A NOTE ON SEMIGROUPS OF MAPPINGS ON BANACH SPACES

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In a series of papers K. D. Magill, Jr. (see [1] and its references) has proved that, in various semigroups of mappings on topological spaces, *every automorphism is inner*, where an *automorphism* ϕ of a semigroup $\mathscr{A}$ is a bijection of $\mathscr{A}$ such that

$$\phi(fg) = \phi(f)\phi(g)$$

for all f and g in $\mathscr{A}$, and it is said to be *inner* if there exists a bijection $h \in \mathscr{A}$ such that h^{-1} (the inverse of h) belongs to $\mathscr{A}$ and

$$\phi(f) = hfh^{-1}$$

for every $f \in \mathscr{A}$.

In this paper, we shall consider the same problem for the semigroups $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ which will be defined in the following sections.

Throughout this paper, E stands for a real Banach space, and the Banach algebra of all continuous linear mappings of E into itself is denoted by $\mathscr{L}$.

1. The semigroups $\mathscr{B}$ and $\mathscr{C}$

For two mappings f and g of E into itself, the product fg is defined by

$$(fg)(x) = f(g(x))$$

for every $x \in E$.

A mapping f of E into itself is said to be *bounded if* $f(B)$ is a bounded subset of E whenever B is a bounded subset of E. The set of all bounded and continuous mappings is denoted by $\mathscr{B}$, which is obviously a semigroup.

A mapping f of E into itself is said to be *completely continuous* if it is continuous and $f(B)$ is contained in a compact subset of E whenever B is a bounded subset of E. The set of all completely continuous mappings of E into itself is denoted by $\mathscr{C}$, which is obviously a semigroup.

A mapping f of E into itself is said to be *constant* if there exists $a \in E$ such that $f(x) = a$ for every $x \in E$. This mapping is denoted by c_a:

$$c_a(x) = a$$

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for every $x \in E$. The set $I(E)$ of all constant mappings is a semigroup, and we have

$$fc_a = c_{f(a)} \quad \text{and} \quad c_a f = c_a, \tag{1}$$

where f is an arbitrary mapping of E into itself.

It is obvious that

$$I(E) \subset \mathscr{C} \subset \mathscr{B}. \tag{2}$$

Throughout this section, we denote by $\mathscr{A}$ an arbitrary semigroup of mappings of E into itself. A subset I of $\mathscr{A}$ is said to be an *ideal* if fg and gf belong to $\mathscr{A}$ whenever $f \in I$ and $g \in \mathscr{A}$. It is easy to see that, if ϕ is an automorphism of $\mathscr{A}$, I is an ideal of $\mathscr{A}$ if and only if $\phi(I)$ is an ideal of $\mathscr{A}$.

The property (1) shows that $I(E)$ is an ideal of $\mathscr{A}$ if $\mathscr{A} \supset I(E)$. Moreover, we can prove the following facts.

(3) *$I(E)$ is the smallest ideal of $\mathscr{A}$ whenever $I(E) \subset \mathscr{A}$.*

PROOF. Let I be an arbitrary ideal of $\mathscr{A}$. Then, by (3), we have

$$c_x = c_x f \in I$$

if $x \in E$ and $f \in I$, which means that $I(E) \subset I$.

REMARK. If $I(E) \subset \mathscr{A}$, the set $\{0\}$ is not an ideal of $\mathscr{A}$, because $c_a 0 \neq 0$ if $a \neq 0$.

(4) $\quad$ *$\phi(I(E)) = I(E)$ for any automorphism ϕ of $\mathscr{A}$ if $I(E) \subset \mathscr{A}$.*

PROOF. Since $I(E)$ is the smallest ideal and $\phi(I(E))$ is an ideal, we have $I(E) \subset \phi(I(E))$. To prove the converse, let a be an arbitrary element. Then, for any $x \in E$,

$$\begin{aligned} \phi(c_a)(x) &= \phi(c_a)c_x(y) \text{ for any } y \in E \\ &= \phi(c_a)\phi(f)(y) \text{ for } f \in \mathscr{A} \text{ such that } \phi(f) = c_x \\ &= \phi(c_a f)(y) \\ &= \phi(c_a)(y), \end{aligned}$$

which means that $\phi(c_a)$ is a constant mapping.

The following theorem is essentially due to K. D. Magill, Jr.

