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WEAKLY NON-ASSOCIATIVE ALGEBRAS AND THE KADOMTSEV–PETVIASHVILI HIERARCHY

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Abstract. On any 'weakly non-associative' algebra there is a universal family of compatible ordinary differential equations (provided that differentiability with respect to parameters can be defined), any solution of which yields a solution of the Kadomtsev–Petviashvili (KP) hierarchy with dependent variable in an associative sub-algebra, the middle nucleus.

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1. Introduction. As explained in the following, the Kadomtsev–Petviashvili (KP) hierarchy (see e.g. [**2**, **10**]) emerges from a simple algebraic problem on non-associative algebras. Let f freely generate a non-associative algebra $\tilde{\mathbb{A}}$ over a commutative ring $\mathcal{R}$ with identity element (see e.g. [**11**] for the algebraic structures used in this work). A derivation of $\tilde{\mathbb{A}}$ is determined by its action on the generator. A family of derivations is then obtained by choosing their action on f as a non-linear homogeneous expression in f. The simplest choice is $\delta_1(f) = f^2$. This extends to $\tilde{\mathbb{A}}$ via the derivation rule. For the second derivation we should choose $\delta_2(f) = \kappa_1 ff^2 - \kappa_2 f^2 f$ with $\kappa_1, \kappa_2 \in \mathcal{R}$, since ff^2 and $f^2 f$ are the only independent monomials cubic in f. Then

$$[\delta_1, \delta_2](f) = (\kappa_1 - \kappa_2) f^2 f^2 - (\kappa_1 + \kappa_2)(f, f^2, f)\,, \tag{1}$$

where $(a, b, c) := (ab)\,c - a\,(bc)$ is the *associator*. Setting $\kappa_1 = \kappa_2 = 1$ and passing over to $\mathbb{A} = \tilde{\mathbb{A}}/\mathcal{I}$ with the ideal generated by $\{(a, bc, d)|\, a, b, c, d \in \tilde{\mathbb{A}}\}$, which is preserved by the derivations δ_1 and δ_2, the latter *commute* on $\mathbb{A}$, and we can look for further commuting derivations. In fact,

$$\delta_3(f) := f(ff^2) - ff^2 f - f^2 f^2 + (f^2 f)f \tag{2}$$

defines another derivation which commutes with the first two. This construction can be continued ad infinitum, and the underlying general building law will be presented in Section 3. The derivations δ_n are subject to algebraic identities. A direct calculation reveals that

$$\delta_1\big(4\,\delta_3(f) - \delta_1^3(f) + 6\,(\delta_1(f))^2\big) - 3\,\delta_2^2(f) + 6\,[\delta_2(f), \delta_1(f)] \equiv 0. \qquad (3)$$

Formally replacing δ_n by the partial derivative ∂_{t_n} with respect to a variable t_n, (3) becomes the potential KP equation (for $-f$, according to our convention). Elements $\delta_{n_1} \cdots \delta_{n_k}(f)$, where $n_1, \ldots, n_k = 1, 2, \ldots$ and $k = 1, 2, \ldots$, satisfy more identities of this kind, and the whole KP hierarchy emerges in this way.

If $\mathbb{A}$ is taken over a commutative ring of smooth functions of independent variables $t_1, t_2, \ldots$, we may consider the hierarchy of first-order ordinary differential equations

$$f_{t_n} := \partial_{t_n}(f) = \delta_n(f), \qquad n = 1, 2, \ldots. \qquad (4)$$

The first equation $f_{t_1} = f^2$, which is the only one that would survive in case of associativity, is the equation of a 'non-associative top' [**8**]. It has the form of a (non-associative) 'quadratic dynamical system' as considered in [**9**], for example. Since (4) turns the identity (3) into the potential KP equation (for $-f$), it follows that any solution of (4) also solves the latter. More generally, we demonstrate that this holds for any algebra $\mathbb{A}$ which is 'weakly non-associative' (WNA), and this yields solutions of the whole KP hierarchy, with dependent variable in an associative (and typically non-commutative) sub-algebra. The KP hierarchy appears in many areas of mathematics and physics, and the above result further adds to its ubiquitousness.

