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Models of Representations and Langlands Functoriality

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Abstract. In this article we explore the interplay between two generalizations of the Whittaker model, namely the Klyachko models and the degenerate Whittaker models, and two functorial constructions, namely base change and automorphic induction, for the class of unitarizable and ladder representations of the general linear groups.

1 Introduction

Let F be a non-archimedean local field. Let G be a quasi-split reductive group with a Borel subgroup B defined over F. Let U denote the unipotent radical of B and ψ a fixed non-degenerate character of it. A smooth irreducible representation (π, V) of G is said to have a Whittaker model, or to be generic, if there exists a non-trivial linear functional ℓ on V such that $\ell(\pi(u)v) = \psi(u)\ell(v)$ for all $u \in U$ and $v \in V$. The importance of Whittaker models in the theory of automorphic forms cannot be overstated. However, not every irreducible unitarizable representation of G admits a Whittaker model. To overcome this, one needs to consider other models that contain non-generic irreducible representations.

In the current article we focus on representations of the general linear groups and two families of models containing the Whittaker model: the degenerate Whittaker models and the Klyachko models. A degenerate Whittaker model is defined by allowing the character ψ of U in the definition of the Whittaker model to be arbitrary. The degenerate Whittaker models were introduced and studied by A. V. Zelevinsky [30, §8.3]. In particular, he showed that given any irreducible representation of $\mathrm{GL}_n(F)$, the representation admits a degenerate Whittaker model and does so with multiplicity one.

Inspired by the work of A. A. Klyachko for the groups $\mathrm{GL}_n(\mathbb{F}_q)$, where $\mathbb{F}_q$ is the finite field with q elements, M. J. Heumos and S. Rallis introduced the second family of models [9] (see Section 7 for the definitions). Although they provided examples of irreducible representations that do not admit any Klyachko model, there are many non-generic irreducible representations that do. For instance, every unitarizable representation of $\mathrm{GL}_n(F)$ admits a Klyachko model [22, Theorem 3.7]. It was shown that any irreducible representation that admits a Klyachko model admits a unique

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Klyachko model, and with multiplicity one [23, Theorem 1]. Thus, to any irreducible representation π of $\mathrm{GL}_n(F)$ that admits a Klyachko model, we can assign a unique integer between 0 and n indicating the precise Klyachko model it admits. We denote this integer by $r(\pi)$ and call it the Klyachko type of π.

The local Langlands correspondence gives a bijection between the set of equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(F)$ and the set of equivalence classes of n-dimensional Weil–Deligne representations of the Weil group of F. Let E be a finite extension of F of degree d. Denote by $\mathcal{A}_F(n)$ and $\mathcal{A}_E(n)$ the set of all equivalence classes of irreducible representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_n(E)$, respectively. The Weil–Deligne group of E, denoted by W'_E, naturally sits as a subgroup of the Weil–Deligne group of F, W'_F. Via the correspondence, one can assign an irreducible representation of the general linear group over one field to a given irreducible representation of the general linear group over the other, by employing functorial constructions on the corresponding Weil–Deligne representations. In this paper, we deal with two such constructions. The base change map $\mathrm{bc}_{E/F} : \mathcal{A}_F(n) \to \mathcal{A}_E(n)$ is obtained by restricting the corresponding Weil–Deligne representation of the Weil group of F. On the other hand, the automorphic induction map $\mathrm{ai}_{E/F} : \mathcal{A}_E(n) \to \mathcal{A}_F(dn)$ is obtained by inducing the corresponding n-dimensional Weil–Deligne representation of the Weil group of E. J. Arthur and L. Clozel investigated the first map [1], while G. Henniart and R. Herb investigated the second [7].

In this paper, we investigate the effect of base change and automorphic induction on the two generalizations of the Whittaker model, *i.e.*, the degenerate Whittaker models and the Klyachko models, for certain classes of irreducible admissible representations.

1.1 Main Results

We now describe our main findings in more detail. In this section we fix a finite cyclic extension E/F of non-archimedean local fields. We will further suppose here that $d = [E : F]$ is prime, although many of the results in this article are true without this assumption (see Section 9 for details).

Before we state our main results we need to introduce some more terminology. Call an irreducible representation *rigid* if it is supported on a single cuspidal line (see Definition 3.3). For $\pi \in \mathcal{A}_F(n)$, two partitions of the integer n, which we denote by $\mathcal{V}(\pi)$ and $\mathbf{d}(\pi)$, were defined in [24] and [30], respectively (see Definitions 6.3 and 6.1). The partition $\mathcal{V}(\pi)$ is called the $\mathrm{SL}(2)$-type of the representation π. We begin with the following result, which investigates its effect on the $\mathrm{bc}_{E/F}$ and $\mathrm{ai}_{E/F}$ maps.

Theorem 1.1 (See Theorem 6.5) (i) *Let* $\pi \in \mathcal{A}_F(n)$ *be a rigid representation. Then* $\mathcal{V}(\pi) = \mathcal{V}(\mathrm{bc}_{E/F}(\pi))$.

(ii) *Let* $\Pi \in \mathcal{A}_E(n)$ *be a rigid representation. Then* $d\mathcal{V}(\Pi) = \mathcal{V}(\mathrm{ai}_{E/F}(\Pi))$.

The degenerate Whittaker model that an irreducible representation π admits as per the prescription in [30] is with respect to the composition $\mathbf{d}(\pi)$. Using Theorem 1.1 we get the following relationship between degenerate Whittaker models and the two maps.

Theorem 1.2 (See Theorem 6.7) (i) *Let* $\pi \in \mathcal{A}_F(n)$ *be a rigid representation. Then* $\mathrm{bc}_{E/F}(\pi)$ *has a degenerate Whittaker model given by the depth sequence* $\mathbf{d}(\pi)$.

(ii) *Let* $\Pi \in \mathcal{A}_E(n)$ *be a rigid representation. Then* $\mathrm{ai}_{E/F}(\pi)$ *has a degenerate Whittaker model given by the depth sequence* $\underbrace{\mathbf{d}(\pi) +_c \cdots +_c \mathbf{d}(\pi)}_{d-times}$.

(Here the composition $\mathbf{d}(\pi) +_c \cdots +_c \mathbf{d}(\pi)$ denotes the composition of nd obtained by coordinate-wise adding d copies of the composition $\mathbf{d}(\pi)$.)

Ladder representations (see Section 5.1.1 for the definition) are a class of irreducible representations of general linear groups over non-archimedean local fields. The building blocks of the unitarizable dual of the general linear groups, the so-called Speh representations (see Section 5.1.2), constitute a subset of the ladders. Recall from above that to any representation $\pi \in \mathcal{A}_F(n)$ that admits a Klyachko model, we assign a unique integer $r(\pi)$ $(0 \le r(\pi) \le n)$ indicating the precise Klyachko model π admits. Next we have the following relationship between the Klyachko models and the two maps.

Theorem 1.3 (See Theorem 7.5) (i) *Let* $\pi \in \mathcal{A}_F(n)$ *be a ladder representation. Then* π *admits a Klyachko model if and only if* $\mathrm{bc}_{E/F}(\pi)$ *admits one. Moreover,* $r(\mathrm{bc}_{E/F}(\pi)) = r(\pi)$.

(ii) *Let* $\Pi \in \mathcal{A}_E(n)$ *be a ladder representation. Then* Π *admits a Klyachko model if and only if* $\mathrm{ai}_{E/F}(\Pi)$ *admits one. Moreover,* $r(\mathrm{ai}_{E/F}(\Pi)) = dr(\Pi)$.

While Theorem 1.3 shows that the two maps "preserve" the Klyachko type of a representation in the ladder class if it exists, we study yet another indicator of compatibility. Let us consider the case of the base change map and let $\Pi \in \mathcal{A}_E(n)$ be a ladder representation in the image of the map that admits a Klyachko model. Any rigid representation in the fiber of Π also admits a Klyachko model of the same type as Π (by Lemma 4.5 (i) and Theorem 7.4), although there are many non rigid representations in the fiber that admit a different Klyachko model or none at all. Thus, given a representation satisfying the conditions that we imposed on Π above, one might ask what proportion of the representations in its fiber admit the corresponding Klyachko model. Our next result addresses this question. For the sake of simplicity, we only state a special case of our result here, and just for the base change map. We refer the reader to Theorem 8.9 for the result in its full generality and for its automorphic induction analogue.

Theorem 1.4 (See Lemma 8.1, Remark 8.2, Lemma 8.3, and Lemma 8.7) *Suppose that* $\Pi = L(\mathfrak{m})$ *(see Section 3.1.1 for the notation) is a rigid representation of* $\mathrm{GL}_n(E)$ *such that it is in the image of the base change map. Denote by s the size of the multi-set* $\mathfrak{m}$. *Then we have the following.*

(i) *The set* $\mathrm{bc}_{E/F}^{-1}(\Pi)$ *has cardinality* d^s.
(ii) *The representation* Π *is generic if and only if every element in its fiber is so.*
(iii) *Further suppose that* Π *is a ladder representation. If it has a symplectic model, then the number of representations in its fiber that admit a symplectic model is* $d^{s/2}$.

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1.2 Context and Related Works

The class of ladder representations plays an important role in this article. This class of irreducible representations was introduced and studied by E. Lapid and A. Mínguez [15]. They proved several results on the structural properties of the standard modules of ladder representations, which makes them easier to deal with than general irreducible representations. For instance, a very useful tool at one's disposal when working with ladders representations (but one that is not available for general irreducible representations) is an explicit description of their Jacquet modules. This was obtained in [13] using the results of [15]. At the same time the ladder class contains many interesting examples of representations, for instance, the Speh representations, as mentioned earlier. This makes it an ideal class of representations to test the plausibility of conjectures for the entire admissible dual. The classification of ladder representations with respect to the Klyachko models was recently obtained in [19].

The result on compatibility of base change and Klyachko models for the class of unitarizable representations was obtained in [24]. There it was shown that the $\mathrm{SL}(2)$-type of a unitarizable representation is preserved under the operation of base change. This statement was then used to show that Klyachko types of unitarizable representations are invariant under base change.

We obtain here independent proofs of the main results of [24]. We also obtain the corresponding results for the automorphic induction map. We prove the statement about $\mathrm{SL}(2)$-type for all rigid representations. However, in this paper, $\mathrm{SL}(2)$-type does not play a role in the proof of the results underlying the connection between Klyachko models and base change. Instead we directly prove that base change preserves Klyachko type for ladder representations. The fact that any unitarizable representation can be obtained by inducing Speh representations is then used to prove the statement for the unitarizable class.

We remark that in a manner similar to this article, the interplay of models of representations and base change was also studied in [20], where the model in question was a special case of the so-called *linear models* for general linear groups.

1.3 Techniques of the Proofs

Recall that Zelevinsky [30] classified the irreducible representations of the general linear groups in terms of the cuspidal representations. We begin by showing that both the base change and automorphic induction maps are compatible with this classification and commute with the Zelevinsky involution, for the class of rigid representations (see Lemma 4.5 and Proposition 4.6, respectively). We prove Theorem 1.1 using Proposition 4.6. In Lemma 6.4, we then observe that for $\pi \in \mathcal{A}_F$, we have $\mathcal{V}(\pi) = \mathbf{d}(\pi)^t$. This lemma is the non-archimedean analogue of [3, Theorem 2.4.2]. Theorem 1.2 is an easy consequence of Theorem 1.1 and Lemma 6.4.

The classification results obtained in [19] for ladders (Theorem 7.4 in this article) play a critical role in the proof of Theorem 1.3, which is the central result of this article. It follows directly from Theorem 7.4 that whether or not a ladder representation admits a Klyachko model is independent of the cuspidal line, it is supported on and depends only on the "shape" of the representation. Lemma 4.5 says that both these

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maps take a ladder representation to a product of ladders, each having the same shape and supported on pairwise disjoint cuspidal lines. The proof of Theorem 1.3 is based on this fact. Our proofs of the analogous results for the unitarizable class are based on the results for ladders, as described in Section 1.2.

Theorem 7.4 is also the key ingredient of the proof of Theorem 1.4 (and that of the more general Theorem 8.9). We obtain a general description of the fiber of a rigid representation under the two maps in Lemma 8.1 which is then used along with Theorem 7.4 to demonstrate Theorem 1.4.

1.4 Organization of the Paper

The rest of the paper is organized as follows. In Section 2 we set up some general notation, while in Section 3 we review some preliminaries on irreducible representations of $\mathrm{GL}_n(F)$ and Weil–Deligne representations. In Section 4 we formally define the base change and automorphic induction maps using the reciprocity map and prove some basic results used in the sequel, including their compatibility with the Zelevinsky classification. In Section 5 we recall the preliminaries of ladder representations. In Section 6 we demonstrate our results on $\mathrm{SL}(2)$-type, namely Theorem 1.1 (see Theorem 6.5). We then use it to study the relationship of the degenerate Whittaker models with the two maps, and in particular, prove Theorem 1.2 (see Theorem 6.7). In Section 7 we prove Theorem 1.3 (see Theorem 7.5). In Section 8 we analyze the fibers of the two maps with respect to the Klyachko models proving a general version of Theorem 1.4 (see Theorem 8.9) and its automorphic induction analogue. In Section 9 we obtain some of the main results of this paper for an arbitrary finite cyclic extension.

2 Notation

We set some primary notation in this section. More particular notation is defined in the section where it first occurs.

2.1 Let F be a non-archimedean local field,[1] $\mathcal{O}_F$ be the ring of integers of F, $\mathfrak{p}_F$ be the unique prime ideal of $\mathcal{O}_F$, and ϖ_F be a fixed choice of a uniformizer of the prime ideal. Let q_F denote the cardinality of its residue field.

Let $|\cdot|_F : F^\times \to \mathbb{C}^\times$ denote the standard absolute value normalized so that $|\varpi_F|_F = q_F^{-1}$. The character of $\mathrm{GL}_n(F)$ given by $g \mapsto |\det(g)|_F$ is denoted by ν_F. Let W_F and W'_F denote the Weil group and the Weil–Deligne group of F, respectively. The reciprocity map $T_F : W_F \to F^\times$, which is given by the local class field theory, is normalized such that geometric Frobenius elements are mapped to uniformizers. The map T_F defines an isomorphism from the topological abelianization W_F^{ab} of W_F to $F^\times$. The composition $|\cdot|_F \circ T_F$ gives the associated absolute value on W_F that we denote by $\|\cdot\|$. We will also denote it sometimes by $\nu'_F(\cdot)$ when we wish to highlight its analogy with the character ν_F. When the underlying field is clear from the context, we will sometimes write ν and ν' for the characters ν_F and ν'_F, respectively.

[1]This article uses the results of [19] where it was assumed that the characteristic of F was different from two. It was later checked by Omer Offen and the second author that this restriction was not necessary [25]. Thus we freely use here the results of [19] for any non-archimedean local field.

2.2 Classes of Representations

The category of smooth complex-valued representations of a topological group G of finite length will be denoted by $\Pi(G)$. Denote by $\mathcal{A}_F(n)$ the class of all irreducible representations in $\Pi(\mathrm{GL}_n(F))$ and by $\mathcal{A}_F$, the union $\bigsqcup_{n\geq 1}\mathcal{A}_F(n)$. Let $\mathcal{A}_F^\circ(n)$ and $\mathcal{A}_F^u(n)$ be the subset of $\mathcal{A}_F(n)$ consisting of the cuspidal representations and the unitarizable representations, respectively. Furthermore, let $\mathcal{A}_F^\circ$ and $\mathcal{A}_F^u$ denote the corresponding unions.

2.3 The Bernstein–Zelevinsky Product

Set $G = \mathrm{GL}_n(F)$. Let M be the F-points of a standard Levi subgroup of GL_n and let P denote the F-points of the corresponding standard parabolic subgroup. We will denote by $\mathbf{i}_{G,M}$ the normalized parabolic induction functor with respect to P from $\Pi(M)$ to $\Pi(G)$. Let $(n_1,\dots,n_k)$ be a composition of n and let $\pi_i \in \Pi(\mathrm{GL}_{n_i}(F))$, $i = 1,\dots,k$. Assume that $M \cong \prod_{i=1}^k \mathrm{GL}_{n_i}(F)$. Let $\pi := \pi_1 \otimes\cdots\otimes \pi_k$. Then $\pi \in \Pi(M)$. Set $\pi_1 \times\cdots\times \pi_k := \mathbf{i}_{G,M}(\pi)$. The functor $\mathbf{i}_{G,M}$ admits a left adjoint that we will denote by $\mathbf{r}_{M,G}$. For a representation $\pi \in \Pi(G)$, the representation $\mathbf{r}_{M,G}(\pi)$ is known as the normalized Jacquet module of π with respect to P.

