J. Appl. Probab. 1–27 (2024)
*doi:*10.1017/jpr.2024.75

STATIONARY MEASURES AND THE CONTINUOUS-STATE BRANCHING PROCESS CONDITIONED ON EXTINCTION

RONGLI LIU,* *Beijing Jiaotong University*
YAN-XIA REN,** *Peking University*
TING YANG,*** *Beijing Institute of Technology*

Abstract

We consider continuous-state branching processes (CB processes) which become extinct almost surely. First, we tackle the problem of describing the stationary measures on $(0, +\infty)$ for such CB processes. We give a representation of the stationary measure in terms of scale functions of related Lévy processes. Then we prove that the stationary measure can be obtained from the vague limit of the potential measure, and, in the critical case, can also be obtained from the vague limit of a normalized transition probability. Next, we prove some limit theorems for the CB process conditioned on extinction in a near future and on extinction at a fixed time. We obtain non-degenerate limit distributions which are of the size-biased type of the stationary measure in the critical case and of the Yaglom distribution in the subcritical case. Finally we explore some further properties of the limit distributions.

Keywords: Continuous-state branching process; stationary measure; vague convergence; conditional limit theorems; size-biased measure

2020 Mathematics Subject Classification: Primary 60J80
Secondary 60F05

1. Introduction

A $[0, +\infty)$-valued strong Markov process $Z = (Z_t)_{t\geq 0}$, with probabilities $\{\mathbb{P}_x : x \geq 0\}$ and absorbing state 0, is called a continuous-state branching process (CB process for short) if it has paths which are right continuous with left limits, $\mathbb{P}_x(Z_0 = x) = 1$ for every $x \geq 0$, and it employs the following branching property: for any $\lambda \geq 0$ and $x, y \geq 0$,

$$\mathbb{P}_{x+y}\big[\mathrm{e}^{-\lambda Z_t}\big] = \mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_t}\big]\mathbb{P}_y\big[\mathrm{e}^{-\lambda Z_t}\big],$$

where $\mathbb{P}_x$ denotes the expectation with respect to the probability $\mathbb{P}_x$. We suppose that Z has branching mechanism ψ, which is specified by the Lévy–Khintchine formula

$$\psi(\lambda) = \alpha\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^{+\infty} (\mathrm{e}^{-\lambda r} - 1 + \lambda r)\pi(\mathrm{d}r), \quad \lambda \geq 0, \tag{1.1}$$

Received 4 September 2023; accepted 8 August 2024.
* Postal address: School of Mathematics and Statistics, Beijing Jiaotong University, Beijing 100044, PR China. Email address: rlliu@bjtu.edu.cn
** Postal address: LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing 100871, PR China. Email address: yxren@math.pku.edu.cn
*** Postal address: School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, PR China. Email address: yangt@bit.edu.cn

where $\alpha \in \mathbb{R}$, $\sigma \geq 0$, and π is a positive Radon measure on $(0, +\infty)$ such that

$$\int_0^{+\infty} (r \wedge r^2)\pi(\mathrm{d}r) < +\infty.$$

We have $\mathbb{P}_x[Z_t] = x\mathrm{e}^{-\psi'(0+)t}$ for all $x, t \geq 0$. Since $\psi'(0+) = \alpha$, the process $(Z_t)_{t\geq 0}$ is called supercritical, critical, and subcritical for $\alpha < 0$, $\alpha = 0$, and $\alpha > 0$, respectively. In this paper we restrict our attention to the cases when the CB processes hit 0 with probability 1, that is, those critical or subcritical CB processes with branching mechanism ψ satisfying $\int^{+\infty} 1/\psi(\lambda)\,\mathrm{d}\lambda < +\infty$.

We are concerned with the stationary measures of CB processes. Since 0 is an absorbing state, the unique (up to a constant multiple) stationary measure on the state space $[0, +\infty)$ is the Dirac measure at 0 (see [13, P23–24]). Therefore we shall exclude the state 0, and call a Radon measure ν on $(0, +\infty)$ a stationary measure for $(Z_t)_{t\geq 0}$ if, for any $t > 0$ and any Borel set $A \subset (0, +\infty)$,

$$\mathbb{P}_\nu(Z_t \in A) = \nu(A),$$

where

$$\mathbb{P}_\nu(Z_t \in A) = \int_{(0,+\infty)} \mathbb{P}_x(Z_t \in A)\nu(\mathrm{d}x).$$

It is well known that a CB process can be viewed as the analogue of the Galton–Watson process (GW process) in continuous time and continuous state space. Before we start, let us first review some classical results concerning stationary measures for GW processes. A standard reference is Athreya and Ney [4]; see also Asmussen and Hering [3], Hoppe [14], Nakagawa [24], and Ogura and Shiotani [28] for related discussions for multitype GW processes. Suppose $(Y_n)_{n\geq 0}$ is a GW process taking values in $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with offspring distribution $(p_k)_{k\geq 0}$. Let $m = \sum_{j=1}^{+\infty} jp_j$ be the reproduction mean and $q = \mathrm{P}(Y_n = 0 \text{ eventually } | Y_0 = 1)$ be the extinction probability. Unless $p_1 = 1$, $q < 1$ if and only if $m > 1$ (supercritical case). Hence extinction occurs almost surely in the critical $(m = 1, p_1 < 1)$ and subcritical $(m < 1)$ cases. We call $(\eta_i)_{i\geq 1}$ a stationary measure for $(Y_n)_{n\geq 0}$ if $\eta_i \geq 0$ for all $i \geq 1$, and

$$\eta_j = \sum_{i=1}^{\infty} \eta_i P(i, j), \qquad j \geq 1,$$

where $(P(i, j))_{i,j\geq 0}$ denote the one-step transition probabilities of $(Y_n)_{n\geq 0}$. Theorem II.1.2 of [4] tells us that $(\eta_i)_{i\geq 1}$ is a stationary measure if and only if its generating function $U(s) = \sum_{i=1}^{+\infty} \eta_i s^i$ is analytic for $|s| < q$, and satisfies Abel's equation

$$U(f(s)) = U(p_0) + U(s), \qquad |s| < q,$$

where f is the generating function of the offspring distribution $(p_k)_{k\geq 0}$. In the supercritical case, if $q = 0$, the only stationary measure is $\eta_i = 0$ for all $i \geq 1$; otherwise, if $q > 0$, then the construction of stationary measures can be handled by reduction to the subcritical case: see [4, II.2]. Now we focus on the critical and subcritical cases. It is proved in [4] that in the critical case a (non-trivial) stationary measure exists and is unique (up to a constant multiple), while in the subcritical case the stationary measure is not unique. In fact, in the critical case, the stationary measure is determined by the ratio limit of the n-step transition probabilities

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

(see [4, Lemma I.7.2 and Theorem II.2.1] and [31]). The continuous-time analogue of this result is due to [15, Lemma 7]. In the subcritical case, the problem of determining all stationary measures is settled by Alsmeyer and Rösler [2], where it is proved that every stationary measure has a unique integral representation in terms of the Martin entrance boundary and a finite measure on [0,1).

For a continuous-time GW process with transition functions $\{p_{ij}(t) : t \geq 0, \ i, j \in \mathbb{Z}^+\}$, a stationary measure is a set of non-negative numbers $\{\nu_j : j \geq 1\}$ satisfying

$$\nu_j = \sum_{i \geq 1} \nu_i p_{ij}(t), \quad j \geq 1, \ t \geq 0.$$

In contrast to the discrete-time situation, for the continuous-time GW process a non-trivial stationary measure exists and is unique (up to a constant multiple) in both critical and subcritical cases; see [15, Lemma 7] for the critical case and [23, Corollary 8] for the subcritical case. A similar phenomenon happens for CB processes; see Ogura [25]. Namely, assuming extinction occurs almost surely, the CB process has a unique non-trivial stationary measure. Indeed, Ogura has established the functional equation satisfied by the Laplace transform of the stationary measure (see [25, Lemma 1.2]), which can be viewed as the continuous counterpart of the above Abel's equation.

In this paper we are interested in the description of the stationary measure of the CB process from different points of view. We extend Ogura's results in the following three respects. First, we establish a representation of the stationary measure for CB processes in terms of the so-called scale functions of the related Lévy processes (Theorem 2.1). Second, we prove that the transition probability on $(0, +\infty)$ of the CB process, when appropriately normalized, converges vaguely, and we obtain the precise limit measure (Theorem 3.1). We shall see from this result that, in the critical case, the stationary measure can be obtained from the vague limit of an appropriately normalized transition probability of the CB process, giving an analogue of the ratio limit theorem (see [4, Lemma I.7.2]). We remark that more regularity properties of the transition probabilities were investigated in [9], [26], and [27] for CB processes (with or without immigration), under additional analytical assumptions on the branching mechanisms. Finally, we obtain a representation of the potential measure of the CB process in terms of the scale functions, and we prove that the stationary measure can also be obtained from the vague limit of the potential measure in both critical and subcritical cases (Theorem 3.2). In the context of GW processes, a result of this type is obtained in [28] for the critical case (under additional assumptions on the reproduction law) and in [2] for the subcritical case. Our proof is based on the relation between CB processes and Lévy processes through the so-called Lamperti transform (see Section 2.1 below), and is easier than the proofs for the discrete state situation. Furthermore, we give equivalent conditions, depending on the branching mechanisms, for the potential measures to be finite (Proposition 3.1).

In this paper we also aim at linking the stationary measure to some conditional limit theorems of CB processes. Conditional limit theorems constitute an important part of the limit theory of branching processes. There has been a lot of work on various conditional limit theorems for branching processes; see e.g. [4], [11], and [29] for the discrete state situation, and [20], [21], [30], and [33] for the continuous state situation. Suppose $(Z_t)_{t\geq 0}$ is a CB process which becomes extinct almost surely. It is usual to condition on extinction after some time t. Let ζ be the extinction time. The asymptotic behavior of Z_t conditioned on $\{\zeta > t\}$ is described in the so-called Yaglom theorem. Namely, in the subcritical case, there is a probability measure

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

ρ on $(0, +\infty)$, called the Yaglom distribution, such that for any $x > 0$ and any Borel set $A \subset (0, +\infty)$,

$$\lim_{t\to+\infty} \mathbb{P}_x(Z_t \in A \mid \zeta > t) = \rho(A). \tag{1.2}$$

The Yaglom distribution belongs to the family of quasi-stationary distributions of CB processes. A brief review of the latter is given at the end of Section 2. By contrast, the critical case is degenerate since all the limits on the left-hand side of (1.2) are 0. However, by taking different conditioning instead of conditioning on non-extinction, one may get non-degenerate results for both critical and subcritical cases. In Section 4, we consider two special conditioning events: $\{t \le \zeta < t+s\}$ and $\{\zeta = t\}$ $(t, s > 0)$. The former is regarded as conditioning on extinction in the near future $[t, t+s)$ and the latter as conditioning on extinction at time t. When the extinction time ζ is finite almost surely, the event $\{t \le \zeta < t+s\}$ is of positive probability and this conditioning can be made in the usual sense. But $\{\zeta = t\}$ is of zero probability, and this conditioning is made by taking the limit of the conditional probability on $\{t \le \zeta < t+s\}$ as $s \to 0+$, or equivalently, by taking a Doob h-transform. The study of the CB process conditioned on $\{\zeta = t\}$ dates back to [1], in which it was shown that the CB process has a spinal decomposition, called a Williams decomposition, under such a conditional probability. Later, a similar property for superprocesses was studied in [10] and [32]. For GW processes, similar conditioning is studied by Esty [11]. We remark that Esty [11] considers only critical GW processes, while we allow the CB process to be either critical or subcritical.

