10
$O(N)$ vector models

The simplest models, which become solvable in the limit of a large number of field components, deal with a field which has N components forming an $O(N)$ vector in an internal symmetry space. A model of this kind was first considered by Stanley [Sta68] in statistical mechanics and is known as the spherical model. The extension to quantum field theory was made by Wilson [Wil73] both for the four-Fermi and φ^4 theories.

Within the framework of perturbation theory, the four-Fermi interaction is renormalizable only in $d = 2$ dimensions and is nonrenormalizable for $d > 2$. The $1/N$-expansion resums perturbation-theory diagrams after which the four-Fermi interaction becomes renormalizable to each order in $1/N$ for $2 \leq d < 4$. An analogous expansion exists for the nonlinear $O(N)$ sigma model. The φ^4 theory remains "trivial" in $d = 4$ to each order of the $1/N$-expansion and has a nontrivial infrared-stable fixed point for $2 < d < 4$.

The $1/N$-expansion of the vector models is associated with a resummation of Feynman diagrams. A very simple class of diagrams – the bubble graphs – survives to the leading order in $1/N$. This is why the large-N limit of the vector models is solvable. Alternatively, the large-N solution is nothing but a saddle-point solution in the path-integral approach. The existence of the saddle point is a result of the fact that N is large. This is to be distinguished from a perturbation-theory saddle point which arises from the fact that the coupling constant is small. Taking into account fluctuations around the saddle-point results in the $1/N$-expansion of the vector models.

We begin this chapter with a description of the $1/N$-expansion of the N-component four-Fermi theory analyzing the bubble graphs. Then we introduce functional methods and construct the $1/N$-expansion of the $O(N)$-symmetric φ^4 theory and nonlinear sigma model. Finally, we discuss the factorization in the $O(N)$ vector models at large N.

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10.1 Four-Fermi theory

The action of the $O(N)$-symmetric four-Fermi theory in a d-dimensional Euclidean space* is defined by

$$S[\bar{\psi}, \psi] \;=\; \int \mathrm{d}^d x \left[\bar{\psi}\, \hat{\partial}\, \psi + m\, \bar{\psi}\psi - \frac{G}{2} \left(\bar{\psi}\psi \right)^2 \right]. \tag{10.1}$$

Here $\hat{\partial} = \gamma_\mu \partial_\mu$ and $\psi = (\psi_1, \ldots, \psi_N)$ is a spinor field which forms an N-component vector in an internal-symmetry space so that

$$\bar{\psi}\psi \;=\; \sum_{i=1}^{N} \bar{\psi}_i \psi_i \,. \tag{10.2}$$

The dimension of the four-Fermi coupling constant G is

$$\dim\,[G] \;=\; m^{2-d}\,. \tag{10.3}$$

For this reason, the perturbation theory for the four-Fermi interaction is renormalizable in $d = 2$ but is nonrenormalizable for $d > 2$ (and, in particular, in $d = 4$). This is why the old Fermi theory of weak interactions was replaced by the modern electroweak theory, where the interaction is mediated by the $W^\pm$ and Z bosons.

The action (10.1) can be rewritten equivalently as

$$S[\bar{\psi}, \psi, \chi] \;=\; \int \mathrm{d}^d x \left(\bar{\psi}\, \hat{\partial}\, \psi + m\, \bar{\psi}\psi - \chi\, \bar{\psi}\psi + \frac{\chi^2}{2G} \right), \tag{10.4}$$

where χ is an auxiliary field. The two forms of the action, (10.1) and (10.4), are equivalent owing to the equation of motion which reads in the operator notation as

$$\boldsymbol{\chi} \;=\; G : \bar{\boldsymbol{\psi}} \boldsymbol{\psi} :, \tag{10.5}$$

where $: \cdots :$ denotes the normal ordering of operators. Equation (10.5) can be derived by varying the action (10.4) with respect to χ.

In the path-integral quantization, where the partition function is defined by

$$Z \;=\; \int \mathcal{D}\chi\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathrm{e}^{-S[\bar{\psi}, \psi, \chi]} \tag{10.6}$$

with $S[\bar{\psi}, \psi, \chi]$ given by Eq. (10.4), the action (10.1) appears after performing the Gaussian integral over χ. Therefore, alternatively one obtains

$$Z \;=\; \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathrm{e}^{-S[\bar{\psi}, \psi]} \tag{10.7}$$

with $S[\bar{\psi}, \psi]$ given by Eq. (10.1).

* In $d = 2$ this model was studied in the large-N limit in [GN74] and is often called the Gross–Neveu model.

The perturbative expansion of the $O(N)$-symmetric four-Fermi theory can be represented conveniently using the formulation (10.4) via the auxiliary field χ. Then the diagrams are of the same type as those in Yukawa theory, and resemble those for QED with $\bar{\psi}$ and ψ being an analog of the electron–positron field and χ being an analog of the photon field.

However, the auxiliary field $\chi(x)$ does not propagate, since it follows from the action (10.4) that

$$D_0(x-y) \;\equiv\; \langle\, \chi(x)\chi(y)\,\rangle_{\text{Gauss}} \;=\; G\,\delta^{(d)}(x-y) \tag{10.8}$$

or

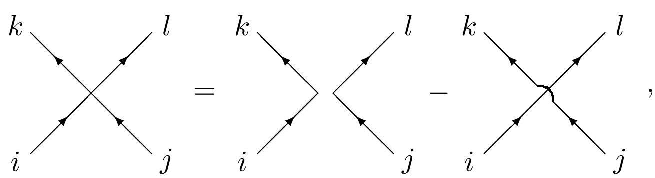
$$D_0(p) \;\equiv\; \langle\, \chi(-p)\chi(p)\,\rangle_{\text{Gauss}} \;=\; G \tag{10.9}$$

in momentum space.

It is convenient to represent the four-Fermi vertex

$$\Gamma^{kl}_{ij} \;=\; G\left(\delta^k_i\delta^l_j - \delta^l_i\delta^k_j\right) \tag{10.10}$$

as the sum of two terms



, (10.11)

where the empty space inside the vertex is associated with the propagator (10.8) (or (10.9) in momentum space). The relative minus sign makes the vertex antisymmetric in both incoming and outgoing fermions as is prescribed by the Fermi statistics.

The diagrams that contribute to second order in G for the four-Fermi vertex are depicted, in this notation, in Fig. 10.1. The $O(N)$ indices propagate through the solid lines so that the closed line in the diagram in Fig. 10.1b corresponds to the sum over the $O(N)$ indices which results in a factor of N. Analogous one-loop diagrams for the propagator of the ψ-field are depicted in Fig. 10.2.

Problem 10.1 Calculate the one-loop Gell-Mann–Low function of the four-Fermi theory in $d=2$.

Solution Evaluating the diagrams in Fig. 10.1 that are logarithmically divergent in $d=2$, and noting that the diagrams in Fig. 10.2 do not contribute to the wave-function renormalization of the ψ-field, which emerges to the next order in G, one obtains

$$\mathcal{B}(G) \;=\; -\frac{(N-1)\,G^2}{2\pi}\,. \tag{10.12}$$

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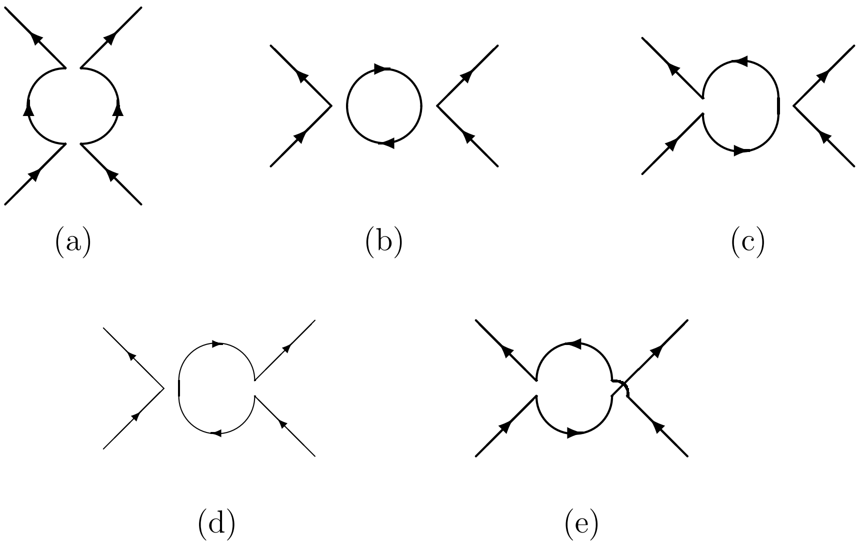


Fig. 10.1. Diagrams of second-order perturbation theory for the four-Fermi vertex. Diagram (b) involves the sum over the $O(N)$ indices.

