Canad. J. Math. Vol. **67** (1), 2015 pp. 3–27
http://dx.doi.org/10.4153/CJM-2013-039-4

On the Local Convexity of Intersection Bodies of Revolution

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Abstract. One of the fundamental results in convex geometry is Busemann's theorem, which states that the intersection body of a symmetric convex body is convex. Thus, it is only natural to ask if there is a quantitative version of Busemann's theorem, *i.e.,* if the intersection body operation actually improves convexity. In this paper we concentrate on the symmetric bodies of revolution to provide several results on the (strict) improvement of convexity under the intersection body operation. It is shown that the intersection body of a symmetric convex body of revolution has the same asymptotic behavior near the equator as the Euclidean ball. We apply this result to show that in sufficiently high dimension the double intersection body of a symmetric convex body of revolution is very close to an ellipsoid in the Banach–Mazur distance. We also prove results on the local convexity at the equator of intersection bodies in the class of star bodies of revolution.

1 Introduction and Notation

A set $S \subset \mathbb{R}^n$ is said to be *symmetric* if it is symmetric with respect to the origin (*i.e.,* $S = -S$) and *star-shaped* if the line segment from the origin to any point in S is contained in S. For a star-shaped set $K \subset \mathbb{R}^n$, the *radial function* of K is defined by

$$\rho_K(u) = \sup\{\lambda \geq 0 : \lambda u \in K\} \quad \text{for every } u \in \mathbb{S}^{n-1}.$$

A *body* in $\mathbb{R}^n$ is a compact set that is equal to the closure of its interior. A body K is called a *star body* if it is star-shaped at the origin and its radial function ρ_K is continuous. We say that a body K is *locally convex* at a point p on the boundary of K if there exists a neighborhood $B(p,\varepsilon) = \{q \in \mathbb{R}^n : |p-q| \leq \varepsilon\}$ of p such that $K \cap B(p,\varepsilon)$ is convex. Furthermore, if p is an extreme point of $K \cap B(p,\varepsilon)$, then K is said to be *strictly convex* at p.

In [10], Lutwak introduced the notion of the *intersection body of a star body*. The intersection body IK of a star body K is defined by its radial function

$$\rho_{IK}(u) = |K \cap u^\perp| \quad \text{for every } u \in \mathbb{S}^{n-1}.$$

Here and throughout the paper, $u^\perp$ denotes the central hyperplane perpendicular to u. By $|A|_k$, or simply $|A|$ when there is no ambiguity, we denote the k-dimensional Lebesgue measure of a set A. From the volume formula in polar coordinates for the

Received by the editors April 12, 2013; revised September 18, 2013.
Published electronically November 26, 2013.
M. A. supported in part by U.S. National Science Foundation grant DMS-1100657. J. K. supported in part by NSERC and U.S. National Science Foundation grant DMS-0652684.
AMS subject classification: **52A20**, 52A38, 44A12.
Keywords: convex bodies, intersection bodies of star bodies, Busemann's theorem, local convexity.

section $K \cap u^{\perp}$, the following analytic definition of the intersection body of a star body can be derived: the radial function of the intersection body IK of a star body K is given by

$$\rho_{IK}(u) = \frac{1}{n-1} \int_{\mathbb{S}^{n-1} \cap u^{\perp}} \rho_K(v)^{n-1} dv = \frac{1}{n-1} \left(\mathcal{R} \rho_K^{n-1} \right)(u), \quad u \in \mathbb{S}^{n-1}.$$

Here $\mathcal{R}$ stands for the spherical Radon transform. The more general class of intersection bodies is defined in the following way (see [4, 8]). A star body K is an *intersection body* if its radial function ρ_K is the spherical Radon transform of an even non-negative measure μ. We refer the reader to [4, 8, 9] for more information on the definition and properties of intersection bodies and their applications in Convex Geometry and Geometric Tomography.

In order to measure the distance between two symmetric bodies K and L, we use the Banach–Mazur distance

$$d_{BM}(K, L) = \inf\left\{ r \geq 1 : K \subset TL \subset rK \text{ for some } T \in GL(n) \right\}.$$

We note that the intersection bodies of linearly equivalent star bodies are linearly equivalent (see [4, Theorem 8.1.6]), in the sense that $I(TK) = |\det T| \, (T^*)^{-1} IK$ for any $T \in GL(n)$. This gives that $d_{BM}(I(TK), I(TL)) = d_{BM}(IK, IL)$ for any $T \in GL(n)$.

A classical theorem of Busemann [2] (see also [4, Theorem 8.1.10]) states that the intersection body of a symmetric convex body is convex. In view of Busemann's theorem it is natural to ask how much of convexity is preserved or improved under the intersection body operation. As a way to measure the "convexity" of a body, we can consider the Banach–Mazur distance from the Euclidean ball. Hensley proved in [6] that the Banach–Mazur distance between the intersection body of any symmetric convex body K and the ball B_2^n is bounded by an absolute constant $c > 1$, that is, $d_{BM}(IK, B_2^n) \leq c$. Compared with John's classical result, $d_{BM}(K, B_2^n) \leq \sqrt{n}$ for any symmetric convex body K, we see that in many cases the intersection body operation improves convexity in the sense of the Banach–Mazur distance from the ball (see [7] for a similar discussion on quasi-convexity).

Given that the intersection body of a Euclidean ball is again a Euclidean ball, another question about the intersection body operator I comes from works of Lutwak [11] and Gardner [4, Prob. 8.6-7] (see also [5]). Are there other fixed points of the intersection body operator? It is shown in [3] that in a sufficiently small neighborhood of the ball in the Banach–Mazur distance there are no other fixed points of the intersection body operator. However, in general this question is still open.

In this paper we concentrate on the symmetric bodies of revolution to study the local convexity properties at the equator for intersection bodies. Throughout the paper we assume that the axis of revolution for any body of revolution is the e_1-axis. In this case $\rho_K(\theta)$ denotes the radial function of a body K of revolution at a direction whose angle from the e_1-axis is θ. Then, following [4, Theorem C.2.9], the radial function $\rho_{IK}(\theta)$ of the intersection body of K is given for $\theta \in (0, \pi/2]$ by

$$\rho_{IK}(\theta) = \frac{c_n}{\sin\theta} \int_{\pi/2-\theta}^{\pi/2} \rho_K(\varphi)^{n-1} \left[1 - \frac{\cos^2\varphi}{\sin^2\theta} \right]^{\frac{n-4}{2}} \sin\varphi \, d\varphi \tag{1.1}$$

and

$$\rho_{IK}(0) = \left|\rho_K(\pi/2)\, B_2^{n-1}\right| = c_n d_n \sqrt{\pi/(2n)}\ \rho_K(\pi/2)^{n-1}, \tag{1.2}$$

where

$$c_n = \frac{n-2}{n-1} \cdot \frac{2\pi^{n/2-1}}{\Gamma(n/2)} \quad \text{and} \quad d_n = \frac{n-1}{n-2} \cdot \frac{\sqrt{n/2}\,\Gamma(n/2)}{\Gamma((n+1)/2)} \to 1 \text{ as } n \to \infty.$$

Since a dilation of the body does not change its regularity or convexity, throughout the paper we will replace c_n by 1.

In Section 2 we introduce several concepts containing the equatorial power type to describe quantitative information about convexity of bodies of revolution.

In Section 3 we investigate the equatorial behavior of symmetric intersection bodies of revolution under the convexity assumption. We prove that if K is a symmetric convex body of revolution, then the intersection body of K has uniform equatorial power type 2, which means that its boundary near the equator is asymptotically the same as the ball. Using this result, we prove in Section 4 that if K is a symmetric convex body of revolution in sufficiently high dimension, then its double intersection body is close, in the Banach–Mazur distance, to the Euclidean ball.

In Section 5 we will study the local convexity of intersection bodies at the equator without the convexity assumption. We prove that the intersection body of a symmetric star body of revolution in dimension $n \geq 5$ is locally convex at the equator, with equatorial power type 2.

2 Equatorial Power Type for Bodies of Revolution

The *equator* of a body K of revolution is the boundary of the section of K by the central hyperplane perpendicular to the axis of revolution, *i.e.*, $\partial K \cap e_1^{\perp}$. The goal of this section is to introduce parameters to measure the local convexity at the equator of bodies of revolution.

Definition 2.1 Let K be a body of revolution in $\mathbb{R}^n$ about the e_1-axis and let $1 \leq p < \infty$. Then the *function* $\psi_K \colon \mathbb{R} \to \mathbb{R}^+$ is defined by

$$\psi_K(x) = \rho_K(\theta)|\sin\theta| \quad \text{for } \theta = \tan^{-1}(1/x). \tag{2.1}$$

See Figure 1.

A body K of revolution is said to have *equatorial power type p* if there exist constants $c_1, c_2 > 0$, depending on K, such that $c_1 < |\psi_K(x) - \psi_K(0)|/x^p < c_2$ for every $0 < x < 1$. If K is a symmetric convex body, then ψ_K is a continuous even function that is non-increasing in $[0, +\infty)$ and with $\psi_K(x) = O(1/x)$ as x tends to infinity.

The local convexity properties of the function ψ_K at $x = 0$ are the same as those of the body K at the equator.