In this paper we prove some limit theorems for CB processes conditioned on the aforementioned two events. Our two principal results, Theorem 4.1 and Theorem 4.2, show that the distributions of Z_{t-q} $(0 < q < t)$ conditioned on extinction in the near future $[t-q, t)$ and on extinction at time t are convergent as t goes to infinity, and we also obtain the precise limit distributions. From these results, we shall see that the limit distributions obtained in the critical (resp. subcritical) case are of the size-biased type of the stationary measure (resp. the Yaglom distribution). As a by-product, in the critical case, we prove that the limit distribution of Z_{t-q} $(0 < q < t)$ conditioned on $\{\zeta = t\}$ is of the size-biased type of the stationary measure, giving an analogue of [4, Theorem I.8.2]. Our proofs of the conditional limit theorems are based on the asymptotic estimates of the log-Laplace functional of CB process derived from the integral equations it satisfies. Moreover, we investigate properties of the limit distribution of Z_{t-q} conditioned on extinction at time t. We show that the limit is infinitely divisible and give a representation of its Lévy–Khintchine triplet in terms of the scale functions (Proposition 4.2). In the subcritical case, we prove that it is weakly convergent as $q \to +\infty$ to a non-degenerate distribution under an additional $L \log L$ condition (Proposition 4.3). As an application of these results, we present a new proof of a limit theorem for the CB process conditioned on non-extinction (Proposition 4.4).

We notice that by conditioning a supercritical CB process to be extinct, we recover a subcritical CB process. To be more specific, if γ is the largest root of $\psi(\lambda) = 0$, then $\gamma > 0$ in the supercritical case, and the supercritical CB process with branching mechanism ψ conditioned on its extinction turns out to be a subcritical CB process with branching mechanism $\psi^*(\lambda) = \psi(\lambda + \gamma)$. As a consequence, our conditional limit theorems obtained for the subcritical case can be applied to supercritical CB processes conditioned to be extinct.

The remainder of this paper is organized as follows. In Section 2 we recall the definition of CB processes and review some classical results concerning CB processes and Lévy processes. Then we give a representation of the stationary measure in terms of the scale functions of the related Lévy process. In Section 3 we prove the vague convergence of the normalized transition probabilities and potential measures of CB processes. Some examples are given to illustrate the results obtained in this section. In Section 4 we study the probabilities of Z_t conditioned on extinction in the near future and on extinction at a fixed time, prove some conditional limit theorems, and explore some properties of the limit distributions. Some minor statements needed along the way are proved in the Appendix.

Throughout this paper, we use ' $:=$ ' to denote definition. For positive functions f, g on $(0, +\infty)$ and constant $c \in [0, +\infty)$, we write $f(x) \sim g(x)$ as $x \to c$ if $\lim_{x\to c} f(x)/g(x) = 1$. For a measure μ on $(0, +\infty)$ and a measurable function f, we write $\langle f, \mu\rangle$ for the integral $\int_{(0,+\infty)} f(x)\mu(\mathrm{d}x)$. Suppose ν_n, ν are measures on $(0, +\infty)$; ν_n, ν can be extended to measures on the larger space $[0, +\infty)$ by setting $\nu_n(\{0\}) = \nu(\{0\}) = 0$. We define the vague convergence following [6]: ν_n is said to converge vaguely to ν if

$$\int_{[0,+\infty)} g(y)\nu_n(\mathrm{d}y) \to \int_{[0,+\infty)} g(y)\nu(\mathrm{d}y)$$

for all continuous functions g on $[0, +\infty)$ vanishing at infinity. If ν_n, ν are finite measures and $\langle f, \nu_n\rangle \to \langle f, \nu\rangle$ for all bounded continuous functions f on $(0, +\infty)$, we say that ν_n converges weakly to ν.

2. Preliminaries

2.1. CB processes and Lévy processes

Let $((Z_t)_{t\geq 0}, \mathbb{P}_x)$ be the CB process with branching mechanism $\psi(\lambda)$ given in (1.1) and initial value $x > 0$. Following [19], such a process is a time-homogeneous strong Markov process taking values in $[0, +\infty)$ with an absorbing state 0, such that for any $\lambda > 0$,

$$\mathbb{P}_x\left[\mathrm{e}^{-\lambda Z_t}\right] = \mathrm{e}^{-xu_t(\lambda)}, \qquad t \geq 0, \tag{2.1}$$

where $u_t(\lambda)$ is the solution to the following ordinary differential equation:

$$\begin{cases} \dfrac{\partial u_t(\lambda)}{\partial t} = -\psi(u_t(\lambda)), \\ u_0(\lambda) = \lambda. \end{cases} \tag{2.2}$$

We assume that $\psi(+\infty) = +\infty$. Thus, by [19, Theorem 12.3]), $(Z_t)_{t\geq 0}$ is conservative in the sense that $\mathbb{P}_x(Z_t < +\infty) = 1$ for all $x > 0$ and $t \geq 0$. Chapter 3 of [22] is also a good reference for continuous state branching processes.

Let $\zeta := \inf\{t > 0 : Z_t = 0\}$ be the extinction time. It follows by (2.1) that

$$\mathbb{P}_x(\zeta \leq t) = \mathbb{P}_x(Z_t = 0) = \mathrm{e}^{-xu_t(+\infty)} \quad \text{for all } x, t > 0.$$

Let $q(x) := \mathbb{P}_x(\zeta < +\infty)$ for $x > 0$. It is proved in [12] that $q(x) > 0$ for some (and then all) $x > 0$ if and only if

$$\int^{+\infty} \frac{1}{\psi(\lambda)}\,\mathrm{d}\lambda < +\infty. \tag{2.3}$$

In this case $q(x) = \mathrm{e}^{-x\gamma}$, where

$$\gamma := \sup\{\lambda \geq 0 : \psi(\lambda) = 0\}.$$

We know that ψ is strictly convex and infinitely differentiable on $(0, +\infty)$ with $\psi(0) = 0$, $\psi(+\infty) = +\infty$ and $\psi'(0+) = \alpha$. So we have $\gamma > 0$ if $\alpha < 0$ (supercritical case) and $\gamma = 0$ if $\alpha \geq 0$ (critical and subcritical cases).

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

Assuming (2.3) holds, we can define a strictly decreasing function ϕ on $(\gamma, +\infty)$ by

$$\phi(\lambda) := \int_{\lambda}^{+\infty} \frac{1}{\psi(u)}\,\mathrm{d}u, \quad \lambda > \gamma.$$

It is easy to see that $\phi(\gamma) = +\infty$ and $\phi(+\infty) = 0$. Let φ be the inverse function of ϕ, which is defined on $(0, +\infty)$ and takes values in $(\gamma, +\infty)$. From (2.2) we have

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)}\,\mathrm{d}u = t, \quad \lambda, t > 0. \tag{2.4}$$

By letting $\lambda \to +\infty$, we have

$$\int_{u_t(+\infty)}^{+\infty} \frac{1}{\psi(u)}\,\mathrm{d}u = t. \tag{2.5}$$

Recall that $u_t(+\infty) = -\log \mathbb{P}_1(\zeta \le t) \ge -\log \mathbb{P}_1(\zeta < +\infty) = \gamma$. By (2.5) we get $u_t(+\infty) = \varphi(t)$ for all $t > 0$, and consequently

$$\mathbb{P}_x(\zeta \le t) = \mathrm{e}^{-x\varphi(t)}, \quad x, t > 0. \tag{2.6}$$

In particular, if $(Z_t)_{t\ge 0}$ is critical or subcritical, then $\gamma = 0$ and (2.4) yields that

$$u_t(\lambda) = \varphi(t + \phi(\lambda)), \quad \lambda, t > 0. \tag{2.7}$$

We note that ψ is also the Laplace exponent of a spectrally positive Lévy process $(X_t)_{t\ge 0}$. We let P_x denote the law of $(X_t)_{t\ge 0}$ started at $x \in \mathbb{R}$ at time 0. Then

$$\mathrm{P}_x\big[\mathrm{e}^{-\lambda X_t}\big] = \mathrm{e}^{-\lambda x + \psi(\lambda)t}, \quad \lambda, t \ge 0.$$

Define $\tau_0^- := \inf\{t \ge 0 : \ X_t < 0\}$ with the convention that $\inf \emptyset = +\infty$. There is a sample-path relationship between the CB process $(Z_t)_{t\ge 0}$ and the Lévy process $(X_t)_{t\ge 0}$ stopped at τ_0^-, called the Lamperti transform (see [19, Theorem 12.2] or [7]). For $t \ge 0$, define

$$\theta_t := \inf\left\{ s > 0 : \ \int_0^s \frac{1}{X_u}\,\mathrm{d}u > t \right\}.$$

Then $\big((X_{\theta_t \wedge \tau_0^-})_{t\ge 0}, \mathrm{P}_x\big)$ is a CB process with branching mechanism ψ and initial value $x > 0$. We refer to [19, Chapter 12] for results on the long-term behavior of the CB process based on the fluctuation theory of spectrally positive Lévy processes.

2.2. Representation of the stationary measure

In what follows and for the remainder of this paper, we assume $(Z_t)_{t\ge 0}$ is a CB process with branching mechanism ψ satisfying (2.3) and $\psi'(0+) = \alpha \ge 0$. In this subsection we shall give a representation of the stationary measure of $(Z_t)_{t\ge 0}$ in terms of the so-called scale function. Recall that the scale function W is the unique strictly increasing and positive continuous function on $[0, +\infty)$ such that

$$\int_0^{+\infty} \mathrm{e}^{-\lambda x} W(x)\,\mathrm{d}x = \frac{1}{\psi(\lambda)}, \quad \lambda > 0. \tag{2.8}$$

We define $W(x) = 0$ for $x < 0$. We refer to [5, Chapter VII] and [18] for the general theory of scale functions.

We write $\int_{0+}^{+\infty}$ for $\int_{(0,+\infty)}$ to emphasize that the integral is on $(0, +\infty)$. For a measure ν on $(0, +\infty)$, we set

$$\widehat{\nu}(\lambda) := \int_{0+}^{+\infty} \mathrm{e}^{-\lambda x} \nu(\mathrm{d}x) \quad \text{for } \lambda \geq 0$$

whenever the right-hand side is well-defined.

Theorem 2.1. *Set*

$$\mu(\mathrm{d}x) := \frac{W(x)}{x}\, \mathrm{d}x \quad \textit{for } x > 0.$$

Then $\mu(\mathrm{d}x)$ is the unique (up to a constant multiple) stationary measure for $(Z_t)_{t\geq 0}$.