In Section 2 we define and characterize WNA algebras. Section 3 introduces a sequence of derived products in such algebras, which mainly serves as a preparation for the construction of a hierarchy of derivations, i.e. a sequence of commuting derivations δ_n, $n = 1, 2, \ldots$, for a sub-class of WNA algebras. Section 4 contains a major result of this work, namely the proof that the derivations δ_n satisfy a sequence of algebraic identities which are in correspondence with the equations of the KP hierarchy (as outlined above). In Section 5 we derive some properties and consequences of the 'non-associative hierarchy' (4). Section 6 treats a class of examples, and Section 7 contains further remarks. A preliminary account of results reported here appeared in our preprint [**5**], which the reader may consult as a supplement, in particular for some proofs omitted in the following.

2. Weakly non-associative algebras. Let $\mathbb{A}$ be a non-associative (and typically non-commutative) algebra over a unital commutative ring $\mathcal{R}$. The *middle nucleus* (see e.g. [**12**])

$$\mathbb{A}' := \{b \in \mathbb{A} \,|\, (a, b, c) = 0 \quad \forall a, c \in \mathbb{A}\} \qquad (5)$$

is then an *associative* sub-algebra. If $\mathbb{A}$ has an identity element, then it belongs to $\mathbb{A}'$.

DEFINITION 2.1. $\mathbb{A}$ is called WNA if $\mathbb{A}^2 \subset \mathbb{A}'$, i.e.

$$(a, bc, d) = 0, \qquad \forall a, b, c, d \in \mathbb{A}. \qquad (6)$$

If $\mathbb{A}$ is WNA, then $\mathbb{A}'$ is a two-sided ideal in $\mathbb{A}$, and the quotient algebra $\mathbb{A}/\mathbb{A}'$ is nilpotent of index 2. If $\mathbb{A}$ is any non-associative algebra and $\mathcal{I}$ the two-sided ideal in $\mathbb{A}$ generated by (a, bc, d) for all $a, b, c, d \in \mathbb{A}$, then $\mathbb{A}/\mathcal{I}$ is a WNA algebra. The WNA condition (6) can also be expressed as $L_a L_b = L_{ab}$ or $R_a R_b = R_{ba}$ for all $a \in \mathbb{A}$ and $b \in \mathbb{A}'$, where L_a and R_a denote, respectively, left and right multiplication by $a \in \mathbb{A}$. The following result characterizes WNA algebras and provides examples.

PROPOSITION 2.1. *(1) Let $\mathcal{A}$ be an associative algebra over $\mathcal{R}$, $L_i, R_i : \mathcal{A} \to \mathcal{A}$, $i = 1, \dots, N$, linear maps such that $[L_i, R_j] = 0$,*

$$L_i(ab) = L_i(a)\, b\,, \qquad R_i(ab) = a\, R_i(b), \qquad \forall a, b \in \mathcal{A} \tag{7}$$

and $g_{ij} \in \mathbb{A}$, $i, j = 1, \dots, N$. For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_N) \in \mathcal{R}^N$, let us define

$$g(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \sum_{i,j=1}^{N} \alpha_i \beta_j\, g_{ij}\,, \qquad L_{\boldsymbol{\alpha}} := \sum_{i=1}^{N} \alpha_i\, L_i\,, \qquad R_{\boldsymbol{\beta}} := \sum_{i=1}^{N} \beta_i\, R_i. \tag{8}$$

The augmented algebra $\mathbb{A} := (\bigoplus_{i=1}^{N} \mathcal{R}) \oplus \mathcal{A}$ becomes WNA when supplied with the product

$$(\boldsymbol{\alpha}, a)(\boldsymbol{\beta}, b) := (\mathbf{0}, g(\boldsymbol{\alpha}, \boldsymbol{\beta}) + L_{\boldsymbol{\alpha}}(b) + R_{\boldsymbol{\beta}}(a) + ab). \tag{9}$$