2.4 Distinguished Representations

This paper is concerned with distinguished representations in the following sense.

Definition 2.1 Let π be a smooth, complex-valued representation of G and let H be a closed subgroup of G.

- We say that π is H-distinguished if the space $\mathrm{Hom}_H(\pi, 1)$ of H-invariant linear forms on π is non-zero.
- More generally, for a character χ of H, we say that π is (H,χ)-distinguished if the space $\mathrm{Hom}_H(\pi,\chi)$ is non-zero.

2.5 Generic Representations

Denote by U_n the F points of the unipotent radical of the standard Borel subgroup of GL_n. Let ψ be a fixed non-trivial additive character of F. We further denote by ψ_n the character of U_n defined by

$$\psi_n(u) = \psi\Big(\sum_{i=1}^{n-1} u_{i,i+1}\Big), \quad u = (u_{i,j}) \in U_n.$$

Definition 2.2 Let π be an irreducible representation of $\mathrm{GL}_n(F)$. We say that π admits a Whittaker model, or is generic, if it is (U_n,ψ_n)-distinguished.

2.6 We henceforth fix a finite cyclic extension E/F of non-archimedean local fields. We will further suppose that $d = [E:F]$ is a prime integer unless mentioned otherwise. (For instance, in Section 9 we do not make this assumption.) Fix $\kappa = \kappa_{E/F}$ to be a character of $F^\times$ coming from the local class field theory with kernel equal to $\mathrm{N}_{E/F}(E^\times)$,

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where $N_{E/F}$ is the norm map from $E^\times$ to $F^\times$. Observe that $\nu_E(\cdot) = |N_{E/F}(\det(\cdot))|_F$ due to the normalization of the absolute values mentioned above.

For $\pi \in \mathcal{A}_E(n)$ and an element $\gamma \in \mathrm{Gal}(E/F)$, denote the representation $\pi^\gamma \in \mathcal{A}_E(n)$ given by $\pi^\gamma(g) = \pi(\gamma(g))$ for all $g \in \mathrm{GL}_n(E)$.

2.7 Multi-sets and Partitions

Denote by $\mathbf{1}_\Omega$ the characteristic function of a set Ω. Let $\mathrm{MS}_{\mathrm{fin}(\Omega)}$ be the set of finite multi-sets of elements in Ω, *i.e.*, the set of functions $f : \Omega \to \mathbb{Z}_{\geq 0}$ of finite support. When convenient, we will also denote f by $\{\omega_1, \dots, \omega_1, \omega_2, \dots, \omega_2, \dots\}$, where $\omega \in \Omega$ is repeated $f(\omega)$ times. Let $\mathcal{P} = \mathrm{MS}_{\mathrm{fin}(\mathbb{Z}_{>0})}$ be the set of partitions of positive integers and let $\mathcal{P}(n) = \{f \in \mathcal{P} : \sum_{k=1}^\infty k f(k) = n\}$ denote the subset of partitions of n. For $n, m \in \mathbb{Z}_{>0}$, let $(n)_m = m\mathbf{1}_n = \{n, \dots, n\}$ be the partition of nm with m parts of size n. As indicated by the definition above, unless otherwise mentioned, we will not suppose a partition to be ordered. We will sometimes use the word *composition* in this article for an ordered partition.

3 Preliminaries on Irreducible Representations of GL_n

We now recall some basics of the representation theory of general linear groups over non-archimedean local fields. For the sake of brevity of notation, in this section we will denote the characters ν_F and ν'_F by ν and ν', respectively.

3.1 Irreducible Representations of $\mathrm{GL}_n(F)$

For an irreducible cuspidal $\sigma \in \mathcal{A}_F^\circ$ define its *cuspidal line* to be

$$\sigma^{\mathbb{Z}} = \{\nu^m \sigma \mid m \in \mathbb{Z}\}.$$

We now recall the combinatorial notion of *segments* as introduced by Zelevinsky [30], and briefly review the description of $\mathcal{A}_F$.

Definition 3.1 Given an irreducible cuspidal representation $\sigma \in \mathcal{A}_F^\circ$ and $a, b \in \mathbb{Z}$ such that $a \leq b+1$, define the segment $[\nu^a\sigma, \nu^b\sigma]$ to be the set $\{\nu^a\sigma, \nu^{a+1}\sigma, \dots, \nu^b\sigma\}$ if $a \leq b$ and the empty set if $a = b+1$. We say that the segment $[\nu^a\sigma, \nu^b\sigma]$ is supported on $\sigma^{\mathbb{Z}}$.

For a segment $\Delta = [\nu^a\sigma, \nu^b\sigma] = [a, b]_{(\sigma)}$, we denote by $b(\Delta) = \nu^a\sigma$ its beginning, by $e(\Delta) = \nu^b\sigma$ its end, and by $\ell(\Delta) = b - a + 1$ its length.

The representation $\nu^a\sigma \times \cdots \times \nu^b\sigma$ has a unique irreducible subrepresentation and a unique irreducible quotient, which we write as $Z(\Delta)$ and $L(\Delta)$, respectively. By convention, if the set Δ is empty, then both $Z(\Delta)$ and $L(\Delta)$ are defined to be the trivial representation of the trivial group.

Definition 3.2 Two segments Δ_1 and Δ_2 are said to be linked if $\Delta_1 \nsubseteq \Delta_2$, $\Delta_2 \nsubseteq \Delta_1$, and $\Delta_1 \cup \Delta_2$ is also a segment. If Δ_1 and Δ_2 are linked and $b(\Delta_1 \cup \Delta_2) = b(\Delta_1)$, then we say that Δ_1 precedes Δ_2 and write $\Delta_1 \prec \Delta_2$.

Let $\mathcal{O}$ be the set of multi-sets of segments. Let $\mathfrak{m} = \{\Delta_1, \dots, \Delta_t\} \in \mathcal{O}$. The integer t will be called the *size* of the multi-set $\mathfrak{m}$ and will be denoted by $|\mathfrak{m}|$. Any permutation ς of the set $\{1, \dots, t\}$ induces an arrangement of the segments of the multi-set $\mathfrak{m}$ that we call an *order* on $\mathfrak{m}$. An order on $\mathfrak{m}$ is of *standard* form if $\Delta_{\varsigma(i)} \nprec \Delta_{\varsigma(j)}$ for all $i < j$. Clearly, every $\mathfrak{m} \in \mathcal{O}$ admits an order that is of standard form.

Let $\mathfrak{m} = \{\Delta_1, \dots, \Delta_t\} \in \mathcal{O}$ be ordered in standard form. Then, the representation $Z(\Delta_1) \times \cdots \times Z(\Delta_t)$ is independent of the choice of order of standard form. It has a unique irreducible submodule that we denote by $Z(\mathfrak{m})$.

The Zelevinsky classification says that the map $(\mathfrak{m} \mapsto Z(\mathfrak{m})) : \mathcal{O} \to \mathcal{A}_F$ is a bijection [30, Theorem 6.1].

3.1.1 The Langlands Classification

Let $\mathfrak{m} = \{\Delta_1, \dots, \Delta_t\} \in \mathcal{O}$ be ordered in standard form. The representation $L(\Delta_1) \times \cdots \times L(\Delta_t)$ is independent of the choice of order of standard form. It has a unique irreducible quotient that we denote by $L(\mathfrak{m})$.

The Langlands classification says that the map $(\mathfrak{m} \mapsto L(\mathfrak{m})) : \mathcal{O} \to \mathcal{A}_F$ is a bijection [14, Theorem 1.2.5].

3.1.2 The Zelevinsky Involution

It follows from the two classifications above that for any $\mathfrak{m} \in \mathcal{O}$ there exists a unique $\mathfrak{m}^t \in \mathcal{O}$ such that $Z(\mathfrak{m}) = L(\mathfrak{m}^t)$. The function $\mathfrak{m} \mapsto \mathfrak{m}^t$ is an involution on $\mathcal{O}$ known as the Zelevinsky involution. For $\pi = Z(\mathfrak{m}) \in \mathcal{A}_F$, let $\pi^t = L(\mathfrak{m})$. Then $\pi \mapsto \pi^t$ is the corresponding involution on $\mathcal{A}_F$.

Given a multi-set $\mathfrak{m}$, an algorithm to compute $\mathfrak{m}^t$ is provided in [21].

3.1.3 The Cuspidal Support

For every $\pi \in \mathcal{A}_F$ there exist $\sigma_1, \dots, \sigma_k \in \mathcal{A}_F^\circ$, unique up to rearrangement, so that π is isomorphic to a subrepresentation of $\sigma_1 \times \cdots \times \sigma_k$ (see [30, Theorem 1.10]). Let $\operatorname{supp}(\pi) = \{\sigma_i : i = 1, \dots, k\}$ be the support of π. For $\mathfrak{m} \in \mathcal{O}$ let $\operatorname{supp}(\mathfrak{m}) = \{\sigma \in \mathcal{A}_F^\circ : \sigma \in \Delta \text{ for some } \Delta \in \mathfrak{m}\}$ be the support of $\mathfrak{m}$.[2]

Definition 3.3 A representation $\pi \in \mathcal{A}_F$ is said to be *rigid* if $\operatorname{supp}(\pi) \subseteq \sigma^{\mathbb{Z}}$ for some $\sigma \in \mathcal{A}_F^\circ$. A multi-set $\mathfrak{m} \in \mathcal{O}$ is called rigid if $\operatorname{supp}(\mathfrak{m}) \subseteq \sigma^{\mathbb{Z}}$ for some $\sigma \in \mathcal{A}_F^\circ$.

3.1.4 Let $\mathfrak{m}$ be a rigid multi-set. Fix a $\sigma \in \mathcal{A}_F^\circ$ such that $\operatorname{supp}(\mathfrak{m}) \subseteq \sigma^{\mathbb{Z}}$. Write $\mathfrak{m} = \{[\nu^{a_1}\sigma, \nu^{b_1}\sigma], \dots, [\nu^{a_t}\sigma, \nu^{b_t}\sigma]\}$ with $a_i, b_i \in \mathbb{Z}$, $i = 1, \dots, t$. In the sequel the rigid multi-set $\{[\nu^{a_1}\sigma, \nu^{b_1}\sigma], \dots, [\nu^{a_t}\sigma, \nu^{b_t}\sigma]\}$ will sometimes be denoted by $\{[a_1, b_1], \dots, [a_t, b_t]\}_{(\sigma)}$. We will write $\mathfrak{m} = \mathfrak{m}_{(\sigma)}$ to emphasize the choice of the fixed cuspidal representation σ in its cuspidal line, in terms of which the multi-set is described.

[2] The support is often considered as a multi-set. However, in this article only the underlying set plays a role and hence we will treat the support, of both a representation and a multi-set of segments, as a set.

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Similarly, let $\pi = L(\mathfrak{m}) \in \mathcal{A}_F$ be a rigid representation. Fix a $\sigma \in \mathcal{A}_F^\circ$ such that $\mathrm{supp}(\pi) \subseteq \sigma^{\mathbb{Z}}$. As above, we will write $\pi = \pi_{(\sigma)}$ when we wish to emphasize the choice of a fixed cuspidal representation σ in its cuspidal line, in terms of which $\mathfrak{m}$ is described.

Let F' be a non-archimedean local field (not necessarily different from F) and let $\sigma' \in \mathcal{A}_{F'}^\circ$. Let $\mathfrak{m}$ be a rigid multi-set of segments with σ a fixed cuspidal representation such that $\mathfrak{m} = \mathfrak{m}_{(\sigma)}$. Write $\mathfrak{m}_{(\sigma)} = \{[a_1, b_1], \dots, [a_t, b_t]\}_{(\sigma)}$. Then define

$$\mathfrak{m}_{(\sigma')} = \{[a_1, b_1], \dots, [a_t, b_t]\}_{(\sigma')}$$

Note that $\{[a_1, b_1], \dots, [a_t, b_t]\}_{(\sigma')}$ is the multi-set

$$\left\{ [\nu_{F'}^{a_1}\sigma', \nu_{F'}^{b_1}\sigma'], \dots, [\nu_{F'}^{a_t}\sigma', \nu_{F'}^{b_t}\sigma'] \right\}.$$

In particular for a segment $\Delta = [a, b]_{(\sigma)}$, we will denote by $\Delta_{(\sigma')}$ the segment $[\nu_{F'}^{a}\sigma', \nu_{F'}^{b}\sigma']$. Similarly, if $\pi \in \mathcal{A}_F$ is a rigid representation with a fixed cuspidal representation σ such that $\pi = \pi_{(\sigma)}$, *i.e.*, $\pi = L(\mathfrak{m}_{(\sigma)})$, for some $\mathfrak{m}_{(\sigma)} \in \mathcal{O}$, then define $\pi_{(\sigma')} = L(\mathfrak{m}_{(\sigma')})$.

3.2 The Weil–Deligne Representations

Definition 3.4 An n-dimensional admissible Weil–Deligne representation of W_F is a pair $((\rho, V), N)$ where (ρ, V) is a semi-simple, smooth and complex-valued representation of W_F of dimension n and the operator $N : V \to V$ is a nilpotent endomorphism such that

$$\rho(w) \circ N \circ \rho(w)^{-1} = \|w\| N, \tag{3.1}$$

where $\|w\|$ is the character of W_F as defined in Section 2.1. A morphism of Weil–Deligne representations $((\rho_1, V_1), N_1) \to ((\rho_2, V_2), N_2)$ is a map of representations $T : (\rho_1, V_1) \to (\rho_2, V_2)$, such that $T \circ N_1 = N_2 \circ T$. For more details on Weil–Deligne representations, we refer the reader to [28].

Let $\mathcal{G}_F(n)$ denote the set of all isomorphism classes of n-dimensional admissible Weil–Deligne representations of W_F and let $\mathcal{G}_F = \bigsqcup_{n \geq 0} \mathcal{G}_F(n)$.

3.2.1 Classification of Weil–Deligne Representations

For a semi-simple, smooth and complex-valued representation (ρ, V) of W_F put $\Delta' = [\nu'^a\rho, \nu'^b\rho]$ $(a \leq b)$ to be the set $\{\nu'^a\rho, \nu'^{(a+1)}\rho, \dots, \nu'^b\rho\}$ as in Section 3.1. Let $\rho(\Delta') = \nu'^a\rho \oplus \nu'^{(a+1)}\rho \oplus \cdots \oplus \nu'^b\rho$. Let V_i be the space on which $\nu'^i\rho$ acts $(a \leq i \leq b)$. Clearly, all these spaces can be identified with the same space V. Define a map $N(\Delta') : \bigoplus_{i=a}^b V_i \to \bigoplus_{i=a}^b V_i$ in the following way. Let $N(\Delta') : V_i \to V_{i+1}$ be the obvious (identity) morphism (for $i = a, \dots, b-1$) and let $N(\Delta')|_{V_b} = 0$. Now assign to each Δ' the Weil–Deligne representation $\tau(\Delta') = (\rho(\Delta'), N(\Delta'))$.

It follows from generalities that the $\tau(\Delta')$ are indecomposable objects. They are mutually non-isomorphic and every indecomposable object is of this form. Thus every Weil–Deligne representation decomposes into a direct sum $\tau(\Delta'_1) \oplus \cdots \oplus \tau(\Delta'_r)$ (for some positive integer r), and moreover, this decomposition is unique up to a permutation.