Proof. Due to [25, Lemma 1.2 and Proposition 1.3], for a CB process which satisfies (2.3), there exists a unique (up to a constant multiple) stationary measure ϱ such that

$$\widehat{\varrho}(\lambda) = \phi(\lambda), \quad \lambda > \gamma.$$

Recall that the CB process we consider in this theorem is critical or subcritical, and then $\gamma = 0$. For any $\lambda > 0$ we have

$$\begin{aligned}
\int_{0+}^{+\infty} \mathrm{e}^{-\lambda x} \mu(\mathrm{d}x) &= \int_{0+}^{+\infty} \mu(\mathrm{d}x) \int_{\lambda}^{+\infty} x\, \mathrm{e}^{-ux}\, \mathrm{d}u \\
&= \int_{0}^{+\infty} W(x)\, \mathrm{d}x \int_{\lambda}^{+\infty} \mathrm{e}^{-ux}\, \mathrm{d}u \\
&= \int_{\lambda}^{+\infty} \mathrm{d}u \int_{0}^{+\infty} W(x)\, \mathrm{e}^{-ux}\, \mathrm{d}x \\
&= \int_{\lambda}^{+\infty} \frac{1}{\psi(u)}\, \mathrm{d}u \\
&= \phi(\lambda).
\end{aligned}$$

Hence μ is the unique (up to a constant multiple) stationary measure. □

Remark 2.1. Suppose $(Z_t)_{t\geq 0}$ is a supercritical CB process satisfying (2.3) and $\gamma = \sup\{\lambda \geq 0 : \psi(\lambda) = 0\} > 0$. Repeating the calculation in the proof of Theorem 2.1, we can show that, for $\mu(\mathrm{d}x) = W(x)x^{-1}\, \mathrm{d}x$,

$$\int_{0+}^{+\infty} \mathrm{e}^{-\lambda x} \mu(\mathrm{d}x) = \phi(\lambda), \quad \lambda > \gamma.$$

Hence the result of Theorem 2.1 also holds for this supercritical CB process.

We notice that $\widehat{\mu}(0) = \phi(0) = +\infty$. So μ is an infinite measure on $(0, +\infty)$. Theorem 2.1 implies that the CB process has no stationary distributions on $(0, +\infty)$. Instead, one may consider a subinvariant distribution, called the quasi-stationary distribution (QSD). For a CB process, a QSD is a probability measure ν on $(0, +\infty)$ satisfying

$$\mathbb{P}_{\nu}(Z_t \in A \mid \zeta > t) = \nu(A) \tag{2.9}$$

for any Borel set $A \subset (0, +\infty)$ and $t > 0$. One can easily show by the Markov property that

$$\mathbb{P}_{\nu}(\zeta > t + s) = \mathbb{P}_{\nu}(\zeta > t)\mathbb{P}_{\nu}(\zeta > s), \quad t, s \geq 0.$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

Hence the extinction time ζ under $\mathbb{P}_\nu$ is exponentially distributed with some parameter $\beta > 0$. So (2.9) is equivalent to

$$\mathbb{P}_\nu(Z_t \in A) = \mathrm{e}^{-\beta t}\nu(A)$$

for any Borel set $A \subset (0, +\infty)$ and $t > 0$. A discrete state analogue is the so-called λ-invariant measure, for which we refer to [23]. Lambert [20] has given a complete characterization of QSDs for CB processes. It is proved in [20] that a subcritical CB process has QSDs while a critical CB process has no QSD. In fact, for a subcritical CB process with $\psi'(0+) = \alpha > 0$, all QSDs form a stochastically decreasing family $\{\nu_\beta\}$ of probabilities indexed by $\beta \in (0, \alpha]$ satisfying

$$\widehat{\nu}_\beta(\lambda) = 1 - \mathrm{e}^{-\beta\phi(\lambda)}, \qquad \lambda > 0.$$

The probability ν_α is the so-called Yaglom distribution in the sense that

$$\lim_{t\to+\infty} \mathbb{P}_x(Z_t \in A \mid \zeta > t) = \nu_\alpha(A) \tag{2.10}$$

for every $x > 0$ and Borel set $A \subset (0, +\infty)$. The conditional limit of (2.10) is due to Li [21, Theorem 4.3], where more general conditioning of the type $\{\zeta > t + r\}$ with $r \geq 0$ is considered. From the theory of Laplace transforms, the QSD ν_β can be expressed by the stationary measure μ as

$$\nu_\beta(\mathrm{d}x) = -\sum_{n=1}^{+\infty} \frac{(-\beta)^n}{n!}\mu^{*n}(\mathrm{d}x),$$

where μ^{*n} denotes the n-fold convolution of μ. On the other hand, since $\widehat{\nu}_\beta(\lambda)/\beta \to \phi(\lambda)$ as $\beta \to 0+$ for all $\lambda > 0$, we get that $\frac{1}{\beta}\nu_\beta$ converges vaguely to μ as $\beta \to 0+$.

Although there is no QSD in the critical case, convergence results are established for the rescaled process Q_tZ_t conditioned on $\{\zeta > t\}$, where $Q_t \to 0$ as $t \to +\infty$. It is proved by [21, Theorem 5.2] that if the critical CB process has finite variance, i.e. $\psi''(0+) < +\infty$, then Z_t/t conditioned on $\zeta > t$ converges in distribution to an exponential distribution random variable with parameter $2/\psi''(0+)$. We refer to [33] for the case allowing infinite variance.

3. Convergence of transition probabilities and potential measures

Let $(P_t(x, \mathrm{d}y); t \geq 0, x, y \geq 0)$ be the transition probability of the CB process $(Z_t)_{t\geq 0}$. First, we shall show that the transition probability $P_t(x, \mathrm{d}y)$ on $(0, +\infty)$, when appropriately normalized, converges vaguely to a precise measure. For notational simplicity, we still use $P_t(x, \mathrm{d}y)$ to denote the restriction of $P_t(x, \mathrm{d}y)$ on $(0, +\infty)$.

Lemma 3.1. *If $\alpha = 0$, then*

$$\lim_{t\to+\infty} \frac{\varphi(t) - \varphi(t + \phi(\lambda))}{\psi(\varphi(t))} = \phi(\lambda), \qquad \lambda > 0.$$

Proof. It follows by the monotone convergence theorem that

$$\psi'(\lambda) = \sigma^2\lambda + \int_0^{+\infty} (1 - \mathrm{e}^{-\lambda r})r\pi(\mathrm{d}r) \to 0 \quad \text{as } \lambda \to 0+.$$

We note that $(\psi(\varphi(t)))' = -\psi'(\varphi(t))\psi(\varphi(t))$ for $t > 0$ and that $t \mapsto \varphi(t)$ is strictly decreasing on $(0, +\infty)$ with $\varphi(+\infty) = 0$. Thus, for any $s > 0$,

$$\ln \frac{\psi(\varphi(t + s))}{\psi(\varphi(t))} = -\int_t^{t+s} \psi'(\varphi(u))\,\mathrm{d}u \to 0 \quad \text{as } t \to +\infty.$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

It follows that

$$\lim_{t\to+\infty} \frac{\psi(\varphi(t+s))}{\psi(\varphi(t))} = 1. \tag{3.1}$$

By the mean value theory, for every $t > 0$ and $\lambda > 0$ there exists $\Delta_t(\phi(\lambda)) \in [0, \phi(\lambda)]$ such that

$$\frac{\varphi(t) - \varphi(t+\phi(\lambda))}{\psi(\varphi(t))} = \frac{\psi(\varphi(t+\Delta_t(\phi(\lambda))))}{\psi(\varphi(t))}\phi(\lambda). \tag{3.2}$$

Since $t \mapsto \psi(\varphi(t))$ is strictly decreasing on $(0, +\infty)$, we have

$$\frac{\psi(\varphi(t+\phi(\lambda)))}{\psi(\varphi(t))} \le \frac{\psi(\varphi(t+\Delta_t(\phi(\lambda))))}{\psi(\varphi(t))} \le 1.$$

By (3.1),

$$\lim_{t\to+\infty} \frac{\psi(\varphi(t+\Delta_t(\phi(\lambda))))}{\psi(\varphi(t))} = 1.$$

Combining this with (3.2), we get

$$\lim_{t\to+\infty} \frac{\varphi(t) - \varphi(t+\phi(\lambda))}{\psi(\varphi(t))} = \lim_{t\to+\infty} \frac{\psi(\varphi(t+\Delta_t(\phi(\lambda))))}{\psi(\varphi(t))}\phi(\lambda) = \phi(\lambda). \qquad \square$$

Theorem 3.1. *If $\alpha > 0$, then for every $x > 0$,*

$$\frac{1}{x\psi(\varphi(t))} P_t(x, \mathrm{d}y)$$

converges weakly to $\frac{1}{\alpha}\nu_\alpha(\mathrm{d}y)$ as $t \to +\infty$. Otherwise, if $\alpha = 0$, then for every $x > 0$,

$$\frac{1}{x\psi(\varphi(t))} P_t(x, \mathrm{d}y)$$

converges vaguely to $\mu(\mathrm{d}y)$ as $t \to +\infty$.

Proof. By Lemma A.1, it suffices to show that for any $x > 0$,

$$\lim_{t\to+\infty} \frac{1}{\psi(\varphi(t))} \int_{0+}^{+\infty} \mathrm{e}^{-\lambda y} P_t(x, \mathrm{d}y) = \begin{cases} \dfrac{x}{\alpha}\widehat{\nu}_\alpha(\lambda) & \text{for all } \lambda \ge 0, \text{ if } \alpha > 0, \\ x\widehat{\mu}(\lambda) & \text{for all } \lambda > 0, \text{ if } \alpha = 0. \end{cases} \tag{3.3}$$

For any $\lambda \ge 0$, we have

$$\int_{0+}^{+\infty} \mathrm{e}^{-\lambda y} P_t(x, \mathrm{d}y) = \mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_t}, Z_t > 0\big] = \mathrm{e}^{-x u_t(\lambda)} - \mathrm{e}^{-x\varphi(t)}.$$

If $\alpha > 0$, then $\psi(\varphi(t)) \sim \alpha\varphi(t)$ and $\mathbb{P}_x(\zeta > t) = 1 - \mathrm{e}^{-x\varphi(t)} \sim x\varphi(t)$ as $t \to +\infty$. Consequently, we have, for $\lambda \ge 0$,

$$\frac{1}{\psi(\varphi(t))} \int_{0+}^{+\infty} \mathrm{e}^{-\lambda y} P_t(x, \mathrm{d}y) \sim \frac{x}{\alpha}\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_t} \mid \zeta > t\big] \quad \text{as } t \to +\infty.$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

Thus, if $\alpha > 0$ the result (3.3) is a reformulation of the limit (2.10). Now suppose $\alpha = 0$. By Lemma 3.1, for any $\lambda > 0$ we have

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{\psi(\varphi(t))}\int_{0+}^{+\infty}\mathrm{e}^{-\lambda y}P_t(x,\mathrm{d}y) &= \lim_{t\to+\infty}\frac{\mathrm{e}^{-x\varphi(t+\phi(\lambda))}-\mathrm{e}^{-x\varphi(t)}}{\psi(\varphi(t))}\\ &= \lim_{t\to+\infty}\frac{x(\varphi(t)-\varphi(t+\phi(\lambda)))}{\psi(\varphi(t))}\\ &= x\phi(\lambda)\\ &= x\widehat{\mu}(\lambda).\end{aligned}$$

Hence we prove (3.3). □

Theorem 3.1 implies that the transition probability $P_t(x, \mathrm{d}y)$ constrained on $(0, \infty)$ is vaguely convergent with rate $x\psi(\varphi(t))$ as $t \to +\infty$. In the following we shall give concrete examples to illustrate the result of Theorem 3.1.

