathbb{C}\setminus\mathbb{R} \to [\hat{\mathcal{K}}]$, $T_{N,+}\colon \mathbb{C}_+ \to [\mathcal{H}_0, \hat{\mathcal{K}}]$ and $T_{N,-}\colon \mathbb{C}_- \to [\mathcal{H}_1, \hat{\mathcal{K}}]$ be the operator functions defined by*

$$\begin{aligned} T_{N,0}(\lambda) &= \hat{N}^* \Omega_{\tau_0}(\lambda)\hat{N}, \quad && \lambda \in \mathbb{C}\setminus\mathbb{R}, \\ T_{N,+}(\lambda) &= \hat{N}^* S_+(\lambda), && \lambda \in \mathbb{C}_+, \\ T_{N,-}(z) &= \hat{N}^* S_-(z), && z \in \mathbb{C}_-. \end{aligned} \tag{4.19}$$

Then, for each collection $\mathcal{P} \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$ given by (3.2)–(3.5), the equality

$$m_{\mathcal{P}}(\lambda) = T_{N,0}(\lambda) + T_{N,+}(\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}C_1(\lambda)T^*_{N,-}(\bar{\lambda}), \quad \lambda \in \mathbb{C}_+, \tag{4.20}$$

together with (4.18), defines a (unique) spectral function $\Sigma_{\mathcal{P},N}(\cdot)$ of the boundary problem (3.13)–(3.17) corresponding to φ_N. Moreover, a spectral function $\Sigma_{\mathcal{P},N}(\cdot)$ is orthogonal if and only if $\mathcal{P} \in \mathrm{TR}^0(\mathcal{H}_0, \mathcal{H}_1)$.

Proof. Let $\mathcal{P} \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$ be defined by (3.2)–(3.5) and let $\tau_+(\lambda)$ $(\in \tilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1))$ be the corresponding linear relation (2.13). Then, by (4.16), (3.56) and the equality

$$-(\tau_+(\lambda) + M_+(\lambda))^{-1} = (C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}C_1(\lambda), \quad \lambda \in \mathbb{C}_+,$$

the m-function $m_{\mathcal{P}}(\lambda)$ can be represented via (4.20). This, together with Theorem 4.11, yields the required statement. □

In the case of equal deficiency indices $n_+(L_0) = n_-(L_0)$, (4.20) can be simplified. Namely, the following corollary is immediate from Theorem 4.12.

Corollary 4.13. *Assume that, under the conditions of Theorem 4.12, $n_+(L_0) = n_-(L_0)$ and $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$, so that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a decomposing boundary triplet (2.22) for L. Moreover, let $M(\lambda)$ be the Weyl function for Π, let $S(\lambda)$ be given by (4.14), let $T_{N,0}(\lambda)$ be the operator function (4.19) and let $T_N(\lambda) = \hat{N}S(\lambda)$. Then, for each operator pair $\mathcal{P} = \{C_0(\cdot), C_1(\cdot)\} \in \mathrm{TR}(\mathcal{H})$ defined by (3.10) and (3.11), the statement of Theorem 4.12 holds with the equality*

$$m_{\mathcal{P}}(\lambda) = T_{N,0}(\lambda) + T_N(\lambda)(C_0(\lambda) - C_1(\lambda)M(\lambda))^{-1}C_1(\lambda)T^*_N(\bar{\lambda}), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{4.21}$$

in place of (4.20).

Next for a given collection $\tau = \{\tau_+, \tau_-\} \in \tilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ defined by (2.13) and (2.36), (2.37) consider the corresponding boundary problem (2.38)–(2.40). Denote by $d_{\min}$ the minimal value of $\dim \tilde{\mathcal{K}}$ for the set of all spectral functions $\Sigma_{\tau,\varphi}\colon \mathbb{R} \to [\tilde{\mathcal{K}}]$ of this boundary problem (recall that according to Definition 4.5 each $\Sigma_{\tau,\varphi}$ corresponds to some operator solution $\varphi(t, \lambda)$ $(\in [\tilde{\mathcal{K}}, H])$ of (2.20)).

