Canad. Math. Bull. Vol. **48** (1), 2005 pp. 80–89

Trivial Units for Group Rings with *G*-adapted Coefficient Rings

Allen Herman, Yuanlin Li, and M. M. Parmenter

Abstract. For each finite group G for which the integral group ring $\mathbb{Z}G$ has only trivial units, we give ring-theoretic conditions for a commutative ring R under which the group ring RG has nontrivial units. Several examples of rings satisfying the conditions and rings not satisfying the conditions are given. In addition, we extend a well-known result for fields by showing that if R is a ring of finite characteristic and RG has only trivial units, then G has order at most 3.

Let R be a ring, and G be a group. We say that the group ring RG has only trivial units if $U(RG) = U(R)G$. In the case where $R = \mathbb{F}$ is a field, the group algebra $\mathbb{F}G$ is known to have only trivial units if $\mathbb{F} = \mathbb{Z}_2$ and $|G| \leq 3$, or $\mathbb{F} = \mathbb{Z}_3$ and $|G| \leq 2$, or if $\mathbb{F}$ is any field and G is a u.p. group (see [6, 8]). It is a famous open problem whether or not the group algebra over any field of a torsion-free group G has only trivial units. (This is known for torsion-free nilpotent groups but not known for torsion-free supersolvable groups [1]).

The current study was motivated by the study of the normalizer property for group rings of finite groups and we will assume from now on that G is finite. This is the question of which finite groups have the property that the only inner automorphisms of $\mathbb{Z}G$ which normalize G are induced from inner automorphisms of G. The question becomes more difficult when $\mathbb{Z}$ is replaced by an arbitrary G-adapted ring—an integral domain of characteristic 0 with the property that no prime divisor of the order of G is invertible. If R is a G-adapted coefficient ring, then in the presence of an abelian normal subgroup N of G such that $R[G/N]$ has only trivial units, the proof that G has the normalizer property in $\mathbb{Z}G$ in [5, Proposition 2.20] can be extended to show that G has the normalizer property in RG. But for which groups G and G-adapted rings R does RG have only trivial units?

Since G-adapted rings contain $\mathbb{Z}$, this can only occur if $\mathbb{Z}G$ contains only trivial units, and such groups were classified by Higman [3]. These groups are abelian groups of exponent dividing 4 or 6, and Hamiltonian 2-groups. Our approach is to analyze what happens if $\mathbb{Z}$ is replaced by a G-adapted ring R in these families of groups, following the elementary proof of Higman's classification given in [4]. We also discuss more general rings R—in particular, the final section of the paper is concerned exclusively with rings of finite characteristic. We observe, however, that if R is an integral domain of characteristic 0 which is not G-adapted, then RG must always contain nontrivial units. To see this, note that in such a situation G must contain an

Received by the editors February 19, 2003; revised June 16, 2003.
All three authours acknowledge the support of NSERC.
AMS subject classification: Primary 16S34, 16U60; Secondary 20C05.

element x of prime order p where p is invertible in R. If $p \geq 5$, RG contains a nontrivial Bass cyclic unit [7, p. 237]. If $p = 3$, $\frac{1}{3} + \frac{2}{3}x$ is a nontrivial unit (with inverse $\frac{1}{3} - \frac{2}{3}x + \frac{4}{3}x^2$). If $p = 2$, $-\frac{1}{2} + \frac{3}{2}x$ is a nontrivial unit (with inverse $\frac{1}{4} + \frac{3}{4}x$).

One example of a G-adapted ring R which properly contains $\mathbb{Z}$ and for which $\mathcal{U}(RG)$ is trivial whenever $\mathcal{U}(\mathbb{Z}G)$ is trivial is the polynomial ring $\mathbb{Z}[t]$. This is because $(\mathbb{Z}[t])G \cong (\mathbb{Z}G)[t]$ and, since $\mathbb{Z}G$ contains no non-zero nilpotent elements whenever $\mathcal{U}(\mathbb{Z}G)$ is trivial [7, p. 231], it follows [2] that $\mathcal{U}((\mathbb{Z}G)[t]) = \mathcal{U}(\mathbb{Z}G) = \pm G$ in these cases. Hence $\mathcal{U}((\mathbb{Z}[t])G) = \pm G = (\mathcal{U}(\mathbb{Z}[t])) \cdot G$, as claimed. Other similar examples are easy to construct, but we have been unable to find "interesting" examples of G-adapted rings satisfying this condition.