3.2.2 The Maps rec and rec°

Let $\mathrm{rec} = \mathrm{rec}_F : \mathcal{A}_F \to \mathcal{G}_F$ be the Langlands reciprocity map established in [18] for the positive characteristic case and in [5,6,26] for characteristic zero. Denote by rec° the restriction of rec to $\mathcal{A}_F^\circ$. The map rec can be described in terms of the map rec° as follows (see [30, §10] for details). If $\pi = L([\nu^{a_1}\sigma_1, \nu^{b_1}\sigma_1], \dots, [\nu^{a_t}\sigma_t, \nu^{b_t}\sigma_t])$, then

$$\begin{aligned}(3.2)\quad &\mathrm{rec}(\pi)\\ &= \bigoplus_{i=1}^{t} \big(\rho([\nu'^{a_i}\,\mathrm{rec}^\circ(\sigma_i), \nu'^{b_i}\,\mathrm{rec}^\circ(\sigma_i)]), N([\nu'^{a_i}\,\mathrm{rec}^\circ(\sigma_i), \nu'^{b_i}\,\mathrm{rec}^\circ(\sigma_i)])\big).\end{aligned}$$

3.2.3 Partition Associated With a Weil–Deligne Representation

Given an n-dimensional admissible Weil–Deligne representation $((\rho, V), N)$, one can associate a partition $f \in \mathcal{P}(n)$ with it in the following manner. Since N is a nilpotent endomorphism, it can be written as a matrix with 1's on the sub-diagonal and 0's elsewhere in a unique way (up to the order of the Jordan blocks). The sizes of the Jordan blocks of N define a partition of n that we will denote by f. In particular, the partition corresponding to $N = 0$ is the partition $n\mathbf{1}_1 = \{1, \dots, 1\}$.

Denote by $P_{F,n} : \mathcal{G}_F(n) \to \mathcal{P}(n)$ the map which takes $((\rho, V), N)$ to the partition f as described above and let $P_F : \mathcal{G}_F \to \mathcal{P}$ be the map such that $P_F|_{\mathcal{G}_F(n)} = P_{F,n}$.

Lemma 3.5 *Let $\pi = L([\nu^{a_1}\sigma_1, \nu^{b_1}\sigma_1], \dots, [\nu^{a_t}\sigma_t, \nu^{b_t}\sigma_t])$, where $\sigma_i \in \mathcal{A}_F^\circ(k_i)$. Then we have*

$$P_F(\mathrm{rec}(\pi)) = \sum_{i=1}^{t} (b_i - a_i + 1)_{k_i}. \tag{3.3}$$

Proof Write $\mathrm{rec}(\pi) = ((\rho, V), N)$ following the description provided in (3.2). By rearranging the basis of V such that N is represented by a matrix with 1's on the sub-diagonal and 0's elsewhere, it is easy to see that (3.3) holds. ∎

Remark 3.6 In [24] it was mistakenly remarked that the map $((\rho, V), N) \mapsto ([\rho], f)$, where $[\rho]$ is the isomorphism class of the representation ρ, is an injection. For a simple counterexample, consider the representations $\pi_1 = L([\nu, \nu^2], [\nu^3])$ and $\pi_2 = L([\nu], [\nu^2, \nu^3])$; evidently both π_1 and π_2 are mapped to $([\nu \oplus \nu^2 \oplus \nu^3], (2,1))$. Although the above statement was used in the proofs of [24, Lemma 4.1 and Proposition 7.1], it was not used in a crucial manner, and the proofs can be rectified simply by working with the description of the reciprocity map that we provide in (3.2) instead of the description given in [24, §3]. In particular, the statements in that paper are correct. In any case, the results on ladder representations that we obtain in this article are used here to provide independent proofs of the main results of [24].

4 Base Change and Automorphic Induction

The base change and the automorphic induction maps were studied in [1] and [7], respectively, before the local Langlands correspondence for the general linear groups over non-archimedean local fields was established. Now that we have the correspondence at our disposal, these two maps can be defined in a much simpler manner. We now recall these definitions. We also obtain some results analyzing the behavior of the class of rigid representations under these two maps. Some of these results (for instance Lemma 4.5) can be found in the aforementioned references, but we provide a proof here using the definitions of the two maps that we use in this article.

4.1 Definition of the Two Maps

For now suppose that E/F is an arbitrary finite extension of non-archimedean local fields such that $[E:F]=d$.

4.1.1 Base Change

Let $\pi \in \mathcal{A}_F(n)$ and $\mathrm{rec}_F(\pi) = ((\rho, V), N)$. Denote by

$$\mathrm{res}_{E/F} : \mathcal{G}_n(F) \to \mathcal{G}_n(E)$$

the map defined by $\mathrm{res}_{E/F}(\rho, N) = (\rho|_{W_E}, N)$. As the restriction to the finite index subgroup W_E is also semi-simple, this defines an irreducible representation of $\mathrm{GL}_n(E)$ via the local Langlands correspondence. The above process of obtaining an irreducible representation of $\mathrm{GL}_n(E)$ from an irreducible representation of $\mathrm{GL}_n(F)$ is known as the *base change*. For $\pi \in \mathcal{A}_F(n)$, its base change will be denoted by $\mathrm{bc}_{E/F}(\pi)$ and is defined by

$$\mathrm{rec}_E(\mathrm{bc}_{E/F}(\pi)) = \mathrm{res}_{E/F}(\mathrm{rec}_F(\pi)).$$

4.1.2 Automorphic Induction

Let $\pi \in \mathcal{A}_E(m)$ and $\mathrm{rec}_E(\pi) = ((\rho, V), N)$. Define now the representation $\mathrm{ind}_{W_E}^{W_F}(\rho)$ of W_F in the following way:

$$\mathrm{ind}_{W_E}^{W_F}(\rho) = \{f : W_F \to V \mid f(hg) = \rho(h)f(g)\ \forall h \in W_E, g \in W_F\}.$$

Since ρ is semi-simple, the induced representation $\mathrm{ind}_{W_E}^{W_F}(\rho)$ is semi-simple as well. Further define $\widetilde{N}$ such that $(\widetilde{N}f)(g) = \|g\|N(f(g))$. It can be easily checked that $\widetilde{N}$ is a nilpotent endomorphism of the induced space satisfying (3.1). Thus define $\mathrm{ind}_{E/F}((\rho, N)) = (\mathrm{ind}_{W_E}^{W_F}(\rho), \widetilde{N})$, which is an element of $\mathcal{G}_F(md)$. This Weil–Deligne representation corresponds to an irreducible representation of $\mathrm{GL}_{md}(F)$, via the reciprocity map. This process of obtaining an irreducible representation of $\mathrm{GL}_{md}(F)$ from an irreducible representation of $\mathrm{GL}_m(E)$ is known as *automorphic induction*. For $\pi \in \mathcal{A}_E(m)$, its automorphic induction will be denoted by $\mathrm{ai}_{E/F}(\pi)$ and is defined by

$$\mathrm{rec}_F(\mathrm{ai}_{E/F}(\pi)) = \mathrm{ind}_{E/F}(\mathrm{rec}_E(\pi)).$$

Our next lemma provides a simplified expression for the nilpotent operator $\widetilde{N}$.

Lemma 4.1 *Let $\{g_1, \dots, g_d\}$ be a fixed set of representatives for the right coset space $H \setminus G$, where $G = W_F$ and $H = W_E$. Let $((\rho, V), N) \in \mathcal{G}_E$ with $\widetilde{N} : \mathrm{ind}_H^G(\rho) \to \mathrm{ind}_H^G(\rho)$ as defined above. Then we can choose bases of V and $\mathrm{ind}_H^G(\rho)$ such that in the*

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matrix form with respect to the two bases, $\widetilde{N} = \mathrm{diag}(\|g_1\|N, \dots, \|g_d\|N)$*. In particular, if the partition corresponding to the operator* N *(via its Jordan canonical form) is* $\mathbf{p}$*, then that corresponding to* $\widetilde{N}$ *is* $d\mathbf{p}$*.*

Proof Let $\dim V = l$ and $(v_1, \dots, v_l)$ be an ordered basis of V such that the matrix of N with respect to it is expressed in its Jordan form. Since N is nilpotent, there exists $\{v_{l_1}, \dots, v_{l_k}\} \subseteq \{v_1, \dots, v_l\}$ such that

$$N(v_j) = \begin{cases} v_{j+1} & \text{if } j \in \{l_1, \dots, l_k\}, \\ 0 & \text{if } j \notin \{l_1, \dots, l_k\}. \end{cases}$$

Now define a standard basis $\{f_{i,j} \mid 1 \le i \le d, 1 \le j \le l\}$ of the space $\mathrm{ind}_{W_E}^{W_F}(\rho)$ in the following manner.

$$f_{i,j} = \begin{cases} \rho(h)v_j & \text{if } g = hg_i, \\ 0 & \text{otherwise.} \end{cases}$$

Fix i. An easy calculation shows that

$$\widetilde{N}(f_{i,j}) = \begin{cases} \|g_i\| f_{i,j+1} & \text{if } j \in \{l_1, \dots, l_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

This proves the lemma. ∎

4.2 Compatibility With Parabolic Induction

Lemma 4.2 (i) *Let* $\pi_1, \dots, \pi_r \in \mathcal{A}_F$ *such that both* $\pi_1 \times \cdots \times \pi_r$ *and* $\mathrm{bc}_{E/F}(\pi_1) \times \cdots \times \mathrm{bc}_{E/F}(\pi_r)$ *are irreducible. Then*

$$\mathrm{bc}_{E/F}(\pi_1 \times \cdots \times \pi_r) = \mathrm{bc}_{E/F}(\pi_1) \times \cdots \times \mathrm{bc}_{E/F}(\pi_r).$$

(ii) *Let* $\pi_1, \dots, \pi_r \in \mathcal{A}_E$ *such that both* $\pi_1 \times \cdots \times \pi_r$ *and* $\mathrm{ai}_{E/F}(\pi_1) \times \cdots \times \mathrm{ai}_{E/F}(\pi_r)$ *are irreducible. Then*

$$\mathrm{ai}_{E/F}(\pi_1 \times \cdots \times \pi_r) = \mathrm{ai}_{E/F}(\pi_1) \times \cdots \times \mathrm{ai}_{E/F}(\pi_r).$$

Proof We first prove (i). The general case reduces to the case when $r = 2$ by induction. Thus, let $r = 2$. Let $\mathrm{rec}_F(\pi_1) = (\rho_1, N_1)$ and $\mathrm{rec}_F(\pi_2) = (\rho_2, N_2)$. Then we have

$$\begin{aligned} \mathrm{rec}_E\big(\mathrm{bc}_{E/F}(\pi_1) \times \mathrm{bc}_{E/F}(\pi_2)\big) &= \mathrm{rec}_E(\mathrm{bc}_{E/F}(\pi_1)) \oplus \mathrm{rec}_E(\mathrm{bc}_{E/F}(\pi_2)) \\ &= (\rho_1|_{W_E}, N_1) \oplus (\rho_2|_{W_E}, N_2) \\ &= ((\rho_1 \oplus \rho_2)|_{W_E}, N_1 \oplus N_2). \end{aligned}$$

This is equal to $\mathrm{rec}_E(\mathrm{bc}_{E/F}(\pi_1 \times \pi_2))$, which demonstrates the statement in the base change case.

Next we consider the statement for the automorphic induction case. As above it is enough to prove the statement for $r = 2$. Let $\mathrm{rec}_E(\pi_1) = (\rho_1, N_1)$ and $\mathrm{rec}_E(\pi_2) = (\rho_2, N_2)$. Then we have

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$$\begin{aligned}\mathrm{rec}_F\big(\mathrm{ai}_{E/F}(\pi_1)\times \mathrm{ai}_{E/F}(\pi_2)\big) &= \mathrm{rec}_F(\mathrm{ai}_{E/F}(\pi_1))\oplus \mathrm{rec}_F(\mathrm{ai}_{E/F}(\pi_2))\\ &= (\mathrm{ind}_{W_E}^{W_F}(\rho_1),\widetilde{N_1})\oplus(\mathrm{ind}_{W_E}^{W_F}(\rho_2),\widetilde{N_2})\\ &= (\mathrm{ind}_{W_E}^{W_F}(\rho_1\oplus\rho_2),\widetilde{N_1}\oplus\widetilde{N_2}).\end{aligned}$$

By Lemma 4.1 this is equal to $\mathrm{rec}_F(\mathrm{ai}_{E/F}(\pi_1\times\pi_2))$ which finishes the proof of the lemma. ∎

4.3 The Cuspidal Case When $[E:F]$ is Prime

We return to the case when E/F is a cyclic extension such that $d=[E:F]$ is prime. We maintain this assumption until the end of Section 8. The following result was obtained in [1, Lemma 6.10] and [7, Proposition 5.5] for the base change and the automorphic induction maps, respectively. Assuming the correspondence, one can obtain these results in an elementary fashion using arguments similar to the ones employed in the proof of [24, Lemma 7.1].

Lemma 4.3 (i) *Let $\sigma\in\mathcal{A}_F^\circ(k)$. Then $\mathrm{bc}_{E/F}(\sigma)=\sigma_1\times\cdots\times\sigma_t$, where $t\,|\,k$ and $\sigma_i\in\mathcal{A}_E^\circ(\frac{k}{t})$ such that $\sigma_i\neq\nu_E^\alpha\sigma_j$, for any $\alpha\in\mathbb{R}$ and $i\neq j$.*

(ii) *Analogously, let $\sigma\in\mathcal{A}_E^\circ(k)$. Then $\mathrm{ai}_{E/F}(\sigma)=\sigma_1\times\cdots\times\sigma_t$, where $t\,|\,kd$ and $\sigma_i\in\mathcal{A}_F^\circ(\frac{kd}{t})$ such that $\sigma_i\neq\nu_F^\alpha\sigma_j$, for any $\alpha\in\mathbb{R}$ and $i\neq j$.*

(iii) *Moreover if $\sigma\in\mathcal{A}_F^u\cap\mathcal{A}_F^\circ$, respectively, $\sigma\in\mathcal{A}_E^u\cap\mathcal{A}_E^\circ$, then each σ_i $(i=1,\dots,t)$ appearing in the cuspidal support of $\mathrm{bc}_{E/F}(\sigma)$, respectively, $\mathrm{ai}_{E/F}(\sigma)$, is in $\mathcal{A}_E^u$, respectively, $\mathcal{A}_F^u$.*

Remark 4.4 Henceforth, in this article, every statement that we make for the base change setting has an automorphic induction analogue and vice versa. The proof in one setting is a verbatim translation of the proof in the other setting. To avoid repetition of arguments, from this point onwards, we will give precise statements for both settings but prove only the one in the base change case.

4.4 Compatibility with the Zelevinsky Classification

The next lemma is a straightforward application of the local Langlands correspondence. We provide a proof for the sake of completeness.

Lemma 4.5 (i) *Let $\sigma\in\mathcal{A}_F^\circ$ and let $\pi=\pi_{(\sigma)}\in\mathcal{A}_F$ be such that $\mathrm{Supp}\,\pi\subseteq\sigma^{\mathbb{Z}}$. Let $\mathrm{bc}_{E/F}(\sigma)=\sigma_1\times\cdots\times\sigma_t$ (see Lemma 4.3 (i)). Then $\mathrm{bc}_{E/F}(\pi)=\pi_{(\sigma_1)}\times\cdots\times\pi_{(\sigma_t)}$ (see Section 3.1.4 for the notation).*

(ii) *Let $\sigma\in\mathcal{A}_E^\circ$ and let $\pi=\pi_{(\sigma)}\in\mathcal{A}_E$ be such that $\mathrm{Supp}\ \pi\subseteq\sigma^{\mathbb{Z}}$. Let $\mathrm{ai}_{E/F}(\sigma)=\sigma_1\times\cdots\times\sigma_t$ (see Lemma 4.3 (ii)). Then*

$$\mathrm{ai}_{E/F}(\pi)=\pi_{(\sigma_1)}\times\cdots\times\pi_{(\sigma_t)}$$

(see Section 3.1.4 for the notation).

Proof Let $\pi=L(\Delta_1,\dots,\Delta_s)$ and $\Pi=\pi_{(\sigma_1)}\times\cdots\times\pi_{(\sigma_t)}$. Note that, since $\sigma_j\neq\nu_E^\alpha\sigma_{j'}$ for any $\alpha\in\mathbb{R}$ if $j\neq j'$, the representation Π is irreducible. We will show that $\mathrm{bc}_{E/F}(\pi)=\Pi$.

Let $\mathrm{rec}_F(\sigma) = \rho$ and $\mathrm{rec}_E(\sigma_j) = \rho_j$ for $j = 1, \dots, t$. Thus for any integer r we have $(\nu_F'^r \rho)|_{W_E} = \nu_E'^r \rho_1 \oplus \cdots \oplus \nu_E'^r \rho_t$. Denote the representation space of $\nu_F'^r \rho$ by V_r and that of $\nu_E'^r \rho_j$ by $V_{r,j}$. In other words, $V_r = \bigoplus_{j=1}^t V_{r,j}$ as a W_E-module.