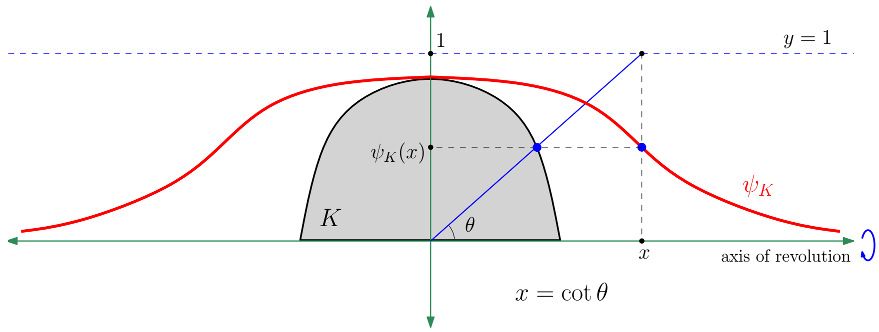

Figure 1: The function ψ_K

Formulas (1.1) and (1.2) provide a very nice relation between ψ_K and ψ_{IK}. Indeed, (1.1) implies

$$\begin{aligned}\rho_{IK}(\theta)\sin\theta &= \int_0^\theta \Big[\rho_K(\pi/2-\phi)\cos\phi\Big]^{n-1}\Big[\frac{1}{\cos^2\phi}\Big]^{\frac{n-4}{2}}\Big[1-\frac{\sin^2\phi}{\sin^2\theta}\Big]^{\frac{n-4}{2}}\frac{d\phi}{\cos^2\phi}\\ &= \int_0^\theta \Big[\rho_K(\pi/2-\phi)\cos\phi\Big]^{n-1}\Big[1-\frac{\tan^2\phi}{\tan^2\theta}\Big]^{\frac{n-4}{2}}\frac{d\phi}{\cos^2\phi}\\ &= \int_0^{\tan\theta}\psi_K(t)^{n-1}\Big[1-t^2\cot^2\theta\Big]^{\frac{n-4}{2}}dt.\end{aligned}$$

Thus we have

$$\psi_{IK}(0) = \rho_{IK}(\pi/2) = \int_0^\infty \psi_K(t)^{n-1}dt, \tag{2.2}$$

$$\psi_{IK}(x) = \int_0^{1/x}\psi_K(t)^{n-1}[1-x^2t^2]^{\frac{n-4}{2}}dt, \quad x\in(0,\infty). \tag{2.3}$$

As another way to describe equatorial power type, we can consider the classical modulus of convexity of a symmetric convex body K defined as

$$\delta_K(\varepsilon) = \inf\left\{1-\left\|\frac{x+y}{2}\right\|_K : x,y\in K, \|x-y\|_K\ge\varepsilon\right\},$$

where $\|\cdot\|_K$ denotes the Minkowski functional of K. However, since we focus on the convexity around the equator for bodies of revolution, it would be better to consider the following related notion.

Definition 2.2 Let K be a symmetric convex body of revolution in $\mathbb{R}^n$ about the e_1-axis. The *modulus of convexity of K at the equator* is defined by

$$\delta_K^e(\varepsilon) = \inf\left\{1-\left\|\frac{x+y}{2}\right\|_K : x,y\in K, \|x-y\|_K\ge 2\varepsilon, x-y\in\operatorname{span}\{e_1\}\right\}.$$

Equivalently δ^e_K can be expressed as

$$(2.4)\qquad \delta^e_K(\varepsilon) = \frac{\rho_K(\pi/2) - \rho_K(\theta)\sin\theta}{\rho_K(\pi/2)} = \frac{\psi_K(0) - \psi_K(\cot\theta)}{\psi_K(0)},$$

where the angle θ is obtained from $\varepsilon = \frac{\rho_K(\theta)}{\rho_K(0)} \cdot \cos\theta$.

It follows from (2.4) that a symmetric convex body K of revolution has equatorial power type p if and only if there exist constants $c_1, c_2 > 0$ such that $c_1 < \delta^e_K(\varepsilon)/\varepsilon^p < c_2$ for all $\varepsilon \in (0,1]$. Moreover, differently from the function ψ_K for a (star) body K of revolution, we notice that the modulus of convexity for a convex body of revolution is invariant for any dilations on the axis of revolution or its orthogonal complement.

For example, if K is the body of revolution in $\mathbb{R}^n$ obtained by rotating a 2-dimensional ℓ_p-ball with respect to the axis e_1, then it has equatorial power type p; more precisely, $\delta^e_K(\varepsilon) = \varepsilon^p/p + o(\varepsilon^p)$.

Definition 2.3 For $1 \le p < \infty$, a collection $\mathcal{C}$ of convex bodies of revolution is said to have *uniform equatorial power type p* if every convex body in $\mathcal{C}$ has equatorial power type p, and moreover there exist uniform constants $c_1, c_2 > 0$ such that

$$c_1 < \frac{\delta^e_K(\varepsilon)}{\varepsilon^p} < c_2 \quad \text{for every } \varepsilon \in (0,1] \text{ and } K \in \mathcal{C}.$$

Now let us show some relation between δ^e_K and ψ_K when K is a symmetric convex body. Fix $\varepsilon \in (0,1]$ and choose the angle $\theta \in (0,\pi/2)$ so that $\varepsilon = (\rho_K(\theta)/\rho_K(0)) \cdot \cos\theta$. Then

$$(2.5)\qquad (1-\varepsilon)\rho_K(\pi/2) \le \rho_K(\theta)\sin\theta \le \rho_K(\pi/2),$$

which can be obtained by applying the convexity property of K to three points on the boundary of K with angles 0, θ, and $\pi/2$. Notice that for small $\varepsilon > 0$ (2.5) gives

$$\cot\theta = \frac{\rho_K(0)\cdot\varepsilon}{\rho_K(\theta)\sin\theta} = \frac{\rho_K(0)}{\rho_K(\pi/2)}\big(\varepsilon + O(\varepsilon^2)\big).$$

Next it follows from (2.4) that

$$(2.6)\qquad \delta^e_K(\varepsilon) = \frac{\psi_K(0) - \psi_K(\delta)}{\psi_K(0)}, \quad \text{for } \delta = \frac{\rho_K(0)}{\rho_K(\pi/2)}\big(\varepsilon + O(\varepsilon^2)\big).$$

In particular, we have $\delta^e_K(\varepsilon) \approx 1 - \psi_K(\varepsilon)$ under the assumption that $\rho_K(0) = \rho_K(\pi/2) = 1$.

In Section 3 we prove that the class of all intersection bodies of symmetric convex bodies of revolution have uniform equatorial power type 2, and we also provide an example showing that the convexity condition cannot be dropped. Thus, for star bodies of revolution, it is not necessary to consider δ^e_K as an invariant quantity under dilations on the axis e_1 or its orthogonal complement; the function ψ_K will be enough for star bodies. For symmetric convex bodies of revolution, we will use the modulus $\delta^e_K(\varepsilon)$ of convexity at the equator to describe the power type or the asymptotic behavior at the equator, and, moreover, the function $\psi_K(x)$ can be used to compute $\delta^e_K(\varepsilon)$ by (2.6).

3 Uniform Equatorial Power Type 2 for Intersection Bodies

In this section we prove that the class of all intersection bodies of symmetric convex bodies of revolution has uniform equatorial power type 2. Namely, if K is a symmetric convex body of revolution, then IK has equatorial power type 2, and, moreover, the coefficient of the quadratic term in the expansion of $\delta^e_{IK}(\varepsilon)$ is bounded above and below by absolute constants.

First we need a specific formula for the function ψ_K in the case that K is a symmetric body of revolution obtained by rotating line segments.

Lemma 3.1 *Let $L_{a,b} \subset \mathbb{R}^n$ be the symmetric body of revolution whose boundary is determined by a line segment $\{(x,y) : ax + by = 1, 0 \le x \le 1/a\}$ for $a, b \ge 0$. Namely, the body $L_{a,b}$ can be given by*

$$L_{a,b} = \Big\{(x,y) \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} : a|x| + b|y| \le 1\Big\}.$$

Then the function $\psi_{L_{a,b}}$, defined in (2.1), is equal to

$$\psi_{L_{a,b}}(x) = \frac{1}{a|x| + b}. \tag{3.1}$$

Moreover, if $K \subset \mathbb{R}^n$ is a symmetric convex body of revolution with $\rho_K(0) = \rho_K(\pi/2) = 1$, then

$$\frac{1}{|x|+1} \le \psi_K(x) \le \min\Big(1, \frac{1}{|x|}\Big). \tag{3.2}$$

Proof Let $x = \cot\theta > 0$, and write $L = L_{a,b}$ to shorten the notation. Then the point

$$\Big(\rho_L(\theta)\cos\theta, \rho_L(\theta)\sin\theta\Big) \in \mathbb{R}^2$$

lies on the straight line $\Big\{(p,q) \in \mathbb{R}^2 : ap + bq = 1\Big\}$. Thus we have

$$x = \cot\theta = \frac{\rho_L(\theta)\cos\theta}{\rho_L(\theta)\sin\theta} = \frac{(1 - b\rho_L(\theta)\sin\theta)/a}{\rho_L(\theta)\sin\theta} = \frac{1 - b\,\psi_L(x)}{a\,\psi_L(x)},$$

which gives $\psi_L(x) = 1/(ax + b)$.

Now, if $K \subset \mathbb{R}^n$ is a symmetric convex body of revolution with $\rho_K(0) = \rho_K(\pi/2) = 1$, then we have $B_1 \subset K \subset B_\infty$ where

$$\begin{aligned} B_1 &= \Big\{(x,y) \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} : |x| + |y| \le 1\Big\}, \\ B_\infty &= \Big\{(x,y) \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} : |x| \le 1, |y| \le 1\Big\}. \end{aligned}$$

We also see that $\psi_{B_1} \le \psi_K \le \psi_{B_\infty}$ by definition of the function ψ. Here ψ_{B_1} and ψ_{B_∞} can be obtained from (3.1):

$$\psi_{B_1}(x) = \psi_{L_{1,1}}(x) = \frac{1}{x+1}$$

and

$$\psi_{B_\infty}(x) = \begin{cases} \psi_{L_{0,1}}(x) = 1, & \text{if } 0 \le x \le 1, \\ \psi_{L_{1,0}}(x) = 1/x, & \text{if } x \ge 1, \end{cases}$$

which imply (3.2). ■

Inequality (3.2) in Lemma 3.1 gives an easy upper or lower bound for the function ψ_K. However, we will need better bounds for high dimension given by the following lemma.