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Definition 4.14. A spectral function $\Sigma(\cdot) = \Sigma_{\tau,\varphi}(\cdot)\colon \mathbb{R} \to [\tilde{\mathcal{K}}]$ will be called minimal if $\dim \tilde{\mathcal{K}} = d_{\min}$.

In the following theorem we give a description of all minimal spectral functions of the 'triangular' boundary problem (3.13)–(3.17).

Theorem 4.15. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet (2.22) for L, let $\mathcal{P} = \{C(\cdot), D(\cdot)\} \in \mathrm{TR}(\mathcal{H}_0, \mathcal{H}_1)$ be a collection of holomorphic pairs (3.2)–(3.5) and let (3.13)–(3.17) be the corresponding boundary problem. Then the following hold.*

1. *$d_{\min} = \dim \hat{\mathcal{K}}$ and the set of all minimal spectral functions $\Sigma_{\min}(\cdot)$ is given by*

$$\Sigma_{\min}(s) = X^* \Sigma_{\mathcal{P},N}(s) X, \tag{4.22}$$

where $\Sigma_{\mathcal{P},N}(s)$ is the (minimal) spectral function defined in Theorem 4.11 and X is an automorphism of the space $\hat{\mathcal{K}}$. Moreover, the minimal spectral function $\Sigma_{\min}(s)$ given by (4.22) corresponds to the operator solution $\varphi_{\min}(t,\lambda) := \varphi_N(t,\lambda)X^{-1*}$ of (2.20).

2. *If $\dim H = \infty$, then $d_{\min}(= \dim \hat{\mathcal{K}}) = \infty$.*

Proof. 1. Let $\Sigma_{\tau,\varphi}\colon \mathbb{R} \to [\tilde{\mathcal{K}}]$ be a spectral function of the problem (3.13)–(3.17) corresponding to the solution $\varphi(t,\lambda)$ $(\in [\tilde{\mathcal{K}}, H])$ with $\tilde{\varphi}(0,\lambda) \equiv \tilde{\varphi}_0 (\in [\tilde{\mathcal{K}}, H^{2n}])$. Since $0 \in \hat{\rho}(\tilde{\varphi}_0)$, there are a Hilbert space $\tilde{\mathcal{K}}^{\perp}$ and an operator $\tilde{\psi}_0 \in [\tilde{\mathcal{K}}^{\perp}, H^{2n}]$ such that the operator $W = (\tilde{\varphi}_0, \tilde{\psi}_0)$ is an isomorphism of the space $\tilde{H} := \tilde{\mathcal{K}} \oplus \tilde{\mathcal{K}}^{\perp}$ onto H^{2n}.

Let $\Omega_{\tau,W}(\lambda)$ be the operator function (4.8) and let $\Sigma_{\tau,W}(\cdot)$ be the spectral function (4.7) corresponding to the solution $Y_W(\cdot,\lambda)$ (see Theorem 4.7). It follows from (2.43) that $s - \lim_{y\to\infty} \Omega_{\tau,W}(\mathrm{i}y)/y = 0$. This and the integral representation of the Nevanlinna function $\Omega_{\tau,W}(\lambda)$ [**2, 17**] yield

$$\operatorname{Ker}\operatorname{Im}\Omega_{\tau,W}(\lambda) = \{\tilde{h} \in \tilde{H}\colon \Sigma_{\tau,W}(s)\tilde{h} = 0,\ s \in \mathbb{R}\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.23}$$

Moreover, by Lemma 4.10, the function $\Sigma_{\tau,W}(s)$ satisfies (4.17), which in view of (4.23) gives the inclusion $\tilde{\mathcal{K}}^{\perp} \subset \operatorname{Ker}\operatorname{Im}\Omega_{\tau,W}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now, letting

$$\tilde{H}_0 := \tilde{H} \ominus \operatorname{Ker}\operatorname{Im}\Omega_{\tau,W}(\lambda),$$

one obtains $\dim \tilde{H}_0 \leqslant \dim \tilde{\mathcal{K}}$.