Set $\Delta_i = [\nu_F^{a_i}\sigma, \nu_F^{b_i}\sigma]$, $\Delta_i' = [\nu_F'^{a_i}\rho, \nu_F'^{b_i}\rho]$, $\Delta_{i,j} = [\nu_E^{a_i}\sigma_j, \nu_E^{b_i}\sigma_j]$, and $\Delta_{i,j}' = [\nu_E'^{a_i}\rho_j, \nu_E'^{b_i}\rho_j]$ (where $1 \le i \le s$ and $1 \le j \le t$). Then

$$\mathrm{rec}_E(\Pi) = \bigoplus_{j=1}^t \Big(\bigoplus_{i=1}^s (\rho(\Delta_{i,j}'), N(\Delta_{i,j}')) \Big).$$

Note that

$$\begin{aligned} \mathrm{res}_{E/F}(\mathrm{rec}_F(\pi)) &= \bigoplus_{i=1}^s \big(\rho(\Delta_i')|_{W_E}, N(\Delta_i')\big) = \bigoplus_{i=1}^s \Big((\bigoplus_{r=a_i}^{b_i} \nu'^r \rho)|_{W_E}, N(\Delta_i') \Big) \\ &= \bigoplus_{i=1}^s \Big((\bigoplus_{r=a_i}^{b_i} (\bigoplus_{j=1}^t \nu'^r \rho_j)), N(\Delta_i') \Big). \end{aligned}$$

From the description of $N(\Delta_i')$ (provided in Section 3.2.1), by rearranging the spaces $V_{r,j}$, we get that $N(\Delta_i') = \bigoplus_{j=1}^t N(\Delta_{i,j}')$ for every i. Thus we get that

$$\mathrm{res}_{E/F}(\mathrm{rec}_F(\pi)) = \bigoplus_{i=1}^s \Big(\bigoplus_{j=1}^t \Big(\bigoplus_{r=a_i}^{b_i} \nu'^r \rho_j \Big), \bigoplus_{j=1}^t N(\Delta_{i,j}') \Big) = \mathrm{rec}_E(\Pi),$$

and we obtain the first statement. ∎

4.4.1 Compatibility With the Zelevinsky Involution

Proposition 4.6 (i) *Let $\pi \in \mathcal{A}_F$ be a rigid representation. Then*

$$\mathrm{bc}_{E/F}(\pi^t) = \mathrm{bc}_{E/F}(\pi)^t.$$

(ii) *Let $\pi \in \mathcal{A}_E$ be a rigid representation. Then* $\mathrm{ai}_{E/F}(\pi^t) = \mathrm{ai}_{E/F}(\pi)^t$.

Proof Suppose that $\mathrm{Supp}\,\pi \subseteq \sigma^{\mathbb{Z}}$ for some $\sigma \in \mathcal{A}_F^\circ$. Since the Zelevinsky involution of a representation preserves its cuspidal support, we have $\mathrm{supp}(\pi^t) \subseteq \sigma^{\mathbb{Z}}$. It was shown in [21] that the action of the Zelevinsky involution on rigid representations is "oblivious" to the cuspidal line on which it is supported. In other words, $(\pi_{(\sigma_i)})^t = (\pi^t)_{(\sigma_i)}$, for all σ_i. Using Lemma 4.5(i) and the multiplicative property of Zelevinsky involution we get that

$$\mathrm{bc}_{E/F}(\pi^t) = \prod_{i=1}^t (\pi^t)_{(\sigma_i)} = \prod_{i=1}^t (\pi_{(\sigma_i)})^t = \mathrm{bc}_{E/F}(\pi)^t. \quad ∎$$

5 Ladder Representations

The class of ladder representations was introduced in [15]. This class of irreducible representations has many interesting properties, for instance, they are precisely the representations in the class of rigid representations whose Jacquet modules are semi-simple [4, Corollary 4.11]. Furthermore, the Jacquet modules of a ladder representation are calculated explicitly in [13, Corollary 2.2]. Moreover, this class is preserved by the Zelevinsky involution, and the algorithm provided in [21] to compute the Zelevinsky involution of an irreducible representation takes a much simpler form when the representation is a ladder [15, §3]. Some of these structural properties make this class more approachable in comparison to the entire admissible dual for the purpose of distinction problems [19]. However the aforementioned properties will not play a direct role in this article.

We will now recall the definition of ladder representations and collect some basic facts about them that we will use in this article. We will show that the rigid representations that are irreducibly induced from ladder representations remain in the class of representations irreducibly induced from ladders under the two maps.

5.1 Preliminaries on Ladder and Unitarizable Representations

5.1.1 Ladders and Proper Ladders

Definition 5.1 Let $\sigma \in \mathcal{A}_F^{\circ}$. Let the set $\mathfrak{m} = \{\Delta_1, \dots, \Delta_k\}$ be such that $\mathrm{supp}(\mathfrak{m}) \subseteq \sigma^{\mathbb{Z}}$ and write $\Delta_i = [\nu_F^{a_i}\sigma, \nu_F^{b_i}\sigma]$ $(a_i, b_i \in \mathbb{Z})$. By renumbering the segments if required, we can assume that $a_1 \geq \cdots \geq a_k$. Then $\mathfrak{m}$ is called a *ladder* if $a_1 > \cdots > a_k$ and $b_1 > \cdots > b_k$. Furthermore, it is called a *proper ladder* if $a_i \leq b_{i+1} + 1$, for all $i = 1, \dots, k-1$.

Definition 5.2
- A representation $\pi \in \mathcal{A}_F$ is called a ladder representation if $\pi = L(\mathfrak{m})$, where $\mathfrak{m}$ is a ladder.
- A representation $\pi \in \mathcal{A}_F$ is called a proper ladder representation if $\pi = L(\mathfrak{m})$, where $\mathfrak{m}$ is a proper ladder.

Example 5.3 Let $\sigma \in \mathcal{A}_F^{\circ}$, $\mathfrak{m}_1 = \{[2,3],[0,1]\}_{(\sigma)}$, and $\mathfrak{m}_2 = \{[3,4],[0,1]\}_{(\sigma)}$. The multi-sets $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are examples of ladders of which only $\mathfrak{m}_1$ is a proper ladder.

Whenever we say that $\mathfrak{m} = \{\Delta_1, \dots, \Delta_k\} \in \sigma^{\mathbb{Z}}$ is a ladder or a proper ladder, we implicitly assume that $\mathfrak{m}$ is already ordered as in the definition above, namely, so that $a_1 > \cdots > a_k$, where $b(\Delta_i) = \nu^{a_i}\sigma$.

We will denote the subset of ladder representations of $\mathcal{A}_F$ by $\mathcal{L} = \mathcal{L}_F$ and the subset of proper ladders by $\mathcal{L}_p = \mathcal{L}_{p,F}$. The class of representations irreducibly induced from ladders will be denoted by $\mathcal{L}_{\mathrm{ind}} = \mathcal{L}_{\mathrm{ind},F}$.

The next proposition follows directly from [15, Theorem 16].

Proposition 5.4 *Let $\pi \in \mathcal{L}_{\mathrm{ind}}$. Then π can be written as $\pi_1 \times \cdots \times \pi_k$, where each π_i is a proper ladder. The decomposition is unique up to a reordering of the π_i.*

We will need the following lemma in the sequel. Let F' be a non-archimedean local field (not necessarily different from F).

Lemma 5.5 *Let $\pi_i = (\pi_i)_{(\sigma)} \in \mathcal{L}_F$ be such that $\mathrm{supp}(\pi_i) \subseteq \sigma^{\mathbb{Z}}$, $i = 1, \dots, k$. Suppose that $\pi_1 \times \cdots \times \pi_k$ is irreducible. Let $\sigma' \in \mathcal{A}_{F'}^{\circ}$. Then the representation $(\pi_1)_{(\sigma')} \times \cdots \times (\pi_k)_{(\sigma')}$ is irreducible (see Section 3.1.4 for the notation).*

Proof By [16, Lemma 6.17], it is enough the prove the statement for the case $k = 2$. Since $(\pi_1)_{(\sigma)} \times (\pi_2)_{(\sigma)}$ is irreducible, so is $(\pi_1)_{(\sigma')} \times (\pi_2)_{(\sigma')}$ (see, for instance, [16, Proposition 6.20, Lemma 6.21]). ■

5.1.2 Tadić's Classification of Unitarizable Representations

An important example of a proper ladder representation is when $a_i = a_{i+1} + 1$ and $b_i = b_{i+1}+1$, for $i = 1, \dots, k-1$. Define a Speh representation to be a proper ladder such that the underlying multi-set satisfies this property. Notice that we are not assuming that Speh representations are unitarizable in general. We will use the term *unitarizable Speh* in this paper for a Speh representation that lies in $\mathcal{A}_F^u$.

For a unitarizable Speh representation τ and a real number $\alpha \in (-\frac{1}{2}, \frac{1}{2})$, define $\pi(\tau, \alpha)$ to be the representation $\nu^\alpha \tau \times \nu^{-\alpha}\tau$. By [30, Proposition 8.5] it is irreducible. We now recall the classification of the unitarizable representations of general linear groups [27, Theorem D].

Theorem 5.6 (i) *The representations $\pi(\tau, \alpha)$ lie in $\mathcal{A}_F^u$.*

(ii) *Every representation $\pi \in \mathcal{A}_F^u$ can be written as $\pi = \pi_1 \times \cdots \times \pi_t$, where each π_i is either a unitarizable Speh representations or a representation of the form $\pi(\tau, \alpha)$.*

In particular, $\mathcal{A}_F^u \subseteq \mathcal{L}_{\text{ind}}$.

5.2 Base Change, Automorphic Induction and Ladder Representations

Proposition 5.7 (i) *Let $\pi \in \mathcal{L}_{\text{ind},F}$ be a rigid representation. Then* $\text{bc}_{E/F}(\pi) \in \mathcal{L}_{\text{ind},E}$.

(ii) *Let $\pi \in \mathcal{L}_{\text{ind},E}$ be a rigid representation. Then* $\text{ai}_{E/F}(\pi) \in \mathcal{L}_{\text{ind},F}$.

Proof The result is obtained by a direct application of Lemma 4.3 and Lemma 4.5. ■

The necessity of the rigidity hypothesis in Proposition 5.7 is shown by the following example. Let $\pi = L([\nu_F^2, \nu_F^3], [1, \nu_F^2], [\kappa_{E/F}])$. It is easy to see that $\text{bc}_{E/F}(\pi) = L([\nu_E^2, \nu_E^3], [1, \nu_E^2], [\nu_E])$ is not in $\mathcal{L}_{\text{ind},E}$. One can easily construct similar examples to demonstrate the failure of the statement without rigidity in the case of automorphic induction as well.

The hypothesis of rigidity can be removed from the above statements if we further assume that the representations we are dealing with are unitarizable.

Proposition 5.8 (i) *Let $\pi \in \mathcal{A}_F^u$. Then* $\text{bc}_{E/F}(\pi) \in \mathcal{A}_E^u$.
(ii) *Let $\pi \in \mathcal{A}_E^u$. Then* $\text{ai}_{E/F}(\pi) \in \mathcal{A}_F^u$.

Proof Suppose $\pi \in \mathcal{A}_F^u$. Write $\pi = \pi_1 \times \cdots \times \pi_k$ such that each π_i is either a unitarizable Speh representation or a representation of the form $\pi(\tau_i, \alpha_i)$ for some unitarizable Speh τ_i and some $\alpha_i \in (-\frac{1}{2}, \frac{1}{2})$. If π_i is a unitarizable Speh, then it is clear by Lemma 4.3, and Lemma 4.5 that $\text{bc}_{E/F}(\pi_i) \in \mathcal{A}_E^u$. Suppose that $\pi_i = \nu^{\alpha_i}\tau_i \times \nu^{-\alpha_i}\tau_i$. By Lemma 4.3, and Lemma 4.5, $\nu^{\alpha_i}\,\text{bc}_{E/F}(\tau_i) \times \nu^{-\alpha_i}\,\text{bc}_{E/F}(\tau_i)$ is a product of Speh representations supported on different cuspidal lines, and is thus irreducible. Hence by Lemma 4.2 (i), $\mathrm{bc}_{E/F}(\pi_i) = \nu^{\alpha_i}\,\mathrm{bc}_{E/F}(\tau_i) \times \nu^{-\alpha_i}\,\mathrm{bc}_{E/F}(\tau_i)$, which is again unitarizable by Theorem 5.6. Thus we have $\mathrm{bc}_{E/F}(\pi_i) \in \mathcal{A}_E^u$ for each i.

Since a representation induced from unitarizable representations is irreducible, appealing to Lemma 4.2 (i), we get that

$$\mathrm{bc}_{E/F}(\pi) = \mathrm{bc}_{E/F}(\pi_1) \times \cdots \times \mathrm{bc}_{E/F}(\pi_k).$$

Since the induced representation is also unitarizable, this proves (i). ■

6 Degenerate Whittaker Models

We now study the degenerate Whittaker models and their relationships with the two maps.

6.1 Definition of Degenerate Whittaker Models

We briefly recall the definition of degenerate Whittaker models as provided in [30, §8.3]. Given a composition $\mathbf{d} = (\lambda_1, \ldots, \lambda_l)$ of n, ordered such that $\lambda_1 \geq \cdots \geq \lambda_l$, define the character $\theta = \theta_{\mathbf{d}}$ of U_n by $\theta((u_{i,j})) = \psi(\sum u_{i,i+1})$, where i runs over $1, \ldots, n-1$ except

$$n - \lambda_1, n - (\lambda_1 + \lambda_2), \ldots, n - (\lambda_1 + \cdots + \lambda_{l-1}).$$

(See Section 2.5 for the definition of U_n and ψ.) Say that a representation $\pi \in \mathcal{A}_F(n)$ has a degenerate Whittaker model with respect to the sequence $\mathbf{d}$ if $\mathrm{Hom}_{U_n}(\pi, \theta_{\mathbf{d}}) \neq 0$.

It was shown in [30, Corollary 8.3] that every $\pi \in \mathcal{A}_F$ has a degenerate Whittaker model.

6.2 The Depth Sequence and $\mathrm{SL}(2)$-type of an Irreducible Representation

For $\pi \in \Pi(\mathrm{GL}_n(F))$ and any $r = 0, \ldots, n$, we denote by $\pi^{(r)}$ the r-th derivative of π as defined in [2, §3.5 and §4.3]. It is a functor from $\Pi(\mathrm{GL}_n(F))$ to $\Pi(\mathrm{GL}_{n-r}(F))$. If the integer r is such that $\pi^{(r)} \neq 0$ and $\pi^{(r+k)} = 0$ for any $k \in \mathbb{Z}_{>0}$, then we call the representation $\pi^{(r)}$ the *highest derivative* of π and the integer r the *depth* of π.

Definition 6.1 Given $\pi \in \mathcal{A}_F(n)$, we recursively define the irreducible representations

$$\tau_0, \tau_1, \ldots, \tau_l$$

and an integer sequence $\mathbf{d}(\pi) = (\lambda_1, \ldots, \lambda_l)$ such that $\tau_0 = \pi$, τ_l is the trivial representation of the trivial group, and $\tau_{i+1} := \tau_i^{(\lambda_{i+1})}$ is the highest derivative of τ_i ($i = 0, \ldots, l-1$). We call this sequence the *depth sequence* of the irreducible representation π.

Clearly $\lambda_1 + \cdots + \lambda_l = n$ and by [30, Theorem 8.1] we get that $\lambda_1 \geq \cdots \geq \lambda_l$. Thus any depth sequence of an element of $\mathcal{A}_F(n)$ can be identified with an element of $\mathcal{P}(n)$.

We now recall [30, Corollary 8.3].

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Theorem 6.2 *Every $\pi \in \mathcal{A}_F$ has a degenerate Whittaker model with respect to its depth sequence $\mathbf{d}(\pi)$ with multiplicity one.*

6.2.1 Definition of an SL(2)-type

The SL(2)-type of an irreducible representation was first defined in [29, Definition 1] for the ones in the unitarizable dual. The definition was then extended to the admissible dual [24, Remark 2]. We recall it below.