1 Abelian Groups of Exponent 2

In this section we start by focussing on the case when G is cyclic of order 2. The first result gives a pair of ring-theoretic conditions on R that are necessary and sufficient for RC_2 to possess nontrivial units.

Proposition 1 *Let R be a commutative ring with unity. The following are equivalent.*

(i) *RC_2 has nontrivial units.*
(ii) *There exist $a, b \neq 0$ in R such that $a^2 - b^2 \in U(R)$.*
(iii) *There exists $a \neq 0, 1$ in R such that $2a - 1 \in U(R)$.*

Proof Suppose $u \in U(RC_2)$ is a nontrivial unit in RC_2. Let $C_2 = \langle x \rangle$. Then $u = a + bx$, for some $a, b \in R$ with $a, b \neq 0$. Since u is a unit, for all $s, t \in R$ there exists $v, w \in R$ such that $(a + bx)(v + wx) = s + tx$. Therefore, the system of equations

$$\begin{aligned} av + bw &= s \\ bv + aw &= t \end{aligned}$$

always has a solution in R. This is equivalent to the matrix

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

having an inverse in $M_2(R)$. Since the determinant property holds over arbitrary commutative rings with unity, this holds if and only if $a^2 - b^2 \in U(R)$. So (i) is equivalent to (ii).

For the other equivalence, suppose $u \in U(RC_2)$ is a nontrivial unit in RC_2. Assuming without loss of generality that u has augmentation 1, we can write $u = a + (1 - a)x$, for some $a \neq 0, 1$ in R. The preceding proof shows that this is a nontrivial unit of RC_2 if and only if $a^2 - (1 - a)^2 = 2a - 1 \in U(R)$. So (i) is also equivalent to (iii). ■

For integral domains of characteristic zero, the next corollary shows that RC_2 having only trivial units forces a condition much stronger than R being C_2-adapted.

https://doi.org/10.4153/CMB-2005-007-1 Published online by Cambridge University Press

Corollary 1 *Let R be a commutative ring with unity of characteristic* 0*. If RC_2 has only trivial units, then no integer prime p can be invertible in R. In particular, $\mathbb{Z}$ is the unique subring R of $\mathbb{Q}$ for which RC_2 has only trivial units.*

Proof Suppose p is an integer prime which is invertible in a commutative ring R containing $\mathbb{Z}$. If p is odd, then there exists a positive integer s such that $(s+1)^2 - s^2 = p$, so RC_2 will have nontrivial units by the preceding proposition. If $p = 2$, then $\mathbb{Z}[\frac{1}{2}] \subseteq R$, and so $\frac{1}{8} \in R$. But $3^2 - 1^2 = 8$, so RC_2 will have nontrivial units by the above proposition. This proves the first statement.

If R is a subring of $\mathbb{Q}$ properly containing $\mathbb{Z}$, then R contains a lowest terms fraction $\frac{m}{n} \notin \mathbb{Z}$. Since m and n are relatively prime, there exists integers k, ℓ such that $km + \ell n = 1$. But then $k(\frac{m}{n}) + \ell(\frac{n}{n}) = \frac{1}{n} \in R$. If p is any prime divisor of n, then $\frac{1}{n} \in R$ implies that $\frac{1}{p} \in R$, and so RC_2 has nontrivial units by the first statement. On the other hand, $\mathbb{Z}C_2$ has only trivial units, so the second statement follows. ■

Example 1 There are commutative rings R of characteristic 0 with no invertible primes for which RC_2 has nontrivial units.

Let $d > 1$ be a square-free integer, and let $R = \mathbb{Z}[\sqrt{d}]$. Then the equation $x^2 - (y\sqrt{d})^2 = 1$ is exactly Pell's equation $x^2 - dy^2 = 1$, which is well-known to have infinitely many integral solutions. Therefore, $\mathbb{Z}[\sqrt{d}]C_2$ always has nontrivial units, for any square-free positive integer d.

Example 2 Nevertheless there are C_2-adapted rings R of algebraic integers which properly contain $\mathbb{Z}$ with the property that RC_2 has only trivial units.