Example 3.1. Suppose $(Z_t)_{t\geq 0}$ is a subcritical CB process with $\psi'(0+) = \alpha > 0$. Let Θ be a positive random variable whose distribution is equal to the Yaglom distribution ν_α. By [20, Lemma 2.1], $\mathrm{E}[\Theta] < +\infty$ if and only if

$$\int^{+\infty} r\ln r\pi(\mathrm{d}r) < +\infty, \tag{3.4}$$

and in this case $\varphi(t) \sim \mathrm{E}[\Theta]^{-1}\,\mathrm{e}^{-\alpha t}$ as $t \to +\infty$. Thus

$$\psi(\varphi(t)) \sim \psi'(0+)\varphi(t) \sim \frac{\alpha}{\mathrm{E}[\Theta]}\,\mathrm{e}^{-\alpha t} \quad \text{as } t \to +\infty.$$

Theorem 3.1 yields that for every $x > 0$, restricted on $(0, +\infty)$,

$$\mathrm{e}^{\alpha t}P_t(x, \mathrm{d}y) \text{ converges weakly to } \frac{x}{\mathrm{E}[\Theta]}\nu_\alpha(\mathrm{d}y) \quad \text{as } t \to +\infty.$$

Otherwise, if (3.4) fails, then $\varphi(t) = \mathrm{o}(\mathrm{e}^{-\alpha t})$ and thus $\psi(\varphi(t)) = \mathrm{o}(\mathrm{e}^{-\alpha t})$. Hence $\mathrm{e}^{\alpha t}P_t(x, \mathrm{d}y)$ converges weakly to the null measure.

Example 3.2. Suppose $(Z_t)_{t\geq 0}$ is a critical CB process with branching mechanism ψ given by

$$\psi(\lambda) = \lambda^{1+p}L(1/\lambda), \qquad \lambda > 0,$$

where $0 < p \leq 1$ and L is a slowly varying function at $+\infty$. For a slowly varying function l, it is known (see [8, Theorem 1.5.13]) that there exists a unique (up to asymptotic equivalence) slowly varying function $l^{\#}$ such that $l(x)l^{\#}(xl(x)) \to 1$ and $l^{\#}(x)l(xl^{\#}(x)) \to 1$ as $x \to +\infty$. $l^{\#}$ is called the de Bruijn conjugate of l.

For $z > 0$, let

$$g(z) := \phi(1/z) = \int_{1/z}^{+\infty}\frac{1}{\psi(\lambda)}\,\mathrm{d}\lambda = \int_0^z \frac{u^{p-1}}{L(u)}\,\mathrm{d}u.$$

Since $p - 1 > -1$, by Karamata's theorem (see [8, Theorem 1.5.11]),

$$g(z) \sim \frac{z^p}{pL(z)} \quad \text{as } z \to +\infty.$$

Note that g is a strictly increasing function on $(0, +\infty)$. Let g^{-1} be its inverse. It follows by [8, Proposition 1.5.15] that

$$g^{-1}(z) \sim pz^{1/p} L^{\diamond}(z^{1/p})^{1/p} \quad \text{as } z \to +\infty,$$

where $L^{\diamond}$ is the de Bruijn conjugate of $1/L$. Recall that $\varphi(t) = \phi^{-1}(t) = 1/g^{-1}(t)$. We get

$$\varphi(t) \sim \frac{1}{p} t^{-1/p} L^{\diamond}(t^{1/p})^{-1/p} \quad \text{as } t \to +\infty.$$

We note that

$$\varphi(t) = -\int_t^{+\infty} \varphi'(s)\, \mathrm{d}s = \int_t^{+\infty} \psi(\varphi(s))\, \mathrm{d}s.$$

We also note that $\psi(\varphi(s))$ is a strictly decreasing function on $(0, +\infty)$. Hence, by the monotone density theorem (see [8, Theorem 1.7.2]),

$$\psi(\varphi(t)) \sim \frac{1}{p^2} t^{-(1/p+1)} L^{\diamond}(t^{1/p})^{-1/p} \quad \text{as } t \to +\infty.$$

Therefore Theorem 3.1 yields that for every $x > 0$,

$$\frac{p^2}{x} t^{1/p+1} L^{\diamond}(t^{1/p})^{1/p} P_t(x, \mathrm{d}y) \text{ converges vaguely to } \mu(\mathrm{d}y) \quad \text{as } t \to +\infty.$$

For every $x > 0$ and Borel set $A \subset (0, +\infty)$, we put

$$G(x, A) := \int_0^{+\infty} \mathbb{P}_x(Z_t \in A)\, \mathrm{d}t \in [0, +\infty],$$

and call the corresponding measure $G(x, \mathrm{d}y)$ on $(0, +\infty)$ the potential measure of $(Z_t)_{t \geq 0}$. Equation (3.3) yields that, if $\alpha \geq 0$ (subcritical or critical case), for every $x > 0$ and $\lambda > 0$,

$$\int_{0+}^{+\infty} \mathrm{e}^{-\lambda y} P_t(x, \mathrm{d}y) \sim c_\lambda x \psi(\varphi(t)) \quad \text{as } t \to +\infty, \tag{3.5}$$

for some positive constant c_λ depending on λ. We note that $\varphi'(t) = -\psi(\varphi(t))$. Thus

$$\int_1^{+\infty} \psi(\varphi(t))\, \mathrm{d}t = \varphi(1) - \varphi(\infty) = \varphi(1) < +\infty.$$

Hence we deduce by (3.5) that

$$\int_{0+}^{+\infty} \mathrm{e}^{-\lambda y} G(x, \mathrm{d}y) = \int_0^{+\infty} \int_{0+}^{+\infty} \mathrm{e}^{-\lambda y} P_t(x, \mathrm{d}y)\, \mathrm{d}t < +\infty \quad \text{for every } x > 0.$$

This implies that $G(x, B) < +\infty$ for every compact subset $B \subset (0, +\infty)$. Thus the potential measure for the CB process $(Z_t)_{t \geq 0}$ is a locally finite measure on $(0, +\infty)$.

Theorem 3.2. *The potential measure $G(x, \mathrm{d}y)$ of $(Z_t)_{t \geq 0}$ has a density with respect to the Lebesgue measure given by*

$$g(x, y) = \frac{W(y) - W(y - x)}{y}$$

for $x, y > 0$. *Moreover,* $G(x, \mathrm{d}y)$ *converges vaguely to the stationary measure* $\mu(\mathrm{d}y)$ *as* $x \to +\infty$.

Proof. Suppose $(X_t)_{t\geq 0}$ is the spectrally positive Lévy process associated with the CB process $(Z_t)_{t\geq 0}$ through the Lamperti transform. Then, for $x > 0$ and $\lambda > 0$, we have

$$\begin{aligned}\int_{0+}^{+\infty} \mathrm{e}^{-\lambda y} G(x, \mathrm{d}y) &= \mathbb{P}_x\left[\int_0^{\zeta} \mathrm{e}^{-\lambda Z_t}\,\mathrm{d}t\right] \\ &= \mathrm{P}_x\left[\int_0^{\tau_0^-} \mathrm{e}^{-\lambda X_s}\frac{1}{X_s}\,\mathrm{d}s\right]. \end{aligned} \tag{3.6}$$

The final equality follows from a change of variables. Let $U(x, \mathrm{d}y)$ be the potential measure of X killed on exiting $[0, +\infty)$ when issued from $x > 0$, that is,

$$U(x, \mathrm{d}y) = \int_0^{+\infty} \mathrm{P}_x\big(X_t \in \mathrm{d}y,\ t < \tau_0^-\big)\,\mathrm{d}t \quad \text{for } y > 0.$$

It follows by (3.6) that

$$G(x, \mathrm{d}y) = \frac{1}{y} U(x, \mathrm{d}y) \quad \text{for } x, y > 0. \tag{3.7}$$

It is proved in [18, Theorem 2.7] that $U(x, \mathrm{d}y)$ has a potential density with respect to the Lebesgue measure given by

$$u(x, y) = \mathrm{e}^{-\gamma x} W(y) - W(y - x), \quad x, y > 0. \tag{3.8}$$

Here $\gamma = 0$ since $\psi'(0+) \geq 0$. Putting this back to (3.7), we prove the first assertion.

We note that for $\lambda > 0$,

$$\begin{aligned}\int_{0+}^{+\infty} \mathrm{e}^{-\lambda y} G(x, \mathrm{d}y) &= \int_0^{+\infty} \mathrm{e}^{-\lambda y}\,\frac{W(y) - W(y - x)}{y}\,\mathrm{d}y \\ &= \int_{0+}^{+\infty} \mathrm{e}^{-\lambda y}\mu(\mathrm{d}y) - \int_x^{+\infty} \mathrm{e}^{-\lambda y}\frac{W(y - x)}{y}\,\mathrm{d}y.\end{aligned}$$

By a change of variables, the second integral in the right-hand side equals

$$\mathrm{e}^{-\lambda x}\int_0^{+\infty} \mathrm{e}^{-\lambda z}\frac{W(z)}{x + z}\,\mathrm{d}z,$$

which converges to 0 as $x \to +\infty$. Hence we get

$$\lim_{x\to+\infty}\int_{0+}^{+\infty} \mathrm{e}^{-\lambda y} G(x, \mathrm{d}y) = \widehat{\mu}(\lambda)$$

for all $\lambda > 0$. Hence we prove the second assertion. □

Remark 3.1. We remark that (3.8) indeed holds for $\gamma \geq 0$. Thus, for a supercritical CB process, applying a similar argument with minor modifications, we can show that the potential density function exists and is given by

$$g(x, y) = \mathrm{e}^{-\gamma x}\frac{W(y)}{y} - \frac{W(y - x)}{y}$$

for $x, y > 0$.

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

A natural question is under what condition $G(x, \mathrm{d}y)$ is a finite measure on $(0, +\infty)$. We give the following equivalent statements.

Proposition 3.1. *The following statements are equivalent:*

(i) $G(x, \mathrm{d}y)$ *is a finite measure on* $(0, +\infty)$ *for some (and then all)* $x > 0$,

(ii) $\mathbb{P}_x[\zeta] < +\infty$ *for some (and then all)* $x > 0$,

(iii) *the branching mechanism* ψ *satisfies*

$$\int_{0+} \frac{u}{\psi(u)}\,\mathrm{d}u < +\infty. \tag{3.9}$$

Proof. (i)$\Longleftrightarrow$(ii) By Fubini's theorem, for every $x > 0$ we have

$$\int_{0+}^{+\infty} G(x, \mathrm{d}y) = \int_0^{+\infty} \mathrm{d}t \int_{0+}^{+\infty} P_t(x, \mathrm{d}y) = \int_0^{+\infty} \mathbb{P}_x(\zeta > t)\,\mathrm{d}t = \mathbb{P}_x[\zeta].$$

Hence (i) and (ii) are equivalent.

(i)$\Longleftrightarrow$(iii) For every $x > 0$ we have

$$\int_{0+}^{+\infty} G(x, \mathrm{d}y) = \int_0^{+\infty} \mathbb{P}_x(\zeta > t)\,\mathrm{d}t = \int_0^{+\infty} \left(1 - \mathrm{e}^{-x\varphi(t)}\right)\mathrm{d}t.$$

Since $\varphi(t) \to 0$ as $t \to +\infty$, we have $1 - \mathrm{e}^{-x\varphi(t)} \sim x\varphi(t)$ as $t \to +\infty$. Hence the final integral is finite if and only if $\int^{+\infty} \varphi(t)\,\mathrm{d}t < +\infty$. Substituting t with $\phi(s)$ in the integral $\int^{+\infty} \varphi(t)\,\mathrm{d}t$, we can deduce that $\int^{+\infty} \varphi(t)\,\mathrm{d}t < +\infty$ if and only if

$$-\int_{0+} s\,\mathrm{d}\phi(s) = \int_{0+} \frac{s}{\psi(s)}\,\mathrm{d}s < +\infty. \qquad \square$$

We will also classify the finiteness of $G(x, \mathrm{d}y)$ through the Lévy measure π.