Definition 6.3 Let $\pi \in \mathcal{A}_F$. Then the SL(2)-type of π is defined to be the partition $P_F(\mathrm{rec}_F(\pi^t))$, where P_F is the map defined in Section 3.2.3. It is denoted by $\mathcal{V}(\pi)$.

6.2.2 Relation Between the Two Partitions

Given $\pi \in \mathcal{A}_F$, we will think of $\mathcal{V}(\pi)$ as a composition by ordering the elements of this partition in a non-increasing manner. For a composition f, denote by f^t its conjugate composition. We have the following non-archimedean analogue of [3, Theorem 2.4.2].

Lemma 6.4 *For $\pi \in \mathcal{A}_F$, $\mathcal{V}(\pi) = \mathbf{d}(\pi)^t$.*

Proof The statement follows directly from Lemma 3.5 and [30, Theorem 8.1]. ∎

6.3 Degenerate Whittaker Models and the Two Maps

We begin by studying the SL(2)-type of the base change (or the automorphic induction) lift of an irreducible representation.

Theorem 6.5 (i) *Let $\pi \in \mathcal{A}_F$ be a rigid representation. Then $\mathcal{V}(\pi) = \mathcal{V}(\mathrm{bc}_{E/F}(\pi))$.*
(ii) *Let $\pi \in \mathcal{A}_E$ be a rigid representation. Then $d\mathcal{V}(\pi) = \mathcal{V}(\mathrm{ai}_{E/F}(\pi))$.*

Proof It is clear from the definition of base change that for $\pi \in \mathcal{A}_F$, $P_F(\mathrm{rec}_F(\pi)) = P_E(\mathrm{rec}_E(\mathrm{bc}_{E/F}(\pi)))$. The result now follows from Proposition 4.6. ∎

The hypothesis of rigidity in Theorem 6.5 is essential as demonstrated by the following example. Take $\pi = L([1, \nu_F], [\kappa\nu_F^2, \kappa\nu_F^3])$. Then $\mathrm{bc}_{E/F}(\pi) = L([1, \nu_E], [\nu_E^2, \nu_E^3])$ and by Lemma 3.5 we get that $\mathcal{V}(\pi) \neq \mathcal{V}(\mathrm{bc}_{E/F}(\pi))$. A similar example can be constructed in the case of automorphic induction.

However, as earlier, the rigidity hypothesis can be removed if we assume that π is unitarizable.

Theorem 6.6 (i) *Let $\pi \in \mathcal{A}_F^u$. Then $\mathcal{V}(\pi) = \mathcal{V}(\mathrm{bc}_{E/F}(\pi))$.*
(ii) *Let $\pi \in \mathcal{A}_E^u$. Then $d\mathcal{V}(\pi) = \mathcal{V}(\mathrm{ai}_{E/F}(\pi))$.*

Proof The result is obtained by applying Theorem 6.5 to the class of Speh representations and arguing as in the proof of Proposition 5.8. ∎

For any two compositions f_1 and f_2 of n_1 and n_2, respectively, of the same length, denote by $f_1 +_c f_2$ the composition of $n_1 + n_2$ given by coordinate-wise addition. Finally we have the following result showcasing the behavior of the two maps with respect to degenerate Whittaker models.

Theorem 6.7 (i) *Let* $\pi \in \mathcal{A}_F$ *be either rigid or unitarizable. Then* $\mathrm{bc}_{E/F}(\pi)$ *has a degenerate Whittaker model given by the sequence* $\mathbf{d}(\pi)$.

(ii) *Let* $\pi \in \mathcal{A}_E$ *be either rigid or unitarizable. Then* $\mathrm{ai}_{E/F}(\pi)$ *has a degenerate Whittaker model given by the sequence* $\underbrace{\mathbf{d}(\pi) +_{\mathrm{c}} \cdots +_{\mathrm{c}} \mathbf{d}(\pi)}_{d-times}$.

Proof The result is an immediate consequence of Theorem 6.5, Theorem 6.6, and Lemma 6.4. ∎

7 Klyachko Models

We begin this section by recalling the definition of Klyachko models and the classification results for ladder representations with respect to these models that were obtained in [19]. We use these results then to show that the two maps preserve the Klyachko type of ladder representations in an appropriate sense.

7.1 Definition of Klyachko Types

For a decomposition $n = 2k + r$, let

$$H_{2k,r} = \left\{ \begin{pmatrix} h & X \\ 0 & u \end{pmatrix} : h \in \mathrm{Sp}_{2k}(F),\ X \in M_{2k\times r}(F),\ u \in U_r \right\}$$

and $\psi = \psi_{2k,r}$ be defined by $\psi\left(\begin{pmatrix} h & X \\ 0 & u \end{pmatrix}\right) = \psi_r(u)$. (See Section 2.5 for the definition of U_r and its character ψ_r.)

Definition 7.1 Let $\pi \in \mathcal{A}_F(n)$. If π is $(H_{2k,r}, \psi)$-distinguished for some decomposition $n = 2k + r$, then we say that it admits a Klyachko model of type r. In this case the integer r is referred to as the Klyachko type of π and denoted by $r(\pi)$.

We remark that $r(\pi)$ is well defined as by [23, Theorem 1] the Klyachko type of an irreducible representation is unique if it exists.

Remark 7.2 Note that for any $\pi \in \Pi(\mathrm{GL}_n(F))$, being $(H_{2k,r}, \psi)$-distinguished is independent of the choice of non-trivial character ψ of F. Indeed, for any other character $\psi' \neq 1$, there is a diagonal matrix $a \in \mathrm{GL}_n(F)$ normalizing $H_{2k,r}$ such that $\psi'_{2k,r}(h) = \psi_{2k,r}(aha^{-1})$ for all $h \in H_{2k,r}$.

7.2 The Classification

We now recall the classification of ladder representations with respect to the Klyachko models.

7.2.1 Right Aligned Segments

We define the following relation on segments of cuspidal representations.

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Definition 7.3 For segments $\Delta = [\nu^a\sigma, \nu^b\sigma]$ and $\Delta' = [\nu^{a'}\sigma, \nu^{b'}\sigma]$, where a, a', b, $b' \in \mathbb{Z}$, we say that Δ' is *right-aligned* with Δ, and write $\Delta' \vdash \Delta$, if $a \geq a' + 1$ and $b = b' + 1$. We label this relation by the integer $r = s(a - a' - 1)$, where $\sigma \in \mathcal{A}_F^\circ(s)$, and write $\Delta' \vdash_r \Delta$.

Note, in particular, that $\Delta' \vdash_0 \Delta$ means that $\Delta = \nu\Delta'$.

We now provide a description of the ladder representations that admit any particular Klyachko model [19, Proposition 14.5 and Theorem 14.7].

Theorem 7.4 (i) *Let* $\mathfrak{m} = \{\Delta_1, \dots, \Delta_t\}$ *be a proper ladder such that* $L(\mathfrak{m}) \in \mathcal{L}_p \cap \mathcal{A}_F(n)$ *and let* $n = 2k + r$. *If* t *is odd, let* s *be such that* $L(\Delta_1) \in \mathcal{A}_F(s)$, *otherwise, set* $s = 0$. *Then* $L(\mathfrak{m})$ *is* $(H_{2k,r}, \psi)$*-distinguished if and only if* $\Delta_{t-2i} \vdash_{r_i} \Delta_{t-2i-1}$ *for some* r_i, $i = 0, \dots, \lfloor t/2 \rfloor - 1$ *and* $r = r_0 + \cdots + r_{\lfloor t/2 \rfloor - 1} + s$.

(ii) *Let* π *be a ladder representation and assume that* $\pi = \pi_1 \times \cdots \times \pi_l$ *is the unique decomposition of* π *as a product of proper ladder representations (see Proposition 5.4). Then* π *admits a Klyachko model if and only if* π_i *admits a Klyachko model for all* $i = 1, \dots, l$. *Furthermore, in that case* $r(\pi) = r(\pi_1) + \cdots + r(\pi_l)$.

7.3 Relationship With the Two Maps

We now prove Theorem 1.3. The analogous result for unitarizable representations and for the case of the base change map (Corollary 7.7 (i)) was obtained in [24, Corollary 6.1]. This was done there by observing that $r(\pi) = \sum_{i=0}^{\infty} \mathcal{V}(\pi)(2i+1)$ (using Lemma 3.5) and then using the fact that SL(2)-type of a unitarizable representation is preserved by base change. Unlike the unitarizable representations though, a ladder representation may not have a Klyachko model, and it is not *a priori* clear, for a representation π admitting a Klyachko model, if even $\mathrm{bc}_{E/F}(\pi)$ will admit one. However Theorem 7.4 allows us to determine precisely which ladders have a Klyachko model and enables us to prove Theorem 1.3 and Corollary 7.7 directly, without resorting to SL(2)-types, as we see below.

Theorem 7.5 (i) *Let* $\pi \in \mathcal{L}_F$. *Then* π *admits a Klyachko model if and only if* $\mathrm{bc}_{E/F}(\pi)$ *admits one. Moreover* $r(\mathrm{bc}_{E/F}(\pi)) = r(\pi)$.

(ii) *Let* $\pi \in \mathcal{L}_E$. *Then* π *admits a Klyachko model if and only if* $\mathrm{ai}_{E/F}(\pi)$ *admits one. Moreover* $r(\mathrm{ai}_{E/F}(\pi)) = dr(\pi)$.

Proof Fix $\sigma \in \mathcal{A}_F^\circ(m)$ such that $\mathrm{supp}(\pi) \in \sigma^{\mathbb{Z}}$ and write $\pi = \pi_{(\sigma)}$. Using Lemma 4.3, we write $\mathrm{bc}_{E/F}(\sigma) = \sigma_1 \times \cdots \times \sigma_t$, where $\sigma_i \in \mathcal{A}_E^\circ(\frac{m}{t})$. By Lemma 4.5, we get that $\mathrm{bc}_{E/F}(\pi) = \pi_{(\sigma_1)} \times \cdots \times \pi_{(\sigma_t)}$ (see Section 3.1.4 for the notation).

If π has a Klyachko model of type $r(\pi)$, then by Theorem 7.4 each $\pi_{(\sigma_i)}$ has a Klyachko model and $r(\pi_{(\sigma_i)}) = \frac{r(\pi)}{t}$. By the hereditary property of Klyachko models [19, Proposition 13.3], $\mathrm{bc}_{E/F}(\pi)$ has the Klyachko model of type $r(\pi)$. This gives us the "only if" part of (i).

For the "if" part, note that the cuspidal lines of σ_i and σ_j are pairwise disjoint if $i \neq j$ and thus, by [19, Proposition 13.4], each $\pi_{(\sigma_i)}$ admits a Klyachko model. Appealing to Theorem 7.4 again, we get that π admits a Klyachko model. ∎

We obtain the following corollary of Theorem 7.5.

Corollary 7.6 (i) *Let $\pi_i \in \mathcal{L}_F$ $(i = 1, \dots, k)$ be such that*

$$\pi_1 \times \cdots \times \pi_k \in \mathcal{L}_{\text{ind}}.$$

Moreover, assume that each π_i admits a Klyachko model and that π is rigid. Then $\mathrm{bc}_{E/F}(\pi)$ admits a Klyachko model with $r(\mathrm{bc}_{E/F}(\pi)) = \sum_{i=1}^{k} r(\pi_i)$.

(ii) *Let $\pi_i \in \mathcal{L}_E$ $(i = 1, \dots, k)$ be such that $\pi := \pi_1 \times \cdots \times \pi_k \in \mathcal{L}_{\text{ind}}$. Moreover, assume that each π_i admits a Klyachko model and that π is rigid. Then $\mathrm{ai}_{E/F}(\pi)$ admits a Klyachko model with $r(\mathrm{ai}_{E/F}(\pi)) = d(\sum_{i=1}^{k} r(\pi_i))$.*

Proof Fix $\sigma \in \mathcal{A}_F^{\circ}$ such that $\mathrm{supp}(\pi) \in \sigma^{\mathbb{Z}}$ and write $\pi = \pi_{(\sigma)}, \pi_i = (\pi_i)_{(\sigma)}$. By Lemma 5.5, for any $\sigma' \in \mathcal{A}_E^{\circ}$, the representation $(\pi_1)_{(\sigma')} \times \cdots \times (\pi_k)_{(\sigma')}$ is irreducible (see Section 3.1.4 for the notation) and is thus equal to $\pi_{(\sigma')}$. Therefore, by Lemma 4.3 and Lemma 4.5, we have

$$\mathrm{bc}_{E/F}(\pi) = \mathrm{bc}_{E/F}(\pi_1) \times \cdots \times \mathrm{bc}_{E/F}(\pi_k).$$

Theorem 7.5 along with the hereditary property of Klyachko models [19, Proposition 13.3] now gives us the corollary. ■

The hypothesis of rigidity is essential for Corollary 7.6 to hold. For example, let $\pi_1 = \kappa$, $\pi_2 = L([\nu_F, \nu_F^2], [\nu_F^2, \nu_F^3])$, and $\pi = \pi_1 \times \pi_2$. By Theorem 7.4 and the hereditary property of Klyachko models [19, Proposition 13.3], the irreducible representation π has a Klyachko model. On the other hand, Theorem 7.4 implies that $\mathrm{bc}_{E/F}(\pi)$ does not have any Klyachko model. Similar counterexamples can be constructed for the case of automorphic induction if π is not assumed to be rigid.

7.3.1 Unitarizable Case

The hypothesis of rigidity can be removed for the case of unitarizable representations. Let π be unitarizable. Write $\pi = \pi_1 \times \cdots \times \pi_k$ such that each π_i is either a unitarizable Speh representation or a representation of the form $\nu^{\alpha}\tau \times \nu^{-\alpha}\tau$, where $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ and τ is a unitarizable Speh representation. Theorem 7.4 shows that each Speh representation admits a Klyachko model and thus by [19, Proposition 13.3], so does π with its Klyachko type equal to $\sum_{i=1}^{k} r(\pi_i)$. Thus by Proposition 5.8, if $\pi \in \mathcal{A}_F^u$, respectively, $\pi \in \mathcal{A}_E^u$, then both π and $\mathrm{bc}_{E/F}(\pi)$, respectively, $\mathrm{ai}_{E/F}(\pi)$, have a Klyachko model. We further have the following.

Corollary 7.7 (i) *Let $\pi \in \mathcal{A}_F^u$. Then $r(\mathrm{bc}_{E/F}(\pi)) = r(\pi)$.*
(ii) *Let $\pi \in \mathcal{A}_E^u$. Then $r(\mathrm{ai}_{E/F}(\pi)) = dr(\pi)$.*

Proof Let $\pi \in \mathcal{A}_F^u$ and $\pi = \pi_1 \times \cdots \times \pi_k$ as above. Arguing as in the proof of Proposition 5.8, we get that $\mathrm{bc}_{E/F}(\pi) = \mathrm{bc}_{E/F}(\pi_1) \times \cdots \times \mathrm{bc}_{E/F}(\pi_k)$, where each $\mathrm{bc}_{E/F}(\pi_i) \in \mathcal{A}_E^u$ and admits a Klyachko model. If π_i is a unitarizable Speh representation, then by Theorem 7.5 $r(\pi_i) = r(\mathrm{bc}_{E/F}(\pi_i))$. Suppose now that $\pi_i = \nu^{\alpha}\tau \times \nu^{-\alpha}\tau$

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for $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ and a unitarizable Speh representation τ. As in the proof of Proposition 5.8, we have

$$\mathrm{bc}_{E/F}(\pi_i) = \mathrm{bc}_{E/F}(\nu^{-\alpha}\tau) \times \mathrm{bc}_{E/F}(\nu^{-\alpha}\tau).$$

By Theorem 7.5, we have $r(\mathrm{bc}_{E/F}(\nu^{-\alpha}\tau)) = r(\nu^{-\alpha}\tau)$ and $r(\mathrm{bc}_{E/F}(\nu^{\alpha}\tau)) = r(\nu^{\alpha}\tau)$. Therefore, even in this case

$$\begin{aligned} r(\mathrm{bc}_{E/F}(\pi_i)) &= r(\mathrm{bc}_{E/F}(\nu^{-\alpha}\tau)) + r(\mathrm{bc}_{E/F}(\nu^{-\alpha}\tau)) \\ &= r(\nu^{-\alpha}\tau) + r(\nu^{\alpha}\tau) = r(\pi_i). \end{aligned}$$

Thus we have $r(\mathrm{bc}_{E/F}(\pi_i)) = r(\pi_i)$ for every $i = 1, \dots, k$.