Lemma 3.2 *Let $K \subset \mathbb{R}^n$ be a convex body of revolution with $\rho_K(\pi/2) = 1$. For every $\sigma > 0$ and $t > 1$,*

$$\psi_K(\sigma t) \le \Big[1 + t\Big(\frac{1}{\psi_K(\sigma)} - 1\Big)\Big]^{-1}. \tag{3.5}$$

Proof Let $\phi_1 = \tan^{-1}(1/\sigma)$ and $\phi_2 = \tan^{-1}(1/\sigma t)$. Choose three points P_0, P_1, $P_2 \in \partial K \cap \operatorname{span}\{e_1, e_2\}$ whose angles from the e_1-axis are $\pi/2$, ϕ_1 and ϕ_2, respectively. That is,

$$\begin{aligned} P_0 &= (0, 1), \\ P_1 &= \big(\rho_K(\phi_1)\cos\phi_1, \rho_K(\phi_1)\sin\phi_1\big) =: (x_1, y_1), \\ P_2 &= \big(\rho_K(\phi_2)\cos\phi_2, \rho_K(\phi_2)\sin\phi_2\big) =: (x_2, y_2). \end{aligned}$$

Since $\frac{1-y_2}{1-y_1} \ge \frac{x_2}{x_1}$ by convexity of K,

$$\frac{1-\rho_K(\phi_2)\sin\phi_2}{1-\rho_K(\phi_1)\sin\phi_1} \ge \frac{\rho_K(\phi_2)\cos\phi_2}{\rho_K(\phi_1)\cos\phi_1} = \frac{\rho_K(\phi_2)\sin\phi_2\cdot\cot\phi_2}{\rho_K(\phi_1)\sin\phi_1\cdot\cot\phi_1},$$

which implies

$$\frac{1-\psi_K(\sigma t)}{1-\psi_K(\sigma)} \ge \frac{\psi_K(\sigma t)\sigma t}{\psi_K(\sigma)\sigma}.$$

Simplifying the above inequality, we have inequality (3.5). ■

The next lemma will be helpful for bounding the integral in (2.2) and controlling its tail.

Lemma 3.3 *For $n \ge 4$, let $K \subset \mathbb{R}^n$ be a convex body of revolution with $\rho_K(\pi/2) = 1$ and let $\sigma_K = \psi_K^{-1}(1 - 1/n)$. Then*

$$c_1 \le \frac{1}{\sigma_K}\int_0^\infty \psi_K(t)^{n-1}\,dt \le c_2, \tag{3.6}$$

where c_1, $c_2 > 0$ are absolute constants. In addition, for every $R > 1$,

$$\int_R^\infty \psi_K(\sigma_K t)^{n-1}\,dt = O\big([1 + R/n]^{2-n}\big).$$

Here, $f(\varepsilon) = O(\varepsilon)$ means that $|f(\varepsilon)| \le c\varepsilon$ for small $\varepsilon > 0$ and an absolute constant $c > 0$.

https://doi.org/10.4153/CJM-2013-039-4 Published online by Cambridge University Press

Proof For any $t \geq R$, Lemma 3.2 gives

$$\psi_K(\sigma_K t) \leq \left[1 + \frac{t}{\psi_K(\sigma_K)} - t\right]^{-1} = \left[1 + \frac{t}{n-1}\right]^{-1}.$$

Thus,

$$\int_R^\infty \psi_K(\sigma_K t)^{n-1} dt \leq \int_R^\infty \left[1 + \frac{t}{n-1}\right]^{1-n} dt = \frac{n-1}{n-2}\left[1 + \frac{R}{n-1}\right]^{2-n}.$$

Next we will show an upper bound in (3.6),

$$\begin{aligned} \int_0^\infty \psi_K(t)^{n-1} dt &= \int_0^{\sigma_K} \psi_K(t)^{n-1} dt + \int_{\sigma_K}^\infty \psi_K(t)^{n-1} dt \\ &\leq \sigma_K + \sigma_K \int_1^\infty \psi_K(\sigma_K t)^{n-1} dt \\ &\leq \sigma_K + \sigma_K \frac{n-1}{n-2}\left[1 + \frac{1}{n-1}\right]^{2-n} \to (1 + 1/e)\sigma_K \quad \text{as } n \to \infty. \end{aligned}$$

For a lower bound,

$$\int_0^{\sigma_K} \psi_K(t)^{n-1} dt \geq \int_0^{\sigma_K} \left[1 - \frac{1}{n}\right]^{n-1} dt \quad \to \frac{\sigma_K}{e} \quad \text{as } n \to \infty.$$

Thus we have that $c_1\sigma_K \leq \int_0^\infty \psi_K(t)^{n-1} dt \leq c_2\sigma_K$ for absolute constants $c_1, c_2 > 0$. ∎

Next, Lemma 3.4 will allow us to estimate the integral in (2.3).

Lemma 3.4 *For $n \geq 4$, let $K \subset \mathbb{R}^n$, for $n \geq 4$, be a symmetric convex body of revolution with $\rho_K(\pi/2) = 1$. Fix $R > 1$ and let $\sigma_K = \psi_K^{-1}(1 - 1/n)$. Then for each $x \leq \frac{1}{R\sigma_K}$,*

$$\frac{\psi_{IK}(x)}{\sigma_K} = \int_0^R \psi_K(\sigma_K t)^{n-1}[1 - \sigma_K^2 x^2 t^2]^{\frac{n-4}{2}} dt + O\big([1 + R/n]^{2-n}\big).$$

Proof If $x \neq 0$, then

$$\begin{aligned} \psi_{IK}(x) &= \int_0^{1/x} \psi_K(t)^{n-1}[1 - x^2t^2]^{\frac{n-4}{2}} dt \\ &= \sigma_K\Big(\int_0^R + \int_R^{(\sigma_K x)^{-1}}\Big) \psi_K(\sigma_K t)^{n-1}[1 - \sigma_K^2 x^2 t^2]^{\frac{n-4}{2}} dt. \end{aligned}$$

Lemma 3.3 gives an upper bound of the second integral, *i.e.,*

$$\begin{aligned} \int_R^{(\sigma_K x)^{-1}} \psi_K(\sigma_K t)^{n-1}\left[1 - \sigma_K^2 x^2 t^2\right]^{\frac{n-4}{2}} dt &\leq \int_R^\infty \psi_K(\sigma_K t)^{n-1} dt \\ &= O\big([1 + R/n]^{2-n}\big). \end{aligned}$$

If $x = 0$, then

$$\psi_{IK}(0) = \int_0^\infty \psi_K(t)^{n-1}dt = \sigma_K\Big[\int_0^R \psi_K(\sigma_K t)^{n-1}dt + O\big([1+R/n]^{2-n}\big)\Big]. \quad \blacksquare$$

Now we are ready to prove the main result of this section.

Theorem 3.5 *The class of intersection bodies of symmetric convex bodies of revolution in dimension $n \geq 4$ has uniform equatorial power type 2. Namely, if K is a symmetric convex body of revolution in $\mathbb{R}^n$ for $n \geq 4$, then its intersection body IK has modulus of convexity at the equator of the form*

$$\delta^e_{IK}(\varepsilon) = c_K\varepsilon^2 + O(\varepsilon^3),$$

where $c_K > 0$ is a constant depending on K and bounded above and below by absolute constants.

Proof The modulus of convexity at the equator is invariant for any dilations on the axis of revolution or its orthogonal complement, so we can start with $\rho_K(\pi/2) = \rho_K(0) = 1$. Fix a small number $\varepsilon > 0$ and choose the angle θ such that

$$\frac{\rho_{IK}(\theta)}{\rho_{IK}(0)}\cos\theta = \varepsilon.$$

Let $\delta = \cot\theta$ and $\sigma_K = \psi_K^{-1}(1 - 1/n)$. By Lemma 3.3, we have

$$c_1 \leq \frac{1}{\sigma_K}\int_0^\infty \psi_K(t)^{n-1}dt \leq c_2$$

for absolute constants $c_1, c_2 > 0$. By convexity of IK, as in (2.5),

$$(1-\varepsilon)\rho_{IK}(\pi/2) \leq \rho_{IK}(\theta)\sin\theta \leq \rho_{IK}(\pi/2). \tag{3.7}$$

Note that $c_1\sigma_K \leq \rho_{IK}(\pi/2) \leq c_2\sigma_K$ and $\rho_{IK}(\theta)\sin\theta = \big(\frac{\rho_{IK}(\theta)}{\rho_{IK}(0)}\cos\theta\big)\rho_{IK}(0)\tan\theta = (\varepsilon/\delta)\rho_{IK}(0)$. Since $\rho_{IK}(0)/\sqrt{\pi/(2n)}$ tends to 1 as $n \to \infty$ by (1.2), the inequality (3.7) implies that there exist absolute constants $c_1', c_2' > 0$ such that

$$c_1'\sigma_K\sqrt{n} \leq \varepsilon/\delta \leq c_2'\sigma_K\sqrt{n}. \tag{3.8}$$

First consider the case of $n \geq 14$. Formula (3.8) and Lemma 3.4 give that, for any R with $1 \leq R \leq (\sigma_K\delta)^{-1}$,

$$\begin{aligned}\frac{\psi_{IK}(\delta)}{\sigma_K} &= \int_0^R \psi_K(\sigma_K t)^{n-1}\Big[1-(\sigma_K\delta t)^2\Big]^{\frac{n-4}{2}}dt + O\Big(\Big[1+R/n\Big]^{2-n}\Big)\\ &= \int_0^R \psi_K(\sigma_K t)^{n-1}\Big[1-\frac{n-4}{2}(\sigma_K\delta t)^2 + O\Big(\big|n(\sigma_K\delta R)^2\big|^2\Big)\Big]dt\\ &\quad + O\Big(\Big[1+R/n\Big]^{2-n}\Big)\end{aligned} \tag{3.9}$$

and

$$\frac{\psi_{IK}(0)}{\sigma_K} = \int_0^R \psi_K(\sigma_K t)^{n-1} dt + O\Big(\Big[1 + R/n\Big]^{2-n}\Big).$$