Corollary 3.1. *If* $\alpha > 0$, *then* $G(x, \mathrm{d}y)$ *is a finite measure on* $(0, +\infty)$ *for every* $x > 0$. *If* $\alpha = 0$, *then* $G(x, \mathrm{d}y)$ *is finite on* $(0, +\infty)$ *for some (then all)* $x > 0$ *if and only if*

$$\int^{+\infty} \frac{1}{s\int_0^s \bar{\bar{\pi}}(r)\,\mathrm{d}r}\,\mathrm{d}s < +\infty, \tag{3.10}$$

where for $r \geq 0$, $\bar{\pi}(r) := \int_r^{+\infty} \pi(\mathrm{d}y)$ *and* $\bar{\bar{\pi}}(r) := \int_r^{+\infty} \bar{\pi}(y)\,\mathrm{d}y$, *or equivalently,*

$$\int^{+\infty} \frac{1}{s\int_0^s r^2\pi(\mathrm{d}r) + s^2\int_s^{+\infty} r\pi(\mathrm{d}r)}\,\mathrm{d}s < +\infty. \tag{3.11}$$

Proof. If $\alpha > 0$, then $u/\psi(u) \sim 1/\alpha$ as $u \to 0$, and (3.9) holds immediately. Now we suppose $\alpha = 0$. In this case

$$\begin{aligned}\frac{\psi(\lambda)}{\lambda} &= \frac{1}{2}\sigma^2\lambda + \frac{1}{\lambda}\int_0^{+\infty} (\mathrm{e}^{-\lambda r} - 1 + \lambda r)\pi(\mathrm{d}r) \\ &= \frac{1}{2}\sigma^2\lambda + \int_0^{+\infty} (1 - \mathrm{e}^{-\lambda r})\bar{\pi}(r)\,\mathrm{d}r\end{aligned}$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

for $\lambda>0$. Obviously $\psi(\lambda)/\lambda$ is the Laplace exponent of a Lévy subordinator. Thus, by [5, Proposition III.1],

$$\frac{\psi(\lambda)}{\lambda} \asymp \lambda\left(\frac{1}{2}\sigma^2+\int_0^{1/\lambda}\bar{\bar{\pi}}(r)\,\mathrm{d}r\right).$$

Consequently we have

$$\int_{0+}\frac{u}{\psi(u)}\,\mathrm{d}u \asymp \int_{0+}\frac{1}{u}\cdot\frac{1}{\frac{1}{2}\sigma^2+\int_0^{1/u}\bar{\bar{\pi}}(r)\,\mathrm{d}r}\,\mathrm{d}u.$$

By a change of variables, the integral on the right-hand side equals

$$\int^{+\infty}\frac{1}{s\left(\frac{1}{2}\sigma^2+\int_0^s\bar{\bar{\pi}}(r)\,\mathrm{d}r\right)}\,\mathrm{d}s.$$

If $\int_0^{+\infty}\bar{\bar{\pi}}(r)\,\mathrm{d}r<+\infty$, then the latter integral equals $+\infty$ and (3.9) fails. Otherwise, if $\int_0^{+\infty}\bar{\bar{\pi}}(r)\,\mathrm{d}r=+\infty$, then

$$\frac{1}{\frac{1}{2}\sigma^2+\int_0^s\bar{\bar{\pi}}(r)\,\mathrm{d}r}\sim\frac{1}{\int_0^s\bar{\bar{\pi}}(r)\,\mathrm{d}r}\quad\text{as } s\to+\infty,$$

and (3.9) holds if and only if (3.10) holds. Next, we prove the equivalence of (3.10) and (3.11). For any $s>0$, by exchanging the order of integration, we obtain

$$\begin{aligned}\int_0^s\bar{\bar{\pi}}(r)\,\mathrm{d}r&=\int_0^{+\infty}\pi(\mathrm{d}r)\int_0^r(u\wedge s)\,\mathrm{d}u\\&=\frac{1}{2}\int_0^s r^2\pi(\mathrm{d}r)+s\int_s^{+\infty}r\pi(\mathrm{d}r)-\frac{s^2}{2}\bar{\pi}(s).\end{aligned}$$

Note that

$$0\le\bar{\pi}(s)\le\frac{\int_s^{+\infty}r\pi(\mathrm{d}r)}{s}.$$

These deduce the following inequalities:

$$\frac{1}{2}\int_0^s r^2\pi(\mathrm{d}r)+\frac{s}{2}\int_s^{+\infty}r\pi(\mathrm{d}r)\le\int_0^s\bar{\bar{\pi}}(r)\,\mathrm{d}r\le\frac{1}{2}\int_0^s r^2\pi(\mathrm{d}r)+s\int_s^{+\infty}r\pi(\mathrm{d}r).$$

Or it can be expressed as

$$\int_0^s\bar{\bar{\pi}}(r)\,\mathrm{d}r\asymp\int_0^s r^2\pi(\mathrm{d}r)+s\int_s^{+\infty}r\pi(\mathrm{d}r).$$

And the equivalence of (3.10) and (3.11) is obtained. □

From this result, we can see that if the critical CB process has finite variance, that is,

$$\int_1^{+\infty}r^2\pi(\mathrm{d}r)<+\infty,$$

then $\mathbb{P}_x[\zeta]=+\infty$ for every $x>0$, though $\mathbb{P}_x(\zeta<+\infty)=1$. However, if the right tail of the Lévy measure π of the critical CB process is heavy enough, e.g. $\pi(\mathrm{d}r)=r^{-(2+p)}\,\mathrm{d}r$ for some $p\in(0,1)$, then the expectation of ζ is finite.

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

4. CB process conditioned on extinction

4.1. Existence of conditional limits

Lemma 4.1. *For any $s > 0$, set*

$$\mu_s(\mathrm{d}x) := \mathrm{e}^{-\varphi(s)x} \frac{W(x)}{sx}\,\mathrm{d}x \tag{4.1}$$

for $x > 0$. Then μ_s is a probability measure on $(0, +\infty)$ with

$$\widehat{\mu}_s(\lambda) = \frac{\phi(\lambda + \varphi(s))}{s}, \quad \lambda > 0.$$

Moreover, μ_s is the size-biased stationary measure given by

$$\mu_s(\mathrm{d}x) = \frac{\mathrm{e}^{-\varphi(s)x}\mu(\mathrm{d}x)}{\int_0^{+\infty} \mathrm{e}^{-\varphi(s)r}\mu(\mathrm{d}r)}.$$

Proof. By (2.8) and Fubini's theorem, for $\lambda \geq 0$ we have

$$\begin{aligned}
\phi(\lambda + \varphi(s)) &= \int_{\lambda+\varphi(s)}^{+\infty} \frac{1}{\psi(u)}\,\mathrm{d}u \\
&= \int_{\lambda+\varphi(s)}^{+\infty} \mathrm{d}u \int_0^{+\infty} \mathrm{e}^{-ux} W(x)\,\mathrm{d}x \\
&= \int_0^{+\infty} W(x)\,\mathrm{d}x \int_{\lambda+\varphi(s)}^{+\infty} \mathrm{e}^{-ux}\,\mathrm{d}u \\
&= \int_0^{+\infty} \mathrm{e}^{-(\lambda+\varphi(s))x} \frac{W(x)}{x}\,\mathrm{d}x \\
&= s \int_0^{+\infty} \mathrm{e}^{-\lambda x} \mu_s(\mathrm{d}x).
\end{aligned}$$

In particular, if $\lambda = 0$,

$$\int_0^{+\infty} \mu_s(\mathrm{d}x) = \phi(\varphi(s))/s = 1.$$

It follows that $\mu_s(\mathrm{d}x)$ is a probability measure on $(0, +\infty)$. The second assertion follows immediately by observing that $\int_0^{+\infty} \mathrm{e}^{-\varphi(s)x}\mu(\mathrm{d}x) = s$. □

Recall that Θ is a random variable distributed as Yaglom distribution ν_α. Then its Laplace function is given by

$$\mathrm{E}\big[\mathrm{e}^{-\lambda\Theta}\big] = 1 - \mathrm{e}^{-\alpha\phi(\lambda)}, \quad \lambda > 0. \tag{4.2}$$

The following result establishes the limit distribution of CB process conditioned on extinction in the near future.

Theorem 4.1. *For any $s > 0$, there is a positive random variable W_s such that for any $\lambda, x > 0$,*

$$\lim_{t\to+\infty} \mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_t} \mid t \leq \zeta < t+s\big] = \mathrm{E}\big(\mathrm{e}^{-\lambda W_s}\big) = \begin{cases} \dfrac{1 - \mathrm{e}^{-\alpha\phi(\lambda+\varphi(s))}}{1 - \mathrm{e}^{-\alpha s}}, & \alpha > 0, \\[2ex] \dfrac{\phi(\lambda + \varphi(s))}{s}, & \alpha = 0. \end{cases}$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

In particular, if $\alpha = 0$, then W_s has the distribution $\mathrm{P}(W_s \in \mathrm{d}r) = \mu_s(\mathrm{d}r)$, where μ_s is the size-biased stationary measure defined in (4.1). Otherwise, if $\alpha > 0$, then W_s has the size-biased Yaglom distribution

$$\mathrm{P}(W_s \in \mathrm{d}r) = \frac{\mathrm{e}^{-\varphi(s)r}\mathrm{P}(\Theta \in \mathrm{d}r)}{\mathrm{E}[\mathrm{e}^{-\varphi(s)\Theta}]}. \tag{4.3}$$

Proof. It follows from the Markov property of $(Z_t)_{t\geq 0}$ that

$$\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_t} \mid t \leq \zeta < t+s\big] = \frac{\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_t} I_{\{\zeta \geq t\}}\mathbb{P}_{Z_t}(\zeta < s)\big]}{\mathbb{P}_x(\zeta < s+t) - \mathbb{P}_x(\zeta < t)}.$$

Making use of (2.6) and (2.7), we obtain

$$\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_t} \mid t \leq \zeta < t+s\big] = \frac{\mathbb{P}_x\big[\mathrm{e}^{-(\lambda+\varphi(s))Z_t} I_{\{\zeta \geq t\}}\big]}{\mathrm{e}^{-x\varphi(t+s)} - \mathrm{e}^{-x\varphi(t)}} = \frac{\mathrm{e}^{-x\varphi(t+\phi(\lambda+\varphi(s)))} - \mathrm{e}^{-x\varphi(t)}}{\mathrm{e}^{-x\varphi(t+s)} - \mathrm{e}^{-x\varphi(t)}}. \tag{4.4}$$

When $\alpha = 0$, since $\lim_{t\to+\infty}\varphi(t) = 0$ and $\varphi'(\lambda) = -\psi(\varphi(\lambda))$, by the integral mean value theorem,

$$\begin{aligned}
&\lim_{t\to+\infty}\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_t} \mid t \leq \zeta < t+s\big] \\
&\quad= \lim_{t\to+\infty}\frac{\int_0^{\phi(\lambda+\varphi(s))}\mathrm{e}^{-x\varphi(t+u)}\psi(\varphi(t+u))\,\mathrm{d}u}{\int_0^s \mathrm{e}^{-x\varphi(t+u)}\psi(\varphi(t+u))\,\mathrm{d}u} \\
&\quad= \lim_{t\to+\infty}\frac{\mathrm{e}^{-x\varphi(t+\xi_{t,\phi(\lambda+\varphi(s))})}\psi(\varphi(t+\xi_{t,\phi(\lambda+\varphi(s))}))}{\mathrm{e}^{-x\varphi(t+\xi_{t,s})}\psi(\varphi(t+\xi_{t,s}))}\,\frac{\phi(\lambda+\varphi(s))}{s},
\end{aligned}$$

where $0 < \xi_{t,\phi(\lambda+\varphi(s))} < \phi(\lambda+\varphi(s))$ and $0 < \xi_{t,s} < s$. Applying (3.1), we obtain

$$\lim_{t\to+\infty}\frac{\psi(\varphi(t+\xi_{t,\phi(\lambda+\varphi(s))}))}{\psi(\varphi(t+\xi_{t,s}))} = 1.$$