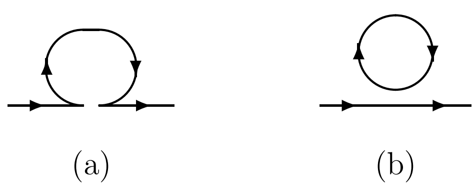


Fig. 10.2. One-loop diagrams for the propagator of the ψ-field. Diagram (b) involves the sum over the $O(N)$ indices.

The four-Fermi theory in two dimensions is asymptotically free as was first noted by Anselm [Ans59] and rediscovered by Gross and Neveu [GN74].

The vanishing of the one-loop Gell-Mann–Low function in the Gross–Neveu model for $N = 1$ is related to the same phenomenon in the Thirring model. The latter model is associated with the vector-like interaction $(\bar{\psi}\gamma_\mu\psi)^2$ of one species of fermions, where γ_μ are the γ-matrices in two dimensions. Since in $d = 2$ a bispinor has only two components ψ_1 and ψ_2, both the vector-like and the scalar-like interaction (10.1) for $N = 1$ reduce to $\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2$, since the square of a Grassmann variable vanishes. Therefore, these two models coincide. For the Thirring model, the vanishing of the Gell-Mann–Low function for any G was shown by Johnson [Joh61] to all loops.

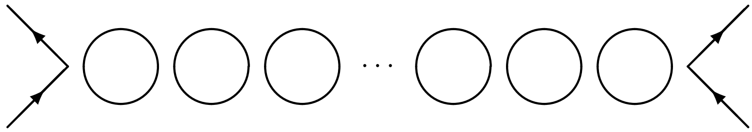


Fig. 10.3. Bubble diagram which survives the large-N limit of the $O(N)$ vector models.

Remark on auxiliary fields

The introduction of the auxiliary field is often called the Hubbard–Stratonovich transformation in statistical mechanics. The proper term used in quantum field theory is just "auxiliary field".

10.2 Bubble graphs as the zeroth order in $1/N$

The perturbation-theory expansion of the $O(N)$-symmetric four-Fermi theory contains, in particular, diagrams of the type depicted in Fig. 10.3, which are called *bubble graphs*. Since each bubble has a factor of N, the contribution of the n-bubble graph is $\propto G^{n+1}N^n$, which is of the order of

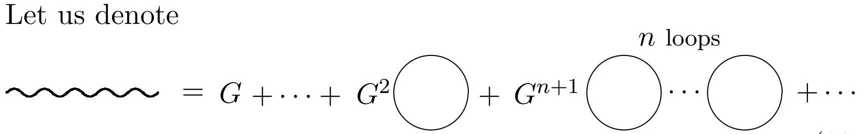
$$G^{n+1}N^n \sim G \tag{10.13}$$

as $N \to \infty$, since

$$G \sim \frac{1}{N}\,. \tag{10.14}$$

Therefore, all the bubble graphs are essential to the leading order in $1/N$.

Let us denote



(10.15)

In fact, the wavy line is nothing but the propagator D of the χ field with the bubble corrections included. The first term G on the RHS of Eq. (10.15) is nothing but the free propagator (10.9).

Summing the geometric series of the fermion-loop chains on the RHS of Eq. (10.15), one obtains analytically*

$$D^{-1}(p) = \frac{1}{G} - N\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{\mathrm{sp}\left[(\hat{k}+\mathrm{i}m)(\hat{k}+\hat{p}+\mathrm{i}m)\right]}{(k^2+m^2)\left[(k+p)^2+m^2\right]}\,. \tag{10.16}$$

* Recall that the free Euclidean fermionic propagator is given by $S_0(p) = (\mathrm{i}\hat{p}+m)^{-1}$ from Eqs. (10.4) and (10.6), and the additional minus sign is associated with the fermion loop.

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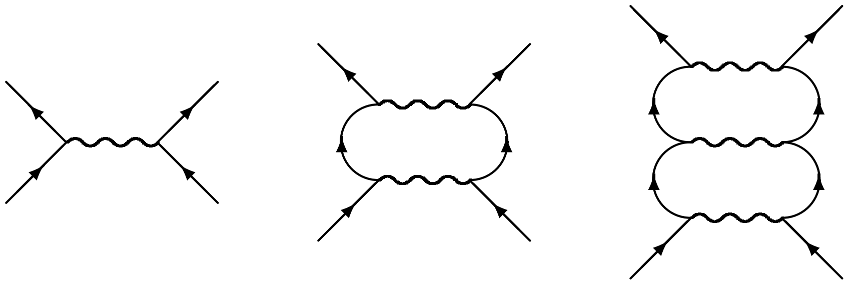

Fig. 10.4. Some diagrams of the $1/N$-expansion for the $O(N)$ four-Fermi theory. The wavy line represents the (infinite) sum of the bubble graphs (10.15).

This determines the exact propagator of the χ field at large N. It is $\mathcal{O}(N^{-1})$ since the coupling G is included in the definition of the propagator.

The idea is now to change the order of summation of diagrams of perturbation theory using $1/N$ rather than G as the expansion parameter. Therefore, the zeroth-order propagator of the expansion in $1/N$ is defined as the sum over the bubble graphs (10.15), which is given by Eq. (10.16).

Some of the diagrams of the new expansion for the four-Fermi vertex are depicted in Fig. 10.4. The first diagram is proportional to G, while the second and third ones are proportional to G^2 or G^3, respectively, and therefore are of order $\mathcal{O}(N^{-1})$ or $\mathcal{O}(N^{-2})$ with respect to the first diagram. The perturbation theory is thus rearranged as a $1/N$-expansion.

The general structure of the $1/N$-expansion is the same for all vector models, say, for the N-component φ^4 theory which is considered in the next section.

The main advantage of the expansion in $1/N$ for the four-Fermi interaction, over the perturbation theory, is that it is renormalizable in $d < 4$, while the perturbation-theory expansion in G is renormalizable only in $d = 2$. Moreover, the $1/N$-expansion of the four-Fermi theory in $2 < d < 4$ demonstrates [Wil73] the existence of an ultraviolet-stable fixed point, i.e. a nontrivial zero of the Gell-Mann–Low function.

Problem 10.2 Show that the $1/N$-expansion of the four-Fermi theory is renormalizable in $2 \le d < 4$ (but not in $d = 4$).

Solution In order to demonstrate renormalizability, let us analyze indices of the diagrams of the $1/N$-expansion.

First of all, we shall remove an ultraviolet divergence of the integral over the d-momentum k in Eq. (10.16). The divergent part of the integral is proportional to Λ^{d-2} (logarithmically divergent in $d = 2$), where Λ is an ultraviolet cutoff. It can be canceled by choosing

$$G = \frac{g^2}{N}\Lambda^{2-d}, \tag{10.17}$$

where g^2 is a proper dimensionless constant which is not necessarily positive since the four-Fermi theory is stable with either sign of G. The power of Λ in Eq. (10.17) is consistent with the dimension of G. This prescription works for $2 < d < 4$ where there is only one divergent term, while another divergence $\propto p^2 \ln \Lambda$ emerges additionally in $d = 4$. This is why the consideration is not applicable in $d = 4$.

The propagator $D(p)$ is therefore finite, and behaves at large momenta $|p| \gg m$ as

$$D(p) \propto |p|^{2-d}. \tag{10.18}$$

The standard power-counting arguments then show that the only divergent diagrams appear in the propagators of the ψ and χ fields, and in the $\bar{\psi}$–χ–ψ three-vertex. These divergences can be removed by a renormalization of the coupling g, mass, and wave functions of ψ and χ.

This completes a demonstration of renormalizability of the $1/N$-expansion for the four-Fermi interaction in $2 \le d < 4$. For more details, see [Par75].

Problem 10.3 Calculate in $d = 3$ the value of g in Eq. (10.17).

Solution To calculate the divergent part of the integral in Eq. (10.16), we set $p = 0$ and $m = 0$. Remembering that the γ-matrices are 2×2 matrices in $d = 3$, we obtain

$$\int^{\Lambda} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{sp}\, \hat{k}\hat{k}}{k^2 k^2} = 2\int^{\Lambda} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{k^2} = \frac{1}{\pi^2}\int_0^{\Lambda} \mathrm{d}|k| = \frac{\Lambda}{\pi^2}. \tag{10.19}$$

Note that the integral is linearly divergent in $d = 3$ and Λ is the cutoff for the integration over $|k|$. This divergence can be canceled by choosing G according to Eq. (10.17) with g equal to

$$g_* = \pi. \tag{10.20}$$

Problem 10.4 Calculate in $d = 3$ the coefficient of proportionality in Eq. (10.18).