Next assume that $W' \in [\hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^{\perp}, H^{2n}]$ is the isomorphism (3.37) and $\Omega_{\tau,W'}(\lambda)$ is the operator function (3.40). It follows from (4.8) that there exists an isomorphism $C \in [\hat{\mathcal{K}} \oplus \hat{\mathcal{K}}^{\perp}, \tilde{H}]$ such that $\Omega_{\tau,W'}(\lambda) = C^* \Omega_{\tau,W}(\lambda) C$. Moreover, by the block representation (3.42) one has $\operatorname{Ker}\operatorname{Im}\Omega_{\tau,W'}(\lambda) = \hat{\mathcal{K}}^{\perp}$. Hence, $\operatorname{Ker}\operatorname{Im}\Omega_{\tau,W}(\lambda) = C\hat{\mathcal{K}}^{\perp}$, and consequently $\hat{\mathcal{K}} = C^*\tilde{H}_0$. Therefore, $\dim \hat{\mathcal{K}} = \dim \tilde{H}_0 \leqslant \dim \tilde{\mathcal{K}}$, which yields the equality $d_{\min} = \dim \hat{\mathcal{K}}$.

To prove the relation (4.22), note that for each automorphism $X \in [\hat{\mathcal{K}}]$ this relation defines the minimal spectral function $\Sigma_{\min}(s) = \Sigma_{\tau,\varphi_{\min}}(s)$, corresponding to the solution $\varphi_{\min}(t,\lambda)\colon = \varphi_N(t,\lambda)X^{-1*}$. Conversely, let $\Sigma_{\min}(s) = \Sigma_{\tau,\varphi_{\min}}(s)$ be a minimal spectral function corresponding to the solution $\varphi_{\min}(t,\lambda)$ $(\in [\hat{\mathcal{K}}, H])$. Since $0 \in \hat{\rho}(\tilde{\varphi}(0,\lambda)) \cap$

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$\hat{\rho}(\tilde{\varphi}_N(0,\lambda))$, there exists an automorphism $X \in [\hat{\mathcal{K}}]$ such that $\varphi_{\min}(t,\lambda) = \varphi_N(t,\lambda)X^{-1*}$, and hence the distribution function $\Sigma(s) := X^*\Sigma_{\mathcal{P},N}(s)X$ is a spectral function corresponding to $\varphi_{\min}$. Since by Lemma 4.10 such a function is unique, it follows that $\Sigma_{\min}(s) = \Sigma(s) = X^*\Sigma_{\mathcal{P},N}(s)X$.

2. This is implied by statement 1 and the inequality (3.12). □

Finally, by using the above results we can estimate the spectral multiplicity of an exit-space extension $\tilde{A} \supset L_0$. Namely, the following corollary is valid.

Corollary 4.16. *Let the assumptions of Theorem 4.15 be satisfied and let $R_{\mathcal{P}}(\lambda) = P_{\mathfrak{H}}(\tilde{A}-\lambda)^{-1} \upharpoonright \mathfrak{H}$ be a generalized resolvent generated by the boundary problem (3.13)–(3.17). Then the spectral multiplicity of the extension $\tilde{A}$ does nod exceed $d_{\min}(= \dim \hat{\mathcal{K}})$.*

Proof. Let $\Sigma = \Sigma_{\mathcal{P},N} : \mathbb{R} \to [\hat{\mathcal{K}}]$ be a spectral function defined in Theorem 4.11 and let $\chi'(s)$ be a bounded linear map in $\mathrm{Hol}(\Sigma,\hat{\mathcal{K}})$ given for all $s \in \mathbb{R}$ by

$$(\chi'(s)f)(\sigma) = \chi_{(-\infty,s)}(\sigma)f(\sigma), \quad f = f(\sigma) \in \mathrm{Hol}(\Sigma,\hat{\mathcal{K}})$$

(here $\chi_{(-\infty,s)}(\cdot)$ is the indicator of the interval $(-\infty,s)$). It is easily seen that the map $\chi'(s)$ admits the continuous extension $\chi(s) \in [L_2(\Sigma;\hat{\mathcal{K}})]$, $s \in \mathbb{R}$, such that $\chi(\cdot)$ is an orthogonal spectral function (resolution of identity) in $L_2(\Sigma;\hat{\mathcal{K}})$.