So from Lemma 4.1, for all $\lambda > 0$,

$$\lim_{t\to+\infty}\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_t} \mid t \leq \zeta < t+s\big] = \frac{\phi(\lambda+\varphi(s))}{s} = \widehat{\mu}_s(\lambda).$$

When $\alpha > 0$, by [20, Lemma 2.1], for any $s \geq 0$ we have

$$\lim_{t\to+\infty}\frac{\varphi(t+s)}{\varphi(t)} = \mathrm{e}^{-\alpha s}. \tag{4.5}$$

Thus, taking limits in (4.4), we get

$$\lim_{t\to+\infty}\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_t} \mid t \leq \zeta < t+s\big] = \lim_{t\to+\infty}\frac{\varphi(t) - \varphi(t+\phi(\lambda+\varphi(s)))}{\varphi(t) - \varphi(t+s)} = \frac{1-\mathrm{e}^{-\alpha\phi(\lambda+\varphi(s))}}{1-\mathrm{e}^{-\alpha s}}.$$

By (4.2), for $\lambda > 0$ we have

$$\int_{0+}^{+\infty}\mathrm{e}^{-\lambda r-\varphi(s)r}\mathrm{P}(\Theta \in \mathrm{d}r) = \mathrm{E}\big[\mathrm{e}^{-(\lambda+\varphi(s))\Theta}\big] = 1 - \mathrm{e}^{-\alpha\phi(\lambda+\varphi(s))}.$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

In particular, $\mathrm{E}\big[\mathrm{e}^{-\varphi(s)\Theta}\big] = 1 - \mathrm{e}^{-\alpha\phi(\varphi(s))} = 1 - \mathrm{e}^{-\alpha s}$. Consequently, we get

$$\lim_{t\to+\infty} \mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_t} \mid t \le \zeta < t+s\big] = \mathrm{E}\big[\mathrm{e}^{-\lambda W_s}\big],$$

where the distribution of W_s is given by (4.3). □

Next we shall define the distribution of Z_{t-q} $(0<q<t)$ conditioned on extinction at a fixed time t by taking the limit of $\mathbb{P}_x(Z_{t-q} \in \cdot \mid t \le \zeta < t+s)$ as $s \to 0+$. Recall that

$$\mathbb{P}_x(\zeta \le t) = \mathrm{e}^{-x\varphi(t)}, \qquad t \ge 0.$$

Since $\varphi'(t) = -\psi(\varphi(t))$, conditioned on $Z_0 = x > 0$, the distribution of ζ has a density function given by

$$f_{\zeta|Z_0}(t|x) = x\mathrm{e}^{-x\varphi(t)}\psi(\varphi(t)), \qquad t > 0. \tag{4.6}$$

For any $s>0$, $0<q<t$, and $\lambda>0$,

$$\begin{aligned}
\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{t-q}} \mid t \le \zeta < t+s\big] &= \frac{\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{t-q}} I_{\{t\le\zeta<t+s\}}\big]}{\mathbb{P}_x(t \le \zeta < t+s)} \\
&= \frac{\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{t-q}} \mathbb{P}_{Z_{t-q}}(q \le \zeta < q+s)\big]}{\mathbb{P}_x(t \le \zeta < t+s)} \\
&= \frac{\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{t-q}} \int_q^{q+s} f_{\zeta|Z_0}(r|Z_{t-q})\,\mathrm{d}r\big]}{\int_t^{t+s} f_{\zeta|Z_0}(r|x)\,\mathrm{d}r} \\
&\to \frac{\mathbb{P}_x\big[Z_{t-q}\,\mathrm{e}^{-(\lambda+\varphi(q))Z_{t-q}}\big]\psi(\varphi(q))}{x\mathrm{e}^{-x\varphi(t)}\psi(\varphi(t))}
\end{aligned} \tag{4.7}$$

as $s\to 0+$. We note that for $\lambda \ge 0$,

$$\begin{aligned}
\mathbb{P}_x\big[Z_{t-q}\,\mathrm{e}^{-(\lambda+\varphi(q))Z_{t-q}}\big] &= x\mathrm{e}^{-x\varphi(t-q+\phi(s))} \frac{\partial}{\partial s} u_{t-q}(s)\,|_{s=\lambda+\varphi(q)} \\
&= x\mathrm{e}^{-x\varphi(t-q+\phi(\lambda+\varphi(q)))} \frac{\psi(\varphi(t-q+\phi(\lambda+\varphi(q))))}{\psi(\lambda+\varphi(q))}.
\end{aligned} \tag{4.8}$$

In particular,

$$\mathbb{P}_x\big[Z_{t-q}\,\mathrm{e}^{-\varphi(q)Z_{t-q}}\big] = x\mathrm{e}^{-x\varphi(t)} \frac{\psi(\varphi(t))}{\psi(\varphi(q))}. \tag{4.9}$$

We can rewrite the limit in (4.7) as

$$\lim_{s\to 0+} \mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{t-q}} \mid t \le \zeta < t+s\big] = \frac{\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{t-q}} \cdot Z_{t-q}\,\mathrm{e}^{-\varphi(q)Z_{t-q}}\big]}{\mathbb{P}_x\big[Z_{t-q}\,\mathrm{e}^{-\varphi(q)Z_{t-q}}\big]}.$$

The term on the right is a Laplace transform of a probability measure on $(0, +\infty)$. For $0<q<t$, we denote this probability by

$$\mathbb{P}_x(Z_{t-q} \in \cdot \mid \zeta = t) := \lim_{s\to 0+} \mathbb{P}_x\big[Z_{t-q} \in \cdot \mid t \le \zeta < t+s\big] = \frac{\mathbb{P}_x\big[Z_{t-q}\,\mathrm{e}^{-\varphi(q)Z_{t-q}}; Z_{t-q} \in \cdot\big]}{\mathbb{P}_x\big[Z_{t-q}\,\mathrm{e}^{-\varphi(q)Z_{t-q}}\big]}. \tag{4.10}$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

Remark 4.1. (*Conditioning on extinction vs. conditioning on non-extinction.*) The above argument justifies the definition of the conditional law $\mathbb{P}_x(Z_{t-q} \in \cdot \mid \zeta = t)$ for $0 < q < t$ and $x > 0$. In fact, applying a similar argument, one can show that the limit

$$\mathbb{P}_x(A \mid \zeta = t) := \lim_{s \to 0+} \mathbb{P}_x(A \mid t \le \zeta > t + s)$$

exists for any $x > 0$, $0 < q < t$, and $A \in \mathcal{F}_{t-q}$. On the other hand, one can also condition the CB process to be extinct at a fixed time in the sense of h-transforms. Given $t > 0$, let

$$M_s^{(t)} := Z_s \,\mathrm{e}^{-\varphi(t-s)Z_s} \psi(\varphi(t-s)) \quad \text{for all } 0 \le s < t.$$

It is known (see [32, Lemma 4.2]) that $(M_s^{(t)})_{0 \le s < t}$ is a non-negative $(\mathcal{F}_s)_{s<t}$-martingale. Moreover, it is proved in [32] that the distribution of $(Z_s)_{s<t}$ under the conditional probability $\mathbb{P}_x(\cdot \mid \zeta = t)$ is the h-transform of $\mathbb{P}_x$ based on this martingale. That is, for any $0 \le s < t$ and $A \in \mathcal{F}_s$,

$$\mathbb{P}_x(A \mid \zeta = t) = \mathbb{P}_x\left[\frac{M_s^{(t)}}{M_0^{(t)}}; A\right]. \tag{4.11}$$

A closely related conditioning for the CB process is conditioning the process on non-extinction. The latter is defined by Lambert [20] in the sense of h-transforms. More precisely, it is shown in [20] that for any $x, t > 0$ and $A \in \mathcal{F}_t$,

$$\lim_{s \to +\infty} \mathbb{P}_x(A \mid \zeta > t + s) = \mathbb{P}_x^{\uparrow}(A),$$

where $\mathbb{P}_x^{\uparrow}$ is the h-transform of $\mathbb{P}_x$ based on the non-negative $(\mathcal{F}_t)$-martingale $M_t := Z_t \,\mathrm{e}^{\alpha t}$, that is,

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{\uparrow}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{M_t}{M_0} \quad \text{for all } t \ge 0. \tag{4.12}$$

The process conditioned on non-extinction is denoted by $Z^{\uparrow}$, and called the Q-process. It is proved in [20] that $Z^{\uparrow}$ is distributed as a CB process with immigration (CBI process). In the remaining of this remark we shall show that for any $x, t > 0$ and $A \in \mathcal{F}_t$,

$$\lim_{s \to +\infty} \mathbb{P}_x(A \mid \zeta = t + s) = \mathbb{P}_x^{\uparrow}(A). \tag{4.13}$$

This implies that the CB process conditioned to be extinct at time $t + s$, as $s \to +\infty$, has the same law as the Q-process $Z^{\uparrow}$. To prove (4.13), we note that for any $t, x > 0$ and $s > 0$,

$$\frac{M_t^{(t+s)}}{M_0^{(t+s)}} = \frac{Z_t \,\mathrm{e}^{-\varphi(s)Z_t} \psi(\varphi(s))}{Z_0 \,\mathrm{e}^{-\varphi(t+s)Z_0} \psi(\varphi(t+s))}.$$

By (3.1), we have $\lim_{s \to +\infty} \psi(\varphi(s))/\psi(\varphi(t+s)) = \mathrm{e}^{\alpha t}$. It follows that

$$\lim_{s \to +\infty} \frac{M_t^{(t+s)}}{M_0^{(t+s)}} = \frac{Z_t \,\mathrm{e}^{\alpha t}}{x} = \frac{M_t}{x}, \qquad \mathbb{P}_x\text{-a.s.}$$

Hence, by the dominated convergence theorem, we get

$$\lim_{s \to +\infty} \mathbb{P}_x(A \mid \zeta = t + s) = \mathbb{P}_x\left(\frac{M_t}{x}; A\right) = \mathbb{P}_x^{\uparrow}(A).$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

In the next result we obtain the distribution of the CB process conditioned to be extinct at a fixed time in the limit of large times.

Theorem 4.2. *For any $q>0$, there is a positive random variable V_q such that for any $\lambda, x>0$,*

$$\lim_{t\to+\infty}\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{t-q}}\mid\zeta=t\big]=\mathrm{E}\big[\mathrm{e}^{-\lambda V_q}\big]=\mathrm{e}^{-\alpha(\phi(\lambda+\varphi(q))-q)}\frac{\psi(\varphi(q))}{\psi(\lambda+\varphi(q))}. \tag{4.14}$$

Moreover, the distribution of V_q satisfies that

$$\mathrm{P}(V_q\in\mathrm{d}r)=\frac{r\mathrm{P}(W_q\in\mathrm{d}r)}{\mathrm{E}[W_q]}, \tag{4.15}$$

where W_q is defined in Theorem 4.1.