Solution Let us choose $G = \pi^2/N\Lambda$, as prescribed by Eqs. (10.17) and (10.20), and in Eq. (10.16) set $m = 0$, since we are interested in the asymptotic behavior at $|p| \gg m$. Then the RHS of Eq. (10.16) can be rearranged as

$$\begin{aligned} D^{-1}(p) &= -2N \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[\frac{k^2 + kp}{k^2 (k+p)^2} - \frac{1}{k^2} \right] \\ &= 2N \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{p^2 + kp}{k^2 (k+p)^2}. \end{aligned} \tag{10.21}$$

This integral is obviously convergent.

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To calculate it, we apply the standard technique of α-parametrization, which is based on the formula

$$\frac{1}{k^2} = \int_0^\infty d\alpha\, e^{-\alpha k^2}. \tag{10.22}$$

We have

$$\int \frac{d^3k}{(2\pi)^3} \frac{p^2 + kp}{k^2(k+p)^2} = \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \int \frac{d^3k}{(2\pi)^3} \left(p^2 + kp\right) e^{-\alpha_1 k^2 - \alpha_2 (k+p)^2} \tag{10.23}$$

after which the Gaussian integral over d^3k can easily be performed. We then obtain

$$D^{-1}(p) = \frac{N}{4\pi^{3/2}} p^2 \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \frac{\alpha_1}{(\alpha_1+\alpha_2)^{5/2}} \exp\left(-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} p^2\right). \tag{10.24}$$

The remaining integration over α_1 and α_2 can easily be performed by introducing the new variables $\alpha \in [0,\infty]$ and $x \in [0,1]$ so that

$$\alpha_1 = \alpha x, \quad \alpha_2 = \alpha(1-x), \quad \frac{\partial(\alpha_1,\alpha_2)}{\partial(x,\alpha)} = \alpha. \tag{10.25}$$

Finally, this gives

$$D(p) = \frac{8}{N\,|p|}. \tag{10.26}$$

Equation (10.26) (or (10.18) in d dimensions) is remarkable since it shows that the scale dimension of the field χ, which is defined in Sect. 3.5 by Eq. (3.65), changes its value from $l_\chi = d/2$ in perturbation theory to $l_\chi = 1$ in the zeroth order of the $1/N$ expansion (remember that the momentum-space propagator of a field with the scale dimension l is proportional to $|p|^{2l-d}$). This appearance of scale invariance in the $1/N$-expansion of the four-Fermi theory at $2 < d < 4$ was first pointed out by Wilson [Wil73] and implies that the Gell-Mann–Low function $\mathcal{B}(g)$ has a zero at $g = g_*$, which is given in $d = 3$ by Eq. (10.20).

Problem 10.5 Find the (logarithmic) anomalous dimensions of the fields ψ and χ, and of the $\bar\psi$–χ–ψ three-vertex in $d = 3$ to order $1/N$.

Solution The $1/N$-correction to the propagator of the ψ-field is given by the diagram depicted in Fig. 10.5a. Since we are interested in the ultraviolet behavior, we can again set $m = 0$. Analytically, we have

$$S^{-1}(p) = i\hat{p} + \frac{8i}{N} \int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{\hat{k}+\hat{p}}{|k|(k+p)^2}. \tag{10.27}$$

The (logarithmically) divergent contribution emerges from the domain of integration $|k| \gg |p|$ so we can expand the integrand in p. The p-independent term

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Fig. 10.5. Diagrams for the $1/N$-correction to the ψ-field propagator (a) and the three-vertex (b).

vanishes after integration over the directions of k so that we obtain

$$S^{-1}(p) \;=\; \mathrm{i}\hat{p}\left(1+\frac{8}{N}\int^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{|k|^3}\right) \;=\; \mathrm{i}\hat{p}\left(1+\frac{2}{3\pi^2 N}\ln\frac{\Lambda^2}{p^2}+\frac{\text{finite}}{N}\right). \tag{10.28}$$

The diagram, which gives a nonvanishing contribution to the three-vertex at order $1/N$, is depicted in Fig. 10.5b. It gives analytically

$$\Gamma(p_1,p_2) \;=\; 1+\frac{8}{N}\int^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{(\hat{k}+\hat{p}_1)(\hat{k}+\hat{p}_2)}{|k|(k+p_1)^2(k+p_2)^2}, \tag{10.29}$$

where p_1 and p_2 are the incoming and outgoing fermion momenta, respectively. The logarithmic domain is $|k| \gg |p|_{\max}$, with $|p|_{\max}$ being the largest of $|p_1|$ and $|p_2|$. This gives

$$\Gamma(p_1,p_2) \;=\; 1-\frac{2}{\pi^2 N}\ln\frac{\Lambda^2}{p_{\max}^2}+\frac{\text{finite}}{N}. \tag{10.30}$$

The analogous calculation of the $1/N$ correction for the field χ is slightly more complicated since it involves three two-loop diagrams (see, for example, [CMS93]). The resulting expression for $D^{-1}(p)$ is given by

$$\Big[ND(p)\Big]^{-1} \;=\; \frac{\Lambda}{g^2}+\left(-\frac{\Lambda}{\pi^2}+\frac{|p|}{8}\right)+\frac{1}{\pi^2 N}\left[2\Lambda-|p|\left(\frac{2}{3}\ln\frac{\Lambda^2}{p^2}+\text{finite}\right)\right]. \tag{10.31}$$

The linear divergence is canceled to order $1/N$, providing g is equal to

$$g_* \;=\; \pi\left(1+\frac{1}{N}\right), \tag{10.32}$$

which determines g_* to order $1/N$. After this $D^{-1}(p)$ takes the form

$$D^{-1}(p) \;=\; \frac{N\,|p|}{8}\left(1-\frac{16}{3\pi^2 N}\ln\frac{\Lambda^2}{p^2}\right). \tag{10.33}$$

To make all three expressions (10.28), (10.30), and (10.33) finite, we need logarithmic renormalizations of the wave functions of ψ- and χ-fields and of the

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vertex Γ. This can be achieved by multiplying them by the renormalization constants

$$Z_i(\Lambda) \;=\; 1 - \gamma_i \ln \frac{\Lambda^2}{\mu^2}\,, \tag{10.34}$$

where μ denotes a reference mass scale and γ_i are anomalous dimensions. The index i denotes ψ, χ, or v for the ψ- and χ-propagators or the three-vertex Γ, respectively. We have, therefore, calculated

$$\gamma_\psi \;=\; \frac{2}{3\pi^2 N}\,, \qquad \gamma_{\rm v} \;=\; -\frac{2}{\pi^2 N}\,, \qquad \gamma_\chi \;=\; -\frac{16}{3\pi^2 N} \tag{10.35}$$

to order $1/N$. Owing to Eq. (10.5) γ_χ coincides with the anomalous dimension of the composite field $\bar\psi\psi$: $\gamma_{\bar\psi\psi} \;=\; \gamma_\chi$.

Note that

$$Z_\psi^2 Z_{\rm v}^{-2} Z_\chi \;=\; 1\,. \tag{10.36}$$

This implies that the effective charge is not renormalized and is given by Eq. (10.32). Thus, the nontrivial zero of the Gell-Mann–Low function persists to order $1/N$ (and, in fact, to all orders of the $1/N$-expansion).

Remark on scale invariance at the fixed point

The renormalization group says that

$$\mu \;=\; \Lambda \exp\left[-\int \frac{{\rm d}g^2}{\mathcal{B}(g^2)}\right], \tag{10.37}$$

which is essentially the same as Eq. (6.85). If $\mathcal{B}$ has a nontrivial fixed point g_*^2 near which

$$\mathcal{B}\big(g^2\big) \;=\; b\left(g^2 - g_*^2\right) \tag{10.38}$$

with $b<0$, then the substitution into Eq. (10.37) gives

$$g^2 \;=\; g_*^2 + \left(\frac{\mu}{\Lambda}\right)^{-b}. \tag{10.39}$$

Therefore, the approach to the critical point is power-like rather than logarithmic as for the case of $g_*^2 = 0$ when

$$\mathcal{B}\big(g^2\big) \;=\; b g^4\,. \tag{10.40}$$

The latter behavior of $\mathcal{B}$ results, after the substitution into Eq. (10.37), in the logarithmic dependence

$$g^2 \;=\; \frac{1}{b\ln(\mu/\Lambda)} \tag{10.41}$$

when $b<0$, which is associated with asymptotic freedom.