Consider the case of $R = \mathbb{Z}[\sqrt{-d}]$ with d either 1 or a square-free positive integer. If $a = x + y\sqrt{-d} \in R$ with $x, y \in \mathbb{Z}$, then $2a - 1$ is a unit in R if and only if its norm is ± 1. But the norm of $2a - 1$ is $4(x^2 - x + dy^2) + 1$, so in order for this to equal ± 1 we must have $x^2 - x + dy^2 = -\frac{1}{2}$ or 0. The first of these is impossible, so we conclude that $x = 1 \pm \sqrt{1 - 4dy^2}/2 \in \mathbb{Z}$, and hence $y = 0$. But then $a = 0, 1$, so condition (iii) is not satisfied. Therefore, $\mathbb{Z}[\sqrt{-d}]C_2$ has only trivial units. (Similar reasoning can be used to show that if R is the ring of algebraic integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, then RC_2 also has only trivial units.)

The next result shows that, in almost all cases, the ring conditions of Proposition 1 apply to finite elementary abelian 2-groups.

Proposition 2 *Let R be a commutative ring with unity. Assume that $R \neq \mathbb{Z}_2$ and also that if R is of characteristic* 3*, then it contains some unit $u \neq \pm 1$. If RC_2 and RG have only trivial units, then $R[G \times C_2]$ has only trivial units.*

Proof Let $u \in U(R[G \times C_2])$. Without loss of generality, assume u has augmentation 1. Then $u = a + bx$, where $C_2 = \langle x \rangle$, $a, b \in RG$, and $\omega(a) + \omega(b) = 1$, with ω denoting the augmentation map into R. Similarly, $u^{-1} = c + dx$, and $\omega(c) + \omega(d) = 1$. Then $(a + bx)(c + dx) = 1$ implies that

$$ac + bd = 1 \text{ and } ad + bc = 0$$

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has a solution in RG. As in the proof of Higman's Lemma this leads to the equations $(a + b)(c + d) = 1$ and $(a - b)(c - d) = 1$ in RG. It follows that $a + b$ is a unit of augmentation 1 in RG, so since RG has only trivial units we have $a+b = g \in G$. Also, $a - b \in U(RG)$, so $a - b = vh$, for some $v \in U(R)$ and $h \in G$. Since $v = 2\omega(a) - 1$ is a unit in R and RC_2 has trivial units, we conclude from Proposition 1 that $\omega(a)$ must equal 0 or 1, and hence $v = \pm 1$.

Now $2a = g + vh$ and $2b = g - vh$ in RG. If $g = h$, then either $2a = 0$ or $2b = 0$. Since our assumptions that $R \neq \mathbb{Z}_2$ and RC_2 has only trivial units imply that 2 is not a zero divisor in R (Proposition 1), we have that $a = 0$ or $b = 0$. If $b = 0$, then $u \in U(RG)$, so u is a trivial unit. If $a = 0$, then $ux \in U(RG)$, so ux is a trivial unit, and hence u is also a trivial unit.

If $g \neq h$, then $2a = g + vh$ implies that the coefficient of g in a is 2^{-1}, so 2 is invertible in R. But then setting $a = -2^{-1}$ in Proposition 1(iii) yields that $-2^{-1} = 1$ so $\operatorname{char}(R) = 3$. However, we can then set $a = 2^{-1}(1 + u)$ where u is as described in the hypothesis, giving a contradiction. ■

To see that both conditions of Proposition 2 are necessary we observe

(a) $\mathbb{Z}_2C_2$ has only trivial units but $\mathbb{Z}_2(C_2 \times C_2)$ has nontrivial units.
(b) If $\operatorname{char}(R) = 3$ and R has only ± 1 as units, then RC_2 has only trivial units but $R(C_2 \times C_2)$ has nontrivial units.

2 Abelian Groups of Exponent 3

In the case when the group G is cyclic of order 3, we again give a ring-theoretic condition equivalent to RC_3 having nontrivial units.

Proposition 3 *Let R be a commutative ring with unity. Then RC_3 has nontrivial units if and only if there exist $a, b \in R$ such that $(a, b) \neq (0, 0)$, $(-1, 0)$, or $(-1, -1)$ and $1 + 3a + 3a^2 + 3b^2 - 3ab \in U(R)$.*

Proof Let $RC_3 = R\langle x\rangle$. Let u be a unit of RC_3 with augmentation 1. Then $u = 1 + (1 - x)(a + bx)$, for some $a, b \in R$, and u is a nontrivial unit if and only if $(a, b) \neq (0, 0), (-1, 0), (-1, -1)$. In the quotient ring $R\langle x\rangle/\langle 1 + x + x^2\rangle \cong R[y]$, where $y^3 = 1$, $y \neq 1$, and $y^2 + y + 1 = 0$, we have that the image of u is the unit $v = 1 + (1 - y)(a + by) = (1 + a + b) + (2b - a)y$.