Next assume that $V \in [\mathfrak{H}, L_2(\Sigma;\hat{\mathcal{K}})]$ is an isometry given by the Fourier transform (4.5) with $\varphi = \varphi_N$ and let $\mathcal{L} := V\mathfrak{H}$, $\tilde{\mathcal{L}} = \mathrm{span}\{\mathcal{L}, \chi(s)\mathcal{L} : s \in \mathbb{R}\}$. As is known, the subspace $\tilde{\mathcal{L}}$ reduces the spectral function $\chi(s)$, and the equality $\tilde{\chi}(s) = \chi(s) \upharpoonright \tilde{\mathcal{L}}$ defines the minimal orthogonal spectral function $\tilde{\chi}(s)$ in $\tilde{\mathcal{L}}$ (actually one can prove that $\tilde{\mathcal{L}} = L_2(\Sigma;\hat{\mathcal{K}})$). Moreover, the relation (4.6) yields

$$F_{\mathcal{P}}(t) = V^*\chi(t)V = V^*(P_{\mathcal{L}}\tilde{\chi}(t) \upharpoonright \mathcal{L})V, \quad t \in \mathbb{R}, \tag{4.24}$$

where $F_{\mathcal{P}}(t) = P_{\mathfrak{H}}\tilde{E}(t) \upharpoonright \mathfrak{H}$ and $\tilde{E}(t)$ is the orthogonal spectral function of $\tilde{A}$. It follows from (4.24) that the spectral functions $F_{\mathcal{P}}(t)$ and $P_{\mathcal{L}}\tilde{\chi}(t) \upharpoonright \mathcal{L}$ are unitary equivalent and, consequently, so are the (minimal) orthogonal spectral functions $\tilde{E}(t)$ and $\tilde{\chi}(t)$. This and the fact that $\tilde{\chi}(t)$ is a part of $\chi(t)$ imply that the spectral multiplicity of $\tilde{E}(t)$ does not exceed the spectral multiplicity of $\chi(t)$, which in turn does not exceed $\dim \hat{\mathcal{K}}$. This proves the required statement. □

Remark 4.17. It follows from Proposition 3.2 that in the case $n_{b+} < \infty$ (in particular, $\dim H < \infty$) the statements of Theorem 4.15 and Corollary 4.16 can naturally be extended to the boundary problems (2.38)–(2.40) generated by a quasi-constant Nevanlinna collection $\{C(\cdot), D(\cdot)\}$.

4.4. Example

To illustrate the results in this section consider the following example. Assume that

$$l[y] = -y'' + q(t)y, \quad t \in [0,\infty), \tag{4.25}$$

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is the Sturm–Liouville expression with the diagonal matrix potential

$$q(t) = \mathrm{diag}(q_1(t), q_2(t), q_3(t)) \ (\in [\mathbb{C}^3]),$$

where $q_j(\cdot)$, $j \in \{1,2,3\}$, is a continuous real function on $[0,\infty)$, and let L_0 and L be the corresponding minimal and maximal operators in $\mathfrak{H} = L_2((0,\infty);\mathbb{C}^3)$. Denote by $L_{0,j}$ (respectively, L_j) the minimal (respectively, maximal) operator generated by the scalar expression

$$l_j[y] = -y'' + q_j(t)y, \quad t \in [0,\infty), \ j \in \{1,2,3\}, \tag{4.26}$$

and suppose that the deficiency indices are $n_\pm(L_{0,1}) = 1$, $n_\pm(L_{0,2}) = n_\pm(L_{0,3}) = 2$. Then $L_0 = L_{0,1} \oplus L_{0,2} \oplus L_{0,3}$ and hence the operator L_0 has equal intermediate deficiency indices $n_\pm(L_0) = 5$.

