Bull. Aust. Math. Soc. (First published online 2025), page 1 of 5*
doi:10.1017/S000497272510021X
*Provisional—final page numbers to be inserted when paper edition is published

ON THE LARGEST CHARACTER DEGREE OF SOLVABLE GROUPS

YONG YANG and MENGTIAN ZHANG

(Received 1 June 2024; accepted 21 March 2025)

Abstract

We strengthen two results of Moretó. We prove that the index of the Fitting subgroup is bounded in terms of the degrees of the irreducible monomial Brauer characters of the finite solvable group G and it is also bounded in terms of the average degree of the irreducible Brauer characters of G that lie over a linear character of the Fitting subgroup.

2020 *Mathematics subject classification*: primary 20C15; secondary 20C20, 20D25.

Keywords and phrases: solvable groups, character degrees, monomial characters.

1. Introduction

Let G be a finite group and let $b_p(G)$ denote the largest degree of an irreducible Brauer character of G. Recently, Moretó (in [7, Theorem 2.1]) showed that if G is solvable, then G has an abelian subgroup of index at most $b_p(G)^{43/4}$, and there exists a characteristic abelian subgroup A of G such that $|G : A| \leq b_p(G)^{43/2}$. We strengthen this result by considering the irreducible monomial Brauer characters and also improve the bound substantially.

Moretó's result is motivated by a result of Gluck [2], who showed that in all finite groups the index of the Fitting subgroup $\mathbf{F}(G)$ in G is bounded by a polynomial function of $b(G)$, where $b(G)$ is the largest degree of an irreducible character of G. For a finite solvable group G, Gluck further showed that $|G : \mathbf{F}(G)| \leq b(G)^{13/2}$; Moretó and Wolf [9] gave the bound $|G : \mathbf{F}(G)| \leq b(G)^3$. As of today, for solvable groups, the best general bound $|G : \mathbf{F}(G)| \leq b(G)^{\alpha}$ was given by Yang in [11] with $\alpha = \ln(6 \cdot (24)^{1/3})/\ln 3 \approx 2.595$.

Let G be a finite solvable group, $\mathrm{Irr}(G)$ be the set of irreducible characters of G, $\mathrm{IBr}(G)$ be the set of irreducible Brauer characters of G and $\mathrm{IBr}_m(G)$ be the set of irreducible monomial Brauer characters of G. We will write $b(G)$ to denote the largest

This work was partially supported by a grant from the Simons Foundation (No. 918096) to Y.Y.

degree of an ordinary irreducible character of G, $b_m(G)$ to denote the largest degree of an irreducible monomial character of G and $b_{mp}(G)$ to denote the largest degree of an irreducible monomial Brauer character of G. We write $\mathrm{acd}(G) = \sum_{\chi\in\mathrm{Irr}(G)} \chi(1)/k(G)$ to denote the average degree of the irreducible characters of G, where $k(G)$ is the number of conjugacy classes of G. In the same way, $\mathrm{acd}_p(G)$ denotes the average degree of the irreducible Brauer characters of G.

For the average degree of the irreducible characters $\mathrm{acd}(G)$, it is not true that the index of the Fitting subgroup is bounded above in terms of $\mathrm{acd}(G)$. However, the index of the Fitting subgroup is bounded in terms of the average degree of the irreducible characters of G that lie over a linear character of the Fitting subgroup, and we write $\mathrm{acd}(G|\lambda) = \sum_{\chi\in\mathrm{Irr}(G|\lambda)} \chi(1)/k(G|\lambda)$. In [8], Moretó proved that there exists a linear $\lambda \in \mathrm{Irr}(\mathbf{F}(G))$ such that $|G : \mathbf{F}(G)| \leq \mathrm{acd}(G|\lambda)^{\alpha}$. We consider the irreducible Brauer character analogues for this case.

2. Main results

THEOREM 2.1. *Let G be a finite solvable group. Then there exists a characteristic abelian subgroup A of G such that $|G : A| \leq b_m(G)^{2\alpha} \cdot b_m(\mathbf{F}(G))^8$.*

PROOF. By [5, Theorem 4.1], $|G : \mathbf{F}(G)| \leq b_m(G)^{\alpha}$. By [3, Theorem 12.26], there exists an abelian group $B \leq \mathbf{F}(G)$ such that $|\mathbf{F}(G) : B| \leq b_m(\mathbf{F}(G))^4$ (note that $b(\mathbf{F}(G)) = b_m(\mathbf{F}(G))$). Thus,

$$|G : B| = |G : \mathbf{F}(G)||\mathbf{F}(G) : B| \leq b_m(G)^{\alpha} \cdot b_m(\mathbf{F}(G))^4.$$