Now, for any element $s+ty \in R[y]$, we can find $p+qy$ such that $v(p+qy) = s+ty$. Since

$$\begin{aligned} v(p + qy) &= \big((1 + a + b) + (2b - a)y\big)\,(p + qy) \\ &= \big((1 + a + b)p - (2b - a)q\big) + \big((2b - a)p + (1 - b + 2a)q\big)\,y, \end{aligned}$$

this implies that the system

$$\begin{aligned} (1 + a + b)p - (2b - a)q &= s \\ (2b - a)p + (1 - b + 2a)q &= t \end{aligned}$$

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in the variables p and q has a solution in R for any $s, t \in R$. Therefore, the determinant d of the coefficient matrix arising from the above system must be a unit of R. This d is equal to $(1+a+b)(1-b+2a)+(2b-a)^2 = 1+3a+3a^2+3b^2-3ab$, so the above condition is necessary for the existence of nontrivial units.

Now assume that a and b are elements of R satisfying the condition. Let $e = a^2+b^2+a-ab$, and let $d = 1+3e$. We are assuming that $d \in U(R)$. Let $u = 1+(1-x)(a+bx)$, and let $w = d^{-1}((1+a+e)+(e-b)x+(e+b-a)x^2)$. Then it is straightforward to check that $uw = 1$, and hence u is a nontrivial unit of RC_3. ■

One can prove easily that $\mathbb{Z}$ does not satisfy the condition. The equation $3e+1 = \pm 1$, has the unique integer solution $e = 0$. Now solving $e = a+a^2+b^2-ab = 0$ for b in terms of a gives

$$b = \frac{a \pm \sqrt{-3a^2-4a}}{2}.$$

In order for $-3a^2-4a$ to be positive, we must have $\frac{-4}{3} \leq a \leq 0$. So the only integer solutions for a are 0 and -1, which correspond to the three pairs of disallowed solutions for the condition.

Corollary 2 *Suppose R is a commutative ring with unity of characteristic* 0. *If RC_3 has only trivial units, then no integer prime p can be invertible in R.*

Proof Assume RC_3 has only trivial units, and let p be an integer prime such that p is invertible in R. If $p = 3$, then the conclusion follows from remarks in the introduction. If $p \neq 3$, then let n be the least power of p so that $p^n \equiv 1 \bmod 3$, and let m be the positive integer for which $p^n = 1+3m$. Then $p^{2n} = 1+3(2m+3m^2)$, and p^{2n} is invertible in R. Letting $a = 2m$ and $b = m$, we have $a^2+b^2+a-ab = 4m^2+m^2+2m-2m^2 = 2m+3m^2$. Thus the condition in Proposition 3 is satisfied, and so RC_3 has nontrivial units. ■

Using the condition of Proposition 3, we have been able to show that if R is the ring of algebraic integers in a quadratic number field $\mathbb{Q}(\sqrt{d})$, d a square-free integer, then RC_3 has nontrivial units unless $d = -3$. (Our arguments are based on an analysis of solutions to certain Pell equations, as in Example 1.) The fact that RC_3 has only trivial units, where R is the ring of algebraic integers of $Q[\sqrt{-3}]$, is a consequence of [3, Theorems 2 and 3].

No problem is encountered extending the condition of Proposition 3 to elementary abelian 3-groups in almost all cases.

Proposition 4 *Let R be a commutative ring with unity such that R is not of characteristic* 2, *and assume that RC_3 has only trivial units. Let G be a finite elementary abelian 3-group. Then RG has only trivial units.*

Proof Induct on the rank r of the elementary abelian 3-group G. The case $r = 1$ holds by assumption.

https://doi.org/10.4153/CMB-2005-007-1 Published online by Cambridge University Press

If $r \geq 2$, then write $G = A \times \langle a \rangle \times \langle b \rangle$, where A is either a trivial group or an elementary abelian 3-group of rank $r - 2$. Let u be a unit in RG.

Then $R[G/\langle b \rangle]$ has only trivial units, so by multiplying by a trivial unit we may assume $u = 1 + (1 - b)(x + yb)$, for some $x, y \in R[A \times \langle a \rangle]$ satisfying the condition $(x, y) \neq (0, 0), (-1, 0)$, or $(-1, -1)$ and $1 + 3x + 3x^2 + 3y^2 - 3xy \in U(R[A \times \langle a \rangle])$. Write $x = x_0 + x_1 a + x_2 a^2$ and $y = y_0 + y_1 a + y_2 a^2$ with $x_i, y_i \in RA$.