Since $\sigma_K\delta$ is comparable to $\varepsilon/\sqrt{n}$ by absolute constants by (3.8), we can take $R = \varepsilon^{-1/4}$ to control error terms of above equation. Then we have

$$\big|n(\sigma_K\delta R)^2\big|^2 = O(\varepsilon^4 R^4) = O(\varepsilon^3),$$

and for $n \geq 14$,

$$(1 + R/n)^{2-n} \leq (1 + R/n)^{-12} = \Big[\frac{n}{1 + n\varepsilon^{1/4}}\Big]^{12}\varepsilon^3,$$
$$(1 + R/n)^{2-n} \to e^{-R} = e^{-\varepsilon^{-1/4}} \quad \text{as } n \to \infty,$$

so the remainder part of (3.9) is $O(\varepsilon^3)$ for $n \geq 14$. Thus,

$$\frac{\psi_{IK}(\delta)}{\sigma_K} = \int_0^R \psi_K(\sigma_K t)^{n-1} dt - \frac{(n-4)(\sigma_K\delta)^2}{2}\int_0^R \psi_K(\sigma_K t)^{n-1} t^2 dt + O(\varepsilon^3)$$

and

$$\frac{\psi_{IK}(0)}{\sigma_K} = \int_0^R \psi_K(\sigma_K t)^{n-1} dt + O(\varepsilon^3).$$

Formula (2.6) gives the modulus of convexity at the equator as follows:

$$\delta^e_{IK}(\varepsilon) = \frac{\psi_{IK}(0) - \psi_{IK}(\delta)}{\psi_{IK}(0)} = \Big[\frac{(n-4)(\sigma_K\delta/\varepsilon)^2}{2}\int_0^R \psi_K(\sigma_K t)^{n-1} t^2 dt\Big]\varepsilon^2 + O(\varepsilon^3).$$

Since $(n-4)(\sigma_K\delta/\varepsilon)^2$ is bounded above and below by absolute constants from (3.8), it is enough to compute $\int_0^R \psi_K(\sigma_K t)^{n-1} t^2 dt$. To get an upper bound, apply Lemma 3.2. Then, for any $t \geq 1$,

$$\psi_K(\sigma_K t) \leq \Big[1 + \frac{t}{\psi_K(\sigma_K)} - t\Big]^{-1} = \Big[1 + \frac{t}{n-1}\Big]^{-1}.$$

Thus,

$$\begin{aligned}
&\int_0^R \psi_K(\sigma_K t)^{n-1} t^2 dx \\
&\leq \int_0^1 dt + \int_1^\infty t^2\Big[1 + \frac{t}{n-1}\Big]^{1-n} dt = 1 + (n-1)^3 \int_{\frac{n}{n-1}}^\infty (s-1)^2 s^{1-n} ds \\
&= 1 + \Big(1 - \frac{1}{n}\Big)^{n-1} \frac{5n^3 - 15n^2 + 12n}{(n-2)(n-3)(n-4)} \to 1 + \frac{5}{e} \quad \text{as } n \to \infty.
\end{aligned}$$

https://doi.org/10.4153/CJM-2013-039-4 Published online by Cambridge University Press

For a lower bound,

$$\int_0^R \psi_K(\sigma_K t)^{n-1} t^2 dt \ge \int_0^1 \psi_K(\sigma_K t)^{n-1} t^2 dt$$
$$\ge \int_0^1 \left[1-\frac{1}{n}\right]^{n-1} t^2 dt \to \frac{1}{3e} \quad \text{as } n\to\infty,$$

which completes the proof for $n \ge 14$.

Now consider the case of $4 \le n < 14$. It follows from (2.2) and (2.3) that

$$\begin{aligned}\psi_{IK}(0)-\psi_{IK}(\delta) &= \int_0^\infty \psi_K(t)^{n-1}dt - \int_0^{1/\delta} \psi_K(t)^{n-1}(1-\delta^2t^2)^{\frac{n-4}{2}}dt\\ &= \int_{1/\delta}^\infty \psi_K(t)^{n-1}dt + \int_0^{1/\delta}\psi_K(t)^{n-1}\left[1-(1-\delta^2t^2)^{\frac{n-4}{2}}\right]dt\\ &= \quad (\mathrm{I}) \quad + \quad (\mathrm{II}).\end{aligned}$$

For (I), use the inequalities (3.2) from Lemma 3.1. Then

$$(\mathrm{I}) \le \int_{1/\delta}^\infty \frac{1}{t^{n-1}}dt = \frac{\delta^{n-2}}{n-2} = O(\delta^{n-2})$$

and

$$(\mathrm{I}) \ge \int_{1/\delta}^\infty \frac{1}{(t+1)^{n-1}}dt = \frac{(1+1/\delta)^{2-n}}{n-2} = O(\delta^{n-2}).$$

Since δ is comparable to ε by (3.8), we have that (I) is $O(\varepsilon^{n-2})$, which is at most $O(\varepsilon^3)$ if $n \ge 5$.

To get an upper bound of (II), use (3.2) again:

$$\begin{aligned}(\mathrm{II}) &\le \int_0^1 \left[1-(1-\delta^2t^2)^{\frac{n-4}{2}}\right]dt + \int_1^{1/\delta}(1/t)^{n-1}\left[1-(1-\delta^2t^2)^{\frac{n-4}{2}}\right]dt\\ &= \quad (\text{II-1}) \quad + \quad (\text{II-2}),\end{aligned}$$

where

$$(\text{II-1}) = \int_0^1 \left[1-\left(1-\frac{n-4}{2}\delta^2t^2\right)\right]dt + O(\delta^4) = \frac{n-4}{6}\,\delta^2 + O(\delta^4)$$

and

$$\begin{aligned}(\text{II-2}) &= \int_1^{1/\delta}(1/t)^{n-1}dt - \int_1^{1/\delta}\frac{(1-\delta^2t^2)^{\frac{n-4}{2}}}{t^{n-4}}\,\frac{dt}{t^3}\\ &= \frac{1-\delta^{n-2}}{n-2} - \frac{1}{2}\int_{\delta^2}^1 (s-\delta^2)^{\frac{n-4}{2}}ds = \frac{1-\delta^{n-2}-(1-\delta^2)^{\frac{n-2}{2}}}{n-2}\\ &= \frac{1}{2}\delta^2 + O(\delta^{n-2}).\end{aligned}$$

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A lower bound of (II) is given by

$$
\begin{aligned}
\text{(II)} &\geq \int_0^{1/\delta} \frac{1-(1-\delta^2t^2)^{\frac{n-4}{2}}}{(t+1)^{n-1}}\,dt \\
&= \int_0^1 \frac{1-(1-\delta^2t^2)^{\frac{n-4}{2}}}{(t+1)^{n-1}}\,dt + \int_1^{1/\delta} \Big(\frac{t}{t+1}\Big)^{n-1} \frac{1-(1-\delta^2t^2)^{\frac{n-4}{2}}}{t^{n-1}}\,dt \\
&\geq \frac{1}{2^{n-1}} \int_0^1 \big[1-(1-\delta^2t^2)^{\frac{n-4}{2}}\big]\,dt + \frac{1}{2^{n-1}} \int_1^{1/\delta} \frac{1-(1-\delta^2t^2)^{\frac{n-4}{2}}}{t^{n-1}}\,dt \\
&= \frac{1}{2^{n-1}}\Big[\frac{n-4}{6}\,\delta^2 + O(\delta^4)\Big] + \frac{1}{2^{n-1}}\text{(II-2)} = \frac{n-1}{3\cdot 2^n}\,\delta^2 + O(\delta^4).
\end{aligned}
$$

In summary, if $n \geq 5$, then (I) is at most $O(\varepsilon^3)$ and (II) is asymptotically $c\delta^2 + O(\delta^3)$. In addition, if $n = 4$, then (II) disappears and (I) is $c\delta^2 + O(\delta^3)$. Note that

$$
\delta^e_{IK}(\varepsilon) = \frac{\psi_{IK}(0) - \psi_{IK}(\delta)}{\psi_{IK}(0)}
$$

and $c_1 < \psi_{IK}(0) < c_2$ by Lemma 3.3. Finally, we get

$$
c_1' < \delta^e_{IK}(\varepsilon)/\varepsilon^2 < c_2',
$$

where c_1', c_2' are positive absolute constants. ∎

Remark 3.6 In general, Theorem 3.5 is not true in dimension 3. For example, the intersection body of the double cone $B_1 \subset \mathbb{R}^3$ does not have equatorial power type 2.

Proof It follows from Lemma 3.1 that the function ψ_K for the double cone $K = B_1$ is given by $\psi_{B_1}(x) = \frac{1}{x+1}$. Let $\varepsilon = \big(\rho_{IB_1}(\theta)/\rho_{IB_1}(0)\big)\cos\theta$ for some angle θ and let $\delta = \cot\theta$. Then

$$
\begin{aligned}
\psi_{IB_1}(0) &= \int_0^\infty \psi_{B_1}(t)^2\,dt = \int_0^\infty (t+1)^{-2}\,dt = 1, \\
\psi_{IB_1}(\delta) &= \int_0^{1/\delta} \psi_{B_1}(t)^2(1-\delta^2t^2)^{-1/2}\,dt = \int_0^{1/\delta} \frac{dt}{(t+1)^2\sqrt{1-\delta^2t^2}}.
\end{aligned}
$$

So

$$
\begin{aligned}
\delta^e_{IB_1}(\varepsilon) = 1 - \psi_{IB_1}(\delta) &= \int_0^\infty \frac{1}{(t+1)^2}\,dt - \int_0^{1/\delta} \frac{dt}{(t+1)^2\sqrt{1-\delta^2t^2}} \\
&= \int_{1/\delta}^\infty \frac{1}{(t+1)^2}\,dt + \int_0^{1/\delta} \frac{1}{(t+1)^2}\Big(1 - \frac{1}{\sqrt{1-\delta^2t^2}}\Big)\,dt \\
&= \Big(\delta - \frac{\delta^2}{1+\delta}\Big) - \delta \int_0^1 \frac{t^2\,dt}{(t+\delta)^2\sqrt{1-t^2}(1+\sqrt{1-t^2})} \\
&= \delta - \delta f(\delta) + O(\delta^2),
\end{aligned}
$$

where

$$f(\delta)=\int_0^1 \frac{t^2dt}{(t+\delta)^2\sqrt{1-t^2}(1+\sqrt{1-t^2})}.$$

Note that

$$f(0)=\int_0^1 \frac{dt}{\sqrt{1-t^2}(1+\sqrt{1-t^2})}=1$$

and

$$\lim_{\delta\to 0}\frac{f(0)-f(\delta)}{\delta}=\int_0^1 \frac{dt}{t\sqrt{1-t^2}(1+\sqrt{1-t^2})}=\infty,$$