If g is chosen exactly at the critical point g_*, then the renormalization-group equations

$$\mu \frac{\mathrm{d} \ln \Gamma_i}{\mathrm{d}\mu} \;=\; \gamma_i(g^2)\,, \tag{10.42}$$

where Γ_i denotes generically either vertices or inverse propagators, possess the scale-invariant solutions

$$\Gamma_i \;\propto\; \mu^{\gamma_i(g_*^2)}\,. \tag{10.43}$$

This complements the heuristic consideration of Sect. 3.5 on the relation between scale invariance and the vanishing of the Gell-Mann–Low function.

For the four-Fermi theory in $d = 3$, Eq. (10.43) yields

$$S(p) \;=\; \frac{1}{\mathrm{i}\hat{p}} \left(\frac{p^2}{\mu^2}\right)^{\gamma_\psi}, \tag{10.44}$$

$$D(p) \;=\; \frac{8}{N\,|p|} \left(\frac{p^2}{\mu^2}\right)^{\gamma_\chi}, \tag{10.45}$$

$$\Gamma(p_1,p_2) \;=\; \left(\frac{\mu^2}{p_1^2}\right)^{\gamma_{\mathrm{v}}} f\left(\frac{p_2^2}{p_1^2}, \frac{p_1 p_2}{p_1^2}\right), \tag{10.46}$$

where f is an arbitrary function of the dimensionless ratios which is not determined by scale invariance. Here the indices obey the relation

$$\gamma_{\mathrm{v}} \;=\; \gamma_\psi + \frac{1}{2}\gamma_\chi \tag{10.47}$$

which guarantees that Eq. (10.36), implied by scale invariance, is satisfied.

The indices γ_i are given to order $1/N$ by Eqs. (10.35). When expanded in $1/N$, Eqs. (10.44) and (10.45) obviously reproduce Eqs. (10.28) and (10.33). Therefore, one obtains the exponentiation of the logarithms which emerge in the $1/N$-expansion. The calculation of the next terms of the $1/N$-expansion for the indices γ_i is given in [Gra91, DKS93, Gra93].

Remark on conformal invariance at fixed point

Scale invariance implies, in a renormalizable quantum field theory, more general conformal invariance as was first pointed out in [MS69, GW70]. The conformal group in a d-dimensional space-time has $(d+1)(d+2)/2$ parameters as illustrated by Table 10.1. More details concerning the conformal group can be found in the lecture by Jackiw [Jac72].

A heuristic proof [MS69] of the fact that scale invariance implies conformal invariance is based on the explicit form of the conformal current

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Table 10.1. Contents and the number of parameters of groups of space-time symmetry.

Group	Transformations		Parameters
Lorentz	$\frac{d(d-1)}{2}$ rotations	$x'_\mu = \Omega_{\mu\nu} x_\nu$	$\frac{d(d-1)}{2}$
Poincaré	$+\,d$ translations	$x'_\mu = x_\mu + a_\mu$	$\frac{d(d+1)}{2}$
Weyl	$+\,1$ dilatation	$x'_\mu = \rho\, x_\mu$	$\frac{d^2+d+2}{2}$
Conformal	$+\,d$ special conformal	$\frac{x'_\mu}{(x')^2} = \frac{x_\mu}{x^2} + \alpha_\mu$	$\frac{(d+1)(d+2)}{2}$

K^α_μ, which is associated with the special conformal transformation, via the energy–momentum tensor:

$$K^\alpha_\mu \;=\; \left(2x_\nu x^\alpha - x^2 \delta^\alpha_\nu\right) \theta_{\mu\nu}\,. \tag{10.48}$$

Differentiating, we obtain

$$\partial_\mu K^\alpha_\mu \;=\; 2x^\alpha \theta_{\mu\mu}\,, \tag{10.49}$$

which is analogous to Eqs. (3.66) and (3.67) for the dilatation current. Therefore, both the dilatation and conformal currents vanish simultaneously when $\theta_{\mu\nu}$ is traceless which is provided, in turn, by the vanishing of the Gell-Mann–Low function.

Conformal invariance completely fixes three-vertices as was first shown by Polyakov [Pol70] for scalar theories. The proper formula for the four-Fermi theory (the same as for Yukawa theory [Mig71]) is given by

$$\begin{aligned}\Gamma(p_1,p_2) \;=\; & \mu^{2\gamma_{\rm v}} \frac{\Gamma(d/2)\Gamma(d/2-\gamma_{\rm v})}{\Gamma(\gamma_{\rm v})} \\ & \times \int \frac{{\rm d}^d k}{\pi^{d/2}} \frac{\hat{k}+\hat{p}_1}{[(k+p_1)^2]^{1+\gamma_\chi/2}} \frac{\hat{k}+\hat{p}_2}{[(k+p_2)^2]^{1+\gamma_\chi/2}} \frac{1}{|k|^{d-2+2\gamma_\psi-\gamma_\chi/2}},\end{aligned} \tag{10.50}$$

where the coefficient in the form of the ratio of the Γ-functions is prescribed by the normalization (10.44) and (10.45), and the indices are related by Eq. (10.47) but can be arbitrary otherwise.*

* The only restriction $\gamma_\psi \geq 0$ is imposed by the Källén–Lehmann representation of the propagator, while there is no such restriction on γ_χ since it is a composite field.

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Equation (10.50), which results from conformal invariance, unambiguously fixes the function f in Eq. (10.46). In contrast to infinite-dimensional conformal symmetry in $d = 2$, the conformal group in $d > 2$ is less restrictive. It fixes only the tree-point vertex while, say, the four-point vertex remains an unknown function of two variables.

Problem 10.6 Calculate the integral on the RHS of Eq. (10.50) in $d = 3$ to order $1/N$.

Solution The integral on the RHS of Eq. (10.50) looks in $d = 3$ very much like that in Eq. (10.29) and can easily be calculated to the leading order in $1/N$ when only the region of integration over large momenta with $|k| \gtrsim |p|_{\max} \equiv \max\{|p_1|, |p_2|\}$ is essential to this accuracy.

Let us first note that the coefficient in front of the integral is $\propto \gamma_{\rm v} \sim 1/N$, so that one is interested in the term $\sim 1/\gamma_{\rm v}$ in the integral for the vertex to be of order 1. This term comes from the region of integration with $|k| \gtrsim |p|_{\max}$. Recalling that $|p_1 - p_2| \lesssim |p|_{\max}$ in Euclidean space, one obtains

$$\int \frac{\mathrm{d}^3 k}{2\pi} \frac{\hat{k} + \hat{p}_1}{[(k+p_1)^2]^{1+\gamma_\chi/2}} \frac{\hat{k} + \hat{p}_2}{[(k+p_2)^2]^{1+\gamma_\chi/2}} \frac{1}{|k|^{1+2\gamma_\psi - \gamma_\chi/2}}$$
$$= \int_{p^2_{\max}}^{\infty} \frac{\mathrm{d}k^2}{[k^2]^{1+\gamma_{\rm v}}} = \frac{1}{\gamma_{\rm v}\, (p^2_{\max})^{\gamma_{\rm v}}}, \tag{10.51}$$

where Eq. (10.47) has been used and

$$\Gamma(p_1, p_2) = \left(\frac{\mu^2}{p^2_{\max}} \right)^{\gamma_{\rm v}}. \tag{10.52}$$

While the integral in Eq. (10.51) is divergent in the ultraviolet for $\gamma_{\rm v} < 0$, this divergence disappears after the renormalization.

Equation (10.30) is reproduced by Eq. (10.51) when expanding in $1/N$. This dependence of the three-vertex solely on the largest momentum is typical for logarithmic theories in the ultraviolet region where one can set, say, $p_1 = 0$ without changing the integral with logarithmic accuracy. This is valid if the integral is quickly convergent in infrared regions which it is in our case.

Remark on broken scale invariance

Scale (and conformal) invariance at a fixed point $g = g_*$ holds only for large momenta $|p| \gg m$. For smaller values of momenta, scale invariance is broken by masses. In fact, any dimensional parameter breaks scale invariance. If the bare coupling g is chosen in the vicinity of g_* according to Eq. (10.39), then scale invariance holds even in the massless case only for $|p| \gg \mu$, while it is broken if $|p| \lesssim \mu$.

10.3 Functional methods for φ^4 theory

The large-N solution of the $O(N)$ vector models, which is given by the sum of the bubble graphs, can be obtained alternatively by evaluating the path integral at large N using the saddle-point method. We shall restrict ourselves to the scalar $O(N)$-symmetric φ^4 theory, while the analysis of the four-Fermi theory is quite analogous.

