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HYPERCOMMUTING VALUES IN ASSOCIATIVE RINGS WITH UNITY

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Abstract

Let K be a commutative ring with unity, R an associative K-algebra of characteristic different from 2 with unity element and no nonzero nil right ideal, and $f(x_1, \ldots, x_n)$ a multilinear polynomial over K. Assume that, for all $x \in R$ and for all $r_1, \ldots, r_n \in R$ there exist integers $m = m(x, r_1, \ldots, r_n) \geq 1$ and $k = k(x, r_1, \ldots, r_n) \geq 1$ such that $[x^m, f(r_1, \ldots, r_n)]_k = 0$. We prove that: (1) if $\mathrm{char}(R) = 0$ then $f(x_1, \ldots, x_n)$ is central-valued on R; and (2) if $\mathrm{char}(R) = p > 2$ and $f(x_1, \ldots, x_n)$ is not a polynomial identity in $p \times p$ matrices of characteristic p, then R satisfies $s_{n+2}(x_1, \ldots, x_{n+2})$ and for any $r_1, \ldots, r_n \in R$ there exists $t = t(r_1, \ldots, r_n) \geq 1$ such that $f^{p^t}(r_1, \ldots, r_n) \in Z(R)$, the center of R.

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1. Introduction

Throughout this paper, R always denotes an associative ring with unity and center $Z(R)$. The kth commutator of $x, y \in R$, denoted by $[x, y]_k$ is defined inductively as follows: for $k = 1$, $[x, y]_1 = [x, y] = xy - yx$, and for $k > 1$, $[x, y]_k = [[x, y]_{k-1}, y]$. In [1] Bergen proved that if R is a ring with no nonzero nil right ideal and $f(x_1, \ldots, x_n)$ is a multilinear polynomial of degree n which is not an identity for the $p \times p$ matrices in characteristic p, and for any $r_1, \ldots, r_n; s_1, \ldots, s_n \in R$ there exist $m = m(r_1, \ldots, r_n; s_1, \ldots, s_n) \geq 1$ and $t = t(r_1, \ldots, r_n; s_1, \ldots, s_n) \geq 1$ such that $[f(r_1, \ldots, r_n)^m, f(s_1, \ldots, s_n)^t] = 0$, then R satisfies the standard identity $s_{n+2}(x_1, \ldots, x_{n+2})$ and the values of $f(x_1, \ldots, x_n)$ are power central. In particular he showed that, if for any $r_1, r_2, s_1, s_2 \in R$, there exist $m = m(r_1, r_2, s_1, s_2) \geq 1$ and $t = t(r_1, r_2, s_1, s_2) \geq 1$ such that $[[r_1, r_2]^m, [s_1, s_2]^t] = 0$, then R satisfies the standard identity $s_4(x_1, \ldots, x_4)$.

Later, Chuang and Lin [5, Theorem 3] proved that if R is a ring with no nonzero nil right ideals and for any $x, y \in R$ there exist $m = m(x, y) \geq 1$ and $t = t(x, y) \geq 1$ and $k = k(x, y) \geq 1$ such that $[x^m, y^t]_k = 0$ then R is commutative.

The aim of this note is to continue this line of investigation, combining in some sense the previous cited results and considering the kth commutators involving the evaluations of a multilinear polynomial. Our main result will be the following theorem.

Theorem 1.1. *Let K be a commutative ring with unity, R an associative K-algebra of characteristic different from 2 with unity element and no nonzero nil right ideal, and $f(x_1, \ldots, x_n)$ a multilinear polynomial over K. Assume that, for all $x \in R$ and for all $r_1, \ldots, r_n \in R$ there exist integers $m = m(x, r_1, \ldots, r_n) \geq 1$ and $k = k(x, r_1, \ldots, r_n) \geq 1$ such that $[x^m, f(r_1, \ldots, r_n)]_k = 0$. We prove the following results:*