Since δ is comparable to ε, we conclude that $\delta^e_{IB_1}(\varepsilon)=o(\varepsilon)$, but $\delta^e_{IB_1}(\varepsilon)\neq O(\varepsilon^2)$. ■

Remark 3.7 The convexity condition of K in Theorem 3.5 is crucial to get the uniform boundedness of the constant c_K. For $t>0$, consider the star body of revolution K_t, defined as the union $K_t=L_t\cup B_\infty$ of two cylinders

$$L_t=\left\{(x,y)\in\mathbb{R}^n=\mathbb{R}\times\mathbb{R}^{n-1}:|x|\le e^{-1/t},|y|\le 1/t\right\}$$

and

$$B_\infty=\left\{(x,y)\in\mathbb{R}^n=\mathbb{R}\times\mathbb{R}^{n-1}:|x|\le 1,|y|\le 1\right\}.$$

If $t>0$ is small enough, then the intersection body of K_t is almost the same as that of B_∞ around the equator. In other words, $\psi_{IK_t}(0)=\psi_{IB_\infty}(0)+O(e^{-1/t}/t^{n-1})$ and $\psi_{IK_t}(\varepsilon)=\psi_{IB_\infty}(\varepsilon)+O(e^{-1/t}/t^{n-1})$ for small $\varepsilon>0$. Nevertheless, note that $\rho_{IB_\infty}(0)=1$, but $\rho_{IK_t}(0)=1/t^{n-1}$, *i.e.,* they have quite different radial functions on the axis as t approaches to zero. So,

$$\begin{aligned}\delta^e_{IK_t}(\varepsilon)&=\frac{\psi_{IK_t}(0)-\psi_{IK_t}(\delta)}{\psi_{IK_t}(0)}=\frac{\psi_{IB_\infty}(0)-\psi_{IB_\infty}(\delta/t^{n-1})}{\psi_{IB_\infty}(0)}+O(e^{-1/t})\\&=\delta^e_{IB_\infty}(\varepsilon/t^{n-1})+O(e^{-1/t}),\end{aligned}$$

where

$$\delta=\frac{\rho_{IK_t}(0)\,\varepsilon}{\rho_{IK_t}(\pi/2)}=\frac{\rho_{IB_\infty}(0)\,\varepsilon}{\rho_{IB_\infty}(\pi/2)}\cdot\frac{1}{t^{n-1}}+O(e^{-1/t}).$$

Thus, $\delta^e_{IK_t}(\varepsilon)/\delta^e_{IB_\infty}(\varepsilon)=O(t^{2-2n})$, which tends to infinity as t tends to zero. Therefore, this example shows that the constant c_K in Theorem 3.5 can be unbounded in the case of star bodies.

4 Double Intersection Bodies of Revolution in High Dimension

Recently, Fish, Nazarov, Ryabogin, and Zvavitch [3] proved that the iterations of the intersection body operator, applied to any symmetric star body sufficiently close to a Euclidean ball B_2^n in the Banach–Mazur distance, converge to B_2^n in the Banach–Mazur distance. Namely, if K is a star body in $\mathbb{R}^n$ with $d_{BM}(K, B_2^n) = 1 + \varepsilon$ for small $\varepsilon > 0$, then

$$\lim_{m\to\infty} d_{BM}(I^m K, B_2^n) = 1.$$

In the case of bodies of revolution in sufficiently high dimension, it turns out that it is enough to apply the intersection body operator twice to get close to the Euclidean ball in the Banach–Mazur distance, which will be shown in this section. The uniform boundedness of the constant c_K by absolute constants in Theorem 3.5 plays an important role in the following result.

Theorem 4.1 *Let K be a symmetric convex body of revolution in $\mathbb{R}^n$. Then the double intersection body I^2K is close to an ellipsoid if the dimension n is large enough. More precisely, for every $\varepsilon > 0$ there exists an integer $N > 0$ such that for every $n \ge N$ and any body $K \subset \mathbb{R}^n$ of revolution,*

$$d_{BM}(I^2K, B_2^n) \le 1 + \varepsilon.$$

Proof By B we denote the unit ball in $\mathbb{R}^n$ (instead of B_2^n). It follows from Theorem 3.5 that $\delta^e_{IK}(\varepsilon) = c_K\varepsilon^2 + O(\varepsilon^3)$ where $c_1 < c_K < c_2$ for absolute constants $c_1, c_2 > 0$. Also note that $\delta^e_B(\varepsilon) = \varepsilon^2/2 + O(\varepsilon^3)$ for the unit ball B. Consider a linear transformation T (dilation), which gives $\rho_{T(IK)}(\pi/2) = 1$ and $\rho_{T(IK)}(0) = 1/\sqrt{2c_K}$. Denote $L := T(IK)$. Then

$$\psi_L(t) = 1 - \delta^e_L\big(t/\sqrt{2c_K}\big) + O(t^3) = 1 - t^2/2 + O(t^3). \tag{4.1}$$

Also, it is not hard to compute the function ψ_B for the ball B,

$$\psi_B(t) = \frac{1}{\sqrt{1+t^2}} = 1 - t^2/2 + O(t^3). \tag{4.2}$$

Let

$$\sigma_L = \psi_L^{-1}(1 - 1/n) = \sqrt{2/n} + o(n^{-1/2}),$$
$$\sigma_B = \psi_B^{-1}(1 - 1/n) = \sqrt{2/n} + o(n^{-1/2}).$$

Fix $\varepsilon > 0$, and let $R = -4\log\varepsilon$, $N = R^2/\varepsilon^4$. Then we claim that for every $n \ge N$,

$$\left|\frac{\rho_{IL}(\theta)}{\rho_{IB}(\theta)} - 1\right| = O(\varepsilon) \quad \forall\theta \in [0, \pi/2].$$

First, consider the case where the angle θ satisfies $\tan\theta \ge \varepsilon$. For $n \ge N$, since $(1 + R/n)^{2-n} \le (e^{\frac{R}{2n}})^{2-n} \le e^{1-R/2} = e\varepsilon^2$, we have

$$\left[1 + \frac{R}{n}\right]^{2-n} = O(\varepsilon^2).$$

Note also that for $n \geq N$, since σ_L, σ_B are bounded by $\sqrt{2/N} = \sqrt{2}\varepsilon^2/R$,

$$\sigma_L R = O(\varepsilon^2) \quad \text{and} \quad \sigma_B R = O(\varepsilon^2).$$

For θ with $\tan\theta \geq \varepsilon$, we have $\cot\theta \leq 1/(\sigma_L R)$. Applying Lemma 3.4 for $x = \cot\theta$, we get

$$\frac{\rho_{IL}(\theta)\sin\theta}{\sigma_L} = \int_0^R \psi_L(\sigma_L t)^{n-1}\Big[1 - \frac{\sigma_L^2 t^2}{\tan^2\theta}\Big]^{\frac{n-4}{2}} dt + O(\varepsilon^2).$$

Note that $\rho_{IL}(\pi/2)$ is comparable to σ_L by Lemma 3.3, and $\rho_{IL}(0) \approx \sqrt{\frac{\pi}{2n}}$ is also comparable to σ_L. So, by convexity of IL, the radial function for IL at any angle is comparable to σ_L. Moreover, since

$$\begin{aligned}\sigma_L\varepsilon^2 &= \frac{\sigma_L}{\rho_{IL}(\theta)} \cdot \frac{\varepsilon^2}{\sin\theta} \cdot \rho_{IL}(\theta)\sin\theta \leq \frac{\sigma_L}{\rho_{IL}(\theta)} \cdot 2\varepsilon \cdot \rho_{IL}(\theta)\sin\theta \\ &= O(\varepsilon)\cdot\rho_{IL}(\theta)\sin\theta,\end{aligned}$$

we have

$$\begin{aligned}\rho_{IL}(\theta)\sin\theta &= \int_0^{\sigma_L R} \psi_L(t)^{n-1}(1 - t^2/\tan^2\theta)^{\frac{n-4}{2}} dt + O(\sigma_L\varepsilon^2) \\ &= (1 + O(\varepsilon))\int_0^{\sigma_L R} \psi_L(t)^{n-1}(1 - t^2/\tan^2\theta)^{\frac{n-4}{2}} dt.\end{aligned}$$

Similarly, we have the same equality for IB. Without loss of generality, we can assume $\sigma_L \geq \sigma_B$. Then

$$\rho_{IL}(\theta)\sin\theta = (1 + O(\varepsilon))\int_0^{\sigma_L R} \psi_L(t)^{n-1}\Big(1 - t^2/\tan^2\theta\Big)^{\frac{n-4}{2}} dt,$$

$$\rho_{IB}(\theta)\sin\theta = (1 + O(\varepsilon))\int_0^{\sigma_L R} \psi_B(t)^{n-1}\Big(1 - t^2/\tan^2\theta\Big)^{\frac{n-4}{2}} dt$$

Moreover, (4.1) and (4.2) give

$$\Big(\frac{\psi_L(t)}{\psi_B(t)}\Big)^{n-1} = \big[1 + O(\sigma_L^3 R^3)\big]^{n-1} = 1 + O(n\sigma_L^3 R^3).$$

Here, since $n\sigma_L^2 \leq 3$, $\varepsilon R^2 = 16\varepsilon(\log\varepsilon)^2 \leq 1$ and $\sigma_L R = O(\varepsilon^2)$, we get

$$n\sigma_L^3 R^3 = (n\sigma_L^2)(\varepsilon R^2)(\sigma_L R/\varepsilon) = O(\varepsilon).$$

Thus,

$$\frac{\rho_{IL}(\theta)}{\rho_{IB}(\theta)} = 1 + O(\varepsilon) \quad \text{for each } \theta \geq \tan^{-1}\varepsilon.$$

Now consider the case of $0 < \theta < \tan^{-1}\varepsilon$. Note that $\rho_{IL}(0) = \rho_{IB}(0)$ and the above statement gives $\frac{\rho_{IL}(\tan^{-1}\varepsilon)}{\rho_{IB}(\tan^{-1}\varepsilon)} = 1 + O(\varepsilon)$. The convexity of IL and IB gives that

$$\frac{\rho_{IL}(\theta)}{\rho_{IB}(\theta)} = 1 + O(\varepsilon) \quad \text{for } 0 < \theta < \tan^{-1}\varepsilon. \qquad \blacksquare$$

Remark 4.2 Theorem 4.1 says that the double intersection body of any body of revolution becomes close to an ellipsoid as the dimension increases to the infinity. However, it is not true if the intersection body operator is applied once, in general. For example, consider the cylinder B_∞. Then the Banach–Mazur distance between IB_∞ and B_2^n does not converge to 1 as n tends to the infinity.