The action of the $O(N)$-symmetric φ^4 theory is given by

$$S[\varphi^a] = \int \mathrm{d}^d x \left[\frac{1}{2}\left(\partial_\mu \varphi^a\right)^2 + \frac{1}{2} m^2 \varphi^a \varphi^a + \frac{\lambda}{8}\left(\varphi^a \varphi^a\right)^2\right], \tag{10.53}$$

where $\varphi^a = \left(\varphi^1, \ldots, \varphi^N\right)$. The coupling λ in the action (10.53) must be positive for the theory to be well-defined. The vertices of Feynman diagrams are associated with $-\lambda$.

Problem 10.7 Calculate the one-loop Gell-Mann–Low function of the $O(N)$-symmetric φ^4 theory in $d = 4$.

Solution The corresponding diagrams are similar to those of Fig. 10.1, though now the arrows are not essential since the field is real. The diagrams are logarithmically divergent in four dimensions. Each diagram contributes with a positive sign, while the diagram in Fig. 10.1b now has an extra combinatoric factor of 1/2. The diagrams in Fig. 10.2 result in a mass renormalization and there is no wave-function renormalization of the φ-field in one loop so that one obtains

$$\mathcal{B}(\lambda) = \frac{(N+8)\lambda^2}{16\pi^2}. \tag{10.54}$$

The positive sign in this formula is the same as for QED and is associated with the "triviality" of the φ^4 theory in four dimensions. It is also worth noting that the coefficient $(N + 8)$ is large even for $N = 1$.

Introducing the auxiliary field $\chi(x)$ as in Sect. 10.1, the action (10.53) can be rewritten as

$$S[\varphi^a, \chi] = \int \mathrm{d}^d x \left[\frac{1}{2}\varphi^a \left(-\partial_\mu^2 + m^2 + \chi\right)\varphi^a - \frac{\chi^2}{2\lambda}\right]. \tag{10.55}$$

The two forms are equivalent owing to the equation of motion

$$\boldsymbol{\chi} = \frac{\lambda}{2} :\boldsymbol{\varphi}^a \boldsymbol{\varphi}^a: . \tag{10.56}$$

In other words, χ is again a composite field.

The correlators of φ and χ are determined by the generating functional

$$Z\left[J^a, K\right] = \int_{\uparrow} \mathcal{D}\chi(x) \int \mathcal{D}\varphi^a(x) \times \exp\left\{-S[\varphi^a, \chi] + \int \mathrm{d}^d x\, J^a(x)\varphi^a(x) + \int \mathrm{d}^d x\, K(x)\chi(x)\right\}, \tag{10.57}$$

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which is a functional of the sources J^a and K for the fields φ^a and χ and extends Eq. (2.49).

To make the path integral over $\chi(x)$ in Eq. (10.57) convergent, at each point x we integrate over a contour that is parallel to the imaginary axis. This is specific to the Euclidean formulation. The propagator of the χ-field in the Gaussian approximation reads

$$D_0(p) \;=\; \langle\, \chi(-p)\chi(p)\,\rangle_{\text{Gauss}} \;=\; -\lambda\,, \tag{10.58}$$

which reproduces the four-boson vertex of perturbation theory.

Since the integral over φ^a is Gaussian, it can be expressed via the Green function

$$G(x,y;\chi) \;=\; \left\langle y \left| \frac{1}{-\partial_\mu^2 + m^2 + \chi} \right| x \right\rangle \tag{10.59}$$

as

$$\begin{aligned} Z\left[J^a, K\right] \;=\; \int_\uparrow \mathcal{D}\chi(x)\, \exp\bigg\{ &\int \mathrm{d}^d x\, \frac{\chi^2}{2\lambda} \\ &+ \frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, J^a(x)\, G(x,y;\chi)\, J^a(y) \\ &+ \int \mathrm{d}^d x\, K(x)\, \chi(x) - \frac{N}{2}\,\mathrm{Tr}\ln G^{-1}[\chi] \bigg\}\,. \end{aligned} \tag{10.60}$$

Here we have used the obvious notation

$$G^{-1}[\chi] \;=\; -\partial_\mu^2 + m^2 + \chi\,. \tag{10.61}$$

It will also be convenient to use the short-hand notation

$$g \circ f \;=\; \langle g|f\rangle \;\equiv\; \int \mathrm{d}^d x\, f(x) g(x)\,. \tag{10.62}$$

Then, Eq. (10.60) can be rewritten as

$$\begin{aligned} Z\left[J^a, K\right] \;=\; \int_\uparrow \mathcal{D}\chi(x)\, \exp\bigg\{ &\frac{\chi \circ \chi}{2\lambda} + \frac{1}{2} J^a \circ G[\chi] \circ J^a \\ &+ K \circ \chi - \frac{N}{2}\,\mathrm{Tr}\ln G^{-1}[\chi] \bigg\}\,. \end{aligned} \tag{10.63}$$

The exponent in Eq. (10.63) is $\mathcal{O}(N)$ at large N so the path integral can be evaluated as $N \to \infty$ by the saddle-point method. The saddle-point field configuration $\chi(x) = \chi_{\text{sp}}(x)$ is determined (implicitly) by the saddle-point equation

$$\chi_{\rm sp}(x) - \frac{\lambda N}{2} G(x,x;\chi_{\rm sp}) \\ + \frac{\lambda}{2} J^a \circ G(\cdot,x;\chi_{\rm sp})\, G(x,\cdot;\chi_{\rm sp}) \circ J^a + \lambda K(x) \;=\; 0\,. \tag{10.64}$$

If $K \sim 1/\lambda$, each term here is $\mathcal{O}(1)$ since

$$\lambda \;\sim\; \frac{1}{N} \tag{10.65}$$

in analogy with Eq. (10.14).

When the sources J^a and K vanish so that the last two terms on the LHS of Eq. (10.64) equal zero, this equation reduces to

$$\chi_{\rm sp} - \frac{\lambda N}{2} G(x,x;\chi_{\rm sp}) \;=\; 0\,. \tag{10.66}$$

Its solution is x-independent owing to translational invariance and can be parametrized as

$$\chi_{\rm sp} \;=\; m_{\rm R}^2 - m^2\,, \tag{10.67}$$

where m and $m_{\rm R}$ are the bare and renormalized mass, respectively. Equation (10.66) then reduces to the standard formula [Wil73]

$$m^2 \;=\; m_{\rm R}^2 - \frac{\lambda N}{2} \int^{\Lambda} \frac{{\rm d}^d k}{(2\pi)^d} \frac{1}{\left(k^2 + m_{\rm R}^2\right)} \tag{10.68}$$

for the mass renormalization at large N.

To take into account fluctuations around the saddle point, we expand

$$\chi(x) \;=\; \chi_{\rm sp} + \delta\chi(x)\,, \tag{10.69}$$

where

$$\delta\chi(x) \;\sim\; \sqrt{\lambda} \;\sim\; N^{-1/2}\,. \tag{10.70}$$

The Gaussian integration over $\delta\chi(x)$ determines the pre-exponential factor in (10.63).

To construct the $1/N$ expansion of the generating functional (10.63), it is convenient to use the generating functional for connected Green functions, which was already introduced in Eq. (2.52). It is usually denoted by $W[J^a,K]$ and is related to the partition function (10.57) by

$$Z[J^a,K] \;=\; {\rm e}^{W[J^a,K]}\,. \tag{10.71}$$

Then we find

$$W[J^a, K] = \frac{1}{2\lambda}\chi_{\rm sp}\circ\chi_{\rm sp} - \frac{N}{2}\,{\rm Tr}\ln G^{-1}[\chi_{\rm sp}] + \frac{1}{2}J^a\circ G\,[\chi_{\rm sp}]\circ J^a + K\circ\chi_{\rm sp} - \frac{1}{2}\,{\rm Tr}\ln\left(\lambda D^{-1}[\chi_{\rm sp}]\right) + \mathcal{O}\left(N^{-1}\right), \tag{10.72}$$

where

$$D^{-1}(x,y;\chi) = -\frac{1}{\lambda}\delta^{(d)}(x-y) - \frac{N}{2}\,G(x,y;\chi)\,G(y,x;\chi) + J^a\circ G(\cdot,x;\chi)\,G(x,y;\chi)\,G(y,\cdot;\chi)\circ J^a\,. \tag{10.73}$$

This operator emerges when integrating over the Gaussian fluctuations around the saddle point. The corresponding (last displayed) term on the RHS of Eq. (10.72) is associated with the pre-exponential factor and, therefore, is ~ 1.

The next terms of the $1/N$ expansion can be calculated in a systematic way by substituting (10.69) in Eq. (10.63) and performing the perturbative expansion in $\delta\chi$.