Let $y_{1,2}(t,\lambda)$ and $y_{2,2}(t,\lambda)$ be solutions of the equation $l_2[y] = \lambda y$ with the initial data $y_{j,2}^{(k-1)}(0,\lambda) = \delta_{jk}$, let $y_{1,3}(t,\lambda)$ and $y_{2,3}(t,\lambda)$ be similar solutions of the equation $l_3[y] = \lambda y$ and let $v_{j,2}(t) = y_{j,2}(t,0)$, $v_{j,3}(t) = y_{j,3}(t,0)$, $j \in \{1,2\}$. By using the results of [**5**, **18**], one can easily prove the following assertions.

1. Let $\Gamma_j \colon \mathcal{D} \to \mathbb{C}^3 \oplus \mathbb{C}^2$, $j \in \{0,1\}$, be the operators defined for any function $y = \{y_1(t), y_2(t), y_3(t)\} \in \mathcal{D}$ by

$$\Gamma_0 y = \{y'(0), \Gamma_0' y\} \ (\in \mathbb{C}^3 \oplus \mathbb{C}^2), \qquad \Gamma_1 y = \{-y(0), \Gamma_1' y\} \ (\in \mathbb{C}^3 \oplus \mathbb{C}^2)$$

with $\Gamma_0' y = \{[y_2, v_{2,2}](\infty), [y_3, v_{2,3}](\infty)\}$ and $\Gamma_1' y = \{[y_2, v_{1,2}](\infty), [y_3, v_{1,3}](\infty)\}$. Then the collection $\Pi = \{\mathbb{C}^5, \Gamma_0, \Gamma_1\}$ is the decomposing boundary triplet for L.

2. The corresponding Weyl function (2.32) for the triplet Π is

$$M(\lambda) = \begin{pmatrix} m(\lambda) & M_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix}$$
$$= \left(\begin{array}{ccc|cc} m_1(\lambda) & 0 & 0 & 0 & 0 \\ 0 & m_2(\lambda) & 0 & M_{22}(\lambda) & 0 \\ 0 & 0 & m_3(\lambda) & 0 & M_{23}(\lambda) \\ \hline 0 & M_{22}(\lambda) & 0 & M_{42}(\lambda) & 0 \\ 0 & 0 & M_{23}(\lambda) & 0 & M_{43}(\lambda) \end{array}\right) : \mathbb{C}^3 \oplus \mathbb{C}^2 \to \mathbb{C}^3 \oplus \mathbb{C}^2, \tag{4.27}$$

where $m_1(\lambda)$ is the Titchmarsh–Weyl function of the boundary problem

$$l_1[y] = \lambda y, \quad y'(0) = 0,$$

$m_k(\lambda)$ is the similar function of the problem

$$l_k[y] = \lambda y, \quad y'(0) = 0, \quad [y, v_{2,k}](\infty) = 0,$$

and the other entries are

$$M_{2k}(\lambda) = -\frac{1}{[y_{1,k}, v_{2,k}](\infty)}, \quad M_{4k}(\lambda) = \frac{[y_{1,k}, v_{1,k}](\infty)}{[y_{1,k}, v_{2,k}](\infty)}, \quad k = 2,3.$$

Next assume that $N=(N_0,N_1)$ with

$$N_0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}:\mathbb{C}^3\to\mathbb{C}^4,\qquad N_1=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\1&0&0\end{pmatrix}:\mathbb{C}^3\to\mathbb{C}^4$$

and let $\mathcal{P}=\{C_0,C_1\}\in \mathrm{TR}(\mathbb{C}^5)$ be an N-triangular self-adjoint operator pair given by

$$C_0=I=\begin{pmatrix}N_0&C'_{01}\\0&C'_{02}\end{pmatrix},\qquad C_1=\begin{pmatrix}N_1&C'_{11}\\0&C'_{12}\end{pmatrix},$$

$$C'_{01}=\begin{pmatrix}0&0\\0&0\\0&0\\1&0\end{pmatrix},\qquad C'_{02}=(01),\qquad C'_{11}=\begin{pmatrix}1&0\\0&0\\0&0\\0&0\end{pmatrix},\qquad C'_{12}=(00).$$