(1) *if $\mathrm{char}(R) = 0$ then $f(x_1, \ldots, x_n)$ is central-valued on R;*
(2) *if $\mathrm{char}(R) = p > 2$ and $f(x_1, \ldots, x_n)$ is not a polynomial identity in $p \times p$ matrices of characteristic p, then R satisfies $s_{n+2}(x_1, \ldots, x_{n+2})$ and for any $r_1, \ldots, r_n \in R$ there exists $t = t(r_1, \ldots, r_n) \geq 1$ such that $f^{p^t}(r_1, \ldots, r_n) \in Z(R)$, the center of R.*

We would like to remark that in the case $\mathrm{char}(R) = p \neq 0$, the assumption that $f(x_1, \ldots, x_n)$ is not an identity in $p \times p$ matrices of characteristic p is inherited from the fundamental work by Herstein *et al.* [8] where the structure of power central polynomials on division rings is determined under this hypothesis. We also note that a ring with no nonzero nil right ideal has no representation as a subdirect product of prime rings with the same property (unlike rings with no nonzero two-sided ideals). In order to circumvent this difficulty we will frequently make use of some methods contained in [1].

Firstly we fix some well-known facts.

Fact 1.2. Let $x, y \in R$. Then $[x, y]_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i y^i x y^{n-i}$ (here we put $[x, y]_0 = x$).

Fact 1.3. Let $x, y, z \in R$. If $[x, y]_n = 0$ for some $n \geq 1$ then $[x, y^m]_n = 0$ for any $m \geq 1$ and $[x, y]_q = 0$ for any $q \geq n$.

Fact 1.4. Let $x, y, z \in R$. If $[x, y^m]_n = 0$ and $[z, y^t]_n = 0$ then $[x, y^{mt}]_n = [z, y^{mt}]_n = 0$.

We will also make use of the following results.

Fact 1.5. Let R be a ring with no nonzero nil right ideal, and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial in n noncommuting variables. Assume that, for all $r_1, \ldots, r_n; u_1, \ldots, u_n \in R$ there exist integers $m = m(r_1, \ldots, r_n; u_1 \ \ldots, u_n) \geq 1$ and $k = k(r_1, \ldots, r_n; u_1, \ldots, u_n) \geq 1$ such that

$$[f(r_1, \ldots, r_n)^m, f(u_1, \ldots, u_n)^k] = 0.$$

If $\mathrm{char}(R) = p \neq 0$ and $f(x_1, \ldots, x_n)$ is not a polynomial identity in $p \times p$ matrices of characteristic p, then R satisfies $s_{n+2}(x_1, \ldots, x_{n+2})$ and for any $r_1, \ldots, r_n \in R$ there exists $t = t(r_1, \ldots, r_n) \geq 1$ such that $f^{p^t}(r_1, \ldots, r_n) \in Z(R)$ [1, Theorem 9].

Fact 1.6. Let R be a K-algebra with no nonzero nil right ideal, and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over K.

(1) If $f(x_1, \ldots, x_n)$ is nil in R, then $f(x_1, \ldots, x_n)$ is a polynomial identity for R [3, Theorem 1].

(2) If I is a right ideal of R such that $f(x_1, \ldots, x_n)$ is nil in I, then $f(x_1, \ldots, x_n)x_{n+1}$ is a polynomial identity for I (it is a consequence of [3, Main Theorem]).

Fact 1.7. Throughout this paper we denote

$$H_R(f) = \{x \in R \mid \forall r_1, \ldots, r_n \in R \; \exists k = k(x, r_1, \ldots, r_n) \text{ such that } [x, f(r_1, \ldots, r_n)]_k = 0\},$$

where $f(x_1, \ldots, x_n)$ is a multilinear polynomial in n noncommuting variables. In particular, in the case where R is a ring with no nonzero nil right ideal, and $\mathrm{char}(R) = 0$, then the following hold.

(1) If R is primitive and R is not a division ring, then either $H_R(f) = Z(R)$ or $f(x_1, \ldots, x_n)$ is central valued in R [6, Lemma 2.4].