Now, by the Chermak–Delgado theorem [4, Theorem 1.41], we conclude that G has a characteristic abelian subgroup A such that

$$|G : A| \leq b_m(G)^{2\alpha} \cdot b_m(\mathbf{F}(G))^8. \qquad \square$$

COROLLARY 2.2. *Suppose that all the irreducible characters of a finite solvable group G have degree at most $b(G)$. Then, there exists a characteristic abelian subgroup A of G such that $|G : A| \leq b(G)^{2\alpha+8}$.*

LEMMA 2.3. *Let N be a normal subgroup of the finite solvable group G and suppose that there exists a Brauer character $\lambda \in \mathrm{IBr}(N)$ such that $\lambda(1) = 1$ and λ is G-invariant.*

(a) *Then, λ is extendible to G if and only if $G' \cap N \leqslant \mathrm{Ker}\lambda$.*
(b) *Suppose that $G = NH$ and $N \cap H = 1$. Then, λ is extendible to G.*

PROOF. For part (a), we write $\bar{G} = G/G'$. Since $N/(G' \cap N) \cong (NG')/G'$ and λ can be viewed as a character in $N/(G' \cap N)$, we have $\bar{\lambda} \in \mathrm{Irr}(NG'/G')$. The group $\bar{G} = G/G'$ is abelian, so $\bar{\lambda}$ is extendible to $\bar{G}$ and it follows that λ is extendible to G.

For part (b), we write

$$\chi : N \to \mathbb{F}^*$$

https://doi.org/10.1017/S000497272510021X Published online by Cambridge University Press

for a module representation of λ which affords χ. For $g \in G$, let $g = ah$, where $a \in N$ and $h \in H$. We define

$$\tilde{\chi}(g) = \chi(a) \in \mathbb{F}^*.$$

We can calculate directly that

$$\begin{aligned}\tilde{\chi}(ah \cdot a'h') &= \tilde{\chi}(ah \cdot a' \cdot hh^{-1} \cdot h') = \tilde{\chi}(a \cdot ha'h^{-1} \cdot hh') \\ &= \chi(a \cdot ha'h^{-1}) = \chi(a) \cdot \chi(ha'h^{-1}) \\ &= \chi(a) \cdot \chi(a') = \tilde{\chi}(ab) \cdot \tilde{\chi}(a'h').\end{aligned}$$

Thus, $\tilde{\chi}$ is an extension of χ and it follows that λ is extendible to G. □

The following result strengthens [7, Theorem 2.1].

THEOREM 2.4. *Let G be a finite solvable group and suppose that $\mathbf{O}_p(G) = 1$. Then, $|G : \mathbf{F}(G)| \le b_{mp}(G)^{\alpha}$. Also, there exists a characteristic abelian subgroup A of G such that $|G : A| \le b_{mp}(G)^{2\alpha} \cdot b_{mp}(\mathbf{F}(G))^8$.*

PROOF. Let $\bar{G} := G/\mathbf{F}(G)$. Since $\mathbf{F}(G/\Phi(G)) = \mathbf{F}(G)/\Phi(G)$, we may assume that $\Phi(G) = 1$. Thus, $\mathbf{F}(G)$ is abelian. Now, G splits over the abelian normal subgroup $\mathbf{F}(G)$. Also, $\mathbf{F}(G)$ is a faithful and completely reducible $\bar{G}$-module by Gaschütz's theorem [6, Theorem 1.12]. By [6, Proposition 12.1], $\mathrm{Irr}(\mathbf{F}(G))$ is a faithful and completely reducible $\bar{G}$-module. Since $\mathbf{O}_p(G) = 1$, we have $\mathrm{Irr}(\mathbf{F}(G)) = \mathrm{IBr}(\mathbf{F}(G))$, and thus $\mathrm{IBr}(\mathbf{F}(G))$ is a faithful and completely reducible $\bar{G}$-module. By [11, Theorem 3.4], there exists $\beta \in \mathrm{IBr}(\mathbf{F}(G))$ such that $|\bar{G}| \le |\bar{G} : \bar{I}|^{\alpha}$, where $\bar{I} = I_{\bar{G}}(\beta) = \{\bar{g} \in \bar{G} \mid \beta^{\bar{g}} = \beta\}$. Let I be the preimage of $\bar{I}$ in G and $I = I_G(\beta) = \{g \in G \mid \beta^g = \beta\}$. By Lemma 2.3, let $\widehat{\beta} \in \mathrm{IBr}(I|\beta)$ be an extension of β and consider $\chi := \widehat{\beta}^G \in \mathrm{IBr}_m(G)$. Then,

$$|G : \mathbf{F}(G)| = |\bar{G}| \le |\bar{G} : \bar{I}|^{\alpha} = |G : I|^{\alpha} \le \chi(1)^{\alpha} \le b_{mp}(G)^{\alpha}.$$