We also have that $R[G/\langle a \rangle]$, $R[G/\langle ab \rangle]$, and $R[G/\langle ab^2 \rangle]$ all have only trivial units. Reducing u modulo $\langle a \rangle$ gives

$$u_a = (1 + x_0 + x_1 + x_2) + (y_0 + y_1 + y_2 - x_0 - x_1 - x_2)b + (-y_0 - y_1 - y_2)b^2,$$

which is a unit of augmentation one in $R[G/\langle a \rangle]$. Since $R[G/\langle a \rangle]$ has only trivial units, we must have that two of the expressions in parentheses are 0 and the other is 1. This gives us three possibilities, either $x_0 + x_1 + x_2 = 0$ and $y_0 + y_1 + y_2 = 0$, or $x_0 + x_1 + x_2 = -1$ and $y_0 + y_1 + y_2 = -1$, or $x_0 + x_1 + x_2 = -1$ and $y_0 + y_1 + y_2 = 0$.

On the other hand, reducing u modulo $\langle ab \rangle$ results in

$$u_{ab} = (1 + x_0 - x_1 + y_1 - y_2) + (-x_0 + x_2 + y_0 - y_1)b + (x_1 - x_2 - y_0 + y_2)b^2,$$

and reducing modulo $\langle a^2 b \rangle$ results in

$$u_{a^2 b} = (1 + x_0 - x_2 - y_1 + y_2) + (-x_0 + x_1 + y_0 - y_2)b + (-x_1 + x_2 - y_0 + y_1)b^2.$$

For each of these we must have that two of the coefficients in RA are 0 and the other is 1.

Suppose $u_{ab} = 1$. Then $x_0 = x_2 + y_0 - y_1$ and $x_1 = x_2 + y_0 - y_2$, so substituting these into $u_{a^2 b}$ we have

$$u_{a^2 b} = (1 + y_0 - 2y_1 + y_2) + (y_0 + y_1 - 2y_2)b + (-2y_0 + y_1 + y_2)b^2.$$

If $y_0 + y_1 + y_2 = 0$, then we have $u_{a^2 b} = (1 - 3y_1) - 3y_2 b - 3y_0 b^2$. If any of y_0, y_1, or y_2 is non-zero, then 3 is either a zero divisor or invertible in RA. Either conclusion contradicts Proposition 3 (if 3 is invertible, use $a = -3^{-1}$ and $b = 0$). So we have $y_0 = y_1 = y_2 = 0$. This implies $x_0 = x_1 = x_2$. If $x_0 + x_1 + x_2 = -1$, then $3x_0 = -1$, so 3 is invertible in RA and we again get a contradiction. We conclude that $x_0 = x_1 = x_2 = 0$, which implies that $u = 1$, so u is a trivial unit.

On the other hand, if $y_0 + y_1 + y_2 = -1$, then we have

$$u_{a^2 b} = (-3y_1) + (-1 - 3y_2)b + (-1 - 3y_0)b^2.$$

This forces 3 to be invertible in R, a contradiction.

Using similar arguments, the case $u_{ab} = b$ leads to the conclusion $u = b$, and the case $u_{ab} = b^2$ leads to the conclusion $u = b^2$. Therefore, all units of RG are trivial, and the result follows by induction. ∎

We note that $\mathbb{Z}_2 C_3$ has only trivial units while $\mathbb{Z}_2(C_3 \times C_3)$ has nontrivial units, explaining the condition in the proposition.

Combining the results so far using Proposition 3, we conclude that if R is a commutative ring with unity, $\operatorname{char}(R) = 0$, and RC_2 and RC_3 have only trivial units, then RG has only trivial units for any finite abelian group of exponent dividing 6. The ring $R = \mathbb{Z}[\sqrt{-3}]$ satisfies these conditions.

3 Abelian Groups of Exponent 4

We start this section with a ring-threoretic condition equivalent to RC_4 having nontrivial units.