By the hereditary property of Klyachko models [19, Proposition 13.3] we get $r(\mathrm{bc}_{E/F}(\pi)) = \sum_{i=1}^{k} r(\mathrm{bc}_{E/F}(\pi_i)) = \sum_{i=1}^{k} r(\pi_i) = r(\pi)$. ■

8 Fiber Under the Two Maps

We now investigate the fibers of the base change and automorphic induction maps. We begin by explicitly describing the fiber of an arbitrary rigid representation in the image.

8.1 Description of the Fiber Under a Rigid Representation

Lemma 8.1 (i) *Suppose that* $\Pi = L(\mathfrak{m})$ *is such that* $\mathrm{supp}(\Pi) \subseteq \sigma^{\mathbb{Z}}$ *for some* $\sigma \in \mathcal{A}_E^{\circ}$. *Write* $\mathfrak{m} = \mathfrak{m}_{(\sigma)}$. *Let* Π *be in the image of the map* $\mathrm{bc}_{E/F}$. *Let* κ *be a character of* $F^{\times}$ *with kernel equal to* $\mathrm{N}_{E/F}(E^{\times})$. *Then there exists* $\sigma' \in \mathcal{A}_F^{\circ}$ *such that* $\mathrm{bc}_{E/F}(\sigma') = \sigma$ *and the fiber* $\mathrm{bc}_{E/F}^{-1}(\Pi)$ *consists of all the representations of the form*

$$L(\mathfrak{m}_1) \times \kappa L(\mathfrak{m}_2) \times \cdots \times \kappa^{d-1} L(\mathfrak{m}_d), \tag{8.1}$$

where the multi-sets $\mathfrak{m}_i$ *are such that each* $\mathfrak{m}_i \subseteq \sigma'^{\mathbb{Z}}$ *and* $\mathfrak{m}_1 + \cdots + \mathfrak{m}_d = \mathfrak{m}_{(\sigma')}$. *(Some of the* $\mathfrak{m}_i$ *can possibly be empty.)*

(ii) *Suppose that* $\Pi = L(\mathfrak{m})$ *is such that* $\mathrm{supp}(\Pi) \subseteq \sigma^{\mathbb{Z}}$ *for some* $\sigma \in \mathcal{A}_F^{\circ}$. *Let* Π *be in the image of the map* $\mathrm{ai}_{E/F}$. *Let* γ *be a fixed non-trivial element of* $\mathrm{Gal}(E/F)$. *Then there exists* $\sigma' \in \mathcal{A}_E^{\circ}$ *such that* $\mathrm{ai}_{E/F}(\sigma') = \sigma$ *and the fiber* $\mathrm{ai}_{E/F}^{-1}(\Pi)$ *consists of all the representations of the form*

$$L(\mathfrak{m}_1) \times L(\mathfrak{m}_2) \times \cdots \times L(\mathfrak{m}_d),$$

where the multi-sets $\mathfrak{m}_i$ *are such that* $\mathfrak{m}_i \subseteq ((\sigma')^{\gamma^i})^{\mathbb{Z}}$, *with*

$$\mathfrak{m}_i = (\mathfrak{m}_i)_{((\sigma')^{\gamma^i})},$$

for each i and $(\mathfrak{m}_1)_{(\sigma)} + \cdots + (\mathfrak{m}_d)_{(\sigma)} = \mathfrak{m}$. *(Some of the* $\mathfrak{m}_i$ *can possibly be empty.)*

Proof Note first that if σ lies in the image of the base change map, then by the local Langlands correspondence there exists a $\sigma' \in \mathcal{A}_F^{\circ}$ such that $\mathrm{rec}(\sigma')|_{W_E} = \mathrm{rec}(\sigma)$. Fix such a σ'. In this case, by the local Langlands correspondence and by a standard result in Clifford theory [12, Proposition 2.8.2] applied to the restriction of $\mathrm{rec}(\sigma')$ to W_E, we have

$$\mathrm{bc}_{E/F}^{-1}(\sigma) = \{\kappa^i \sigma' \mid i = 0, \dots, d-1\}. \tag{8.2}$$

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Let $\pi \in \mathrm{bc}_{E/F}^{-1}(\Pi)$. Write $\Pi = L(\Delta_1, \dots, \Delta_t)$ and $\pi = L(\Delta'_1, \dots, \Delta'_{t'})$. Thus

$$\mathrm{rec}(\pi) = \bigoplus_{i=1}^{t'} (\rho(\Delta'_i), N(\Delta'_i)).$$

If there exists a $\sigma'' \in \mathrm{supp}(\pi)$ such that $\mathrm{bc}_{E/F}(\sigma'')$ is not cuspidal, then we get a contradiction to the rigidity of Π (using Lemma 4.3). In other words, $(\rho(\Delta'_i)|_{W_E}, N(\Delta'_i))$ is an indecomposable Weil–Deligne representation of W_E for every i. Thus $t = t'$ and, renumbering the segments if necessary, we can assume that $(\rho(\Delta'_i)|_{W_E}, N(\Delta'_i)) = (\rho(\Delta_i), N(\Delta_i))$. Hence the segments Δ and Δ' are of the same length, and if $\Delta_i = [\nu_E^{a_i}\sigma, \nu_E^{b_i}\sigma]$, then Δ'_i can be written as $[\nu_F^{a_i}\sigma_i, \nu_F^{b_i}\sigma_i]$ where $\mathrm{rec}(\sigma_i)|_{W_E} = \mathrm{rec}(\sigma)$. Therefore $\mathrm{bc}_{E/F}(\sigma_i) = \sigma$, and as noted in (8.2), this implies that $\sigma_i = \kappa^{k_i}\sigma'$ for some integer k_i. Thus we have that π is a representation of the form described in (8.1).

The converse statement that every representation in $\mathcal{A}_F$ of the form described in (8.1) lies in $\mathrm{bc}_{E/F}^{-1}(\Pi)$ follows directly from the definition of base change and the observation in (8.2). ∎

Remark 8.2 In the statement of Lemma 8.1 (i), we have $\sigma' \ncong \kappa^i\sigma'$ for $i = 1, \dots, d-1$. To see this, observe that, by Frobenius reciprocity, we have

$$\begin{aligned}
&\mathrm{Hom}_{W_E}(\mathrm{rec}(\sigma')|_{W_E}, \mathrm{rec}(\sigma')|_{W_E})\\
&\qquad \cong \mathrm{Hom}_{W_F}(\mathrm{rec}(\sigma'), \mathrm{ind}_{W_E}^{W_F}(\mathrm{rec}(\sigma')|_{W_E}))\\
&\qquad \cong \mathrm{Hom}_{W_F}(\mathrm{rec}(\sigma'), \mathrm{rec}(\sigma') \oplus \cdots \oplus \mathrm{rec}(\sigma')\kappa^{d-1}),
\end{aligned}$$

and we get a contradiction to the fact that $\mathrm{rec}(\sigma')|_{W_E}$ is irreducible if $\sigma' \cong \kappa^i\sigma'$ for any $i \in \{1, \dots, d-1\}$. Similarly, in the statement of Lemma 8.1 (ii), we have $\sigma' \ncong (\sigma')^{\gamma^i}$ for $i = 1, \dots, d-1$.

In particular, if s is the size of the multi-set $\mathfrak{m}$, then the cardinality of the set $\mathrm{bc}_{E/F}^{-1}(\Pi)$ and $\mathrm{ai}_{E/F}^{-1}(\Pi)$ in the respective situations is d^s.

8.1.1 The Case of Generic Rigid Representations

Lemma 8.3 (i) *Suppose that $\Pi = L(\mathfrak{m})$ is rigid and in the image of the base change map. Then Π is generic if and only if every representation in $\mathrm{bc}_{E/F}^{-1}(\Pi)$ is generic.*

(ii) *Suppose that $\Pi = L(\mathfrak{m})$ is rigid and in the image of the automorphic induction map. Then Π is generic if and only if every representation in $\mathrm{ai}_{E/F}^{-1}(\Pi)$ is generic.*

Proof Suppose first that Π is generic. Let $\mathfrak{m} = \{\Delta_1, \dots, \Delta_s\}$ be such that $\mathrm{supp}(\mathfrak{m}) \subseteq \sigma^{\mathbb{Z}}$ and write $\mathfrak{m} = \mathfrak{m}_{(\sigma)}$. Fix a $\sigma' \in \mathrm{bc}_{E/F}^{-1}(\sigma)$. By [30, Theorem 9.7], we have $\Delta_i \nprec \Delta_j$ for every i and j. Thus $\kappa^{k_i}(\Delta_i)_{(\sigma')} \nprec \kappa^{k_j}(\Delta_j)_{(\sigma')}$ for any integers k_i and k_j, and so by [30, Theorem 9.7] and Lemma 8.1, every representation in $\mathrm{bc}_{E/F}^{-1}(\Pi)$ is generic. The converse is obtained in a similar manner by applying [30, Theorem 9.7] to a rigid representation in the fiber. ∎

The next result follows immediately from Lemma 8.1.

Proposition 8.4 (i) *Let* $\Pi \in \mathcal{L}_E$. *Then* $\mathrm{bc}_{E/F}^{-1}(\Pi) \subseteq \mathcal{L}_{\mathrm{ind},F}$.
(ii) *Let* $\Pi \in \mathcal{L}_F$. *Then* $\mathrm{ai}_{E/F}^{-1}(\Pi) \subseteq \mathcal{L}_{\mathrm{ind},E}$. ∎

8.2 Estimates for Klyachko Types

Definition 8.5 (i) Suppose $\Pi = \mathrm{bc}_{E/F}(\pi)$ for some $\pi \in \mathcal{A}_F(n)$, is rigid, and admits a Klyachko model. Set $\mathcal{H}_{\pi,r} = \mathrm{Hom}_{H_{2k,r}}(\pi,\psi)$, where $2k + r = n$. Define $d_{\Pi,\mathrm{bc}} = \sum_{\pi \in \mathrm{bc}_{E/F}^{-1}(\Pi)} \dim_{\mathbb{C}}(\mathcal{H}_{\pi,r(\Pi)})$.

(ii) Suppose $\Pi = \mathrm{ai}_{E/F}(\pi)$ for some $\pi \in \mathcal{A}_E(n)$, is rigid, and admits a Klyachko model. Set $\mathcal{H}_{\pi,r} = \mathrm{Hom}_{H_{2k,r}}(\pi,\psi)$, where $2k + r = n$. Define

$$d_{\Pi,\mathrm{ai}} = \sum_{\pi \in \mathrm{ai}_{E/F}^{-1}(\Pi)} \dim_{\mathbb{C}}(\mathcal{H}_{\pi,\frac{r(\Pi)}{d}}). \tag{8.3}$$

Since an irreducible representation admits a Klyachko model with multiplicity at most one [23, Theorem 1], the integer $d_{\Pi,\mathrm{bc}}$ (or $d_{\Pi,\mathrm{ai}}$) is equal to the number of elements in the preimage of the respective maps that have the corresponding Klyachko type. For example, if $\Pi = L(\mathfrak{m})$ is a rigid generic representation in the image of the base change map, then by Lemma 8.3 and Remark 8.2, $d_{\Pi,\mathrm{bc}} = d^s$, where s is the size of the multi-set $\mathfrak{m}$.

Remark 8.6 Consider the representation Π in Definition 8.5 (ii). Let $G = \mathrm{GL}_{nd}(F)$, $r = r(\Pi)$, $k = \frac{nd-r}{2}$, and

$$M_{(2k,r)} = \{\mathrm{diag}(g_1, g_2) \mid g_1 \in \mathrm{GL}_{2k}(F),\ g_2 \in \mathrm{GL}_r(F)\}.$$

Moreover, let $P_{(2k,r)}$ be the standard parabolic subgroup of G corresponding to $M_{(2k,r)}$. Note that $\mathrm{Ind}_{\mathrm{Sp}_{2k}(F)\times U_r}^{M_{(2k,r)}}(1 \otimes \psi) = \mathrm{Ind}_{H_{2k,r}}^{P_{(2k,r)}}(\psi)|_{M_{(2k,r)}}$. By Frobenius reciprocity and transitivity of induction, we have

$$\begin{aligned}\mathrm{Hom}_{H_{2k,r}}(\Pi,\psi) &\simeq \mathrm{Hom}_G(\Pi, \mathrm{Ind}_{H_{2k,r}}^G(\psi)) \\ &\simeq \mathrm{Hom}_{M_{(2k,r)}}(\mathbf{r}_{M_{(2k,r)},G}(\Pi), \mathrm{Ind}_{\mathrm{Sp}_{2k}(F)\times U_r}^{M_{(2k,r)}}(1\otimes\psi)),\end{aligned}$$

where $\mathbf{r}_{M_{(2k,r)},G}(\Pi)$ is the normalized Jacquet module of Π with respect to $P_{(2k,r)}$. By assumption $\mathrm{Hom}_{H_{2k,r}}(\Pi,\psi) \neq 0$ which further implies that

$$\mathbf{r}_{M_{(2k,r)},G}(\Pi) \neq 0.$$

Note that every cuspidal representation in $\mathrm{supp}(\Pi)$ lies in $\mathcal{A}_F(m)$ for some m such that $d \mid m$. Thus $r(\Pi)$ has to be divisible by d and the expression in the right-hand side of (8.3) is well defined.

Lemma 8.7 (i) *Suppose that* $\Pi = L(\mathfrak{m}) \in \mathcal{L}_E \cap \mathcal{A}_E(2n)$ *such that* Π *admits the symplectic model. Further suppose that* Π *is in the image of the base change map. Denote by* s *the size of the multi-set* $\mathfrak{m}$. *Then*

$$d_{\Pi,\mathrm{bc}} = d^{s/2}.$$

(ii) *Suppose that* $\Pi = L(\mathfrak{m}) \in \mathcal{L}_F \cap \mathcal{A}_F(2dn)$ *such that* Π *admits the symplectic model. Further suppose that* Π *is in the image of the automorphic induction map. Denote by s the size of the multi-set* $\mathfrak{m}$. *Then*

$$d_{\Pi,\mathrm{ai}} = d^{s/2}.$$

Proof Since $r(\Pi) = 0$, by Theorem 7.4 s is even. Let $s = 2s'$ and let $\mathfrak{m} = \{\Delta_1, \dots, \Delta_{2s'}\}$. Appealing to Theorem 7.4 again, we get that $\Delta_{2j+1} = \nu\Delta_{2j+2}$ for every $j = 0, \dots, s'-1$. Let $\pi \in \mathrm{bc}_{E/F}^{-1}(\Pi)$. Then by Lemma 8.1 (i) π is a representation of the form described in (8.1), and by [19, Lemma 5.9], the representations $\kappa^{i-1}L(\mathfrak{m}_i)$ has a symplectic model for every i, if π does so. Theorem 7.4 applied to each $\kappa^{i-1}L(\mathfrak{m}_i)$ gives us that if $(\Delta_1)_{(\sigma')} \in \mathfrak{m}_i$, then $\nu_F^{-1}(\Delta_1)_{(\sigma')} \in \mathfrak{m}_i$ as well. Since $\mathfrak{m}$ is a ladder multi-set, the segment $\nu_F^{-1}(\Delta_1)_{(\sigma')}$ can only be equal to $(\Delta_2)_{(\sigma')}$. An easy induction gives us that for all $j = 1, \dots, s'$ if $(\Delta_{2j-1})_{(\sigma')} \in \mathfrak{m}_i$ for some i, then $(\Delta_{2j})_{(\sigma')} \in \mathfrak{m}_i$ as well. Thus any multi-set $\mathfrak{m}_i$ is of the form $\{\Delta_{2j_1+1}, \Delta_{2j_1+2}, \dots, \Delta_{2j_a+1}, \Delta_{2j_a+2}\}_{(\sigma')}$ for some distinct integers $\{j_1, j_2, \dots, j_a\} \subseteq \{0, \dots, s'-1\}$. By the observation in Remark 8.2, we get that $d_{\Pi,\mathrm{bc}} = d^{\frac{s}{2}}$. ∎

Thus for a ladder representation Π having Whittaker or symplectic models, the integers $d_{\Pi,\mathrm{bc}}$ and $d_{\Pi,\mathrm{ai}}$ depend only on the degree of the field extension and the size of the underlying multi-set when Π is expressed using the Langlands classification. For other Klyachko models this is not the case. For example, consider the ladder representations $\Pi_1 = L([\nu_E^2, \nu_E^3], [\nu_E, \nu_E^2], [1, \nu_E])$ and $\Pi_2 = L([\nu_E^3, \nu_E^4], [\nu_E, \nu_E^2], [1, \nu_E])$. By Theorem 7.4 both Π_1 and Π_2 admit Klyachko models and $r(\Pi_1) = r(\Pi_2) = 2$. Using Lemma 8.1, Theorem 7.4, and [19, Proposition 13.4], it is easy to check that $d_{\Pi_1,\mathrm{bc}} > d_{\Pi_2,\mathrm{bc}}$. Similar examples can be constructed in the case of automorphic induction as well.