(2) If R is a domain such that $R = H_R(f)$, then $f(x_1, \ldots, x_n)$ is central valued in R [6, Lemma 2.8].

2. The results

We begin with the following easy reduction.

Lemma 2.1. *Let* $\mathrm{char}(R) = p > 2$. *If* $f(x_1, \ldots, x_n)$ *is not a polynomial identity in* $p \times p$ *matrices of characteristic* p, *then* R *satisfies* $s_{n+2}(x_1, \ldots, x_{n+2})$ *and for any* $r_1, \ldots, r_n \in R$ *there exists* $t = t(r_1, \ldots, r_n) \geq 1$ *such that* $f^{p^t}(r_1, \ldots, r_n) \in Z(R)$.

Proof. Given $x, r_1, \ldots, r_n \in R$, there exist suitable m, k positive integers such that $[x^m, f(r_1, \ldots, r_n)]_k = 0$. Hence, for $t \geq 1$ such that $p^t \geq k$,

$$0 = [x^m, f(r_1, \ldots, r_n)]_{p^t} = [x^m, f(r_1, \ldots, r_n)^{p^t}],$$

and the conclusion follows from Fact 1.5. □

In all that follows we will always assume that $\mathrm{char}(R) = 0$, and moreover that R has the unity element. Let $R_{\mathbb{Z}}$ be the localization of R at $\mathbb{Z}$. By the multilinearity of $f(x_1, \ldots, x_n)$, our hypotheses on R carry over to $R_{\mathbb{Z}}$. Therefore we may assume that R is a $\mathbb{Q}$-algebra.

Lemma 2.2. *Let* R *be a domain. Then* $f(x_1, \ldots, x_n)$ *is central valued on* R.

Proof. Pick $x \in R$ and $u = f(r_1, \ldots, r_n)$, with $r_1, \ldots, r_n \in R$. Notice that the set

$$R_u = \{r \in R \mid \exists k = k(r, u) \geq 1 \text{ such that } [r, u]_k = 0\}$$

is a subring of R. Moreover we observe that for any $x \in R$, there exists $m = m(x, u) \geq 1$ such that $x^m \in R_u$, that is R is radical over R_u. By [2, Theorem 2], we have $R = R_u$. Therefore by the arbitrariness of $r_1, \ldots, r_n \in R$, it follows that for all $x \in R$ and for all $r_1, \ldots, r_n \in R$ there exists suitable $k \geq 1$ such that $[x, f(r_1, \ldots, r_n)]_k = 0$. Hence by Fact 1.7 we have that $f(x_1, \ldots, x_n)$ is central valued on R. □

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Lemma 2.3. *If R is primitive, then $f(x_1, \dots, x_n)$ is central valued on R.*

Proof. If R is a division ring, then we conclude by Lemma 2.2. However we know that R is a ring dense of D-linear transformations over V, where D is a division ring and V is a faithful irreducible right R-module with endomorphisms ring D; moreover we may assume $\dim_D V \geq 2$. Firstly we consider the case where $\dim_D V = t$ is finite. Thus R contains some nontrivial idempotent element $e = e^2$ $(e \neq 0, 1)$. Of course, since $e(1-e) = 0$ then e is a zero-divisor, so $e \notin Z(R)$. By our main hypothesis, for all $r_1, \dots, r_n \in R$ there exist $m = m(e, r_1, \dots, r_n) \geq 1$ and $k = k(e, r_1, \dots, r_n) \geq 1$ such that $[e^m, f(r_1, \dots, r_n)]_k = 0$, that is $[e, f(r_1, \dots, r_n)]_k = 0$. Hence, by the definition contained in Fact 1.7, $e \in H_R(f)$. Moreover, again by Fact 1.7, we have that either $f(x_1, \dots, x_n)$ is central valued on R, or $H_R(f) = Z(R)$. In this last case we have the contradiction $e \in Z(R)$.