Then the corresponding boundary problem (3.18), (3.19) is

$$l[y]-\lambda y=f, \tag{4.28}$$

$$\left.\begin{aligned}N_0y'(0)+N_1y(0)+C'_{01}\Gamma'_0y-C'_{11}\Gamma'_1y=0,\\ C'_{02}\Gamma'_0y-C'_{12}\Gamma'_1y=0.\end{aligned}\right\} \tag{4.29}$$

To find the m-function $m_{\mathcal{P}}(\cdot)$ for the problem (4.28), (4.29) we make use of the equality (4.21). The immediate calculations show that in our case

$$T_{N,0}(\lambda)=\begin{pmatrix}m_1&0&0&\frac12\\0&m_2&0&0\\0&0&m_3&0\\\frac12&0&0&0\end{pmatrix},\qquad T_N(\lambda)=\begin{pmatrix}m_1&0&0&0&0\\0&m_2&0&M_{22}&0\\0&0&m_3&0&M_{23}\\1&0&0&0&0\end{pmatrix},$$

where $m_j=m_j(\lambda)$, $M_{22}=M_{22}(\lambda)$ and $M_{23}=M_{23}(\lambda)$ are taken from (4.27). Therefore, by (4.21), one has

$$m_{\mathcal{P}}(\lambda)=T_{N,0}(\lambda)+\Phi_{\mathcal{P}}(\lambda), \tag{4.30}$$

where

$$\begin{aligned}\Phi_{\mathcal{P}}(\lambda)&=T_N(\lambda)(C_0-C_1M(\lambda))^{-1}C_1T_N^*(\bar\lambda)\\ &=\frac{1}{1-m_1(\lambda)M_{42}(\lambda)}\cdot\begin{pmatrix}m_1^2(\lambda)M_{42}(\lambda)&m_1(\lambda)M_{22}(\lambda)&0&m_1(\lambda)M_{42}(\lambda)\\m_1(\lambda)M_{22}(\lambda)&m_1(\lambda)M_{22}^2(\lambda)&0&M_{22}(\lambda)\\0&0&0&0\\m_1(\lambda)M_{42}(\lambda)&M_{22}(\lambda)&0&M_{42}(\lambda)\end{pmatrix}.\end{aligned}$$

Let $\varphi_0(t,\lambda)$ ($\in[\mathbb{C}^3]$) be the operator solution of the equation $l[y]=\lambda y$ with the initial data $\varphi_0(0,\lambda)=-I$, $\varphi'_0(0,\lambda)=0$. It follows from (4.30) that the (minimal) orthogonal

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spectral function $\Sigma_{\mathcal{P},N}(s)$ is the sum

$$\Sigma_{\mathcal{P},N}(s) = \begin{pmatrix} \Sigma_0(s) & 0 \\ 0 & 0 \end{pmatrix} + \Sigma_1(s), \tag{4.31}$$

where $\Sigma_0(s)$ is the 3×3 spectral function of the decomposing boundary problem

$$l[y] - \lambda y = f, \quad y'(0) = 0, \quad [y_2, v_{2,2}](\infty) = [y_3, v_{2,3}](\infty) = 0 \tag{4.32}$$

corresponding to $\varphi_0(t, \lambda)$, and $\Sigma_1(s)$ is the 4×4 function obtained from $\Phi_{\mathcal{P}}(\lambda)$ by

$$\Sigma_1(s) = \lim_{\delta \to +0} \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \operatorname{Im} \Phi_{\mathcal{P}}(\sigma + \mathrm{i}\varepsilon) \, \mathrm{d}\sigma.$$

Note that $\Sigma_0(s)$ corresponds to the decomposing boundary problem (4.32), and therefore its dimension 3×3 is less than the dimension 4×4 of $\Sigma_{\mathcal{P},N}(s)$. Formula (4.31) shows that such a growth in the dimension of $\Sigma_{\mathcal{P},N}(s)$ is caused by the second term, $\Sigma_1(s)$. At the same time, according to Theorem 4.15, the dimension of $\Sigma_{\mathcal{P},N}(s)$ is minimal among all spectral functions of the boundary problem (4.28), (4.29). Observe also that, by Corollary 4.16, the boundary conditions (4.29) define the self-adjoint extension $\tilde{A}$ of L_0 with a spectral multiplicity of not more than 4.

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