Proof. Combining (4.7) and (4.8), for all $\lambda>0$ we have

$$\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{t-q}}\mid\zeta=t\big]=\mathrm{e}^{-x(\varphi(t-q+\phi(\lambda+\varphi(q)))-\varphi(t))}\frac{\psi(\varphi(t-q+\phi(\lambda+\varphi(q))))}{\psi(\varphi(t))}\,\frac{\psi(\varphi(q))}{\psi(\lambda+\varphi(q))}. \tag{4.16}$$

If $\alpha>0$, then by (4.5) as $t\to+\infty$,

$$\frac{\psi(\varphi(t-q+\phi(\lambda+\varphi(q))))}{\psi(\varphi(t))}\sim\frac{\alpha\varphi(t-q+\phi(\lambda+\varphi(q)))}{\alpha\varphi(t)}\to\mathrm{e}^{-\alpha(\phi(\lambda+\varphi(q))-q)}.$$

Otherwise, if $\alpha=0$, by (3.1), we have

$$\lim_{t\to+\infty}\frac{\psi(\varphi(t-q+\phi(\lambda+\varphi(q))))}{\psi(\varphi(t))}=1.$$

In either case, we have

$$\lim_{t\to+\infty}\frac{\psi(\varphi(t-q+\phi(\lambda+\varphi(q))))}{\psi(\varphi(t))}=\mathrm{e}^{-\alpha(\phi(\lambda+\varphi(q))-q)}.$$

Hence we get (4.14) by letting $t\to+\infty$ in (4.16). It follows by the first conclusion of Theorem 4.1 that for any $\lambda>0$

$$\mathrm{E}\big[W_q\mathrm{e}^{-\lambda W_q}\big]=-\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathrm{e}\big[\mathrm{e}^{-\lambda W_q}\big]=\begin{cases}\dfrac{\alpha}{1-\mathrm{e}^{-\alpha q}}\,\dfrac{1}{\psi(\lambda+\varphi(q))}\,\mathrm{e}^{-\alpha\phi(\lambda+\varphi(q))}, & \alpha>0,\\[2mm] \dfrac{1}{q\psi(\lambda+\varphi(q))}, & \alpha=0.\end{cases}$$

By letting $\lambda\to 0+$, we have

$$\mathrm{E}[W_q]=\begin{cases}\dfrac{\alpha}{\mathrm{e}^{\alpha q}-1}\,\dfrac{1}{\psi(\varphi(q))}, & \alpha>0,\\[2mm] \dfrac{1}{q\psi(\varphi(q))}, & \alpha=0.\end{cases}$$

Thus we get

$$\frac{1}{\mathrm{E}[W_q]}\int_0^{+\infty}\mathrm{e}^{-\lambda r}r\mathrm{P}(W_q\in\mathrm{d}r)=\frac{\mathrm{E}\big[W_q\mathrm{e}^{-\lambda W_q}\big]}{\mathrm{E}[W_q]}=\mathrm{e}^{-\alpha(\phi(\lambda+\varphi(q))-q)}\frac{\psi(\varphi(q))}{\psi(\lambda+\varphi(q))}.$$

This yields (4.15). $\square$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

There is another way to obtain the distribution of V_q for the critical CB process by reversing the process from the extinction time ζ.

Proposition 4.1. *Suppose* $(Z_t)_{t\geq 0}$ *is a critical CB process. For any* $q>0$, *under* $\mathbb{P}_x$, $Z_{\zeta-q}I_{\{\zeta>q\}}$ *converges in distribution to* V_q *as* $x\to+\infty$.

Proof. For any $\lambda>0$, by the total probability formula,

$$\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{\zeta-q}}I_{\{\zeta>q\}}\big]=\int_q^{+\infty} f_{\eta|Z_0}(t|x)\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{\zeta-q}}\mid \zeta=t\big]\,\mathrm{d}t.$$

Here $f_{\eta|Z_0}(t|x)$ is the probability density function of ζ given that $Z_0=x$. By (4.6), (4.9), and (4.10), we get

$$\begin{aligned}\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{\zeta-q}}I_{\{\zeta>q\}}\big]&=\psi(\varphi(q))\int_q^{+\infty}\mathbb{P}_x\big[Z_{t-q}\,\mathrm{e}^{-(\lambda+\varphi(q))Z_{t-q}}\big]\,\mathrm{d}t\\&=\psi(\varphi(q))\int_0^{+\infty}\mathbb{P}_x\big[Z_t\,\mathrm{e}^{-(\lambda+\varphi(q))Z_t}\big]\,\mathrm{d}t\\&=\psi(\varphi(q))\int_{0+}^{+\infty}y\,\mathrm{e}^{-(\lambda+\varphi(q))y}G(x,\,\mathrm{d}y).\end{aligned}$$

It follows from Theorem 3.2 that

$$\begin{aligned}\lim_{x\to+\infty}\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{\zeta-q}}I_{\{\zeta>q\}}\big]&=\psi(\varphi(q))\lim_{x\to+\infty}\int_{0+}^{+\infty}y\,\mathrm{e}^{-(\lambda+\varphi(q))y}G(x,\,\mathrm{d}y)\\&=\psi(\varphi(q))\int_{0+}^{+\infty}y\,\mathrm{e}^{-(\lambda+\varphi(q))y}\mu(\mathrm{d}y)\\&=\psi(\varphi(q))\int_{0+}^{+\infty}\mathrm{e}^{-(\lambda+\varphi(q))y}W(y)\,\mathrm{d}y\\&=\frac{\psi(\varphi(q))}{\psi(\lambda+\varphi(q))}\\&=\mathrm{E}\big[\mathrm{e}^{-\lambda V_q}\big].\end{aligned}$$

We observe that

$$\lim_{x\to+\infty}\mathbb{P}_x(\zeta\leq q)=\lim_{x\to+\infty}1-\mathrm{e}^{-x\varphi(q)}=0.$$

Thus, for every $\lambda>0$,

$$\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{\zeta-q}I_{\{\zeta>q\}}}\big]=\mathbb{P}_x\big[\mathrm{e}^{-\lambda Z_{\zeta-q}}I_{\{\zeta>q\}}\big]+\mathbb{P}_x(\zeta\leq q)\to\mathrm{E}\big[\mathrm{e}^{-\lambda V_q}\big]\quad\text{as } x\to+\infty.$$

We complete the proof. □

Finally, we give some examples to illustrate the results obtained in this subsection.

Example 4.1. Suppose $(Z_t)_{t\geq 0}$ is a critical CB process with branching mechanism $\psi(\lambda)=\lambda^\beta$ $(1<\beta\leq 2)$. Then the corresponding scale function $W(x)=x^{\beta-1}/\,\Gamma(\beta)$ for $x>0$, and $\varphi(t)=((\beta-1)t)^{-1/(\beta-1)}$ for $t>0$. So the stationary measure on $(0,+\infty)$ is given by

$$\mu(\mathrm{d}x)=\frac{x^{\beta-2}}{\Gamma(\beta)}\mathrm{d}x\quad\text{for } x>0.$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

By Theorem 4.1, for any $q>0$, conditioned on $\{t-q\le\zeta<t\}$, Z_{t-q} converges in distribution to a positive random variable W_q as $t\to+\infty$, where W_q has a Gamma($[q(\beta-1)]^{-1/(\beta-1)}$, $\beta-1$)-distribution with the probability density function given by

$$g_q(x)=\frac{x^{\beta-2}}{q\Gamma(\beta)}\exp\left\{-\frac{x}{[q(\beta-1)]^{1/(\beta-1)}}\right\},\quad x>0.$$

By Theorem 4.2, for any $q>0$, conditioned on $\{\zeta=t\}$, Z_{t-q} converges in distribution to a positive random variable V_q as $t\to+\infty$, where V_q has a Gamma($[q(\beta-1)]^{-1/(\beta-1)}$, β)-distribution with the probability density function given by

$$p_q(x)=\frac{x^{\beta-1}}{\Gamma(\beta)[q(\beta-1)]^{\beta/(\beta-1)}}\exp\left\{-\frac{x}{[q(\beta-1)]^{1/(\beta-1)}}\right\},\quad x>0.$$

In particular, when $\beta=2$, W_q is distributed according to the exponential distribution with parameter $1/q$, and V_q is distributed according to Gamma distribution with parameter $(1/q, 2)$.

Example 4.2. Suppose $(Z_t)_{t\ge0}$ is a subcritical CB process with branching mechanism $\psi(\lambda)=\lambda+\lambda^2$. Then, by elementary calculation, we get $W(x)=1-\mathrm{e}^{-x}$ for $x>0$, $\phi(\lambda)=\ln(1+\lambda^{-1})$ for $\lambda>0$ and $\varphi(t)=(\mathrm{e}^t-1)^{-1}$ for $t>0$. The Laplace transform of the Yaglom distribution $\nu_1(\mathrm{d}x)$ is given by

$$\widehat{\nu}_1(\lambda)=1-\mathrm{e}^{-\phi(\lambda)}=\frac{1}{\lambda+1}\quad\text{for all }\lambda>0.$$

So the corresponding Yaglom distribution is the exponential distribution with parameter 1. It follows by Theorem 4.1 that for any $q>0$, conditioned on $\{t-q\le\zeta<t\}$, Z_{t-q} converges in distribution to a positive random variable W_q as $t\to+\infty$, where W_q is exponentially distributed with parameter $1+(\mathrm{e}^q-1)^{-1}$. Moreover, by Theorem 4.2, for any $q>0$, conditioned on $\{\zeta=t\}$, Z_{t-q} converges in distribution to a positive random variable V_q as $t\to+\infty$, where V_q is distributed according to the Gamma distribution with parameter $(1+(\mathrm{e}^q-1)^{-1}, 2)$.

4.2. Further properties of the limiting distributions

In this subsection we will investigate properties of the distribution of V_q obtained in Theorem 4.2. We show that it is infinitely divisible, and give a representation of its Lévy–Khintchine triplet. Then we show that the distribution of V_q is weakly convergent as $q\to+\infty$, and give a necessary and sufficient condition for the limit distribution to be non-degenerate.

Recall that $(X_t)_{t\ge0}$ is a spectrally positive Lévy process with Laplace exponent ψ and W is a corresponding scale function. Under the assumption (2.3), X has unbounded variation. Hence by [18, Lemma 3.1], $W(0)=0$. Moreover, by [19, Lemma 8.2] (and the reference therein), the restriction of W to $(0,+\infty)$ is continuously differentiable.

Proposition 4.2. *For any $q>0$, the distribution of V_q is infinitely divisible and its Laplace exponent $l_q(\lambda):=-\ln\mathrm{E}[\mathrm{e}^{-\lambda V_q}]$ is given by*

$$l_q(\lambda)=\int_{\varphi(q)}^{\lambda+\varphi(q)}\frac{\psi'(s)-\alpha}{\psi(s)}\,\mathrm{d}s,\quad\lambda>0.\tag{4.17}$$

Moreover, $l_q(\lambda)$ has the Lévy–Khintchine decomposition

$$l_q(\lambda)=b_q\lambda+\int_0^{+\infty}(1-\mathrm{e}^{-\lambda x})\frac{\nu_q(x)}{x}\,\mathrm{d}x,$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

where $b_q = 0$*,*

$$v_q(x) = \mathrm{e}^{-\varphi(q)x}\left[\sigma^2 W'(x) + \int_{(0,+\infty)} (W(x) - W(x-r))r\pi(\mathrm{d}r)\right], \quad x > 0, \tag{4.18}$$

and W'(x) denotes the derivative of W(x).

Proof. By Theorem 4.2, we have

$$l_q(\lambda) = \ln \frac{\psi(\lambda + \varphi(q))}{\psi(\varphi(q))} + \alpha(\phi(\lambda + \varphi(q)) - q).$$

Consequently

$$l'_q(\lambda) = \frac{\psi'(\lambda + \varphi(q)) - \alpha}{\psi(\lambda + \varphi(q))} \quad \text{for all } \lambda > 0.$$

Thus (4.17) follows by taking integrals on both sides of the above equation. Note that $l_q(\lambda) \to 0$ as $\lambda \to 0+$. So to show that the distribution of V_q is infinitely divisible, it suffices to show that $l_q(\lambda)$ is a Bernstein function, or equivalently, the first derivative of $l_a(\lambda)$ is completely monotone, i.e. $(-1)^n l_q^{(n+1)}(\lambda) \geq 0$ for all $\lambda > 0$ and $n = 0, 1, 2, \ldots$.