Proposition 5 *Let R be a commutative ring with unity. Then RC_4 has nontrivial units if and only if either RC_2 has nontrivial units or there exist $a, b \in R$ such that $2a^2 + 2b^2 + 2a = 0$ with $(a, b) \neq (0, 0), (-1, 0)$.*

Proof Let $C_4 = \langle x \rangle$. Let $f \colon RC_4 \to RC_2$ be the R-linear extension of the natural group homomorphism $C_4 \to C_2$. Suppose u is a nontrivial unit of RC_4 and RC_2 has only trivial units. By multiplying by a group element if necessary, we may assume that $f(u) = 1$, $u = 1 + (1 - x^2)(a + bx)$ for some $a, b \in R$ with $(a, b) \neq (0, 0), (-1, 0)$.

Now $u^* = 1 + (1 - x^2)(a + bx^3) = 1 + (1 - x^2)(a - bx)$, and so $uu^* = 1 + (1 - x^2)(2a + 2a^2 + 2b^2) \in U(R\langle x^2 \rangle)$. Since $R\langle x^2 \rangle$ has only trivial units, we must either have $2a + 2a^2 + 2b^2 = 0$ or $2a + 2a^2 + 2b^2 = -1$. However the latter implies that 2 is invertible, contradictory to Proposition 1 unless R is of characteristic 3. But in that case $a = 1, b = 1$ satisfy $2a^2 + 2b^2 + 2a = 0$. So existence of nontrivial units in RC_4 implies the condition.

On the other hand, if $a, b \in R$ are such that $2a^2 + 2b^2 + 2a = 0$ and $(a, b) \neq (0, 0), (-1, 0)$, then by the above $u = 1 + (1 - x^2)(a + bx)$ satisfies $uu^* = 1$. Therefore, u is a nontrivial unit of RC_4. ∎

Again using Pell equations, we have been able to use the condition of Proposition 5 to show that if R is the ring of integers in a quadratic number field $\mathbb{Q}(\sqrt{d})$, d a square-free negative integer, then RC_4 has only trivial units if and only if $d = -1$. (Note that cases where $d > 0$ are irrelevant by Example 1.)

Proposition 6 *Let R be a commutative ring with unity. Assume that RC_4 has only trivial units, and G is the direct product of finitely many copies of C_4. Then RG has only trivial units.*

Proof First note that $\mathbb{Z}_2C_4$ and $\mathbb{Z}_3C_4$ have nontrivial units by Proposition 5, so we can assume $\operatorname{char}(R) \neq 2, 3$. Induct on the rank r of the abelian group G of exponent 4. The case $r = 1$ holds by assumption.

If $r \geq 2$, then write $G = A \times \langle a \rangle \times \langle b \rangle$, where A is either a trivial group or a direct product of $r - 2$ copies of C_4.

It follows from the inductive hypothesis and Proposition 2 that $R[G/\langle b^2 \rangle]$ has only trivial units. Therefore, if u is a nontrivial unit in $R[G]$, then up to multiplication by a trivial unit we have

$$u = 1 + (1 - b^2)(x_0 + x_1 a + x_2 a^2 + x_3 a^3 + y_0 b + y_1 ab + y_2 a^2 b + y_3 a^3 b),$$

for some $x_i, y_i \in R[A]$.

Since $R[G/\langle a^2 \rangle]$ has only trivial units, by writing u with the ensuing factorization we conclude that $y_2 = -y_0$, $y_3 = -y_1$, $x_3 = -x_1$, and $x_2 = -x_0$ or $-x_0 - 1$. But

$R[G/\langle a^2b^2\rangle]$ also has only trivial units, so in a similar fashion we also obtain $y_2 = y_0$, $y_3 = y_1$, $x_3 = x_1$, and $x_2 = x_0$ or $x_0 + 1$. Proposition 1(iii) now allows us to conclude that $x_1 = x_2 = x_3 = y_0 = y_1 = y_2 = y_3 = 0$ and $x_0 = 0$ or -1. This is a contradiction since it implies that $u = 1$ or b^2, a trivial unit. ∎

4 Hamiltonian 2-Groups

The last case to consider is where G is a Hamiltonian 2-group, *i.e.,* the direct product of the quaternion group Q_8 with a finite elementary abelian 2-group. We start with Q_8.