THEOREM 1. *Let ϕ be an automorphism of $\mathscr{A}$ such that $I(E) \subset \mathscr{A}$. Then, there exists a bijection $h = h(\phi)$ of E such that*

$$\phi(f) = hfh^{-1} \tag{5}$$

for every $f \in \mathscr{A}$.

PROOF. The mapping h is defined by

$$\phi(c_x) = c_{h(x)} \tag{6}$$

for every $x \in E$. Then, h is injective, because, if $h(x) = h(y)$, we have

$$\phi(c_x) = c_{h(x)} = c_{h(y)} = \phi(c_y),$$

from which it follows that $c_x = c_y$, or $x = y$. To prove that h is surjective, let a be an arbitrary element of E. By (4), we can find $x \in E$ such that $c_a = \phi(c_x) = c_{h(x)}$, from which it follows that $a = h(x)$. Finally, to prove (5), let f be an arbitrary element of $\mathscr{A}$. Then, for any $x \in E$,

$$\begin{aligned} \phi(f)(x) &= \phi(f)c_x(y) \text{ for any } y \in E \\ &= \phi(f)\phi(c_{h^{-1}(x)})(y) && \text{by (6)} \\ &= \phi(fc_{h^{-1}(x)})(y) && \text{by (1)} \\ &= \phi(c_{fh^{-1}(x)})(y) && \text{by (1)} \\ &= c_{hfh^{-1}(x)}(y) && \text{by (6)} \\ &= hfh^{-1}(x), \end{aligned}$$

from which (5) follows.

This theorem means that every automorphism of the semigroup of all mappings of E into itself is inner. On the other hand, if the semigroup is 'small', an automorphism is not always inner.

(7) *In the semigroup $I(E)$ no automorphism is inner,*

because the mapping c_a does not have an inverse.

(8) *If E is infinite dimensional, no automorphism of the semigroup $\mathscr{C}$ is inner.*

PROOF. If an automorphism ϕ is inner, the bijection $h = h(\phi)$ and its inverse h^{-1} belong to $\mathscr{C}$. This means that the closed unit sphere is contained in a compact set, which is true only if E is finite dimensional.

(9) *If E is infinite dimensional, in the semigroup $1+\mathscr{C} = \{1+f | f \in \mathscr{C}\}$ where 1 is the identity mapping, some automorphisms are inner and some are not inner.*

PROOF. We assume that E is a Hilbert space and consider the one-dimensional mapping ℓ:

$$\ell(x) = (a, x)a,$$

where a is a fixed non-zero element and (a, x) is the scalar product of a and x. Then, $h = 1+\ell$ is a bijection and

$$h^{-1} = 1-(1+||a||^2)^{-1}\ell \in 1+\mathscr{C}.$$

Therefore, the automorphism defined by this h is inner. (It is easy to see

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that we have the same conclusion if the mapping l is replaced by any completely continuous mapping which is *monotone* in the sense of G. Minty and F. E. Browder, [1] and [4].)

On the other hand, let h be a bicontinuous linear bijection (which obviously is not in $\mathscr{C}$) and let ϕ be the mapping such that $\phi(f) = hfh^{-1}$. Then, ϕ is an automorphism of $1+\mathscr{C}$, because, if $f \in \mathscr{C}$,

$$\phi(1+f) = h(1+f)h^{-1} = 1+hfh^{-1}$$

and $hfh^{-1} \in \mathscr{C}$. Therefore, this ϕ is an automorphism which is not inner.

On the other hand, we can prove the following theorem.

THEOREM 2. *Every automorphism of the semigroup $\mathscr{B}$ is inner.*

PROOF. Let ϕ be an automorphism of $\mathscr{B}$. By Theorem 1, there exists a bijection h which satisfies (5). Therefore, we have only to prove that h and h^{-1} belong to $\mathscr{B}$.