Assume now that $\dim_D V = \infty$. In [4] it is proved that the range of the polynomial $f(x_1, \dots, x_n)$ is dense in $\mathrm{Hom}_D(V, V)$. So, given D-independent elements $u, v \in V$, there exist $x, r_1, \dots, r_n \in R$ such that $ux = u$, $vx = 0$, $uf(r_1, \dots, r_n) = v$ and $vf(r_1, \dots, r_n) = v$. Then, for $k \geq 1$,

$$0 = u[x^m, f(r_1, \dots, r_n)]_k = uf(r_1, \dots, r_n)^k = v,$$

which is a contradiction. □

Lemma 2.4. *Let R be semiprime. If R satisfies some polynomial identity, then $f(x_1, \dots, x_n)$ is central valued on R.*

Proof. Suppose first that R is prime. Since R is a PI-ring, then $Z(R) \neq \{0\}$ and the ring of central quotients of R, denoted by $Q = RZ^{-1} = \{r\alpha^{-1} : r \in R, \alpha \in Z(R) - \{0\}\}$, is a central simple algebra finite dimensional over its center. Thus Q is primitive and satisfies the following condition: for all $x \in Q$ and for all $r_1, \dots, r_n \in Q$ there exist integers $m = m(x, r_1, \dots, r_n) \geq 1$ and $k = k(x, r_1, \dots, r_n) \geq 1$ such that $[x^m, f(r_1, \dots, r_n)]_k = 0$. Hence by Lemma 2.3, Q satisfies the polynomial identity $[f(x_1, \dots, x_n), x_{n+1}]$, as well as R.

If R is a semiprime ring, then R is a subdirect sum of prime rings R_i. By the previous argument each R_i satisfies $[f(x_1, \dots, x_n), x_{n+1}]$, which implies that R satisfies $[f(x_1, \dots, x_n), x_{n+1}]$. □

Lemma 2.5. *If R is semisimple then $f(x_1, \dots, x_n)$ is central valued on R.*

Proof. Since the Jacobson's radical $J(R)$ is zero, then R is a subdirect product of primitive rings $R_\gamma = R/P_\gamma$, where any P_γ is a prime ideal of R. By Lemmas 2.2 and 2.3, it follows that $f(x_1, \dots, x_n)$ is central valued in every R_γ. Therefore for all $r_1, \dots, r_{n+1} \in R$ we have that $[f(r_1, \dots, r_n), r_{n+1}] \in P_\gamma$, for any γ. Thus $[f(r_1, \dots, r_n), r_{n+1}] \in \cap P_\gamma = (0)$, that is R satisfies $[f(x_1, \dots, x_n), x_{n+1}]$ and $f(x_1, \dots, x_n)$ is central valued on R. □

Remark 2.6. In all that follows we may assume $J(R) \neq (0)$.

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LEMMA 2.7. *Let P be a prime ideal of R and assume that there exists $0 \neq a \in R$, such that $a^2 = 0$ and such that $a \notin P$. Then R/P has no nonzero nil right ideal.*

PROOF. Let $r_1, \ldots, r_n \in R$. Then there exist $m_1 = m_1(a, r_1, \ldots, r_n) \geq 1$, $k_1 = k_1(a, r_1, \ldots, r_n) \geq 1$, $m_2 = m_2(a, r_1, \ldots, r_n) \geq 1$ and $k_2 = k_2(a, r_1, \ldots, r_n) \geq 1$ such that both

$$0 = [f(r_1a, \ldots, r_na)^{m_1}, f(ar_1, \ldots, ar_n)]_{k_1} = f(ar_1, \ldots, ar_n)^{k_1} \cdot f(r_1a, \ldots, r_na)^{m_1} \tag{2.1}$$

and

$$\begin{aligned} 0 &= [(af(r_1a, \ldots, r_na) + f(r_1a, \ldots, r_na))^{m_2}, f(ar_1, \ldots, ar_n)]_{k_2} \\ &= f(ar_1, \ldots, ar_n)^{k_2} \cdot (af(r_1a, \ldots, r_na)^{m_2} + f(r_1a, \ldots, r_na)^{m_2}). \end{aligned} \tag{2.2}$$