If, in addition, the equations

$$\begin{aligned} L_{\boldsymbol{\beta}} L_{\boldsymbol{\alpha}}(a) = g(\boldsymbol{\beta}, \boldsymbol{\alpha})\, a,\ \ R_{\boldsymbol{\beta}} R_{\boldsymbol{\alpha}}(a) = a\, g(\boldsymbol{\alpha}, \boldsymbol{\beta}), \\ R_{\boldsymbol{\gamma}}(g(\boldsymbol{\beta}, \boldsymbol{\alpha})) = L_{\boldsymbol{\beta}}(g(\boldsymbol{\alpha}, \boldsymbol{\gamma})),\ \ R_{\boldsymbol{\alpha}}(a)\, b = a\, L_{\boldsymbol{\alpha}}(b), \end{aligned} \tag{10}$$

for all $a, b \in \mathcal{A}$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{R}^N$, imply $\boldsymbol{\alpha} = \mathbf{0}$, then $\mathbb{A}' = \mathcal{A}$, and $\mathbb{A}/\mathbb{A}'$ is N-dimensional.
(2) Any WNA algebra $\mathbb{A}$, for which $\mathbb{A}/\mathbb{A}'$ is finite-dimensional, is isomorphic to a WNA algebra of the type described in (1).

Proof. One easily verifies that the construction in (1) satisfies (6). Defining $f_i := (0, \dots, 1, 0, \dots, 0, 0)$ (with identity of $\mathcal{R}$ at the ith position), (10) implies that $[f_i] \in \mathbb{A}/\mathbb{A}'$, $i = 1, \dots, N$, are independent. Conversely, let $\mathbb{A}$ be WNA and $f_i \in \mathbb{A}$, $i = 1, \dots, N$, such that $[f_i]$, $i = 1, \dots, N$, freely generate $\mathbb{A}/\mathbb{A}'$. Then $g_{ij} := f_i f_j \in \mathbb{A}'$, and $L_i(a) := f_i\, a$ and $R_i(a) := a f_i$ define linear maps $\mathbb{A}' \to \mathbb{A}'$. The WNA property implies $[L_i, R_j] = 0$ and (7) with $\mathcal{A} := \mathbb{A}'$. Since $[f_i]$, $i = 1, \dots, N$, are independent, (10) holds. Furthermore, $\iota(a) := (\mathbf{0}, a)$ for all $a \in \mathbb{A}'$, and $\iota(f_i) := (0, \dots, 1, 0, \dots, 0, 0)$ (with identity at the ith position), $i = 1, \dots, N$, determines an isomorphism $\iota : \mathbb{A} \to (\bigoplus_{i=1}^{N} \mathcal{R}) \oplus \mathbb{A}'$, where the target is supplied with the product (9). □

In the following, $\mathbb{A}(f)$ denotes the sub-algebra of a WNA algebra $\mathbb{A}$ generated by an element f.

PROPOSITION 2.2. *$\mathbb{A}(f)$ is spanned by f and products of the elements $L_f^n R_f^m(f^2)$, $m, n = 0, 1, 2, \dots$.*

EXAMPLE: free WNA algebra. Let $\mathcal{A}_{\text{free}}$ be the *free associative* algebra over $\mathcal{R}$, generated by elements $c_{m,n}$, $m, n = 0, 1, \dots$. We define linear maps $L, R : \mathcal{A}_{\text{free}} \to \mathcal{A}_{\text{free}}$ by $L(c_{m,n}) := c_{m+1,n}$, $R(c_{m,n}) := c_{m,n+1}$ and $L(ab) = L(a)\, b$, $R(ab) = a\, R(b)$. As a consequence, $c_{m,n} = L^m R^n(c_{0,0})$. The *free WNA algebra* $\mathbb{A}_{\text{free}}(f)$ over $\mathcal{R}$ is then defined as the algebra $\mathcal{A}_{\text{free}}$ augmented with an element f, such that $ff = c_{0,0}$, $fa = L(a)$, $af = R(a)$. It is easily seen that $f \notin \mathbb{A}_{\text{free}}(f)'$; thus $\mathbb{A}_{\text{free}}(f)' = \mathcal{A}_{\text{free}}$, and f generates $\mathbb{A}_{\text{free}}(f)$. Any other WNA algebra $\mathbb{A}(f')$ over $\mathcal{R}$, with a single generator, is the homomorphic image of $\mathbb{A}_{\text{free}}(f)$ by the linear map given by $f \mapsto f'$ and $c_{m,n} \mapsto L_{f'}^m R_{f'}^n(f'^2)$ (cf. proposition 2.2). The derivations defined in Section 1 are well defined on $\mathbb{A}_{\text{free}}(f)$, and the reader can check the identity (3). □