(10) *h is continuous.*

PROOF. Let a be an arbitrary element. We take $b \in E$ such that $b \neq h(a)$. Let ε be an arbitrary positive number, and put

$$S = S(h(a), \varepsilon) = \{x \in E \mid ||x-h(a)|| < \varepsilon\}.$$

Then, since E is completely regular as a topological space, there exists a continuous function $\alpha(x)$ such that

$$\alpha(h(a)) = 1, \ \alpha(x) = 0 \text{ if } x \notin S \text{ and } 0 \leqq \alpha(x) \leqq 1 \quad (x \in E).$$

We consider the mapping

$$g(x) = \alpha(x)(b-x)+h(a).$$

Since $g \in \mathscr{B}$, we can take $f \in \mathscr{A}$ such that $\phi(f) = g$. We have $f(a) \neq a$, because, if $f(a) = a$, since $fc_a = c_a$, we have

$$\begin{aligned} c_{h(a)} &= \phi(c_a) = \phi(fc_a) = \phi(f)\phi(c_a) \\ &= gc_{h(a)} = c_{g(h(a))}, \end{aligned}$$

from which it follows that

$$h(a) = g(h(a)) = \alpha(h(a))(b-h(a))+h(a) = b,$$

which is a contradiction. Therefore, there exists $\delta > 0$ such that

$$f(x) \neq a \text{ if } ||x-a|| < \delta.$$

For this δ, we can prove that

$$||h(a)-h(x)|| < \varepsilon \text{ if } ||x-a|| < \delta.$$

Assume that there exists $x \in E$ such that

$$||h(a)-h(x)|| \geqq \varepsilon \text{ and } ||x-a|| < \delta.$$

Then, since $\alpha(h(x)) = 0$, it follows from the definition of g that $g(h(x)) = h(a)$. Therefore, since $\phi(f) = g$,

$$f(x) = h^{-1}gh(x) = h^{-1}(g(h(x))) = h^{-1}(h(a)) = a,$$

which is a contradiction.

(11) *h is bounded.*

PROOF. Let B be an arbitrary bounded subset of E and we assume that $h(B)$ is not bounded. Then, there exists a sequence $x_n \in B$ such that

$$||y_n||+1 < ||y_{n+1}||$$

where $y_n = h(x_n)$. Let us consider

$$S_n = S(y_n, \tfrac{1}{3}) = \{x \in E | \; ||x-y_n|| < \tfrac{1}{3}\}.$$

Obviously, $y_n \in S_n$ for each n and $S_n \cap S_m =$ empty if $n \neq m$.

Next, we consider continuous functions $\alpha_n(x)$ such that

$$\alpha_n(y_n) = 1, \; \alpha_n(x) = 0 \text{ if } x \notin S_n \text{ and } 0 \leqq \alpha_n(x) \leqq 1,$$

and define a mapping g by

$$g(x) = \sum_{n=1}^{\infty} \alpha_n(x)(x-y_n+z_n)$$

where $z_n = h(y_n)$.

g is defined for all $x \in E$.

To see this, let a be an arbitrary element. If $\alpha_n(a) = 0$ for all n, we have $g(a) = 0$. If $\alpha_k(a) \neq 0$, it follows from the definition of $\alpha_k(x)$ that $a \in S_k$. Then, since $\alpha_n(a) = 0$ for $n \neq k$, we have

$$g(a) = \alpha_k(a)(a-y_k+z_k).$$

g is continuous.

Let us assume that $\lim_{i\to\infty} a_i = a$. Then, there exists i_0 such that $a_i \in S(a, \frac{1}{6})$ if $i \geqq i_0$. If $S(a, \frac{1}{6}) \cap S_k =$ *empty* for all n, we have $g(a_i) = 0$ $(i \geqq i_0)$ and $g(a) = 0$. If $S(a, \frac{1}{6}) \cap S_k \neq$ *empty* for some k, since $S(a, \frac{1}{6}) \cap S_n =$ *empty* for $n \neq k$, we have

$$g(a_i) = \alpha_k(a_i)(a_i-y_k+z_k) \text{ for } i \geqq i_0,$$
$$g(a) = \alpha_k(a)(a-y_k+z_k),$$

hence it follows that $\lim_{i\to\infty} g(a_i) = g(a)$.

g is bounded.

For any number $\gamma > 0$ there exists k such that $||y_k|| > \gamma$. Then, $x \notin S_n$ if $n > k$ and $||x|| < \gamma$, because

$$||x-y_n|| \geqq |\, ||y_n||-||x||\, | = ||y_n||-||x|| \geqq ||y_k||+1-||x|| > \gamma+1-\gamma = 1,$$

which means that $\alpha_n(x) = 0$ if $n > k$ and $||x|| < \gamma$. Therefore, if $||x|| < \gamma$,

$$\begin{aligned} g(x) &= || \sum_{n=1}^{k} \alpha_n(x)(x-y_n+z_n)|| \\ &\leqq \sum_{n=1}^{k} ||x-y_n+z_n|| \leqq \sum_{n=1}^{k} (||x||+||y_n-z_n||) \\ &\leqq k\gamma + \sum_{n=1}^{k} ||y_n-z_n|| \end{aligned}$$

which means that g is bounded.