However we can say the following.

Lemma 8.8 (i) *Suppose that* $\Pi = L(\mathfrak{m}) \in \mathcal{L}_E$ *such that* Π *admits a Klyachko model other than the Whittaker and the symplectic models. Further suppose that* Π *is in the image of the base change map. Denote by s the size of the multi-set* $\mathfrak{m}$. *Then* $d^{s/2} \le d_{\Pi,\mathrm{bc}}$.

(ii) *Suppose that* $\Pi = L(\mathfrak{m}) \in \mathcal{L}_F$ *such that* Π *admits a Klyachko model other than the Whittaker and the symplectic models. Further suppose that* Π *is in the image of the automorphic induction map. Denote by s the size of the multi-set* $\mathfrak{m}$. *Then* $d^{s/2} \le d_{\Pi,\mathrm{ai}}$.

Proof Let us first consider the case when $\Pi \in \mathcal{L}_{p,E}$. Suppose first that $s = 2s'+1$ for some integer s' and let $\mathfrak{m} = \{\Delta_1, \dots, \Delta_{2s'+1}\}$. Let π be a representation of the form described in (8.1) such that each $\mathfrak{m}_i$ is a multi-set of the form

$$\{\Delta_{2i_1}, \Delta_{2i_1+1}, \dots, \Delta_{2i_a}, \Delta_{2i_a+1}\}_{(\sigma')}$$

or

$$\{\Delta_1, \Delta_{2i_1}, \Delta_{2i_1+1}, \dots, \Delta_{2i_a}, \Delta_{2i_a+1}\}_{(\sigma')}$$

for some distinct integers $\{i_1, i_2, \dots, i_a\} \subseteq \{1, \dots, s'\}$, and $\mathfrak{m}_1 + \cdots + \mathfrak{m}_d = \mathfrak{m}_{(\sigma')}$. Note that by Lemma 8.1 (i), the representation $\pi \in \mathrm{bc}_{E/F}^{-1}(\Pi)$, while by Theorem 7.4

and [19, Proposition 13.4] it admits a Klyachko model with $r(\pi) = r(\Pi)$. By the observation in Remark 8.2, we get that $d_{\Pi,\mathrm{bc}} \geq d^{s'+1}$, which proves the lemma when s is an odd integer. The case when s is even is dealt with similarly, which proves the statement for all proper ladders.

Now let $\Pi \in \mathcal{L}_E$. Using Proposition 5.4, write $\Pi = \Pi_1 \times \cdots \times \Pi_k$, where each $\Pi_i \in \mathcal{L}_{p,E}$ for $1 \leq i \leq k$. It follows easily from Lemma 8.1 (i) that

$$\mathrm{bc}_{E/F}^{-1}(\Pi) = \{\pi_1 \times \cdots \times \pi_k \mid \pi_i \in \mathrm{bc}_{E/F}^{-1}(\Pi_i),\ \forall\, 1 \leq i \leq k\}. \tag{8.4}$$

Using (8.4) and Theorem 7.4 (ii) we get that $d_{\Pi,\mathrm{bc}} \geq \prod_{i=1}^{k} d_{\Pi_i,\mathrm{bc}}$. The statement for proper ladders proved above now implies the statement for $\Pi \in \mathcal{L}_E$. ■

We have the following theorem summarizing the results of this section.

Theorem 8.9 (i) *Suppose that $\Pi = L(\mathfrak{m}) \in \mathcal{L}_E$ such that Π admits a Klyachko model and is in the image of the base change map. Denote by s the size of the multi-set $\mathfrak{m}$. Then the set $\mathrm{bc}_{E/F}^{-1}(\Pi)$ has cardinality d^s and*

$$\frac{1}{2} \leq \frac{\log_d(d_{\Pi,\mathrm{bc}})}{s} \leq 1.$$

If Π has the symplectic model, then the lower bound is an equality. The upper bound is an equality if and only if Π has the Whittaker model.

(ii) *Suppose that $\Pi = L(\mathfrak{m}) \in \mathcal{L}_F$ such that Π admits a Klyachko model and is in the image of the automorphic induction map. Denote by s the size of the multi-set $\mathfrak{m}$. Then the set $\mathrm{ai}_{E/F}^{-1}(\Pi)$ has cardinality d^s and*

$$\frac{1}{2} \leq \frac{\log_d(d_{\Pi,\mathrm{ai}})}{s} \leq 1.$$

If Π has the symplectic model, then the lower bound is an equality. The upper bound is an equality if and only if Π has the Whittaker model.

Remark 8.10 It would be an interesting problem to find invariants for ladder representations, and, more generally, for rigid representations, that determine the integers $d_{\Pi,\mathrm{bc}}$,respectively, $d_{\Pi,\mathrm{ai}}$, completely for Π with a given Klyachko type, and study their asymptotic behavior in the manner of Theorem 8.9.

8.2.1 Example of a Speh Representation

Let $\Pi = L(\mathfrak{m}) \in \mathcal{A}_E$ be a Speh representation in the image of the base change map and as above, let $|\mathfrak{m}| = s$. The simple structure of the representation allows us to obtain a precise value for the integer $d_{\Pi,\mathrm{bc}}$ in this case for any Klyachko model. If s is even, then by Theorem 7.4 (i) Π has the symplectic model. Thus, in this case $d_{\Pi,\mathrm{bc}} = d^{\frac{s}{2}}$ (by Lemma 8.7 (i)). So let $s = 2s'+1$. Let Δ be an arbitrary segment in $\mathfrak{m}$ and let the integer m be such that $L(\Delta) \in \mathcal{A}_E(m)$. Then applying Theorem 7.4 (i) again, we get that Π admits a Klyachko model and $r(\Pi) = m$. Arguing as in the proof of Lemma 8.7, in this case it is easy to see that $d_{\Pi,\mathrm{bc}} = (s'+1)d^{(s'+1)} - s'd^{s'}$. For a Speh representation $\Pi \in \mathcal{A}_F$ in the image of the automorphic induction map, the integer $d_{\Pi,\mathrm{ai}}$ can be calculated similarly.

9 The Case of a General Cyclic Extension

In this section let E/F denote a finite cyclic extension of non-archimedean local fields. We will now prove some of our results for the case when $[E:F]$ is a prime number in this more general setting. This will be done using the transitivity of the base change and automorphic induction maps. To that end, fix a chain of non-archimedean local fields $F = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_l = E$ such that E_{i+1}/E_i is a cyclic extension of prime degree for all $i = 0, \dots, l-1$.

9.1 The Unitarizable Case

Theorem 9.1 (i) *Let $\pi \in \mathcal{A}_F^u$. Then we have the following.*
- (a) *The representation* $\mathrm{bc}_{E/F}(\pi) \in \mathcal{A}_E^u$.
- (b) $\mathcal{V}(\pi) = \mathcal{V}(\mathrm{bc}_{E/F}(\pi))$.
- (c) $r(\pi) = r(\mathrm{bc}_{E/F}(\pi))$.

(ii) *Let $\pi \in \mathcal{A}_E^u$. Then we have the following.*
- (a) *The representation* $\mathrm{ai}_{E/F}(\pi) \in \mathcal{A}_F^u$.
- (b) $\mathcal{V}(\mathrm{ai}_{E/F}(\pi)) = d\mathcal{V}(\pi)$.
- (c) $r(\mathrm{ai}_{E/F}(\pi)) = dr(\pi)$.

Proof Note that by the transitivity of the base change map, we have

$$\mathrm{bc}_{E_i/F}(\pi) = \mathrm{bc}_{E_i/E_{i-1}} \circ \mathrm{bc}_{E_{i-1}/E_{i-2}} \circ \cdots \circ \mathrm{bc}_{E_1/F}(\pi),$$

for any $i = 1, \dots, l$. We have that $\mathrm{bc}_{E_i/F}(\pi) \in \mathcal{A}_{E_i}^u$, for any $i = 1, \dots, l$, by repeatedly using Proposition 5.8. This demonstrates part (a). Parts (b) and (c) now follow similarly by repeatedly using Theorem 6.6 and Corollary 7.7, respectively. ■

Remark 9.2 Part (i) of Theorem 9.1 was also proved in [24].

9.2 The Case of Representations Induced From the Ladder Class

9.2.1 Some Preliminary Lemmas

In order to deal with the case of irreducible representations that are induced from ladder representations, we need a few preliminary lemmas. We will obtain these lemmas for a slightly larger class of representations that we now introduce. For an arbitrary non-archimedean local field K, denote by $\mathcal{A}_K^{\square}$ the set of all $\pi \in \mathcal{A}_K$ such that $\pi \times \pi$ is irreducible. For $\pi \in \mathcal{L}_{\mathrm{ind},K}$ we have that $\pi \times \pi$ is irreducible (see [16, Lemma 6.17, Proposition 6.20, Lemma 6.21]), and thus $\mathcal{L}_{\mathrm{ind},K} \subseteq \mathcal{A}_K^{\square}$. The set $\mathcal{A}_K^{\square}$ has recently been studied; in particular, a combinatorial criterion to characterize the irreducibility of $\pi \times \pi$ ($\pi \in \mathcal{A}_K$) has been provided, which makes this class more explicit [17, Theorem 7.1]. The purpose of dealing with this larger class for the next few lemmas is to emphasize to the readers that this is precisely the property of ladder representations (and of those that are irreducibly induced from them) that goes into the proofs of these lemmas.

Lemma 9.3 (i) *Let $\sigma \in \mathcal{A}_F^{\circ}$. Fix an i such that $1 \le i \le l$. Then*

$$\mathrm{bc}_{E_i/F}(\sigma) = \sigma_{i,1} \times \cdots \times \sigma_{i,t_i}, \tag{9.1}$$

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where $\sigma_{i,j}$, $j = 1, \ldots, t_i$ are cuspidal representations in $\mathcal{A}^{\circ}_{E_i}$ such that $\sigma_{i,j} \neq \nu^{\alpha}_{E_i}\sigma_{i,j'}$, for any $\alpha \in \mathbb{R}^{\times}$ and $1 \leq j, j' \leq t_i$.

(ii) *Let $\sigma \in \mathcal{A}^{\circ}_{E}$. Fix an i such that $0 \leq i \leq l - 1$. Then*

$$\mathrm{ai}_{E/E_i}(\sigma) = \sigma_{i,1} \times \cdots \times \sigma_{i,t_i}, \tag{9.2}$$

where $\sigma_{i,j}$, $j = 1, \ldots, t_i$ are cuspidal representations in $\mathcal{A}^{\circ}_{E_i}$ such that $\sigma_{i,j} \neq \nu^{\alpha}_{E_i}\sigma_{i,j'}$, for any $\alpha \in \mathbb{R}^{\times}$ and $1 \leq j, j' \leq t_i$.

Proof Suppose first that σ is unitarizable. Since the base change map is transitive, by repeatedly applying Lemma 4.3 (iii) and Lemma 4.2 to σ, for cyclic extensions E_{j+1}/E_j, $j = 0, \ldots, i-1$, we get that $\mathrm{bc}_{E_i/F}(\sigma) = \sigma_{i,1} \times \cdots \times \sigma_{i,t_i}$ for some $t_i \in \mathbb{N}$ and some cuspidals $\sigma_{i,1}, \ldots, \sigma_{i,t_i} \in \mathcal{A}^{u}_{E_i}$. (Note that the hypothesis of Lemma 4.2 follows from the fact that the parabolic induction of unitarizable representations is irreducible.) The result for unitarizable cuspidal representations follows. The general case follows from the unitarizable case by the fact that $\mathrm{bc}_{E_i/F}(\nu^{\alpha}_{F}\sigma) = \nu^{\alpha}_{E_i}\,\mathrm{bc}_{E_i/F}(\sigma)$ for any $\alpha \in \mathbb{R}$. ∎

Remark 9.4 Lemma 9.3 is a slightly weakened variant of Lemma 4.3 in this general setting of an arbitrary finite cyclic extension. Unlike the prime case, here the cuspidals appearing in the right-hand side of (9.1) (or that of (9.2)) can lie in the same cuspidal line; however, we are able to show that in case they do, they have to be isomorphic.

Lemma 9.5 (i) *Let $\sigma \in \mathcal{A}^{\circ}_{F}$ and let $\pi = \pi_{(\sigma)} \in \mathcal{A}^{\square}_{F}$ be such that* Supp $\pi \subseteq \sigma^{\mathbb{Z}}$*. Fix an i such that $1 \leq i \leq l$. Let $\mathrm{bc}_{E_i/F}(\sigma) = \sigma_{i,1} \times \cdots \times \sigma_{i,t_i}$ (see Lemma 9.3* (i)*). Then*

$$\mathrm{bc}_{E_i/F}(\pi) = \pi_{(\sigma_{i,1})} \times \cdots \times \pi_{(\sigma_{i,t_i})}.$$

(ii) *Let $\sigma \in \mathcal{A}^{\circ}_{E}$ and let $\pi = \pi_{(\sigma)} \in \mathcal{A}^{\square}_{E}$ be such that* Supp $\pi \subseteq \sigma^{\mathbb{Z}}$*. Fix an i such that $0 \leq i \leq l - 1$. Let $\mathrm{ai}_{E/E_i}(\sigma) = \sigma_{i,1} \times \cdots \times \sigma_{i,t_i}$ (see Lemma 9.3* (ii)*). Then $\mathrm{ai}_{E/E_i}(\pi) = \pi_{(\sigma_{i,1})} \times \cdots \times \pi_{(\sigma_{i,t_i})}$.*

Proof We will prove the statement by induction on the index i. The case $i = 1$ follows directly from Lemma 4.5. Suppose now that the statement is true for $1, \ldots, i-1$. Note that by the transitivity of the base change map and Lemma 9.3 we have

$$\begin{aligned} \mathrm{bc}_{E_i/E_{i-1}}(\sigma_{i-1,1} \times \cdots \times \sigma_{i-1,t_{i-1}}) &= \mathrm{bc}_{E_i/E_{i-1}} \circ \mathrm{bc}_{E_{i-1}/F}(\sigma) \\ &= \mathrm{bc}_{E_i/F}(\sigma) = \sigma_{i,1} \times \cdots \times \sigma_{i,t_i}. \end{aligned} \tag{9.3}$$

Also, using the transitivity of the base change map again and the induction hypothesis, we have

$$\begin{aligned} \mathrm{bc}_{E_i/F}(\pi) &= \mathrm{bc}_{E_i/E_{i-1}} \circ \mathrm{bc}_{E_{i-1}/F}(\pi) \\ &= \mathrm{bc}_{E_i/E_{i-1}}(\pi_{(\sigma_{i-1,1})} \times \cdots \times \pi_{(\sigma_{i-1,t_{i-1}})}). \end{aligned} \tag{9.4}$$

Consider the pair of representations $\pi_{(\sigma_{i,j})}$ and $\pi_{(\sigma_{i,j'})}$, $1 \leq j, j' \leq t_i$. Now either $\sigma_{i,j} \cong \sigma_{i,j'}$, in which case the representation $\pi_{(\sigma_{i,j})} \cong \pi_{(\sigma_{i,j'})}$, or $\pi_{(\sigma_{i,j})}$ and $\pi_{(\sigma_{i,j'})}$ are supported on different cuspidal lines. Since $\pi \times \pi$ is irreducible, so is $\pi_{(\sigma_{i,j})} \times \pi_{(\sigma_{i,j})}$ (follows from [17, Theorem 3.8]. In fact, by [17, Corollary 2.7], $\pi_{(\sigma_{i,j})} \times \pi_{(\sigma_{i,j})} \times \cdots \times \pi_{(\sigma_{i,j})}$ (n times) is irreducible for any $n \in \mathbb{N}$. This statement, along with [30, Proposition 8.5],