If the sources J^a and K vanish so that the saddle-point value $\chi_{\rm sp}$ is given by the constant (10.67), then the RHS of Eq. (10.73) simplifies to

$$D^{-1}(x,y;\chi_{\rm sp}) = -\frac{1}{\lambda}\delta^{(d)}(x-y) - \frac{N}{2}\,G(x,y;\chi_{\rm sp})\,G(y,x;\chi_{\rm sp})\,. \tag{10.74}$$

Remembering the definition (10.59) of G and passing to the momentum-space representation, we obtain

$$D^{-1}(p) = -\frac{1}{\lambda} - \frac{N}{2}\int\frac{{\rm d}^dk}{(2\pi)^d}\,\frac{1}{\left(k^2+m_{\rm R}^2\right)\left[(k+p)^2+m_{\rm R}^2\right]}\,. \tag{10.75}$$

The sign of the first term on the RHS is consistent with Eq. (10.58).

Equation (10.75) is analogous to Eq. (10.16) in the fermionic case and can be obtained alternatively by summing bubble graphs of the type shown in Fig. 10.3 for

$$D(p) = \langle\,\chi(-p)\chi(p)\,\rangle\,. \tag{10.76}$$

The extra symmetry factor of $1/2$ in Eq. (10.75) is the usual combinatoric one for bosons. Therefore, the large-N saddle-point calculation of the propagator (10.76) results precisely in the zeroth order of the $1/N$-expansion.

We see from Eq. (10.72) the difference between perturbation theory and the $1/N$-expansion. The perturbation theory in λ can be constructed as an expansion (10.69) around the saddle point $\chi_{\rm sp}$ given again by Eq. (10.64), with the omitted second term on the LHS, which is now justified by the fact that λ is small (even for $N \sim 1$). The second term on the RHS of Eq. (10.72), which is associated with a one-loop diagram, appears in perturbation theory as a result of Gaussian fluctuations around this saddle point.

Remark on the effective action

The effective action is a functional of the mean values of fields

$$\varphi^a_{\rm cl}(x) \;=\; \frac{\delta W}{\delta J^a(x)}\,, \qquad \chi_{\rm cl}(x) \;=\; \frac{\delta W}{\delta K(x)} \tag{10.77}$$

in the presence of the external sources. The effective action is defined as the Legendre transformation of $W[J^a,K]$ by

$$\Gamma[\varphi^a_{\rm cl},\chi_{\rm cl}] \;\equiv\; -W + J^a\circ\varphi^a_{\rm cl} + K\circ\chi_{\rm cl}\,, \tag{10.78}$$

where the sources J^a and K, which are regarded as functionals of $\varphi^a_{\rm cl}$ and $\chi_{\rm cl}$, are to be determined by an inversion of Eq. (10.77). To the leading order in $1/N$ we obtain

$$\left.\begin{aligned} J^a(x) \;&=\; G^{-1}[\chi_{\rm cl}]\,\varphi^a_{\rm cl}(x) + \mathcal{O}\big(N^{-1}\big)\,, \\ \chi_{\rm cl}(x) \;&=\; \chi_{\rm sp}(x) + \mathcal{O}\big(N^{-1}\big)\,. \end{aligned}\right\} \tag{10.79}$$

When Eq. (10.79) (with the $1/N$ correction included) is substituted into Eq. (10.78) and account is taken of the $1/N$ terms, most of them cancel and we arrive at the relatively simple formula

$$\begin{aligned} \Gamma[\varphi^a_{\rm cl},\chi_{\rm cl}] \;=\; & -\frac{1}{2\lambda}\chi_{\rm cl}\circ\chi_{\rm cl} + \frac{N}{2}\,{\rm Tr}\ln G^{-1}[\chi_{\rm cl}] \\ & + \frac{1}{2}\varphi^a_{\rm cl}\circ G^{-1}[\chi_{\rm cl}]\circ\varphi^a_{\rm cl} + \frac{1}{2}\,{\rm Tr}\ln\big(\lambda D^{-1}[\chi_{\rm cl}]\big) + \mathcal{O}\big(N^{-1}\big)\,, \end{aligned} \tag{10.80}$$

where

$$\begin{aligned} D^{-1}(x,y;\chi_{\rm cl}) \;=\; & -\frac{1}{\lambda}\delta^{(d)}(x-y) - \frac{N}{2}\,G(x,y;\chi_{\rm cl})\,G(y,x;\chi_{\rm cl}) \\ & + \varphi^a_{\rm cl}(x)\,G(x,y;\chi_{\rm cl})\,\varphi^a_{\rm cl}(y) \end{aligned} \tag{10.81}$$

coinciding with (10.73) to the leading order in $1/N$.

The second and fourth terms on the RHS of Eq. (10.80), which involve Tr, are associated with one-loop diagrams of the fields φ^a and χ, respectively, in the classical background fields $\varphi^a_{\rm cl}$ and $\chi_{\rm cl}$. Higher orders in $1/N$

are given by diagrams which are one-particle irreducible with respect to both φ and χ.

It follows immediately from the definitions (10.77) and (10.78) that

$$\frac{\delta\Gamma}{\delta\varphi^a_{\rm cl}(x)} = J^a(x)\,, \qquad \frac{\delta\Gamma}{\delta\chi_{\rm cl}(x)} = K(x)\,. \tag{10.82}$$

Therefore, $\varphi^a_{\rm cl}(x)$ and $\chi_{\rm cl}(x)$ are determined in the absence of external sources by the equations

$$\frac{\delta\Gamma[\varphi^a_{\rm cl},\chi_{\rm cl}]}{\delta\varphi^b_{\rm cl}(x)} = 0\,, \qquad \frac{\delta\Gamma[\varphi^a_{\rm cl},\chi_{\rm cl}]}{\delta\chi_{\rm cl}(x)} = 0\,. \tag{10.83}$$

Substituting (10.80) into Eqs. (10.83), we get to the leading order in $1/N$, respectively, the equations

$$\left[-\partial^2_\mu + m^2 + \chi_{\rm cl}(x)\right]\varphi^a_{\rm cl}(x) = 0 \tag{10.84}$$

and

$$\chi_{\rm cl}(x) = \frac{\lambda}{2}\varphi^a_{\rm cl}(x)\varphi^a_{\rm cl}(x) + \frac{\lambda N}{2}G(x,x;\chi_{\rm cl})\,. \tag{10.85}$$

The first equation is just a classical equation of motion in an external field $\chi_{\rm cl}(x)$, while the second one is just the average of the (quantum) equation (10.56). Equation (10.85) is often called the *gap equation*.

A solution to Eqs. (10.84) and (10.85) depends on what initial (or boundary) conditions are imposed.

Problem 10.8 Find translationally invariant solutions to Eqs. (10.84) and (10.85) and calculate the corresponding effective potential.

Solution The effective potential $V(\varphi^a_{\rm cl},\chi_{\rm cl})$ is defined via the integrand in the effective action $\Gamma[\varphi^a_{\rm cl},\chi_{\rm cl}]$ for translationally invariant

$$\varphi^a_{\rm cl}(x) = \bar\varphi^a\,, \qquad \chi_{\rm cl} = \bar\chi\,, \tag{10.86}$$

i.e. it is given by Γ divided by the volume of Euclidean space. From Eq. (10.80), at large N we find

$$V = -\frac{1}{2\lambda}\bar\chi^2 + \frac{N}{2}\int^\Lambda \frac{{\rm d}^d k}{(2\pi)^d}\ln\left(k^2+m^2+\bar\chi\right) + \frac{1}{2}\left(m^2+\bar\chi\right)\bar\varphi^2\,, \tag{10.87}$$

which obviously recovers Eqs. (10.84) and (10.85) after varying with respect to constant $\bar\varphi^a$ and $\bar\chi$.