Thus, it has been shown that $g \in \mathscr{B}$ and

$$g(y_n) = z_n = h(y_n).$$

Then, for $f \in \mathscr{B}$ such that $\phi(f) = g$, we have

$$f(x_n) = h^{-1}gh(x_n) = h^{-1}g(y_n) = y_n,$$

which is a contradiction.

Thus, from (10) and (11) it follows that $h \in \mathscr{B}$. The fact that $h^{-1} \in \mathscr{B}$ can be proved in the same way if we consider ϕ^{-1} instead of ϕ.

2. The semigroup $\mathscr{D}$

A mapping f of E into itself is said to be (Fréchet)-*differentiable at* $a \in E$ if there exists $\ell \in \mathscr{L}$ such that

$$\lim_{||x|| \to 0} \frac{f(a+x)-f(a)-\ell(x)}{||x||} = 0.$$

This mapping ℓ is determined uniquely for each a and is denoted by $f'(a)$. If f is differentiable at every point of E, it is said to be *differentiable*. We denote the set of all differentiable mappings of E into itself by $\mathscr{D}$. This set $\mathscr{D}$ is a semigroup because $fg \in \mathscr{D}$ whenever $f \in \mathscr{D}$ and $g \in \mathscr{D}$. Moreover, in this case, we have

$$(fg)'(x) = f'(g(x))g'(x)$$

for every $x \in E$. It is easy to see that

$$I(E) \subset \mathscr{D} \text{ and } c'_a(x) = 0 \qquad (x, a \in E),$$

$$\mathscr{L} \subset \mathscr{D} \text{ and } \ell'(x) = \ell \qquad (x \in E, \ell \in \mathscr{L}).$$

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In [3], K. D. Magill, Jr. has proved that, *when E is the field of real numbers, every automorphism of $\mathscr{D}$ is inner*. In the proof, he has used the fact that a bijection of E is a monotone function, which is differentiable at countably many points. When E is a general Banach space, this is no longer true. For example, in a Banach space with non-differentiable norm, the bijection $h(x) = ||x||x$ is differentiable only at the origin. We have to leave the following problem unsolved: *is every bijection of a Banach space differentiable at at least one point?*

In this section, we shall prove that some automorphisms of $\mathscr{D}$ are inner. At first, we prove the following theorem.

THEOREM 3. *Let $\mathscr{A}$ be a semigroup of mappings of E into E such that $I(E) \subset \mathscr{A}$ and $\mathscr{L} \subset \mathscr{A}$, and ϕ be an automorphism of $\mathscr{A}$ such that $\phi(\mathscr{L}) = \mathscr{L}$. Then, ϕ is inner and $h(\phi) \in \mathscr{L}$.*

PROOF. By Theorem 1, there exists a bijection h such that (5) is satisfied. We have only prove that $h \in \mathscr{L}$.

We denote the mapping $x \to \xi x$ by ξ. Then, the mappings $\phi(\xi)$ belong to the centre of the primitive Banach algebra $\mathscr{L}$, because, if $\ell \in \mathscr{L}$, since $\phi^{-1}(\ell) \in \mathscr{L}$, for any $x \in E$ and $y = h^{-1}(x)$, we have

$$\begin{aligned} \ell\phi(\xi)(x) &= \ell\phi(\xi)h(y) = \ell h\xi h^{-1}h(y) = \ell h(\xi y) \\ &= hh^{-1}\ell h(\xi y) = h\phi^{-1}(\ell)(\xi y) = h\xi\phi^{-1}(\ell)(y) \\ &= h\xi h^{-1}\ell h(y) = \phi(\xi)\ell(x). \end{aligned}$$

Therefore, by Corollary 2.4.5, p. 61, of [5], there exists a real-valued function $\lambda(\xi)$ of a real variable ξ such that

$$\phi(\xi)(x) = \lambda(\xi)x \text{ if } x \in E \text{ and } -\infty < \xi < \infty.$$