Proposition 7 *Let R be a commutative ring with unity. Then RQ_8 has nontrivial units if and only if either RC_2 has nontrivial units or there exist $a, b, c, d \in R$ with $(a, b, c, d) \neq (0, 0, 0, 0), (-1, 0, 0, 0)$ such that $a + a^2 + b^2 + c^2 + d^2 = 0$.*

Proof Let $Q_8 = \langle x, y \mid x^4 = y^4 = 1, x^2 = y^2, yx = x^{-1}y\rangle$ and assume RQ_8 has nontrivial units. Note if $\operatorname{char}(R) = 2$ or 3 it is easy to find suitable a, b, c, d, so we assume this is not the case. We also assume RC_2 has only trivial units. Since $Q_8/\langle x^2\rangle \cong C_2 \times C_2$, we have that $R(Q_8/\langle x^2\rangle)$ has only trivial units. Suppose u is a nontrivial unit in RQ_8. Multiplying by a trivial unit, we may assume that $u = 1 + (1 - x^2)(a + bx + cy + dxy)$, and $(a, b, c, d) \neq (0, 0, 0, 0), (-1, 0, 0, 0)$. Note that $uu^* = (1 + (1 - x^2)(a + bx + cy + dxy))(1 + (1 - x^2)(a - bx - cy - dxy)) = 1 + (1 - x^2)2(a + a^2 + b^2 + c^2 + d^2)$ is in $R\langle x^2\rangle$. However, $R\langle x^2\rangle$ has only trivial units, so we must have $2(a + a^2 + b^2 + c^2 + d^2) = -1$ or 0. The first implies that 2 is invertible which is contradictory to Proposition 1. Thus we have $2(a + a^2 + b^2 + c^2 + d^2) = 0$. But then Proposition 1 again implies that 2 is not a zero divisor in R, and it follows that $a + a^2 + b^2 + c^2 + d^2 = 0$.

Since the condition implies that $u = 1 + (1 - x^2)(a + bx + cy + dxy)$ satisfies $uu^* = 1$ and $u \notin \pm Q_8$, the proof is complete. ∎

Extending the above conclusion to finite Hamiltonian 2-groups is immediate from Proposition 2. Recall that $\mathbb{Z}_2Q_8$ and $\mathbb{Z}_3Q_8$ have nontrivial units by Proposition 5.

Corollary 3 *Let A be a finite elementary abelian 2-group. Let R be a commutative ring with unity such that RQ_8 has only trivial units. Then $R[Q_8 \times A]$ has only trivial units.*

For $R = \mathbb{Z}[i]$, we have that $\mathbb{Z}[i]C_4$ has only trivial units. However, we can satisfy the above condition by taking $a = -1$, $b = 1 + i$, $c = 1 - i$, and $d = 0$, since $(-1)^2 - 1 + (1 + i)^2 + (1 - i)^2 + 0^2 = 1 - 1 + 2i - 2i = 0$. So $\mathbb{Z}[i]Q_8$ has nontrivial units.

5 Trivial Units in Finite Characteristic

Up until now, our primary interest has been G-adapted rings R (though some results were proved in a more general setting), and we have focussed on Higman's classification of finite groups G for which $\mathcal{U}(\mathbb{Z}G)$ is trivial. As described in the introduction, the other setting where this question has been settled is when R is a field, and it only happens there in a very limited number of cases when R is of finite characteristic. In this section, we extend these considerations to commutative rings of finite characteristic which are not necessarily fields. The next result shows that it is still the case that $\mathcal{U}(RG)$ trivial implies $|G| \leq 3$.

Proposition 8 *Let R be a commutative ring with unity of finite characteristic $\ell > 1$, and let G be a finite group such that RG has only trivial units. Then G is cyclic of order* 2 *or* 3.

Proof If G has a non-normal cyclic subgroup, then $\mathbb{Z}_\ell G$ has a nontrivial bicyclic unit. (For a definition of bicyclic and Bass cyclic units, see [7, Section 8.1].) Suppose H is a proper normal subgroup of G, and let $\hat{H} = \sum_{h \in H} h$. If $(|H|, \ell) = 1$, then $e = \frac{1}{|H|}\hat{H}$ is a central idempotent of RG, and for all $g \in G \setminus H$, $e + g(1-e)$ is a nontrivial unit with inverse $e + g^{-1}(1-e)$. If $(|H|, \ell) = r > 1$ and $\ell = rm$, then $m\hat{H} \neq 0$ in RG, $(m\hat{H})^2 = m^2|H|\hat{H} = 0$, so $1 + m\hat{H}$ is a nontrivial unipotent unit in RG. So the only possibility is that G is cyclic of prime order p.