It is convenient to perform renormalization by introducing, in $d=4$, the renormalized coupling $\lambda_{\rm R}$ given by

$$\frac{1}{\lambda_{\rm R}} = \frac{1}{\lambda} + \frac{1}{2}\int^\Lambda \frac{{\rm d}^4 k}{(2\pi)^4}\frac{1}{k^2\left(k^2+m^2_{\rm R}\right)} = \frac{1}{\lambda} + \frac{N}{32\pi^2}\ln\frac{\Lambda^2}{m^2_{\rm R}} \tag{10.88}$$

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and $\bar{\chi}_{\mathrm{R}} = \bar{\chi} + m^2$. Assuming that $\bar{\chi}_{\mathrm{R}} \ll \Lambda^2$ (also $m_{\mathrm{R}} \ll \Lambda$ as usual) and representing Eq. (10.68) in the form

$$\frac{m_{\mathrm{R}}^2}{\lambda_{\mathrm{R}}} = \frac{m^2}{\lambda} - \frac{N}{32\pi^2}\Lambda^2 \,, \tag{10.89}$$

we rewrite Eqs. (10.84) and (10.85) as [Sch74, CJP74]

$$\bar{\chi}_{\mathrm{R}}\bar{\varphi}^a = 0 \tag{10.90}$$

and

$$\bar{\chi}_{\mathrm{R}}\left(1 - \frac{\lambda_{\mathrm{R}} N}{32\pi^2}\ln\frac{\bar{\chi}_{\mathrm{R}}}{m_{\mathrm{R}}^2}\right) = m_{\mathrm{R}}^2 + \frac{\lambda_{\mathrm{R}}}{2}\bar{\varphi}^2 \,. \tag{10.91}$$

Equation (10.87) then gives the renormalized effective potential

$$V_{\mathrm{R}} = -\frac{1}{2\lambda_{\mathrm{R}}}\bar{\chi}_{\mathrm{R}}^2 + \frac{m_{\mathrm{R}}^2\bar{\chi}_{\mathrm{R}}}{\lambda_{\mathrm{R}}} + \frac{N}{64\pi^2}\bar{\chi}_{\mathrm{R}}^2\left(-\frac{1}{2} + \ln\frac{\bar{\chi}_{\mathrm{R}}}{m_{\mathrm{R}}^2}\right) + \frac{1}{2}\bar{\chi}_{\mathrm{R}}\bar{\varphi}^2 \,, \tag{10.92}$$

which obviously reproduces Eqs. (10.90) and (10.91).

Equations (10.90) and (10.91) possess the solutions

$$\bar{\varphi}^a = 0 \,, \qquad \bar{\chi}_{\mathrm{R}} = m_{\mathrm{R}}^2 \qquad \text{for } m_{\mathrm{R}}^2 > 0 \,, \tag{10.93}$$

$$\bar{\varphi}^2 = -\frac{2m_{\mathrm{R}}^2}{\lambda_{\mathrm{R}}} \,, \qquad \bar{\chi}_{\mathrm{R}} = 0 \qquad \text{for } m_{\mathrm{R}}^2 < 0 \,. \tag{10.94}$$

The first of them is associated with an unbroken $O(N)$ symmetry, while the second one corresponds to a spontaneous breaking of $O(N)$ down to $O(N-1)$. Both formulas look like the proper tree-level ones, while the only effect of loop corrections at large N is the renormalization of the coupling constant and mass.

A subtle point is the question of the stability of these solutions. For small deviations of $\bar{\varphi}^2$ from the mean value given by Eqs. (10.93) and (10.94), the effective potential V_{R} is a monotonically increasing function of $\bar{\varphi}^2$, as can be shown for $\lambda_{\mathrm{R}} N < 32\pi^2$ by eliminating the auxiliary field $\bar{\chi}_{\mathrm{R}}$ from Eq. (10.92) by solving the gap equation (10.91) iteratively in $\bar{\varphi}^2$, and the solutions are locally stable. Both solutions are, however, unstable globally with respect to large fluctuations of the fields. This can be seen by eliminating $\bar{\varphi}^2$ from V_{R} by solving the gap equation (10.91) for $\bar{\varphi}^2$ which yields

$$V_{\mathrm{R}} = \frac{1}{2}\bar{\chi}_{\mathrm{R}}^2\left(\frac{1}{\lambda_{\mathrm{R}}} - \frac{N}{32\pi^2}\ln\frac{\bar{\chi}_{\mathrm{R}}}{m_{\mathrm{R}}^2}\right) - \frac{N}{128\pi^2}\bar{\chi}_{\mathrm{R}}^2 \,. \tag{10.95}$$

This function is monotonically decreasing for very large

$$\bar{\chi}_{\mathrm{R}} > m_{\mathrm{R}}^2\,\mathrm{e}^{32\pi^2/\lambda_{\mathrm{R}} N} \,, \tag{10.96}$$

where the theory becomes unstable. This is related to the usual problem of "triviality" of the φ^4 theory, which makes sense only for small couplings $\lambda_{\mathrm{R}} N$ as an effective theory and cannot be fundamental at very small distances of the order of

$$r \sim m_{\mathrm{R}}^{-1}\,\mathrm{e}^{-16\pi^2/\lambda_{\mathrm{R}} N} \,. \tag{10.97}$$

Problem 10.9 Find a solution to Eqs. (10.84) and (10.85) which decreases exponentially as

$$\varphi^a_{\rm cl}(x) \;=\; \xi^a m_{\rm R}\,{\rm e}^{m_{\rm R}\tau} \qquad \text{for} \;\; \tau \to -\infty\,, \tag{10.98}$$

where $\tau \equiv x_4$ and ξ^a is an $O(N)$ vector.

Solution The difference with respect to the previous Problem is that $\varphi_{\rm cl}$ is no longer translationally invariant along the time-variable owing to the initial condition (10.98). Let us denote

$$\varphi^a_{\rm cl}(x) \;\equiv\; \Phi^a(\tau)\,, \qquad \chi_{\rm cl}(x) \;\equiv\; v(\tau)\,. \tag{10.99}$$

The the saddle-point equations (10.84) and (10.85) can be rewritten as

$$\left[-D^2 + m^2 + v(\tau)\right]\Phi^a(\tau) \;=\; 0 \tag{10.100}$$

and

$$v(\tau) \;=\; \frac{\lambda}{2}\Phi^a(\tau)\Phi^a(\tau) + \frac{\lambda N}{2}\int \frac{{\rm d}^3 k}{(2\pi)^3} G_\omega(\tau,\tau;v)\,, \tag{10.101}$$

where

$$D \;\equiv\; \frac{\rm d}{{\rm d}\tau}\,, \qquad \omega \;=\; \sqrt{k^2+m^2} \tag{10.102}$$

and we have introduced the Fourier image of the Green function (10.59)

$$\begin{aligned} G_\omega(\tau,\tau;v) \;&\equiv\; \int {\rm d}^3\vec{x}\,{\rm e}^{{\rm i}\vec{k}\vec{x}}\, G\big((\tau,\vec{x}),(\tau,\vec{0});v\big) \\ &=\; \left\langle \tau \left| \frac{1}{-D^2+\omega^2+v} \right| \tau \right\rangle \end{aligned} \tag{10.103}$$

with respect to the spatial coordinate.

The solution to Eqs. (10.100) and (10.101) can be easily found to be

$$\Phi^a(\tau) \;=\; \frac{\xi^a m_{\rm R}\,{\rm e}^{m_{\rm R}\tau}}{1-\frac{\bar\lambda_{\rm R}\xi^2}{16}\,{\rm e}^{2m_{\rm R}\tau}}\,, \qquad v(\tau) \;=\; \frac{\bar\lambda_{\rm R}}{2}\Phi^a(\tau)\Phi^a(\tau)\,, \tag{10.104}$$

where the renormalized coupling

$$\bar\lambda_{\rm R} \;=\; \frac{\lambda_{\rm R}}{1+\frac{\lambda_{\rm R}N}{16\pi^2}} \tag{10.105}$$

differs from Eq. (10.88) only by an additional final renormalization and the renormalized mass $m_{\rm R}$ is defined in Eq. (10.68). This solution is nontrivial for $\xi^2 \sim N$ and obviously satisfies the initial condition (10.98).

The solution is so simple because the diagonal resolvent (10.103) takes the very simple form

$$G_\omega(\tau,\tau;v) \;=\; \frac{1}{2\omega} - \frac{v\,(\tau)}{4\omega(\omega^2-m_{\rm R}^2)} \tag{10.106}$$

for the potential $v(\tau)$ given by Eq. (10.104). This can be verified by substituting into the Gel'fand–Dikii equation (1.127) with $\mathcal{G}=1$. This is a feature of an integrable potential, which was already discussed in Problem 4.4 on p. 73.

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The function $\Phi^a(\tau)$ given by Eq. (10.104) describes large-N amplitudes of multiparticle production at a threshold [Mak94].

10.4 Nonlinear sigma model

The nonlinear $O(N)$ sigma model* in two Euclidean dimensions is defined by the partition function

$$Z = \int \mathcal{D}\vec{n}\, \delta\left(\vec{n}^2 - \frac{1}{g^2}\right) \exp\left[-\frac{1}{2}\int \mathrm{d}^2x\, (\partial_\mu \vec{n})^2\right], \tag{10.107}$$

where $\vec{n} = (n_1, \ldots, n_N)$ is an $O(N)$ vector. While the action is pure Gaussian, the model is not free owing to the constraint

$$\vec{n}^2(x) = \frac{1}{g^2}, \tag{10.108}$$

which is imposed on the $\vec{n}$ field via the (functional) delta-function.