We shall prove that $\lambda(\xi) = \xi$, or

$$\phi(\xi) = \xi \text{ for all } \xi. \tag{12}$$

Now, from the definition of $\lambda(\xi)$ we have

$$\lambda(\xi\eta)x = \phi(\xi\eta)(x) = \phi(\xi)\phi(\eta)(x) = \lambda(\xi)(\phi(\eta)(x)) = \lambda(\xi)\lambda(\eta)x$$

for every x which means that

$$\lambda(\xi\eta) = \lambda(\xi)\lambda(\eta).$$

Next, we have

$$\lambda(-1) = -1,$$

because

$$1 = \phi(1) = \phi(-1 \times -1) = \phi(-1)\phi(-1) = \lambda(-1)^2$$

and, since $\phi(-1) \neq \phi(1)$, $\lambda(-1) \neq 1$. Moreover, $\lambda(\xi)$ is a bijection of the real number field. The fact that $\lambda(\xi)$ is injective follows immediately from the injectivity of ϕ. To show that $\lambda(\xi)$ is surjective, let α be an arbitrary number. Then, there exists $\ell_0 \in \mathscr{L}$ such that $\phi(\ell_0) = \alpha$, and

$$\phi(\ell_0\ell) = \phi(\ell_0)\phi(\ell) = \alpha\phi(\ell) = \phi(\ell)\alpha = \phi(\ell)\phi(\ell_0) = \phi(\ell\ell_0),$$

from which it follows that $\ell_0\ell = \ell\ell_0$ for every $\ell \in \mathscr{L}$. This means that ℓ_0 belongs to the centre of $\mathscr{L}$. There exists β such that $\ell_0 = \beta$, which is equivalent to the fact that $\lambda(\beta) = \alpha$.

Thus, $\lambda(\xi)$ is continuous at at least one point, $\lambda(-1) = -1$ and the relation $\lambda(\xi\eta) = \lambda(\xi)\lambda(\eta)$ is satisfied. Therefore, there exists α such that

$$\lambda(\xi) = \xi^\alpha \quad (= (\text{sign } \xi)|\xi|^\alpha).$$

To prove that $\alpha = 1$, we consider the one-dimensional linear mapping $x \otimes \bar{x}$ ($x \in E$ and $\bar{x} \in \bar{E}$ (*the conjugate space of* E)) defined by

$$x \otimes \bar{x}(y) = \bar{x}(y)x \text{ for every } y \in E.$$

Then, since

$$\begin{aligned}\phi(x \otimes \bar{x})(y) &= h(x \otimes \bar{x})h^{-1}(y) = h(\bar{x}(h^{-1}(y))x) \\ &= \phi(\bar{x}(h^{-1}(y)))h(x) = (\bar{x}(h^{-1}(y)))^\alpha h(x)\end{aligned}$$

and $\phi(x \otimes \bar{x})(y)$ is linear with respect to y, $(\bar{x}(h^{-1}(y)))^\alpha$ is a linear functional on E for each $\bar{x} \in \bar{E}$, in other words,

$$(\bar{x}(h^{-1}(a+b)))^\alpha = (\bar{x}(h^{-1}(a)))^\alpha + (\bar{x}(h^{-1}(b)))^\alpha$$

for any $\bar{x} \in \bar{E}$. This means that $h^{-1}(a+b)$ belongs to the subspace spanned by $h^{-1}(a)$ and $h^{-1}(b)$, because

$$\bar{x}(h^{-1}(a)) = \bar{x}(h^{-1}(b)) = 0$$

implies $\bar{x}(h^{-1}(a+b)) = 0$. Therefore,

$$h^{-1}(a+b) = \mu h^{-1}(a) + \rho h^{-1}(b)$$

for some numbers μ and ρ. Now, we take a and b such that $h^{-1}(a)$ and $h^{-1}(b)$ are linearly independent. We can take $\bar{x} \in \bar{E}$ such that $\bar{x}(h^{-1}(a)) = 1$ and $\bar{x}(h^{-1}(b)) = 0$. Then,

$$\begin{aligned}1 &= (\bar{x}(h^{-1}(a)))^\alpha + \bar{x}((h^{-1}(b)))^\alpha = (\bar{x}(h^{-1}(a+b)))^\alpha \\ &= (\mu\bar{x}(h^{-1}(a)) + \rho\bar{x}(h^{-1}(b)))^\alpha \\ &= \mu^\alpha,\end{aligned}$$

from which it follows that $\mu = 1$, because $\mu^\alpha = \lambda(\mu)$ and $\lambda(-1) = -1$. Similarly, we have $\rho = 1$. Therefore,

$$h^{-1}(a+b) = h^{-1}(a) + h^{-1}(b). \tag{13}$$

Next, we take $\bar{x} \in \bar{E}$ such that $\bar{x}(h^{-1}(a)) = \bar{x}(h^{-1}(b)) = 1$. This can be