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3. A sequence of products and derivations of WNA algebras. Let $\mathbb{A}$ be any (non-associative) algebra. With respect to a fixed element $f \in \mathbb{A}$ we define a sequence of products $\circ_n$, $n = 1, 2, \ldots$, in $\mathbb{A}$ recursively by $a \circ_1 b := a\, b$ and

$$a \circ_{n+1} b := a\,(f \circ_n b) - (a f) \circ_n b, \qquad n = 1, 2, \ldots . \tag{11}$$

If $f \in \mathbb{A}'$, then $a \circ_n b = 0$ for $n > 1$. Some properties of these products are stated below. We omit the proofs which are straightforward using induction.

PROPOSITION 3.1. *Let $\mathbb{A}$ be a WNA algebra. Then the products $\circ_n$ only depend on the equivalence class $[f] \in \mathbb{A}/\mathbb{A}'$ and, for all $m, n \in \mathbb{N}$ and $a, c \in \mathbb{A}$, satisfy the identities*

$$(a \circ_n b) \circ_m c = a \circ_n (b \circ_m c) \qquad \text{if} \quad b \in \mathbb{A}', \tag{12}$$

$$a \circ_{m+n} c = a \circ_m (f \circ_n c) - (a \circ_m f) \circ_n c. \tag{13}$$

Next we note a general property of derivations of WNA algebras and construct a family of commuting derivations for a special class of WNA algebras.

PROPOSITION 3.2. *Any derivation δ of a WNA algebra $\mathbb{A}$ with the property $\delta(\mathbb{A}) \subset \mathbb{A}'$ is also a derivation with respect to any of the products $\circ_n$, $n \in \mathbb{N}$.*

Proof. By induction. The induction step can be formulated as follows:

$$\begin{aligned}
&\delta(a \circ_{n+1} b) = \delta(a\,(f \circ_n b) - (a f) \circ_n b)\\
&= \delta(a)\,(f \circ_n b) + a\,(\delta(f) \circ_n b) + a\,(f \circ_n \delta(b)) - (\delta(a) f) \circ_n b - (a\,\delta(f)) \circ_n b - (a f) \circ_n \delta(b)\\
&= \delta(a)\,(f \circ_n b) + a\,(f \circ_n \delta(b)) - (\delta(a) f) \circ_n b - (a f) \circ_n \delta(b) = \delta(a) \circ_{n+1} b + a \circ_{n+1} \delta(b)\,,
\end{aligned}$$

where we used the definition (11) and also (12). □

DEFINITION 3.1. We call $\mathbb{A}(f)$ δ*-compatible* if it admits derivations δ_n, $n = 1, 2, \ldots$, such that

$$\delta_n(f) \equiv f \circ_n f, \qquad n = 1, 2, \ldots . \tag{14}$$

For $n = 1, 2, 3$, (14) reproduces the derivations considered in Section 1. Clearly, $\mathbb{A}_{\text{free}}(f)$ is δ-compatible. If $\mathcal{I}$ is a two-sided ideal in $\mathbb{A}_{\text{free}}(f)$ with the property $\delta_n(\mathcal{I}) \subset \mathcal{I}$, $n = 1, 2, \ldots$, then the derivations δ_n, $n = 1, 2, \ldots$, of $\mathbb{A}_{\text{free}}(f)$ project to derivations of $\mathbb{A}_{\text{free}}(f)/\mathcal{I}$, which is then also δ-compatible.