In particular for $k = max\{k_1, k_2\}$ and $m = max\{m_1, m_2\}$, and from (2.1) and (2.2)

$$\begin{aligned} 0 &= f(ar_1, \ldots, ar_n)^k \cdot (af(r_1a, \ldots, r_na)^m + f(r_1a, \ldots, r_na)^m) \\ &= f(ar_1, \ldots, ar_n)^k \cdot af(r_1a, \ldots, r_na)^m = f(ar_1, \ldots, ar_n)^{k+m}a \end{aligned}$$

, and therefore $f(ar_1, \ldots, ar_n)^{k+m+1} = 0$. By Fact 1.6, the right ideal $\varrho = aR$ satisfies the identity $f(x_1, \ldots, x_n)x_{n+1}$. Since $a \notin P$ then $\varrho' = \varrho/P$ is also a nonzero right ideal of R/P which satisfies a polynomial identity. Suppose that R/P has a nonzero nil right ideal N. Since R/P is a prime ring, then there exists $b \in \varrho'$ such that bN is a nonzero nil right ideal. Moreover $bN \subseteq \varrho'$ satisfies a polynomial identity, and this is a contradiction in a prime ring. Therefore R/P has no nonzero nil right ideal, for all $P \in A$. □

2.1. A reduced result. Here we prove a result which will be useful in the sequel. Firstly we state the following one, which is contained in [7, Lemma 1].

LEMMA 2.8. *Let R be a prime ring and let ϱ be a nonzero right ideal of R such that the left annihilator $l(\varrho) = \{x \in R : x\varrho = (0)\}$ is zero. If ϱ satisfies a polynomial identity then R also satisfies some polynomial identity.*

LEMMA 2.9. *Let R be a prime ring and suppose that for any $r_1, \ldots, r_n \in R$ there exists $m = m(r_1, \ldots, r_n) \geq 1$ such that $f(r_1, \ldots, r_n)^m$ is either zero or regular. If R is not a domain, then R satisfies some polynomial identity.*

PROOF. Firstly we note that if for any $r_1, \ldots, r_n \in R$ there exists $m = m(r_1, \ldots, r_n) \geq 1$ such that $f(r_1, \ldots, r_n)^m = 0$, then by Fact 1.6, $f(x_1, \ldots, x_n)$ is a polynomial identity for R and the lemmas are proved. Assume that R is not a domain. Hence there exists $0 \neq a \in R$ such that $a^2 = 0$. Let $\varrho = aR$ and notice that the left annihilator $l(\varrho)$ is not zero. Thus ϱ does not contain any regular element and so for any $r_1, \ldots, r_n \in \varrho$, there exists $m = m(r_1, \ldots, r_n) \geq 1$ such that $f(r_1, \ldots, r_n)^m = 0$. In particular this also holds in $R_1 = \varrho/l(\varrho) \cap \varrho$, which is a prime ring with no nonzero nil right ideal. Again by Fact 1.6, $f(x_1, \ldots, x_n)$ is a polynomial identity for R_1, that is $f(s_1, \ldots, s_n) \subseteq l(\varrho)$, for all $s_1, \ldots, s_n \in \varrho$. Therefore ϱ satisfies the polynomial identity $f(x_1, \ldots, x_n)x_{n+1}$.

By Zorn's lemma there exists a nonzero right ideal M of R which is maximal with respect to the property that it satisfies $f(x_1, \ldots, x_n)x_{n+1}$.

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Now let $r \in R$ and $t_1, \ldots, t_{n+1} \in M$. Since $t_i r \in M$ (for all i), we have that $f(rt_1, \ldots, rt_n)rt_{n+1} = rf(t_1 r, \ldots, t_n r)t_{n+1} = 0$. This means that the right ideal rM satisfies $f(x_1, \ldots, x_n)x_{n+1}$. Hence by [9, Theorem 6], $M + rM$ also satisfies some polynomial identity.