Proof The function ψ_{B_∞} for the cylinder B_∞ is given by $\psi_{B_\infty}(t) = \min(1, 1/t)$ as in the proof of Lemma 3.1. Note that $\rho_{IB_\infty}(0) = \sqrt{\pi/(2n)}$ by (1.2), and

$$\rho_{IB_\infty}(\pi/2) = \int_0^\infty \psi_{B_\infty}(t)^{n-1} = \int_0^1 1dt + \int_1^\infty t^{1-n}dt = \frac{n-1}{n-2}.$$

Choose the angle θ with $\tan\theta = \frac{\rho_{IB_\infty}(\pi/2)}{\rho_{IB_\infty}(0)}$, and let $x = \cot\theta$. Then

$$x = \frac{\rho_{IB_\infty}(0)}{\rho_{IB_\infty}(\pi/2)} = \sqrt{\frac{\pi}{2n}} \cdot \frac{n-2}{n-1} = O(1/\sqrt{n}).$$

In addition,

$$\rho_{IB_\infty}(\theta)\sin\theta = \psi_{IB_\infty}(x) = \int_0^{1/x} \psi_{B_\infty}(t)^{n-1}(1 - x^2t^2)^{\frac{n-4}{2}}dt = (\mathrm{I}) + (\mathrm{II}),$$

where

$$(\mathrm{I}) = \int_0^1 (1 - x^2t^2)^{\frac{n-4}{2}}dt \quad \text{and} \quad (\mathrm{II}) = \int_1^{1/x} t^{1-n}(1 - x^2t^2)^{\frac{n-4}{2}}dt.$$

For the first term, note that

$$\begin{aligned}\lim_{n\to\infty}(\mathrm{I}) &= \lim_{n\to\infty}\int_0^1 \Big(1 - \frac{\pi}{2}\frac{(n-2)^2}{(n-1)^2}\cdot\frac{t^2}{n}\Big)^{\frac{n-4}{2}}dt \\ &= \int_0^1 e^{-\frac{\pi}{4}t^2}dt \geq \int_0^1 (1 - \pi t^2/4)dt = 1 - \pi/12.\end{aligned}$$

The second term

$$\begin{aligned}(\mathrm{II}) &= \int_1^{1/x} t^{1-n}(1 - x^2t^2)^{\frac{n-4}{2}}dt = \int_1^{1/x}\frac{1}{t^3}\Big(\frac{1}{t^2} - x^2\Big)^{\frac{n-4}{2}}dt \\ &= \frac{1}{2}\int_0^{1-x^2} s^{\frac{n-4}{2}}ds = \frac{1}{n-2}(1-x^2)^{\frac{n-4}{2}} = \frac{1}{n-2}\Big(1 - \frac{\pi}{2}\frac{(n-2)^2}{(n-1)^2}\cdot\frac{1}{n}\Big)^{\frac{n-4}{2}}\end{aligned}$$

converges to zero as n tends to infinity. Let L be the body of revolution obtained by shrinking IB_∞ by $\rho_{IB_\infty}(0)$ on span $\{e_1\}$ and by $\rho_{IB_\infty}(\pi/2)$ on $e_1^\perp$. That is, $\rho_L(0) = \rho_L(\pi/2) = 1$. Then

$$\rho_L(\pi/4)\sin(\pi/4) = \frac{\rho_{IB_\infty}(\theta)\sin\theta}{\rho_{IB_\infty}(\pi/2)},$$

and its limit as $n \to \infty$ is given by

$$\lim_{n\to\infty} \frac{(\mathrm{I}) + (\mathrm{II})}{(n-1)/(n-2)} \geq 1 - \pi/12.$$

Thus, we get $\rho_L(\pi/4) \geq \sqrt{2}(1 - \pi/12) > 1$ for large n. Note that L is a symmetric body of revolution about the axis e_1 satisfying $\rho_L(0) = \rho_L(\pi/2) = 1$ and $\rho_L(\pi/4) = c > 1$, which implies that $d_{BM}(L, B_2^n) \geq c > 1$ for large n. Therefore,

$$\lim_{n\to\infty} d_{BM}(IB_\infty, B_2^n) = \lim_{n\to\infty} d_{BM}(L, B_2^n) \geq c > 1. \qquad \blacksquare$$

5 Local Convexity of Intersection Bodies of Star Bodies of Revolution

Let K be a symmetric star body of revolution in $\mathbb{R}^n$. Following the definition in [4, Section 0.7], the radial function ρ_K is continuous, but it can attain the value zero, and hence the origin need not be an interior point of K. In this section we will study the local convexity at the equator of ρ_{IK}. The main result is an analogue of Theorem 3.5. In dimensions five and higher, the intersection body of a star body is locally convex at the equator with equatorial power type 2. As observed in Remark 3.7, in the convex case the constant c_K is uniformly bounded, but if K is not convex c_K can be made arbitrarily big or close to zero by choosing K appropriately. As for the four-dimensional case, IK is still locally convex at the equator, but it may not be strictly convex (Example 5.5), or if it is, its modulus of convexity may not be of power type 2 (Example 5.6). However, if the origin is an interior point of K, then IK has equatorial power type 2 (Theorem 5.7).

If ρ_K is not identically zero, we have from (1.1) that $\rho_{IK}(\pi/2)$ is a positive number. Applying a dilation, we will assume that $\rho_{IK}(\pi/2) = 1$. Thus, to study the equatorial power type of IK, we can use the function ψ_{IK} as in Section 2.

We start with a characterization of local convexity at the equator in terms of the radial function.

Lemma 5.1 *Let K be a symmetric star body of revolution such that $\rho_K \in C^2[0, \pi]$. If $\rho_K(\pi/2) - \rho_K''(\pi/2) > 0$, then K is locally convex at the equator.*

Proof We express the boundary of K parametrically by $\langle x(\theta), y(\theta)\rangle$, where $x(\theta) = \rho_K(\theta)\cos(\theta)$ and $y(\theta) = \rho(\theta)\sin(\theta)$. With this representation, the equator corresponds to the point $(0, \rho_K(\pi/2))$. Since the boundary of K is of the class C^2, we can study the local convexity at the equator by means of the second derivative

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{d\theta}/\frac{dx}{d\theta}\right)}{\frac{dx}{d\theta}} = -\frac{\rho_K(\theta)^2 + 2(\rho_K'(\theta))^2 - \rho_K(\theta)\rho''(\theta)}{\left(\sin(\theta)\rho_K(\theta) - \cos(\theta)\rho_K'(\theta)\right)^3}.$$

At the equator, $\theta = \pi/2$. Also, it follows from the central and axial symmetries that $\rho_K'(\pi/2) = 0$. Therefore, the above expression simplifies to

$$\frac{d^2y}{dx^2} = -\frac{\left(\rho_K(\pi/2) - \rho_K''(\pi/2)\right)}{(\rho_K(\pi/2))^2}$$

https://doi.org/10.4153/CJM-2013-039-4 Published online by Cambridge University Press

Hence K is locally convex at the equator if $\rho_K(\pi/2) - \rho_K''(\pi/2) > 0$. ■

Lemma 5.2 *Let K be a symmetric star body of revolution in $\mathbb{R}^n$, $n \geq 5$, with radial function ρ_K. Assume that the one-sided derivatives of ρ_K are finite for every $\theta \in (0, \pi/2)$ (i.e., K has no spikes, except maybe at the axis of revolution or the equator). Let IK be the intersection body of K. Then $\rho_{IK} \in C^2(0, \pi)$.*

Proof The part of this lemma corresponding to even values of n was proven in [1, Proposition 8], where it was shown that ρ_{IK} has continuous $(n-2)/2$-th derivative at every $\theta \in (0, \pi/2)$, and a continuous $(n-2)$-th derivative at the point $\theta = \pi/2$. We will thus assume that n is odd, $n \geq 5$.

At the point $\theta = \pi/2$, we will use Definition 2.1 and prove that ψ_{IK} has a continuous second derivative at $x = 0$. If $n \geq 7$, differentiating with respect to x twice in equation (2.3) gives

$$\psi_{IK}''(x) = -(n-4)\int_0^{1/x} \psi_K(t)^{n-1}\, t^2 [1 - x^2t^2]^{\frac{n-6}{2}}\, dt \\ + (n-4)(n-6)x^2 \int_0^{1/x} \psi_K(t)^{n-1}\, t^4 [1 - x^2t^2]^{\frac{n-8}{2}}\, dt.$$

Since $n \geq 7$ and ψ_K is a bounded function that satisfies $\psi_K(t) = O(1/t)$ as t tends to infinity, the integrals are convergent at infinity and

$$\psi_{IK}''(0) = -(n-4)\int_0^{\infty} \psi_K(t)^{n-1}\, t^2\, dt < +\infty.$$

Thus, $\psi_{IK}''(0)$ is finite, and since $\psi_{IK}(x)$ is extended evenly for negative values of x, ψ_{IK}'' is continuous at 0. As for $n = 5$, the first derivative of ψ_{IK} is

$$\psi_{IK}'(x) = -x \int_0^{1/x} \psi_K(t)^4\, t^2 \Big[1 - x^2t^2\Big]^{-1/2} dt,$$

we have

$$\frac{\psi_{IK}'(x) - \psi_{IK}'(0)}{x} = -\int_0^{1/x} \psi_K(t)^4\, t^2 \Big[1 - x^2t^2\Big]^{-1/2} dt,$$

which, when x approaches zero, tends to the convergent integral $-\int_0^{\infty} \psi_K(t)^4\, t^2 dt$. We have thus shown that $\psi_{IK}(x)$ has continuous second derivative at zero for every $n \geq 5$, which implies that ρ_{IK} has continuous second derivative at $\pi/2$.