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done because $h^{-1}(a)$ and $h^{-1}(b)$ are linearly independent. Then,

$$\begin{aligned} 2 &= (\bar{x}(h^{-1}(a))+\bar{x}(h^{-1}(b)))^{\alpha} = (\bar{x}(h^{-1}(a+b)))^{\alpha} \\ &= (\bar{x}(h^{-1}(a)))^{\alpha}+(\bar{x}(h^{-1}(b)))^{\alpha} \\ &= 2^{\alpha}, \end{aligned}$$

from which it follows that $\alpha = 1$. Thus, the proof of (12) is completed.

Now, we can prove that h is linear. If x and y are linearly independent, it follows from (13) that

$$h^{-1}(h(x)+h(y)) = x+y,$$

which is equivalent to

$$h(x)+h(y) = h(x+y).$$

If x and y are linearly dependent, since $y = \xi x$ for a number ξ,

$$\begin{aligned} h(x+y) &= h((1+\xi)x) = h(1+\xi)h^{-1}h(x) \\ &= \phi(1+\xi)h(x) = (1+\xi)h(x) \\ &= h(x)+\xi h(x) = h(x)+\phi(\xi)h(x) \\ &= h(x)+h\xi h^{-1}h(x) = h(x)+h(\xi x) \\ &= h(x)+h(y). \end{aligned}$$

Finally, we prove that h and h^{-1} are continuous. Since h is a bijection, we have only to prove that it is closed. Let us assume that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} h(x_n) = y$. Then, for $x \neq 0$ and an arbitrary $\bar{x} \in \bar{E}$, since $\phi(x \otimes \bar{x})$ is a continuous linear mapping,

$$\lim_{n\to\infty} \phi(x \otimes \bar{x})(h(x_n)) = \phi(x \otimes \bar{x})(y).$$

On the other hand,

$$\phi(x \otimes \bar{x})(h(x_n)) = \bar{x}(x_n)h(x)$$

and

$$\phi(x \otimes \bar{x})(y) = \bar{x}(h^{-1}(y))h(x).$$

Therefore, $\{x_n\}$ converges weakly to $h^{-1}(y)$, hence it follows $y = h(x_0)$.

Now, we return to the semigroup $\mathscr{D}$. For $f \in \mathscr{D}$, we define the set $d(f)$ by

$$d(f) = \{f'(x) | x \in E\}.$$

In [6], we have introduced the notion of *d-ideals*. Here we introduce the notion of *d-automorphisms* in the same way.

An automorphism ϕ of $\mathscr{D}$ is said to be a *d-automorphism* if

$$d\phi = \phi d,$$

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in other words, ϕ is a d-automorphism if

$$\{\phi(f)'(x) | x \in E\} = \{\phi(f'(x)) | x \in E\}$$

for each $f \in \mathscr{D}$.

Then the following theorem can easily be proved.

THEOREM 4. *Every d-automorphism of $\mathscr{D}$ is inner.*

PROOF. By THEOREM 3, we have only to prove that $\phi(\mathscr{L}) = \mathscr{L}$. If $\ell \in \mathscr{L}$, there exists $f \in \mathscr{D}$ such that $\ell = \phi(f)$. Then,

$$\{\ell\} = d\phi(f) = \phi d(f) = \{\phi(f'(x)) | x \in E\},$$

from which it follows that $f'(x)$ is constant with respect to x. Therefore, $f \in \mathscr{L}$, and $\mathscr{L} \subset \phi(\mathscr{L})$ was proved.

If $f \in \phi(\mathscr{L})$, since $f = \phi(\ell)$ for some $\ell \in \mathscr{L}$,

$$\mathrm{d}(f) = d\phi(\ell) = \phi d(\ell) = \{\phi(\ell)\}.$$

This means that $f \in \mathscr{L}$. Thus, the proof is completed.

REMARK. If we do not assume $\phi(\mathscr{L}) = \mathscr{L}$ in Theorem 3, the problem becomes almost equivalent to the problem of finding the infinitesimal generator of the one-parameter semigroup $\phi(e^{\xi})$ of purely non-linear mappings.

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