If $l(M + rM) = (0)$, then Lemma 2.9 is proved.

Suppose now that $l(M + rM) \neq (0)$. Then by the previous argument we have that $M + rM$ satisfies $f(x_1, \ldots, x_n)x_{n+1}$. Moreover by the maximality of M, it follows that $M + rM \subseteq M$, that is $rM \subseteq M$. This holds for all $r \in R$, implying that M is a two-sided ideal of R, which satisfies a polynomial identity. Therefore R also satisfies a polynomial identity. □

2.2. Proof of main theorem. In light of previous lemmas, we can now continue with the proof of our main results.

PROPOSITION 2.10. *If R is a prime ring (without nil one-sided ideals), then $f(x_1, \ldots, x_n)$ is central valued on R.*

PROOF. By Lemmas 2.2 and 2.5 we may consider the case where R is not a domain and $J(R) \neq (0)$. In addition, since R and $J(R)$ satisfy the same polynomial identities, in order to prove that $f(x_1, \ldots, x_n)$ is central valued on R, we may replace R by $J(R)$ (without loss of generality we consider $R = J(R)$). If $f(r_1, \ldots, r_n)$ is nilpotent for all $r_1, \ldots, r_n \in R$, then Fact 1.6 shows that $f(x_1, \ldots, x_n)$ is a polynomial identity for R. Hence we may suppose that there exist $r_1, \ldots, r_n \in R$ such that $c = f(r_1, \ldots, r_n)$ is not nilpotent, in other words $c^m \neq 0$ for all $m \geq 1$. Here we denote by $f(R)$ the set of all the evaluations of $f(x_1, \ldots, x_n)$ on R, that is $f(R) = \{f(r_1, \ldots, r_n) : r_i \in R\}$.

We divide the proof into two cases.

Firstly we suppose that there exists an ideal H of R such that, for any $n \geq 1$, $c^n \notin H$. By Zorn's lemma there is an ideal P_c of R which is maximal with respect to the exclusion of all powers of c. In particular the ideal P_c is a prime ideal of R and, for any ideal I of R such that $P_c \subsetneq I \subseteq R$, there exists $n = n(c) \geq 1$ such that $c^n \in I$.

Let $F = \{P_c : c \in f(R) \text{ is not nilpotent}\}$ and consider the following partition of F:

$$C = \{P_c \in F : \exists 0 \neq x \in R \text{ such that } x^2 = 0,\ x \notin P_c\}$$
$$D = \{P_c \in F : \forall x \in R \text{ such that } x^2 = 0, \text{ then } x \in P_c\}.$$

Let $M = \cap P_c$ for all $P_c \in F$, $\overline{C} = \cap P_c$ for all $P_c \in C$ and $\overline{D} = \cap P_c$ for all $P_c \in D$.

Note that if $r_1, \ldots, r_n \in M$ then $f(r_1, \ldots, r_n)$ is nilpotent (since if not, then $c = f(r_1, \ldots, r_n) \notin P_c$, whereas $c \in M \subseteq P_c$). Moreover if $M \neq (0)$, then $f(x_1, \ldots, x_n)$ is nilpotent in the nonzero ideal M of the prime ring R. Therefore, again from Fact 1.6, $f(x_1, \ldots, x_n)$ is central valued on M, as well as in R.

On the other hand if $M = (0)$, then $\overline{C} \cap \overline{D} = (0)$. Therefore $\overline{D}$ contains all the square-zero elements of R, and $\overline{C}$ contains no nonzero square-zero element. Since a ring with no nonzero square-zero element is a subdirect sum of a domain, then the ideal $\overline{C}$ is a subdirect sum of domains and by Lemma 2.2 we have that $f(x_1, \ldots, x_n)$ is central valued in $\overline{C}$, and so also in R. We may hence assume $\overline{C} = 0$.