Finally, we will show that ρ_{IK} has continuous second derivative at every $\theta \in (0, \pi/2)$. Setting $x = \sin\theta$, $t = \cos\phi$, $r(t) = \rho_K^{n-1}(\arccos(t))$ and $F(x) = \rho_{IK}(\arcsin x)$ in equation (1.1), we obtain the expression

$$F(x) = \frac{1}{x^{n-3}} \int_0^{x} r(t)(x^2 - t^2)^{(n-4)/2}\, dt, \tag{5.1}$$

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for $x \in (0,1]$, and $F(0) = c_n\, r(0)$, where c_n is a constant depending only on n. Consider a point $a \in (0,1)$ such that $r(x)$ is continuous but not differentiable at a. Differentiating equation (5.1) $(n-3)/2$ times, we obtain

$$F^{((n-3)/2)}(x) = \sum_{\substack{-1\le k\le n-4\\ k \text{ odd}}} p_k(x) \int_0^x r(t)(x^2-t^2)^{k/2}\,dt,$$

where each $p_k(x)$ is a rational function of the form $c(n,k)/x^{b(n,k)}$, for some constants c, b. Note that the denominator of $p_k(x)$ is nonzero if $x \in (0,1)$. Therefore, every term with positive k is a continuous function at a. For the term corresponding to $k = -1$, we change variables by setting $t = x\sin u$, so that $\int_0^x \frac{r(t)}{\sqrt{x^2-t^2}}\,dt$ becomes $\int_0^{\pi/2} r(x\sin u)\,du =: G(x)$. Then

$$\lim_{x\to a-} \frac{G(x)-G(a)}{x-a} = \int_0^{\pi/2} \lim_{x\to a-} \Big(\frac{r(x\sin u) - r(a\sin u)}{x-a} \Big)\,du. \tag{5.2}$$

The limit inside the integral exists because of the assumption that ρ_K has finite one-sided derivatives at every $\theta \in (0,\pi/2)$.

On the other hand, if $x > a$,

$$\begin{aligned}\lim_{x\to a+} \frac{G(x)-G(a)}{x-a} = \lim_{x\to a+} \Big(&\frac{1}{x-a} \int_0^{\arcsin(\frac{a}{x})} \big(r(x\sin u) - r(a\sin u)\big)\,du \\ &+ \frac{1}{x-a}\int_{\arcsin(\frac{a}{x})}^{\pi/2} \big(r(x\sin u) - r(a\sin u)\big)\,du \Big).\end{aligned}$$

Note that the first term tends to the right-hand side of (5.2), while the second term tends to zero, because r is continuous at $x = a$. However, for the second derivative, the corresponding term will tend to $r'_+(a) - r'_-(a)$, which is not zero, and thus G does not have a continuous second derivative at a. We have, in fact, shown that ρ_{IK} has $(n-1)/2$ continuous derivatives at any point $\theta \in (0,\pi/2)$. ■

Remark 5.3 We wish to note that the local convexity and regularity properties of IK at the axis of revolution are the same as those of K at the equator. It is easily seen by calculating the intersection body of a double cone that, in general, $\rho_{IK}(\theta)$ is not differentiable at $\theta = 0$. The general argument is as follows. Setting $t = x\sin u$ in (5.1) gives

$$F(x) = \int_0^{\pi/2} r(x\sin u)(\cos u)^{n-3}\,du$$

for $x \in (0,1]$. At $x = 0$, $F(0) = r(0)(\int_0^{\pi/2} (\cos u)^{n-3}\,du)$. Then

$$\lim_{x\to 0+} \frac{F(x)-F(0)}{x} = \int_0^{\pi/2} \Big(\lim_{x\to 0+} \frac{r(x\sin u) - r(0)}{x} \Big) (\cos u)^{n-3}\,du \tag{5.3}$$

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and similarly for the left-hand side limit. If the function ρ_K (and hence r) is differentiable at zero, then so is F. However, if the right and left-hand side limits of r take different values, then the same will be the case for F. Observe also that (5.3) implies that the local convexity of r at the equator and F at the axis present the same behavior. In particular, if the body K is not locally convex at the equator, then IK will not be locally convex at the axis of revolution.

Now we are ready to prove the result on local convexity of IK at the equator.

Theorem 5.4 *Let K be a symmetric star body of revolution in $\mathbb{R}^n$, $n \geq 5$, whose radial function ρ_K has finite one-sided derivatives for every $\theta \in (0, \pi/2)$. Then its intersection body IK is strictly convex at the equator, with equatorial power type 2.*

Proof By Lemma 5.2, $\rho_{IK}(\theta)$ has continuous second derivative for every $\theta \in (0, \pi/2]$. Observe that $(1 - (\cos^2\phi)/(\sin^2\theta)) < (1 - \cos^2\phi)$ for every $\theta < \pi/2$. Using this estimate in equation (1.1), we obtain

$$\begin{aligned}\rho_{IK}(\theta)\sin\theta &= \int_{\pi/2-\theta}^{\pi/2} \rho_K(\phi)^{n-1}\Big[1 - \frac{\cos^2\phi}{\sin^2\theta}\Big]^{\frac{n-4}{2}} \sin\phi\, d\phi\\ &< \int_{\pi/2-\theta}^{\pi/2} \rho_K(\phi)^{n-1}(1-\cos^2\phi)^{\frac{n-4}{2}} \sin\phi\, d\phi\\ &\leq \int_{0}^{\pi/2} \rho_K(\phi)^{n-1}(1-\cos^2\phi)^{\frac{n-4}{2}} \sin\phi\, d\phi = \rho_{IK}(\pi/2) = 1.\end{aligned}$$

Therefore, $1 - \psi_{IK}(\delta) = 1 - \rho_{IK}(\theta)\sin(\theta)$ has a local minimum at $\delta = 0$, and thus $-\psi_{IK}''(0) = \rho_{IK}(\pi/2) - \rho_{IK}''(\pi/2) > 0$. By Lemma 5.1, IK is locally convex at the equator.

Assume that $\rho_{IK}(\pi/2) - \rho_{IK}''(\pi/2) = 0$. We claim that this contradicts the fact that IK is an intersection body by using a variation of Koldobsky's Second Derivative test ([8, Theorem 4.19]). Since ρ_{IK} may not be C^2 at the axis of revolution, we cannot use the Second Derivative Test directly. Instead, we proceed as in the proof of [1, Proposition 6], where regularity is not needed everywhere.

Since IK is a body of revolution, if we consider the coordinates $(x_1, \overline{x})$ in $\mathbb{R}^n$, where $\overline{x} = (x_2, \ldots, x_n)$ and x_1 is in the direction of the axis of revolution, then the Minkowski functional of IK is given by

$$\|(x_1, \overline{x})\|_{IK}^{-1} = \frac{1}{\sqrt{x_1^2 + \overline{x}^2}}\, \rho_{IK}\Big(\arccos\Big(\frac{x_1}{\sqrt{x_1^2+\overline{x}^2}}\Big)\Big),$$

and the condition of the Second Derivative test,

$$\frac{\partial^2(\|(x_1,\overline{x})\|_{IK})}{\partial x_1^2}(0, x_2, \ldots, x_n) = 0, \tag{5.4}$$

is easily computed to be equivalent to $\rho_{IK}(\pi/2) - \rho_{IK}''(\pi/2) = 0$. Besides, the convergence of $(\partial^2(\|x\|_{IK}))/(\partial x_1^2)$ to 0 as x_1 approaches zero is uniform in a neighborhood of the equator by Lemma 5.2. Hence, letting

$$u(\bar{x}) = \frac{1}{(2\pi)^{(n-1)/2}} e^{-\|\bar{x}\|_2^2/2} \quad \text{and} \quad h_m(x_1) = \frac{m}{\sqrt{2\pi}} e^{-m^2 x_1^2/2},$$

we have

$$\left\langle \|(x_1,\bar{x})\|_{IK}^{-1}, u(\bar{x}) h_m''(x_1) \right\rangle = \int_{\mathbb{R}^{n-1}\setminus\{0\}} u(\bar{x}) \int_{-\infty}^{\infty} h_m''(x_1) \|(x_1,\bar{x})\|_{IK}^{-1}\, dx_1 d\bar{x}.$$

Integrating by parts twice, the terms at infinity vanish, and this equals

$$\begin{aligned} -2\int_{\mathbb{R}^{n-1}\setminus\{0\}} u(\bar{x}) \int_{-\infty}^{\infty} h_m(x_1) \Big(&\frac{\partial^2(\|(x_1,\bar{x})\|_{IK})}{\partial x_1^2} \|(x_1,\bar{x})\|_{IK}^{-2} \\ &- 2\frac{\partial(\|(x_1,\bar{x})\|_{IK})}{\partial x_1} \|(x_1,\bar{x})\|_{IK}^{-3} \Big)\, dx_1 d\bar{x}. \end{aligned}$$

The second term of this integral is positive. We only need to check that the first one approaches zero as m goes to infinity, but this follows from (5.4) and the fact that the convergence is uniform in a neighborhood of the equator. We have now proved the equivalent of [8, Lemma 4.20], and the rest of the argument follows now exactly as in the proof of the Second Derivative test [8, p. 89]. Therefore, we get that IK is not an intersection body, obviously a contradiction. Therefore, $\rho_{IK}(\pi/2) - \rho_{IK}''(\pi/2) > 0$, and IK is strictly convex at the equator, with equatorial power type 2. ∎

In four dimensions, the result of Theorem 5.4 is not necessarily true, as the following two examples show.