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implies that the representation $\pi_{(\sigma_{i,1})} \times \cdots \times \pi_{(\sigma_{i,t_i})}$ is irreducible. By Lemma 4.5 and (9.3) we have that the representation

$$\mathrm{bc}_{E_i/E_{i-1}}(\pi_{(\sigma_{i-1,1})}) \times \cdots \times \mathrm{bc}_{E_i/E_{i-1}}(\pi_{(\sigma_{i-1,t_{i-1}})})$$

equals a rearrangement of the representation $\pi_{(\sigma_{i,1})} \times \cdots \times \pi_{(\sigma_{i,t_i})}$. Since the latter is irreducible, we have that

$$\mathrm{bc}_{E_i/E_{i-1}}(\pi_{(\sigma_{i-1,1})}) \times \cdots \times \mathrm{bc}_{E_i/E_{i-1}}(\pi_{(\sigma_{i-1,t_{i-1}})}) \cong \pi_{(\sigma_{i,1})} \times \cdots \times \pi_{(\sigma_{i,t_i})}. \tag{9.5}$$

Therefore by (9.4), (9.5), and Lemma 4.2 we have

$$\mathrm{bc}_{E_i/F}(\pi) = \pi_{(\sigma_{i,1})} \times \cdots \times \pi_{(\sigma_{i,t_i})},$$

which finishes the proof of the induction step. ∎

Remark 9.6 Let π be a rigid representation in $\mathcal{A}_F^\square$, respectively, $\mathcal{A}_E^\square$. It follows easily from Lemma 9.5 that $\mathrm{bc}_{E/F}(\pi) \times \mathrm{bc}_{E/F}(\pi)$, respectively, $\mathrm{ai}_{E/F}(\pi) \times \mathrm{ai}_{E/F}(\pi)$), is irreducible. In other words, the base change, respectively, automorphic induction, map takes rigid representations in $\mathcal{A}_F^\square$, respectively, $\mathcal{A}_E^\square$, to a representation in $\mathcal{A}_E^\square$, respectively, $\mathcal{A}_F^\square$. As in the case of representations that are irreducibly induced from ladder representations, the hypothesis of rigidity is essential. Examples of representations π, with $\pi \times \pi$ irreducible and $\mathrm{bc}_{E/F}(\pi) \times \mathrm{bc}_{E/F}(\pi)$, respectively, $\mathrm{ai}_{E/F}(\pi) \times \mathrm{ai}_{E/F}(\pi)$, reducible can easily be constructed along the lines of the example provided in Section 5.2. However, since these facts have no bearing on this article we will not provide any further details.

We will need the following auxiliary result on irreducibility. The proof of the lemma is similar to that of [16, Lemma 6.17] but for the sake of completeness we provide an argument here.[3]

Lemma 9.7 *Let K be a non-archimedean local field and let $\pi_1, \dots, \pi_k \in \mathcal{A}_K$. Suppose that all but at most one of the π_i's lie in the set $\mathcal{A}_K^\square$. Then $\pi_1 \times \cdots \times \pi_k$ is irreducible if and only if $\pi_i \times \pi_j$ is irreducible for all $i < j$.*

Proof The "only if" part of the statement is obvious. We will prove the other direction by induction on k. The case $k = 2$ is clear. For the induction step, suppose that the statement holds for $k-1$. Note that without loss of generality we can assume that $\pi_k \in \mathcal{A}_K^\square$. Set $\tau = \pi_1 \times \cdots \times \pi_{k-1}$. We wish to demonstrate that $\tau \times \pi_k \in \mathcal{A}_K$. By the induction hypothesis $\tau \in \mathcal{A}_K$. By [16, Theorem 4.6, Remark 4.4] (see also [10, 11]), we have that $\pi_k \times \tau$ is irreducible if and only if $\pi_k \times \tau \cong \tau \times \pi_k$. Since $\pi_k \times \pi_i$ is irreducible for all $i = 1, \dots, k-1$, we have that $\pi_k \times \pi_i \cong \pi_i \times \pi_k$ [30, Theorem 1.9] for all $i = 1, \dots, k-1$. This proves the induction step and the statement. ∎

Lemma 9.8 (i) *Let $\pi_j \in \mathcal{A}_F^\square$, $j = 1, \dots, k$, be such that*

$$\pi := \pi_1 \times \cdots \times \pi_k \in \mathcal{A}_F.$$

[3] We were informed that a more general version of Lemma 9.7 also follows from certain results on quantum affine algebra obtained in [8]. However we still provide the proof since we feel that this proof is more transparent. We are grateful to Erez Lapid for informing us of this reference.

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Moreover suppose that π is rigid. Fix an i such that $1 \le i \le l$. Then

$$\mathrm{bc}_{E_i/F}(\pi) = \mathrm{bc}_{E_i/F}(\pi_1) \times \cdots \times \mathrm{bc}_{E_i/F}(\pi_k).$$

(ii) *Let $\pi_j \in \mathcal{A}_E^{\square}$, $j = 1, \dots, k$ be such that $\pi := \pi_1 \times \cdots \times \pi_k \in \mathcal{A}_E$. Moreover, suppose that π is rigid. Fix an i such that $0 \le i \le l - 1$. Then*

$$\mathrm{ai}_{E/E_i}(\pi) = \mathrm{ai}_{E/E_i}(\pi_1) \times \cdots \times \mathrm{ai}_{E/E_i}(\pi_k).$$

Proof In view of Lemma 4.2, it is enough to show that

$$\mathrm{bc}_{E_i/F}(\pi_1) \times \cdots \times \mathrm{bc}_{E_i/F}(\pi_k)$$

is irreducible. Fix $\sigma \in \mathcal{A}_F^{\circ}$ such that $\mathrm{supp}(\pi) \subseteq \sigma^{\mathbb{Z}}$ and write $\pi_j = (\pi_j)_{(\sigma)}$ for all $j = 1, \dots, k$. Using Lemma 9.3 write $\mathrm{bc}_{E_i/F}(\sigma) = \sigma_{i,1} \times \cdots \times \sigma_{i,t_i}$. By Lemma 9.5, we have that

$$\mathrm{bc}_{E_i/F}(\pi_1) \times \cdots \times \mathrm{bc}_{E_i/F}(\pi_k) = \prod_{j=1}^{t_i} (\pi_1)_{(\sigma_{i,j})} \times \cdots \times \prod_{j=1}^{t_i} (\pi_k)_{(\sigma_{i,j})}. \tag{9.6}$$

Consider an arbitrary pair of representations $(\pi_k)_{(\sigma_{i,j})}$ and $(\pi_{k'})_{(\sigma_{i,j'})}$ appearing in the right-hand side of (9.6). Now either $\sigma_{i,j} \cong \sigma_{i,j'}$ in which case, as the representation $\pi_k \times \pi_{k'}$ is irreducible, the representation $(\pi_k)_{(\sigma_{i,j})} \times (\pi_{k'})_{(\sigma_{i,j'})}$ is also irreducible (follows from [17, Theorem 3.8]), or $\sigma_{i,j} \ncong \sigma_{i,j'}$ in which case we have that $(\pi_k)_{(\sigma_{i,j})}$ and $(\pi_{k'})_{(\sigma_{i,j'})}$ are supported on different cuspidal lines. In the latter case the representation $(\pi_k)_{(\sigma_{i,j})} \times (\pi_{k'})_{(\sigma_{i,j'})}$ is irreducible [30, Proposition 8.5]. By Lemma 9.7, the representation in the right-hand side of (9.6) is irreducible, which proves the result. ■

9.2.2 Main Results

Using the above lemmas we now extend some of the results in Sections 5 and 7 to this general setting.

Proposition 9.9 (i) *Let $\pi \in \mathcal{L}_{\mathrm{ind},F}$ be a rigid representation. Then $\mathrm{bc}_{E/F}(\pi) \in \mathcal{L}_{\mathrm{ind},E}$.*

(ii) *Let $\pi \in \mathcal{L}_{\mathrm{ind},E}$ be a rigid representation. Then $\mathrm{ai}_{E/F}(\pi) \in \mathcal{L}_{\mathrm{ind},F}$.*

Proof The result is a direct application of Lemma 9.8 and Lemma 9.5. ■

Theorem 9.10 (i) *Let $\pi_j \in \mathcal{L}_F$, $j = 1, \dots, k$, be such that*

$$\pi := \pi_1 \times \cdots \times \pi_k \in \mathcal{L}_{\mathrm{ind},F}.$$

Moreover, assume that each π_j admits a Klyachko model and that π is rigid. Then $\mathrm{bc}_{E/F}(\pi)$ admits a Klyachko model with $r(\mathrm{bc}_{E/F}(\pi)) = \sum_{j=1}^{k} r(\pi_j)$. In particular, if $\pi \in \mathcal{A}_F$ is a ladder representation that admits a Klyachko model, then $\mathrm{bc}_{E/F}(\pi)$ admits a Klyachko model with $r(\mathrm{bc}_{E/F}(\pi)) = r(\pi)$.

(ii) *Let $\pi_j \in \mathcal{L}_E$, $j = 1, \dots, k$, be such that $\pi := \pi_1 \times \cdots \times \pi_k \in \mathcal{L}_{\mathrm{ind},E}$. Moreover, assume that each π_j admits a Klyachko model and that π is rigid. Then $\mathrm{ai}_{E/F}(\pi)$ admits a Klyachko model with $r(\mathrm{ai}_{E/F}(\pi)) = d(\sum_{j=1}^{k} r(\pi_j))$. In particular, if $\pi \in \mathcal{A}_E$ is a*

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ladder representation that admits a Klyachko model, then $\mathrm{ai}_{E/F}(\pi)$ *admits a Klyachko model with* $r(\mathrm{ai}_{E/F}(\pi)) = dr(\pi)$.

Proof We first treat the case when $k = 1$. If π has a Klyachko model, then by Theorem 7.4 each $\pi_{(\sigma_{l,j})}$, $j = 1,\dots,t_l$, has a Klyachko model. The statement now follows from Lemma 9.5 and by the hereditary property of Klyachko models [19, Proposition 13.3].

The statement in the case of a general k follows directly from the statement in the case $k = 1$, Lemma 9.8, and the hereditary property of Klyachko models. ∎

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References

[1] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*. Annals of Mathematics Studies, 120, Princeton University Press, Princeton, 1989.

[2] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups. I*. Ann. Sci. École Norm. Sup. (4) 10(1977), no. 4, 441–472.

[3] D. Gourevitch, O. Offen, S. Sahi, and E. Sayag, *Existence of Klyachko models for* $\mathrm{GL}(n,\mathbb{R})$ *and* $\mathrm{GL}(n,\mathbb{C})$. J. Funct. Anal. 262(2012), no. 8, 3585–3601. https://doi.org/10.1016/j.jfa.2012.01.023.

[4] M. Gurevich, *Decomposition rules for the ring of representations of non-archimedean* GL_n. International Mathematics Research Notices, rnz006. https://doi.org/10.1093/imrn/rnz006.

[5] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*. Annals of Mathematics Studies, 151, Princeton University Press, Princeton, 2001.

[6] G. Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*. Invent. Math. 139(2000), no. 2, 439–455. https://doi.org/10.1007/s002220050012.

[7] G. Henniart and R. Herb, *Automorphic induction for* $\mathrm{GL}(n)$ *(over local nonarchimedean fields)*. Duke Math. J. 78(1995), 131–192. https://doi.org/10.1215/S0012-7094-95-07807-7.

[8] D. Hernandez, *Simple tensor products*. Invent. Math. 181(2010), 649–675. https://doi.org/10.1007/s00222-010-0256-9.

[9] M. J. Heumos and S. Rallis, *Symplectic-Whittaker models for* GL_n. Pacific J. Math. 146(1990), no. 2, 247–279.

[10] S.-J. Kang, M. Kashiwara, M. Kim, and S. Oh, *Monoidal categorification of cluster algebras*. arxiv:1412.8106.

[11] S.-J. Kang, M. Kashiwara, M. Kim, and S. Oh, *Simplicity of heads and socles of tensor products*. Compos. Math. 151(2015), no. 2, 377–396. https://doi.org/10.1112/S0010437X14007799.

[12] E. Kowalski, *An introduction to the representation theory of groups*. Graduate Studies in Mathematics, 155, American Mathematical Society, Providence, RI, 2014.

[13] Arno Kret and Erez Lapid, *Jacquet modules of ladder representations*. C. R. Math. Acad. Sci. Paris 350(2012), no. 21–22, 937–940. https://doi.org/10.1016/j.crma.2012.10.014.

[14] Stephen S. Kudla, *The local Langlands correspondence: the non-Archimedean case*. Proc. Sympos. Pure Math., 55, Part 2, Amer. Math. Soc., Providence, RI, 1994, pp. 365–391.

[15] E. Lapid and A. Mínguez, *On a determinantal formula of Tadić*. Amer. J. Math. 136(2014), no. 1, 111–142. https://doi.org/10.1353/ajm.2014.0006.

[16] E. Lapid and A. Mínguez, *On parabolic induction on inner forms of the general linear group over a non-archimedean local field*. Selecta Math. (N.S.) 22(2016), no. 4, 2347–2400. https://doi.org/10.1007/s00029-016-0281-7.

[17] E. Lapid and A. Mínguez, *Geometric conditions for □-irreducibility of certain representations of the general linear group over a non-archimedean local field*. Adv. Math. 339(2018), 113–190. https://doi.org/10.1016/j.aim.2018.09.027.
[18] G. Laumon, M. Rapoport, and U. Stuhler, *D-elliptic sheaves and the Langlands correspondence*. Invent. Math. 113(1993), no. 2, 217–338. https://doi.org/10.1007/BF01244308.
[19] A. Mitra, O. Offen, and E. Sayag, *Klyachko models for ladder representations*. Documenta Math. 22(2017), 611–657.
[20] A. Mitra and C. G. Venketasubramanian, *Base change and* $(\mathrm{GL}_n(F), \mathrm{GL}_{n-1}(F))$*-distinction*. J. Ramanujan Math. Soc. 31(2016), no. 2, 109–124.
[21] C. Mœglin and J.-L. Waldspurger, *Sur l'involution de Zelevinski*. J. Reine Angew. Math. 372(1986), 136–177. https://doi.org/10.1515/crll.1986.372.136.
[22] O. Offen and E. Sayag, *Global mixed periods and local Klyachko models for the general linear group*. Int. Math. Res. Not. (2008), Art. ID rnm 136. https://doi.org/10.1093/imrn/rnm136.
[23] O. Offen and E. Sayag, *Uniqueness and disjointness of Klyachko models*. J. Funct. Anal. 254(2008), no. 11, 2846–2865. https://doi.org/10.1016/j.jfa.2008.01.004.
[24] O. Offen and E. Sayag, *The* $\mathrm{SL}(2)$*-type and base change*. Represent. Theory 13(2009), 228–235. https://doi.org/10.1090/S1088-4165-09-00353-7.
[25] O. Offen and E. Sayag, *Klyachko periods for Zelevinsky modules*. Ramanujan J. (2018). https://doi.org/10.1007/s11139-018-0040-9.
[26] P. Scholze, *The local Langlands correspondence for* GL_n *over p-adic fields*. Invent. Math. 192(2013), no. 3, 663–715. https://doi.org/10.1007/s00222-012-0420-5.
[27] M. Tadić, *Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case)*. Ann. Sci. École Norm. Sup. 19(1986), no. 3, 335–382.
[28] T. Wedhorn, *The local Langlands correspondence for* $\mathrm{GL}(n)$ *over p-adic fields*. In: ICTP Lecture Notes Series, 21, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2008, pp. 237–320.
[29] A. Venkatesh, *The Burger-Sarnak method and operations on the unitary dual of* $\mathrm{GL}(n)$. Represent. Theory 9(2005), 268–286. https://doi.org/10.1090/S1088-4165-05-00226-8.
[30] A. V. Zelevinsky, *Induced representations of reductive* $\mathfrak{p}$*-adic groups. II. On irreducible representations of* $\mathrm{GL}(n)$. Ann. Sci. École Norm. Sup. 13(1980), no. 2, 165–210.

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