The sigma model in $d = 2$ is sometimes considered as a toy model for QCD since it possesses:

(1) asymptotic freedom [Pol75];
(2) instantons for $N = 3$ [BP75].

The action in Eq. (10.107) is $\sim N$ as $N \to \infty$ but the entropy, i.e. the contribution from the measure of integration, is also $\sim N$ so that a straightforward saddle point is not applicable.

To overcome this difficulty, we proceed as in the previous section, introducing an auxiliary field $u(x)$, which is ~ 1 as $N \to \infty$, and rewrite the partition function (10.107) as

$$Z \propto \int_{\uparrow} \mathcal{D}u(x) \int \mathcal{D}\vec{n}(x) \exp\left\{-\frac{1}{2}\int \mathrm{d}^2x \left[(\partial_\mu \vec{n})^2 - u\left(\vec{n}^2 - \frac{1}{g^2}\right)\right]\right\}, \tag{10.109}$$

where the contour of integration over $u(x)$ is parallel to the imaginary axis.

Performing the Gaussian integration over $\vec{n}$, we find

$$Z \propto \int_{\uparrow} \mathcal{D}u(x) \exp\left\{-\frac{N}{2}\mathrm{Tr}\ln\left[-\partial_\mu^2 + u(x)\right] + \frac{1}{2g^2}\int \mathrm{d}^2x\, u(x)\right\}. \tag{10.110}$$

* The name comes from elementary particle physics where a nonlinear sigma model in four dimensions is used as an effective Lagrangian for describing low-energy scattering of the Goldstone π-mesons.

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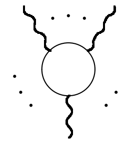
The first term in the exponent is as before nothing but the sum of one-loop diagrams in two dimensions,

$$\frac{N}{2}\mathrm{Tr}\ln\left[-\partial_\mu^2+u(x)\right] \;=\; \sum_n \frac{1}{n}$$



$$, \qquad (10.111)$$

where the auxiliary field u is again denoted by a wavy line. Equation (10.110) looks very much like Eq. (10.63) if we set $J^a = K = 0$. The difference is that the exponent in (10.110) involves the term which is linear in u, while the analogous term in (10.63) is quadratic in χ.

Now the path integral over $u(x)$ in Eq. (10.110) is a typical saddle-point one: the action $\sim N$, while the entropy ~ 1 since only one integration over u is left. The saddle-point equation for the nonlinear sigma model

$$\frac{1}{g^2} - NG(x,x;u_{\mathrm{sp}}) \;=\; 0 \qquad (10.112)$$

is quite analogous to Eq. (10.66) for the φ^4 theory, while G is defined by

$$G(x,y;u) \;=\; \left\langle y\left|\frac{1}{-\partial_\mu^2+u}\right|x\right\rangle, \qquad (10.113)$$

which is an analog of Eq. (10.59).

Introducing sources for the $\vec{n}$ and u fields, we can derive the analogs of Eqs. (10.84) and (10.85) for φ^4 theory which are given for the sigma model by

$$\left[-\partial_\mu^2+u_{\mathrm{cl}}(x)\right]\vec{n}_{\mathrm{cl}}(x) \;=\; 0\,, \qquad (10.114)$$

and

$$\frac{1}{g^2} \;=\; \vec{n}^2_{\mathrm{cl}}(x) + NG(x,x;u_{\mathrm{cl}})\,. \qquad (10.115)$$

For a translationally invariant solution when $\vec{n}_{\mathrm{cl}}(x)=0$ and $u_{\mathrm{cl}}(x)=u_{\mathrm{sp}}$, we recover Eq. (10.112).

The coupling g^2 in Eq. (10.112) is $\sim 1/N$, as prescribed by the constraint (10.108), which involves a sum over N terms on the LHS. This guarantees that a solution to Eq. (10.112) exists. Next orders of the $1/N$-expansion for the two-dimensional sigma model can be constructed analogously to the previous section.

The $1/N$-expansion of the two-dimensional nonlinear sigma model has many advantages over perturbation theory, which is usually constructed

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by solving the constraint (10.108) explicitly, say, by choosing

$$n_N = \frac{1}{g}\sqrt{1 - g^2 \sum_{a=1}^{N-1} n_a^2} \tag{10.116}$$

and expanding the square root in g^2. Only $N-1$ dynamical degrees of freedom are left so that the $O(N)$-symmetry is broken in perturbation theory down to $O(N-1)$. The particles in perturbation theory are massless (like Goldstone bosons) and it suffers from infrared divergences.

In contrast, the solution to Eq. (10.112) has the form

$$u_{\rm sp} = m_{\rm R}^2 \equiv \Lambda^2 \, {\rm e}^{-4\pi/Ng^2}\,, \tag{10.117}$$

where Λ is an ultraviolet cutoff. Therefore, all N particles acquire the same mass $m_{\rm R}$ in the $1/N$-expansion so that the $O(N)$ symmetry is restored. This appearance of mass is a result of the dimensional transmutation which says in this case that the parameter $m_{\rm R}$ rather than the renormalized coupling constant $g_{\rm R}^2$ is observable. The emergence of the mass cures the infrared problem.

Problem 10.10 Show that (10.117) is a solution to Eq. (10.112).

Solution Let us look for a translationally invariant solution $u_{\rm sp}(x) = m_{\rm R}^2$. Then Eq. (10.112) in the momentum space gives

$$\begin{aligned} \frac{1}{g^2} &= N \int^{\Lambda} \frac{{\rm d}^2 k}{(2\pi)^2} \frac{1}{k^2 + m_{\rm R}^2} = \frac{N}{4\pi} \int_0^{\Lambda^2} \frac{{\rm d}k^2}{k^2 + m_{\rm R}^2} \\ &= \frac{N}{4\pi} \ln \frac{\Lambda^2}{m_{\rm R}^2}\,. \end{aligned} \tag{10.118}$$

The exponentiation results in Eq. (10.117).

Equation (10.118) relates the bare coupling g^2 and the cutoff Λ and allows us to calculate the Gell-Mann–Low function, yielding

$$\mathcal{B}(g^2) \equiv \Lambda \frac{{\rm d}g^2}{{\rm d}\Lambda} = -\frac{Ng^4}{2\pi}\,. \tag{10.119}$$

The analogous one-loop perturbation-theory formula for any N is given by [Pol75]

$$\mathcal{B}(g^2) = -\frac{(N-2)g^4}{2\pi}\,. \tag{10.120}$$

Thus, the sigma model is asymptotically free in two dimensions for $N > 2$, which is the origin of the dimensional transmutation. There is no asymptotic freedom for $N = 2$ since $O(2)$ is Abelian.

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10.5 Large-N factorization in vector models

The fact that a path integral has a saddle point at large N implies a very important feature of large-N theories – the factorization. It is a general property of the large-N limit and holds not only for the $O(N)$ vector models. However, it is useful to illustrate it by these solvable examples.

The factorization at large N holds for averages of *singlet* operators, for example

$$\begin{aligned}\langle\, u(x_1)\cdots u(x_k)\,\rangle \\ \equiv\ & Z^{-1}\int_{\uparrow}\mathcal{D}u\,\exp\left[-\frac{N}{2}\mathrm{Tr}\ln\left(-\partial_\mu^2+u\right)+\frac{1}{2g^2}\int \mathrm{d}^2x\,u\right] \\ & \times\ u(x_1)\cdots u(x_k) \end{aligned} \tag{10.121}$$

in the two-dimensional sigma model.

Since the path integral has a saddle point at some configuration $u(x) = u_{\rm sp}(x)$ (which is, in fact, x-independent owing to translational invariance), we obtain to the leading order in $1/N$:

$$\langle\, u(x_1)\cdots u(x_k)\,\rangle \ =\ u_{\rm sp}(x_1)\cdots u_{\rm sp}(x_k)+\mathcal{O}\big(N^{-1}\big)\,, \tag{10.122}$$

which can be written in the factorized form

$$\langle\, u(x_1)\cdots u(x_k)\,\rangle \ =\ \langle\, u(x_1)\,\rangle\cdots\langle\, u(x_k)\,\rangle+\mathcal{O}\big(N^{-1}\big)\,. \tag{10.123}$$

Therefore, u becomes "classical" as $N\to\infty$ in the sense of the $1/N$-expansion. This is an analog of the WKB-expansion in $\hbar = 1/N$. "Quantum" corrections are suppressed as $1/N$.

We shall return to discussing large-N factorization in the next chapter when considering the large-N limit of QCD.

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