PROPOSITION 3.3. *If $\mathbb{A}(f)$ is δ-compatible, the derivations δ_n, $n = 1, 2, \ldots$, commute on $\mathbb{A}(f)$.*

Proof. (13) implies $f \circ_m \delta_n(f) - \delta_m(f) \circ_n f = \delta_{m+n}(f) = f \circ_n \delta_m(f) - \delta_n(f) \circ_m f$. Hence $\delta_m \delta_n(f) = \delta_m(f \circ_n f) = \delta_m(f) \circ_n f + f \circ_n \delta_m(f) = \delta_n(f) \circ_m f + f \circ_m \delta_n(f) = \delta_n \delta_m(f)$. □

4. KP identities. In this section we consider a δ-compatible sub-algebra $\mathbb{A}(f)$ of a WNA algebra $\mathbb{A}$, derive identities for the elements $\delta_{n_1} \cdots \delta_{n_r}(f)$ and establish a correspondence with the equations of the (potential) KP hierarchy. Since, according to propositions 3.2 and 3.3, the δ_n are commuting derivations of $\mathbb{A}(f)$ with respect to all products $\circ_k$, the formal power series $\exp(\sum_{n \geq 1} (\lambda^n/n)\, \delta_n)$ with an indeterminate λ defines

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a homomorphism. Here $\mathcal{R}$ has to be extended to the ring $\mathcal{R}[[\lambda]]$ of formal power series in λ. On $\mathbb{A}(f)$ we can now define an *algebraic analogue* of a *Miwa shift* [**13**],

$$a_{\pm[\lambda]} := \exp\left(\pm \sum_{n\geq 1} \frac{\lambda^n}{n} \delta_n \right) a \,. \tag{15}$$

LEMMA 4.1.

$$h(\lambda) := \sum_{n\geq 0} \lambda^n L_f^n(f) = f_{[\lambda]} \,, \qquad e(\lambda) := \sum_{n\geq 0} \lambda^n R_f^n(f) = f_{-[-\lambda]} \,. \tag{16}$$

Proof. Setting $h_n := L_f^n(f)$ and using (13), one first proves by induction

$$\delta_n(f) = h_n - \sum_{k=1}^{n-1} \delta_k(f)\, h_{n-1-k}$$

and with its help, again by induction,

$$n\, h_n = \sum_{k=1}^{n} \delta_k(h_{n-k}) \,, \qquad n = 1, 2, \ldots .$$

In terms of $h(\lambda)$, this can be expressed as

$$\frac{d}{d\lambda} h(\lambda) = \delta_\lambda(h(\lambda)) \,, \qquad \delta_\lambda := \sum_{n\geq 1} \lambda^{n-1} \delta_n \,,$$

which integrates to (note that $h(0) = f$)

$$h(\lambda) = \exp\left(\sum_{n\geq 1} \frac{\lambda^n}{n} \delta_n \right) f = f_{[\lambda]} \,.$$

The second formula in (16) can be verified in a similar way. □

In terms of the elementary Schur polynomials $\mathbf{p}_n$ and $\tilde{\delta} := (\delta_1, \delta_2/2, \delta_3/3, \ldots)$, (16) reads

$$L_f^n(f) = \mathbf{p}_n(\tilde{\delta})(f) \,, \qquad R_f^n(f) = (-1)^n \mathbf{p}_n(-\tilde{\delta})(f) \,, \qquad n = 1, 2, \ldots . \tag{17}$$

THEOREM 4.1.

$$\begin{aligned} -\delta_1(f_{[\lambda_1]} - f_{[\lambda_2]}) = {} & (\lambda_1^{-1} - \lambda_2^{-1} + f_{[\lambda_1]} - f_{[\lambda_2]})\,(f_{[\lambda_1]+[\lambda_2]} - f_{[\lambda_1]} - f_{[\lambda_2]} + f) \\ & + [f_{[\lambda_1]} - f, f_{[\lambda_2]} - f]. \end{aligned} \tag{18}$$

Proof. The trivial identities $L_f^{n+1}(f) = f\, L_f^n(f)$ are combined into $h(\lambda) = f + \lambda f\, h(\lambda)$. By use of (16) and $\delta_1(f) = f^2$, this leads to

$$(\lambda^{-1} - f)(f_{[\lambda]} - f) = \delta_1(f) \,. \tag{19}$$

We rename λ to λ_1. After application of an algebraic Miwa shift with λ_2 and subtraction of the original equation, anti-symmetrization in λ_1, λ_2 eliminates terms not in $\mathbb{A}'$ and leads to (18). □

COROLLARY 4.1.

$$\sum_{i,j,k=1}^{3} \varepsilon_{ijk} \left(\lambda_i^{-1} - f_{[\lambda_k]} + f\right) (f_{[\lambda_i]} - f)_{[\lambda_k]} = 0, \tag{20}$$

where ε_{ijk} is totally anti-symmetric with $\varepsilon_{123} = 1$.