Example 5.5 When the origin is not an interior point of K, IK need not be strictly convex at the equator. Assume that $\rho_K(\phi) = 0$ for all $\phi \in [0,\alpha]$, $\alpha > 0$. Then if $\theta \in [\pi/2-\alpha, \pi/2]$,

$$\rho_{IK}(\theta)\sin\theta = \int_{\pi/2-\theta}^{\pi/2} \rho_K(\phi)^3 \sin\phi\, d\phi = \int_0^{\pi/2} \rho_K(\phi)^3 \sin\phi\, d\phi = C.$$

Therefore, $\rho_{IK}(\theta) = C/\sin\theta$ for all $\theta \in [\pi/2-\alpha, \pi/2]$, which means that IK is cylindrical around the equator. Hence, IK is locally convex at the equator, but not strictly convex.

Example 5.6 In this example we present a four-dimensional intersection body of a star body, which is strictly convex but does not have modulus of convexity of power type 2. Figure 2 shows the body of revolution K and its four-dimensional intersection body IK. The radial function of K is

$$\rho_K(\theta) = \begin{cases} (4\sin^2 u/\cos^5 u)^{1/3}, & \text{if } 0 \le \theta \le \pi/2 - \arctan(\sqrt[4]{5}), \\ A/\sin u, & \text{if } \pi/2 - \arctan(\sqrt[4]{5}) \le \theta \le \pi/2. \end{cases}$$

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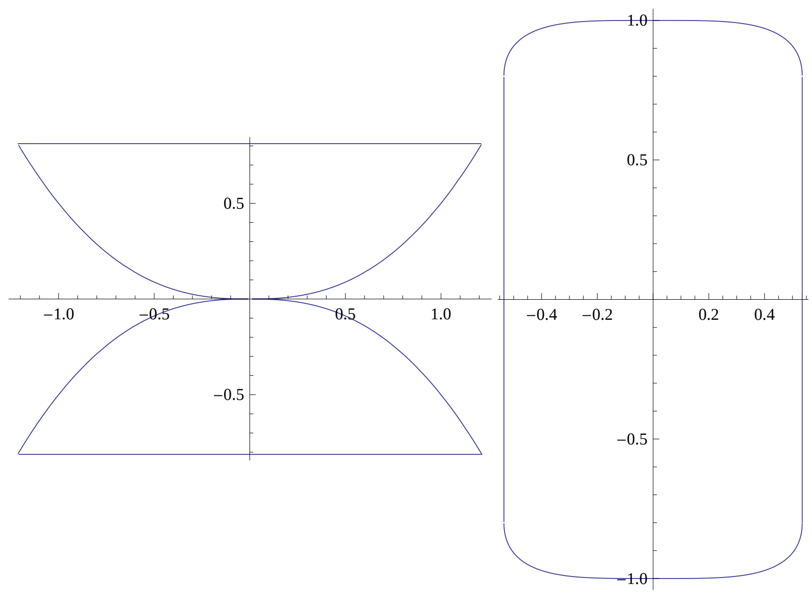

Figure 2: The bodies K (left) and IK (right) in Example 5.6.

The radial function of IK, which we calculated using *Mathematica*, is

$$\rho_{IK}(\theta) = \begin{cases} B/\cos\theta, & \text{if } 0 \le \theta \le \arctan(\sqrt[4]{5}), \\ (2(\sin\theta)^2 - 1)/(\sin\theta)^5, & \text{if } \arctan(\sqrt[4]{5}) \le \theta \le \pi/2, \end{cases}$$

where A, B are constants chosen appropriately so that ρ_K, ρ_{IK} are continuous. If we compute the modulus of convexity at the equator for IK, we obtain

$$1 - \psi_{IK}(\delta) = 1 - \rho_{IK}(\theta)\sin\theta = 1 - \frac{2(\sin\theta)^2 - 1}{(\sin\theta)^4} = (\cot\theta)^4 = \delta^4,$$

where the last step comes from the Figure 1 of ψ_{IK}. Hence IK is strictly convex at the equator, but has equatorial power type 4.

The bodies in Examples 5.5 and 5.6 have the common feature that the origin is not an interior point of K. With the additional hypothesis that the origin is an interior point of K, the intersection body of K has equatorial power type 2, even in the four-dimensional case, as shown in the next theorem. Note that neither Theorem 5.4 nor 5.7 implies the other one. Theorem 5.4 assumes dimension $n \ge 5$, but allows the origin to be a boundary point of K, while the result of Theorem 5.7 applies for $n \ge 4$, while needing that the origin is interior to K. The proof of Theorem 5.7 uses similar ideas as those in the proof of Theorem 3.5.

Theorem 5.7 *Let K be a symmetric star body of revolution in $\mathbb{R}^n$, for $n \ge 4$, such that the origin is an interior point of K. Then IK has equatorial power type 2.*

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Proof Since the origin is an interior point, we can assume that $\rho_K(0) = 1$ and $rB_\infty \subset K \subset RB_\infty$ for some constants $r, R > 0$ depending on K where

$$B_\infty = \left\{(x, y) \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} : |x| \le 1, |y| \le 1\right\}.$$

For $\theta \in [0, \pi/2]$, consider the symmetric convex body K_θ defined by

$$\rho_{K_\theta}(\varphi) = \begin{cases} \rho_K(\varphi), & 0 \le \varphi \le \theta, \\ \rho_{L_\theta}(\varphi), & \theta \le \varphi \le \pi/2, \end{cases}$$

where L_θ is a body of revolution obtained by rotating the line containing two points of angles $0, \theta$ on the boundary of K, *i.e.*,

$$L_\theta = \left\{(x, y) \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} : |x| + b|y| = 1\right\} \text{ for } b = b(\theta) = \frac{1 - \rho_K(\theta)\cos\theta}{\rho_K(\theta)\sin\theta}.$$

Then by (3.1) in Lemma 3.1, we have

$$\psi_{K_\theta}(x) = \psi_{L_\theta}(x) = \frac{1}{x + b}, \quad \text{for every } x \ge \cot\theta,$$

and, moreover, from $rB_\infty \subset K \subset RB_\infty$,

$$\frac{r}{x} \le \psi_{K_\theta}(x) \le \frac{R}{x}, \quad \text{for every } x \ge 1. \tag{5.5}$$

We need to compute $\psi_{IK_\theta}(0) - \psi_{IK_\theta}(\delta)$ for small δ:

$$\begin{aligned}\psi_{IK_\theta}(0) - \psi_{IK_\theta}(\delta) &= \int_{1/\delta}^{\infty} \psi_{K_\theta}(t)^{n-1} dt + \int_0^{1/\delta} \psi_{K_\theta}(t)^{n-1}\left[1 - (1 - \delta^2 t^2)^{\frac{n-4}{2}}\right] dt \\ &= \quad (\mathrm{I}) \quad + \quad (\mathrm{II}).\end{aligned}$$

If $\theta \le \pi/2$ and $\delta < \tan\theta$, then (5.5) gives upper/lower bounds of the first term:

$$\int_{1/\delta}^{\infty} (r/t)^{n-1} dt \le (\mathrm{I}) \le \int_{1/\delta}^{\infty} (R/t)^{n-1} dt.$$

So, the first term is bounded by $(r^{n-1}/(n-2))\delta^{n-2}$ and $(R^{n-1}/(n-2))\delta^{n-2}$, which are independent of θ. If $n = 4$, then (I) is asymptotically equivalent to δ^2, and the second term (II) is equal to zero. Assume $n \ge 5$. Then the second term (II) is divided into two parts as follows:

$$\begin{aligned}(\mathrm{II}) &= \int_0^{\cot\theta} + \int_{\cot\theta}^{1/\delta} \psi_{K_\theta}(t)^{n-1}\left[1 - (1 - \delta^2 t^2)^{\frac{n-4}{2}}\right] dt \\ &= (\mathrm{II}\text{-}1) \ + \ (\mathrm{II}\text{-}2),\end{aligned}$$

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where

$$
\begin{aligned}
\text{(II-1)} &= \int_0^{\cot\theta} \psi_{K_\theta}(t)^{n-1}\Big[1-(1-\frac{n-4}{2}\delta^2t^2+O(\delta^4))\Big]\,dt\\
&= \Big(\frac{n-4}{2}\int_0^{\cot\theta}\psi_{K_\theta}(t)^{n-1}t^2dt\Big)\,\delta^2+O(\delta^4)
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(II-2)} &\asymp \int_{\cot\theta}^{1/\delta}\frac{1-(1-\delta^2t^2)^{\frac{n-4}{2}}}{t^{n-1}}dt = \int_{\cot\theta}^{1/\delta}\frac{1}{t^{n-1}}dt-\int_{\cot\theta}^{1/\delta}\frac{(1-\delta^2t^2)^{\frac{n-4}{2}}}{t^{n-4}}\,\frac{dt}{t^3}\\
&= \frac{(\tan\theta)^{n-2}-\delta^{n-2}}{n-2}-\frac{1}{2}\int_{\delta^2}^{\tan^2\theta}(s-\delta^2)^{\frac{n-4}{2}}ds\\
&= \frac{(\tan\theta)^{n-2}-\delta^{n-2}-(\tan^2\theta-\delta^2)^{\frac{n-2}{2}}}{n-2}=\frac{(\tan\theta)^{n-4}}{2}\delta^2+O(\delta^4).
\end{aligned}
$$

(Note that $\psi_{K_\theta}(t)$ in the second integral (II-2) is comparable to $1/t$ by (5.5)). Furthermore, when $n \geq 5$, the integral of (II-1) is bounded above and below by positive constants independent of θ:

$$
\begin{aligned}
&\int_0^{\cot\theta}\psi_{K_\theta}(t)^{n-1}t^2dt \leq \int_0^1 R^{n-1}t^2dt+\int_1^\infty (R/t)^{n-1}t^2dt=\frac{n-1}{3(n-4)}R^{n-1},\\
&\int_0^{\cot\theta}\psi_{K_\theta}(t)^{n-1}t^2dt \geq \int_0^1 r^{n-1}t^2dt=\frac{1}{3}r^{n-1}.
\end{aligned}
$$

Finally, letting θ go to zero, we have equatorial power type 2 for the body K. ■

Acknowledgments It is with great pleasure that we thank Dmitry Ryabogin and Artem Zvavitch for helpful advice during the preparation of this paper. We also thank the referee for his useful suggestions.

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