By [3, Theorem 12.26], there exists an abelian group $B \le \mathbf{F}(G)$ such that $|\mathbf{F}(G) : B| \le b(\mathbf{F}(G))^4$. Since $\mathbf{O}_p(G) = 1$, we have $b(\mathbf{F}(G)) = b_p(\mathbf{F}(G)) = b_{mp}(\mathbf{F}(G))$. Thus,

$$|G : B| = |G : \mathbf{F}(G)||\mathbf{F}(G) : B| \le b_{mp}(G)^{\alpha} \cdot b_{mp}(\mathbf{F}(G))^4.$$

By the Chermak–Delgado theorem [4, Theorem 1.41], we conclude that G has a characteristic abelian subgroup A such that

$$|G : A| \le b_{mp}(G)^{2\alpha} \cdot b_{mp}(\mathbf{F}(G))^8.$$ □

COROLLARY 2.5. *Suppose that all the irreducible p-Brauer characters of a finite solvable group G have degree at most $b_p(G)$ and $\mathbf{O}_p(G) = 1$. Then, there exists a characteristic abelian subgroup A of G such that $|G : A| \le b_p(G)^{2\alpha+8}$.*

THEOREM 2.6. *Let G be a finite solvable group such that $\mathbf{O}_p(G) = 1$. Then, there exists a linear $\lambda \in \mathrm{IBr}(\mathbf{F}(G))$ such that $|G : \mathbf{F}(G)| \le \mathrm{acd}_p(G|\lambda)^{\alpha}$.*

https://doi.org/10.1017/S000497272510021X Published online by Cambridge University Press

PROOF. By Gaschütz's theorem [6, Theorem 1.12], $\bar{G} = G/\mathbf{F}(G)$ acts faithfully and completely reducibly on $V = \mathbf{F}(G)/\Phi(G)$. By [6, Proposition 12.1], $\mathrm{Irr}(V)$ is a faithful and completely reducible $\bar{G}$-module. Since $\mathbf{O}_p(G) = 1$, we have $p \nmid |\mathbf{F}(G)|$ and $\mathrm{Irr}(V) = \mathrm{IBr}(V)$. Applying [11, Theorem 3.4] to this action, we deduce that there exists $\lambda \in \mathrm{IBr}(V)$ such that

$$|\bar{G}| \le |G : I_G(\lambda)|^{\alpha}.$$

By Clifford's correspondence [10, Theorem 8.9], all the characters in $\mathrm{IBr}(G|\lambda)$ are induced from irreducible Brauer characters of $I_G(\lambda)$. In particular, if $\chi \in \mathrm{IBr}(G|\lambda)$, then

$$|\bar{G}| \le |G : I_G(\lambda)|^{\alpha} \le \chi(1)^{\alpha}.$$

It follows that

$$|G : \mathbf{F}(G)| \le \mathrm{acd}_p(G|\lambda)^{\alpha}. \qquad \square$$

As usual, if G is a group, then $\mathbf{O}_{p'}(G)$ is the largest normal p'-subgroup of G. We define $\mathbf{O}_{p'p}(G)$ to be the subgroup of G such that $\mathbf{O}_{p'p}(G)/\mathbf{O}_{p'}(G) = \mathbf{O}_p(G/\mathbf{O}_{p'}(G))$. If $N \trianglelefteq G$ and $\lambda \in \mathrm{Irr}(N)$, we write $C_G(\lambda)$ to be the inertia subgroup in G of λ. We write $\mathrm{Irr}_{mp'}(G)$ to be the set of all the irreducible monomial p'-degree characters of G and $b_{mp'}(G)$ to be the largest degree of the irreducible characters in $\mathrm{Irr}_{mp'}(G)$.