Proof. This follows by adding (18) three times with cyclically permuted indeterminates $\lambda_1, \lambda_2, \lambda_3$. □

It is important to note that all terms appearing in (18) and (20) lie in the associative sub-algebra $\mathbb{A}(f)'$. (Hence the bare f's appear only spuriously and actually drop out.) The first non-trivial identity which results from expanding these functional equations in powers of the indeterminates is (3), which has the form of the potential KP equation. In fact, formally replacing δ_n by the partial derivative ∂_{t_n} with respect to a variable t_n, $n = 1, 2, \ldots$, equation (20) becomes a functional representation of the potential KP hierarchy [**1**, **4**], and the equivalent formula (18) is turned into a non-commutative version of the differential Fay identity (cf. [**4**]).

REMARK. In the proof of theorem 4.1, one arrives at the same result by starting alternatively from the identities $R_f^{n+1}(f) = R_f^n(f)f$, which translate into

$$(f_{-[\lambda]} - f)(\lambda^{-1} + f) = -\delta_1(f) \,. \tag{21}$$

5. From a non-associative hierarchy of ODEs to the KP hierarchy. In this section, $\mathbb{A}$ denotes a WNA algebra with the property that its elements are (infinitely often) differentiable with respect to variables $\mathbf{t} = (t_1, t_2, \ldots)$. The ordinary differential equations

$$f_{t_n} = f \circ_n f, \qquad n = 1, 2, \ldots \tag{22}$$

then constitute a 'non-associative hierarchy' according to the following proposition. We shall assume that $f \notin \mathbb{A}'$, since otherwise (22) would reduce to a single equation. In the following, $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$ and $\mathbb{A}(f, \mathbb{K})$ denotes the WNA algebra generated in $\mathbb{A}$ by $f \in \mathbb{A}$ with coefficients in $\mathbb{K}$.

PROPOSITION 5.1.
(1) The flows (22) *commute.*
(2) For any solution f of (22)*, $\mathbb{A}(f, \mathbb{K})$ is δ-compatible.*

Proof. Since (22) implies $f_{t_n} \in \mathbb{A}'$, it follows (cf. the proof of proposition 3.2) that the flow derivatives ∂_{t_n} act as derivations of the products $\circ_m$ in $\mathbb{A}(f)$. The commutativity of the flows can now be checked directly as follows, by use of (13):

$$\begin{aligned}(f_{t_m})_{t_n} &= (f \circ_m f)_{t_n} = f_{t_n} \circ_m f + f \circ_m f_{t_n} = (f \circ_n f) \circ_m f + f \circ_m (f \circ_n f) \\ &= f \circ_n (f \circ_m f) - f \circ_{m+n} f + (f \circ_m f) \circ_n f + f \circ_{m+n} f \\ &= (f \circ_m f) \circ_n f + f \circ_n (f \circ_m f) = (f_{t_n})_{t_m} \,.\end{aligned}$$

Since ∂_{t_n} in particular extends as a derivation to $\mathbb{A}(f, \mathbb{K})$, (22) guarantees the consistency of extending $\delta_n(f) := f \circ_n f$ to $\mathbb{A}(f, \mathbb{K})$ via the derivation property. □

Now we formulate the main result of this work.