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In the last case, via the subdirect sum of R/P_c for $P_c \in F$, we suppose that for any ideal $H \neq 0$ of R, there exists $m = m(H) \geq 1$ such that $c^m \in H$. Let $b \in f(R)$ and suppose that b is neither nilpotent nor regular. We also define

$$L_b = \{x \in R : xb^n = 0, n = n(x) \geq 1\}$$
$$T_b = \{x \in R : b^n x = 0, n = n(x) \geq 1\}.$$

Let $a \in L_b$. Since $b \in f(R)$, by the assumption of Theorem 1.1, there exist suitable $m = m(a, b) \geq 1$ and $k = k(a, b) \geq 1$ such that

$$[b^m, (1 + a)b(1 + a)^{-1}]_k = 0 \quad \text{with } ab^n = 0 \text{ for some } n \geq 1,$$

and in light of Fact 1.3 it also holds that

$$[b^m, (1 + a)b^n(1 + a)^{-1}]_k = 0 \quad \text{with } ab^n = 0.$$

Therefore

$$\sum_{h=0}^{k} \binom{k}{h} (-1)^h((1 + a)b^{nh}(1 + a)^{-1})b^m((1 + a)b^{n(k-h)}(1 + a)^{-1}) = 0$$

in other words

$$\sum_{h=0}^{k-1} \binom{k}{h} (-1)^h((1 + a)b^{nh}(1 + a)^{-1})b^m((1 + a)b^{n(k-h)}(1 + a)^{-1}) = b^{m+nk}$$

and easy computations show that $b^{m+nk}(1 + a)^{-1} = b^{m+nk}$, that is $b^{m+nk} = b^{m+nk}(1 + a)$, that is $b^{m+nk}a = 0$, that is $a \in T_b$. Analogously we can prove that $T_b \subseteq L_b$. Thus $T_b = L_b = I$ is a two-sided ideal of R. Since there exists a suitable $m \geq 1$ such that $c^m \in I$, it follows that c is neither nilpotent nor regular. So by the above argument, there exists $m_1 \geq 1$ such that $c^{m_1} \in L_c$, therefore there exists $m_2 \geq 1$ such that $c^{m_1}c^{m_2} = 0$, a contradiction.

Hence any element $b \in f(R)$ is either nilpotent or regular. Since we are considering the case when R is not a domain, by Lemma 2.9, R satisfies a polynomial identity and by Lemma 2.4 $f(x_1, \ldots, x_n)$ is central valued on R. □

The Proof of Theorem 1.1. By Lemma 2.1, we may consider only the case char$(R) = 0$. Let

$$A = \{P \mid \text{there exists } 0 \neq x \in R \text{ such that } x^2 = 0, x \notin P\}$$
$$B = \{P \mid \text{for any } x \in R \text{ such that } x^2 = 0 \text{ then } x \in P\}$$

and $\overline{A} = \cap_A P$, $\overline{B} = \cap_B P$. Consider the diagonal map $\varphi : R \to \prod_{P \in A} R/P$. Since by Proposition 2.10 $f(x_1, \ldots, x_n)$ is central valued on R/P, for all $P \in A$,

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then $f(x_1, \ldots, x_n)$ is central valued on $R/\mathrm{Ker}(\varphi)$, where $\mathrm{Ker}(\varphi) = \bigcap_A P = \overline{A}$ and $\overline{A} \cap \overline{B} = (0)$, by the semi-primeness of R. Therefore $\overline{B}$ contains all the square-zero elements of R, and $\overline{A}$ contains no nonzero square-zero element. In particular $\overline{A}$ is a subdirect sum of domains, so by Lemma 2.2 $f(x_1, \ldots, x_n)$ is central valued on $\overline{A}$. Since $R/\overline{A}$ and $\overline{A}$ satisfy some polynomial identities, so does R and we obtain the required conclusion by Lemma 2.4. □

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