LEMMA 2.7 [7, Lemma 3.1]. *Let $G = NH$, where $N \trianglelefteq G$ and $N \cap H = 1$. If $\lambda \in \mathrm{Irr}(N)$ is linear and G-invariant, then λ is extendible to G.*

LEMMA 2.8. *Let $G = HV$, where $V \trianglelefteq G$ is an elementary abelian p-group, $H \cap V = 1$, and H acts faithfully and completely reducibly on V. Then, $|\mathbf{F}(H)| \le b_{mp'}(G)^2$.*

PROOF. Let P be a Sylow p-subgroup of H and let $F = \mathbf{F}(H)$. Set $K = PF$. We note that K acts faithfully on V, so $\mathbf{F}(KV) = V$. This implies that K acts faithfully and completely reducibly on $\mathrm{Irr}(V)$. By [1, Theorem 1.1], there exists a P-invariant $\lambda \in \mathrm{Irr}(V)$ such that $|F : C_F(\lambda)| \ge \sqrt{|F|}$. Clifford's correspondence implies that if $\widehat{\lambda} \in \mathrm{Irr}(G|\lambda)$, then $\widehat{\lambda}(1) \ge \sqrt{|F|}$.

By Lemma 2.7, λ can be extended to $C_G(\lambda)$. Since $P \le C_G(\lambda)$, by using Clifford's correspondence again, all the characters in $\mathrm{Irr}(G|\lambda)$ are induced from irreducible characters of $C_G(\lambda)$. We deduce that there exists a p'-degree character $\widehat{\lambda} \in \mathrm{Irr}(G|\lambda)$. Then, we have a p'-degree character $\chi := \widehat{\lambda}^G \in \mathrm{Irr}_m(G)$ and $\chi(1) \ge \sqrt{|F|}$. Hence,

$$\sqrt{|F|} \le \chi(1) \le b_{mp'}(G).$$

Thus, we deduce that $|\mathbf{F}(H)| \le b_{mp'}(G)^2$. $\square$

THEOREM 2.9. *Let G be a solvable group. Then, $|G : \mathbf{O}_{p',p}(G)| \le b_{mp'}(G)^{13/2}$.*

PROOF. Without loss of generality, we may assume that $\mathbf{O}_{p'}(G) = 1$. We write $V = \mathbf{O}_p(G)$ and note that V is the Fitting subgroup G. By Gaschütz's theorem, we may assume that $\Phi(G) = 1$, so that V is elementary abelian. We write $G = HV$

https://doi.org/10.1017/S000497272510021X Published online by Cambridge University Press

with $H \cap V = 1$ and note that H acts faithfully and completely reducibly on V. Let $F = \mathbf{F}(H)$. Then, $|\mathbf{F}(H)| \leq b_{mp'}(G)^2$ by Lemma 2.8.

From [6, Theorem 3.5], $|H : F| \leq |F|^{9/4}$, and thus

$$|H| = |H : F||F| \leq |F|^{13/4} \leq b_{mp'}(G)^{13/2}. \qquad \square$$

References

[1] S. Dolfi and G. Navarro, 'Large orbits of elements centralized by a Sylow subgroup', *Arch. Math.* **93** (2009), 299–304.

[2] D. Gluck, 'The largest irreducible character degree of a finite group', *Canad. J. Math.* **37** (1985), 442–451.

[3] I. M. Isaacs, *Character Theory of Finite Groups* (Dover, New York, 1994).

[4] I. M. Isaacs, *Finite Group Theory* (American Mathematical Society, Providence, RI, 2008).

[5] J. Li, F. Bian and X. Zhang, 'Monomial characters and the index of the Fitting subgroup of a solvable group', *Comm. Algebra* **51** (2023), 4866–4870.

[6] O. Manz and T. R. Wolf, *Representations of Solvable Groups* (Cambridge University Press, Cambridge, 1993).

[7] A. Moretó, 'On the largest Brauer and p'-character degrees', *J. Algebra* **630** (2023), 1–8.

[8] A. Moretó, 'The average character degree of finite groups and Gluck's conjecture', *J. Group Theory* **26** (2023), 803–815.

[9] A. Moretó and T. R. Wolf, 'Orbit sizes, character degrees and Sylow subgroups', *Adv. Math.* **184** (2004), 18–36.

[10] G. Navarro, *Characters and Blocks of Finite Groups* (Cambridge University Press, Cambridge, 1998).

[11] Y. Yang, 'On Gluck's conjucture', *Bull. Sci. Math.* **183** (2023), Article no. 103235, 17 pages.

YONG YANG, Department of Mathematics,
Texas State University, San Marcos, TX 78666, USA
e-mail: yang@txstate.edu

MENGTIAN ZHANG, College of Science,
China Three Gorges University, Yichang, Hubei 443002, PR China
e-mail: zmt1110@hotmail.com

https://doi.org/10.1017/S000497272510021X Published online by Cambridge University Press