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THEOREM 5.1. *If f solves* (22)*, then $u := -f_{t_1} \in \mathbb{A}'$ solves the KP hierarchy in $\mathbb{A}'$.*

Proof. Since $\mathbb{A}(f, \mathbb{K})$ is δ-compatible by proposition 5.1, the identity (20) holds. As a consequence of (22), the algebraic Miwa shifts can be replaced by the usual ones satisfying $f_{[\lambda]}(\mathbf{t}) = f(\mathbf{t} + [\lambda])$ with $[\lambda] := (\lambda, \lambda^2/2, \lambda^3/3, \ldots)$. This results in a well-known functional representation of the potential KP hierarchy [**1**, **4**], which means that u solves the KP hierarchy. □

We refer to [**6**, **7**] for exact solutions of the matrix KP hierarchy obtained with the help of this theorem. The following proposition provides us with a formal solution of the initial value problem for (22) for a subclass of WNA algebras.

PROPOSITION 5.2. *Let $\mathbb{A}$ be a WNA algebra over $\mathbb{K}[[\mathbf{t}]]$ and $f_0 \in \mathbb{A}$ constant, $f_0 \notin \mathbb{A}'$, generating a δ-compatible sub-algebra $\mathbb{A}(f_0, \mathbb{K})$. Then*

$$f := \mathcal{S}(f_0) \qquad \textit{with} \quad \mathcal{S} := \exp\left(\sum_{n \geq 1} t_n \, \delta_n\right) \tag{23}$$

(where the δ_n are defined in terms of f_0) solves the non-associative hierarchy (22)*.*

Proof. Since the δ_n are commuting derivations with respect to all the products $\circ_m$, $m = 1, 2, \ldots$, the linear operator $\mathcal{S}$ on $\mathbb{A}(f_0, \mathbb{K})$ is an automorphism with respect to all these products (which are defined via (11) in terms of f_0). Hence

$$f_{t_n} = \partial_{t_n} \mathcal{S}(f_0) = \mathcal{S}(\delta_n(f_0)) = \mathcal{S}(f_0 \circ_n f_0) = \mathcal{S}(f_0) \circ_n \mathcal{S}(f_0) = f \circ_n f \; .$$

Since $\delta_n(f) \in \mathbb{A}(f_0)'$, we have $f - f_0 \in \mathbb{A}(f_0)'$; hence $[f] = [f_0] \in \mathbb{A}(f_0)/\mathbb{A}(f_0)'$. The products $\circ_n$ (and then also the derivations δ_n) are thus equivalently defined in terms of f (proposition 3.1). This proves our assertion. □

The solution given by proposition 5.2 has the property

$$f = \nu - \phi \qquad \text{with constant } \nu \text{ and} \quad \phi \in \mathbb{A}' \; . \tag{24}$$

Inserting this decomposition in (22), turns it into the Riccati-type hierarchy

$$\phi_{t_n} = -\nu \circ_n \nu + \nu \circ_n \phi + \phi \circ_n \nu - \phi \circ_n \phi \, , \qquad n = 1, 2, \ldots . \tag{25}$$

If ϕ solves (25), then also the potential KP hierarchy. Splitting off a constant term in (24) is natural from the point of view that the potential ϕ is obtained from the proper KP variable u by integration with respect to t_1, so ν plays the role of a constant of integration.

6. A simplified case and a class of solutions of the KP hierarchy. Let $(\mathcal{A}, \circ)$ be any associative algebra over $\mathcal{R}$, and L, R commuting linear maps such that $L(a \circ b) = L(a) \circ b$ and $R(a \circ b) = a \circ R(b)$ for all $a, b \in \mathcal{A}$. We write $La := L(a)$ and $aR := R(a)$, for short. A new associative product in $\mathcal{A}$ is then given by

$$a \circ_1 b := (aR) \circ b - a \circ (Lb) \; . \tag{26}$$

Augmenting $(\mathcal{A}, \circ_1)$ with an element ν such that $\nu \circ_1 \nu := 0$, $\nu \circ_1 a := La$, $a \circ_1 \nu := -aR$, we obtain a WNA algebra $(\mathbb{A}, \circ_1)$ with $\mathbb{A}' = \mathcal{A}$. Restricted to $\mathbb{A}'$, we have $L_\nu = L$ and $R_\nu = -R$. For the products (11), defined with respect to ν, one easily proves by induction