Suppose p is a prime greater than 3, and let $G = \langle x \rangle$. Then

$$u = (1+g)^{p-1} - \frac{2^{p-1}-1}{p}(1 + g + g^2 + \cdots + g^{p-1})$$

is a nontrivial Bass cyclic unit of $\mathbb{Z}G$. Letting $c := 2^{p-1} - 1/p$, we have

$$u = (1-c) + (p-1-c)g + \left(\binom{p-1}{2} - c\right) g^2 + \cdots + (p-1-c)g^{p-2} + (1-c)g^{p-1}.$$

Note that for any $\ell > 2$, the ring homomorphism $\mathbb{Z}G \to \mathbb{Z}_\ell G$ induced by reducing the coefficients mod ℓ sends u to a unit, which must also have augmentation 1. If the image is a trivial unit, then the coefficient of each pair g^i, g^{p-1-i} must be sent to 0 for $i = 0, 1, \ldots, \frac{p-1}{2} - 1$. In particular, both $1 - c \equiv 0 \bmod \ell$ and $p - 1 - c \equiv 0 \bmod \ell$. Therefore, $p - 2 \equiv 0 \bmod \ell$.

In the case $p = 5$ this would force $\ell = 3$. But then $\mathbb{Z}_\ell = \mathbb{Z}_3$ is a field, and so from the field case we know that $\mathbb{Z}_3 C_5$ has nontrivial units. So we may assume $p > 5$. Then the coefficient of g^2 and g^{p-3} in u, namely $\binom{p-1}{2} - c$, must also be congruent to 0 mod ℓ. But $p-2$ divides $\binom{p-1}{2}$, so $\binom{p-1}{2} \equiv 0 \bmod \ell$. It follows that $c \equiv 0 \bmod \ell$. But then it follows from $1 - c \equiv 0 \bmod \ell$ that $\ell = 1$, a contradiction, which completes the proof of the proposition. ■

Propositions 1 and 3 can now be used to say more about which commutative rings R of finite characteristic have the property that $\mathcal{U}(RG)$ is trivial when $G = C_2$ or C_3.

First assume $G = C_2$. If char(R) is even then there exists $a \neq 0$ such that $2a = 0$, and Proposition 1(iii) guarantees the existence of a nontrivial unit in RC_2 except for the case where $R = \mathbb{Z}_2$ (recall $\mathbb{Z}_2 C_2$ has only trivial units). If char$(R) = \ell > 3$ is odd, then 2 is invertible and $|\mathcal{U}(R)| \geq 3$ ($1, 2, \ell - 1$ are all units in R), so again Proposition

https://doi.org/10.4153/CMB-2005-007-1 Published online by Cambridge University Press

1(iii) shows $\mathcal{U}(RC_2)$ is nontrivial. A similar remark applies when $\mathrm{char}(R) = 3$ as long as $|\mathcal{U}(R)| \geq 3$, but when $\mathrm{char}(R) = 3$ and $\mathcal{U}(R) = \pm 1$, RC_2 has only trivial units.

Next assume $G = C_3$. If $\mathrm{char}(R)$ is divisible by 3, then there exists $r \neq 0$ in R such that $3r = 0$, and $a = 0, b = r$ in Proposition 3 guarantees the existence of a nontrivial unit in RC_3. But if $\mathrm{char}(R)$ is not divisible by 3, then 3 is invertible in R and $a = -3^{-1}, b = 3^{-1}$ satisfies the condition of Proposition 3. The only time this does not force nontrivial units in RC_3 is when $\mathrm{char}(R) = 2$. We do not know precisely when $\mathcal{U}(RC_3)$ is trivial if $\mathrm{char}(R) = 2$; the result does hold for $\mathbb{Z}_2$ and $\mathbb{Z}_2[t]$, but not for finite proper extension fields of $\mathbb{Z}_2$ or for the rational polynomial field $\mathbb{Z}_2(t)$.

Note that the remarks just made show that $\mathcal{U}(RC_2)$ and $\mathcal{U}(RC_3)$ can be trivial (when R is commutative with unity and of finite characteristic) only in cases which are excluded in Propositions 2 and 4. So these propositions are not helpful when dealing with rings of finite characteristic.

Acknowledgement The authors wish to express their thanks to the referee for his suggestions in preparing the revised version of the manuscript.

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Department of Mathematics and Statistics
University of Regina
Regina, SK
S4S 0A2
e-mail: aherman@math.uregina.ca

Department of Mathematics
Brock University
St. Catharines, ON
L2S 3A1
e-mail: Yuanlin.Li@brocku.ca

Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, NF
A1C 5S7
e-mail: michael1@math.mun.ca