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that

$$\nu \circ_n \nu = 0\,, \qquad \nu \circ_n a = L^n a\,, \qquad a \circ_n \nu = -aR^n\,,$$
$$a \circ_n b = \sum_{k=0}^{n-1} (-1)^k \left(R_\nu^k a\right) \circ_1 L_\nu^{n-k-1} b = (aR^n) \circ b - a \circ L^n b\,, \tag{27}$$

for all $a, b \in \mathcal{A}$. The telescoping sum in (27) is a consequence of (26). Now (25) simplifies to

$$\phi_{t_n} = L^n \phi - \phi R^n + \phi \circ L^n \phi - \phi R^n \circ \phi\,, \qquad n = 1, 2, \ldots. \tag{28}$$

According to our general results, *any* solution of (28) is a solution of the potential KP hierarchy in $(\mathcal{A}, \circ_1)$.

Now we choose $\mathcal{A} = \mathcal{A}_{\text{free}}$ (see the example in Section 2) and $\mathcal{R} = \mathbb{K}[[\mathbf{t}, \epsilon]]$. Then

$$\delta_n(c_{r,s}) := L^n c_{r,s} - c_{r,s} R^n = c_{r+n,s} - c_{r,s+n}\,, \qquad n = 1, 2, \ldots, \tag{29}$$

determines derivations δ_n. They extend to $(\mathbb{A}, \circ_1)$ by setting $\delta_n(\nu) = 0$ and are derivations with respect to all products $\circ_n$ (proposition 3.2). For $c := c_{0,0}$ we find

$$\delta_n(c^{\circ m}) = \nu \circ_n c^{\circ m} + c^{\circ m} \circ_n \nu - \sum_{k=1}^{m-1} c^{\circ k} \circ_n c^{\circ (m-k)}, \tag{30}$$

where $c^{\circ n}$ denotes the nth power of c using the product $\circ$. This implies

$$\delta_n(f_0) = f_0 \circ_n f_0\,, \quad \text{where} \quad f_0 := \nu - \phi_0\,, \quad \phi_0 := \sum_{n \geq 1} \epsilon^n c^{\circ n}. \tag{31}$$

By proposition 5.2 and theorem 5.1,

$$\phi = \mathcal{S}(\phi_0) = \sum_{n \geq 1} \epsilon^n \mathcal{S}(c)^{\circ n} \quad \text{with} \quad \mathcal{S}(c) = e^{\xi(\mathbf{t}, L)}\, c\, e^{-\xi(\mathbf{t}, R)}\,, \tag{32}$$

where $\xi(\mathbf{t}, L) := \sum_{n \geq 1} t_n L^n$, solves the potential KP hierarchy in $(\mathcal{A}_{\text{free}}, \circ_1)$. Any homomorphism ρ that commutes with the partial derivatives ∂_{t_n} induces a corresponding solution in $\rho(\mathcal{A}_{\text{free}})$ (see also [**5**]).

7. Further remarks. Another possibility to derive from (19), respectively (21), an equation in $\mathbb{A}'$, is via a decomposition (24), assuming in addition that $\nu\, a = 0$, respectively $a\, \nu = 0$, for all $a \in \mathbb{A}'$. Then (22) implies $(\lambda^{-1} + \phi)(\phi_{[\lambda]} - \phi) = -\partial_{t_1}(\phi)$, respectively $(\phi_{-[\lambda]} - \phi)(\lambda^{-1} - \phi) = \partial_{t_1}(\phi)$. These are functional representations of (non-commutative) *Burgers hierarchies*. The simplest equations derived from them are $\phi_{t_2} + \phi_{t_1 t_1} + 2\phi\phi_{t_1} = 0$, respectively $\phi_{t_2} - \phi_{t_1 t_1} - 2\phi_{t_1}\phi = 0$.

Further examples and some applications of WNA algebras in the context of KP hierarchies appeared in [**3**, **6**, **7**]. There is a WNA algebra such that (22) reproduces the Gelfand–Dickey–Sato formulation of the (potential) KP hierarchy [**6**]. The free WNA algebra described in Section 2 has a representation in terms of the algebra of quasi-symmetric functions, which therefore also exhibits KP identities. This will be elaborated elsewhere.

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