We note that

$$\psi'(u) - \alpha = \sigma^2 u + \int_{(0,+\infty)} (1 - \mathrm{e}^{-ur})r\pi(\mathrm{d}r) \quad \text{for all } u > 0$$

is the Laplace exponent of a Lévy subordinator. Applying [17, (3.15), (3.16)] by taking $F(u) = \psi'(u) - \alpha$ and $R(u) = -\psi(u)$ (and correspondingly $b = \sigma^2$ and $m(\mathrm{d}r) = r\pi(\mathrm{d}r)$), we get

$$\begin{aligned}\frac{F(u)}{\psi(u)} &= \sigma^2 W(0) + \sigma^2 \int_0^{+\infty} \mathrm{e}^{-ux} W'(x)\,\mathrm{d}x \\ &\quad + \int_0^{+\infty} \mathrm{e}^{-ux}\left[\int_{(0,+\infty)} (W(x) - W(x-r))r\pi(\mathrm{d}r)\right]\mathrm{d}x \quad \text{for all } u > 0.\end{aligned}$$

It follows that for $\lambda > 0$,

$$\begin{aligned}l'_q(\lambda) &= \frac{F(\lambda + \varphi(q))}{\psi(\lambda + \varphi(q))} \\ &= \sigma^2 W(0) + \sigma^2 \int_0^{+\infty} \mathrm{e}^{-\lambda x}(\mathrm{e}^{-\varphi(q)x} W'(x))\,\mathrm{d}x \\ &\quad + \int_0^{+\infty} \mathrm{e}^{-\lambda x}\left[\mathrm{e}^{-\varphi(q)x} \int_{(0,+\infty)} (W(x) - W(x-r))r\pi(\mathrm{d}r)\right]\mathrm{d}x.\end{aligned} \tag{4.19}$$

One can easily show by the above identity that $l'_q(\lambda)$ is completely monotone. Suppose the Lévy–Khintchine decomposition of $l_q(\lambda)$ is given by

$$l_q(\lambda) = b_q\lambda + \int_{(0,+\infty)} (1 - \mathrm{e}^{-\lambda x})\Gamma_q(\mathrm{d}x), \quad \lambda > 0,$$

where $b_q \geq 0$ and Γ_q is a measure on $(0, +\infty)$ such that $\int_{(0,+\infty)} (1 \wedge x)\Gamma_q(\mathrm{d}x) < +\infty$. Then

$$l'_q(\lambda) = b_q + \int_{(0,+\infty)} \mathrm{e}^{-\lambda x} x\Gamma_q(\mathrm{d}x).$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

Comparing the right-hand side with that of (4.19), we deduce that $b_q = \sigma^2 W(0) = 0$ and $\Gamma_q(\mathrm{d}x) = v_q(x)x^{-1}\,\mathrm{d}x$, with $v_q(x)$ being given by (4.18). $\square$

Proposition 4.3. *If*

$$\alpha > 0 \quad \textit{and} \quad \int^{+\infty} r \ln r\pi(\mathrm{d}r) < +\infty, \tag{4.20}$$

then V_q converges in distribution as $q \to +\infty$ to a positive random variable V_∞. The distribution of V_∞ has the following properties:

(i) *it is of the size-biased Yaglom distribution*

$$\mathrm{P}(V_\infty \in \mathrm{d}r) = \frac{r\mathrm{P}(\Theta \in \mathrm{d}r)}{\mathrm{E}[\Theta]},$$

(ii) *it is infinitely divisible,*

(iii) *its Laplace exponent $l_\infty(\lambda) := -\ln \mathrm{E}\big[\mathrm{e}^{-\lambda V_\infty}\big]$ is given by*

$$l_\infty(\lambda) = \int_0^\lambda \frac{\psi'(s) - \alpha}{\psi(s)}\,\mathrm{d}s, \qquad \lambda > 0,$$

(iv) *$l_\infty(\lambda)$ has the Lévy–Khintchine decomposition*

$$l_\infty(\lambda) = b_\infty \lambda + \int_0^{+\infty} (1 - \mathrm{e}^{-\lambda x}) \frac{v_\infty(x)}{x}\,\mathrm{d}x,$$

where $b_\infty = 0$, and

$$v_\infty(x) = \sigma^2 W'(x) + \int_0^{+\infty} (W(x) - W(x-r)) r\pi(\mathrm{d}r), \qquad x > 0.$$

Otherwise, if (4.20) fails, then V_q converges in probability as $q \to +\infty$ to infinity.

Proof. First we claim that (4.20) holds if and only if

$$\int_{0+} \frac{\psi'(s) - \alpha}{\psi(s)}\,\mathrm{d}s < +\infty.$$

In fact, if $\alpha = 0$, then

$$\int_{0+} \psi'(s)/\psi(s)\,\mathrm{d}s = \int_{0+} \mathrm{d}\ln\psi(s) = +\infty.$$

On the other hand, if $\alpha > 0$, we have

$$\frac{s\psi'(s)}{\psi(s)} = \frac{\alpha + \sigma^2 s + \int_{(0,+\infty)} (1 - \mathrm{e}^{-sr}) r\pi(\mathrm{d}r)}{\alpha + \frac{1}{2}\sigma^2 s + \int_{(0,+\infty)} \big(\frac{\mathrm{e}^{-sr} - 1 + sr}{sr}\big) r\pi(\mathrm{d}r)} \to 1 \quad \text{as } s \to 0+.$$

Hence $\psi'(s)/\psi(s) \sim 1/s$ as $s \to 0+$. This implies further that

$$\int_{0+} \frac{\psi'(s)}{\psi(s)} - \frac{\alpha}{\psi(s)}\,\mathrm{d}s < +\infty \quad \text{if and only if} \quad \int_{0+} \frac{1}{s} - \frac{\alpha}{\psi(s)}\,\mathrm{d}s < +\infty.$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

By [20, Lemma 2.1], the latter holds if and only if (4.20) holds. Hence we prove the claim.

Let $l_q(\lambda)$ be the Laplace exponent of V_q. It follows by (4.17) and the above claim that

$$\lim_{q\to+\infty} l_q(\lambda)=\lim_{q\to+\infty}\int_{\varphi(q)}^{\lambda+\varphi(q)}\frac{\psi'(s)-\alpha}{\psi(s)}\,\mathrm{d}s=\begin{cases}\int_0^\lambda \frac{\psi'(s)-\alpha}{\psi(s)}\,\mathrm{d}s & \text{if (4.20) holds,}\\ +\infty & \text{otherwise.}\end{cases}$$

So V_q converges in distribution as $q\to+\infty$ to some random variable V_∞ if (4.20) holds, and V_q converges in probability to infinity if (4.20) fails. When (4.20) holds, it follows by (4.15) and (4.3) that

$$\mathrm{P}(V_q\in\mathrm{d}r)=\frac{r\mathrm{e}^{-\varphi(q)r}\mathrm{P}(\Theta\in\mathrm{d}r)}{\mathrm{E}[\Theta\mathrm{e}^{-\varphi(q)\Theta}]}.$$

Hence (i) follows by letting $q\to+\infty$. The statements (ii)–(iv) follow directly from Proposition 4.2. □

Recall that $(Z_t^\uparrow)_{t\ge0}$ is the Q-process defined in Remark 4.1. The next result shows that $Z_t^\uparrow$ converges in distribution as $t\to+\infty$, and its limit distribution is equal to that of V_q as $q\to+\infty$. Since $(Z_t^\uparrow)_{t\ge0}$ is a CBI process, criteria for convergence in distribution and properties of the limiting distribution can readily be found in [17], but since they follow very easily from Theorem 4.2 and then Proposition 4.3, we present the proof here for the sake of being more self-contained.

Proposition 4.4. *If (4.20) holds, then $Z_t^\uparrow$ converges in distribution as $t\to+\infty$ to a positive random variable $Z_\infty^\uparrow$ which is equal in distribution to V_∞ defined in Proposition 4.3. Otherwise, if (4.20) fails, $Z_t^\uparrow$ converges in probability as $t\to+\infty$ to infinity.*

Proof. Fix an arbitrary $x>0$. We shall prove the following. For all $\lambda>0$,

$$\lim_{t\to+\infty}\mathbb{P}_x^\uparrow\left[\mathrm{e}^{-\lambda Z_t^\uparrow}\right]=\begin{cases}\mathrm{E}[\mathrm{e}^{-\lambda V_\infty}] & \text{if (4.20) holds,}\\ 0 & \text{otherwise.}\end{cases}\tag{4.21}$$

Fix $\lambda>0$. Suppose $s>0$ is sufficiently large such that $\varphi(s)<\lambda$. Suppose $t\in(s,+\infty)$. Recall the definitions of the martingales $(M_r^{(t)})_{0\le r<t}$ and $(M_r)_{r\ge0}$ given in (4.11) and (4.12) respectively. It is easy to see that for $t>s$,

$$\frac{M_{t-s}}{M_0}=\frac{\psi(\varphi(t))}{\psi(\varphi(s))}\,\mathrm{e}^{\alpha(t-s)+\varphi(s)Z_{t-s}-\varphi(t)x}\frac{M_{t-s}^{(t)}}{M_0^{(t)}},\qquad \mathbb{P}_x\text{-a.s.}$$

Thus we have for $t>s$

$$\begin{aligned}\mathbb{P}_x^\uparrow\left[\mathrm{e}^{-\lambda Z_{t-s}^\uparrow}\right]&=\mathbb{P}_x\left[\frac{M_{t-s}}{M_0}\,\mathrm{e}^{-\lambda Z_{t-s}}\right]\\&=\frac{\psi(\varphi(t))}{\psi(\varphi(s))}\,\mathrm{e}^{\alpha(t-s)-\varphi(t)x}\mathbb{P}_x\left[\frac{M_{t-s}^{(t)}}{M_0^{(t)}}\,\mathrm{e}^{-(\lambda-\varphi(s))Z_{t-s}}\right]\\&=\mathrm{I}(\alpha,t,s)\times\mathrm{II}(\lambda,t,s),\end{aligned}\tag{4.22}$$

where

$$\mathrm{I}(\alpha,t,s):=\frac{\psi(\varphi(t))}{\psi(\varphi(s))}\,\mathrm{e}^{\alpha(t-s)-\varphi(t)x}\quad\text{and}\quad \mathrm{II}(\lambda,t,s):=\mathbb{P}_x\left[\mathrm{e}^{-(\lambda-\varphi(s))Z_{t-s}}\mid\zeta=t\right].$$

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

If $\alpha = 0$, we have
$$\lim_{r\to 0+} \psi(r)\,\mathrm{e}^{\alpha\phi(r)} = \lim_{r\to 0+} \psi(r) = 0. \tag{4.23}$$
Otherwise, if $\alpha > 0$, we note that by (4.2)
$$\mathrm{E}[\Theta \mathrm{e}^{-r\Theta}] = -\alpha\phi'(r)\,\mathrm{e}^{-\alpha\phi(r)} = \frac{\alpha}{\psi(r)\,\mathrm{e}^{\alpha\phi(r)}} \quad \text{for all } r > 0.$$

Consequently we have
$$\lim_{r\to 0+} \psi(r)\,\mathrm{e}^{\alpha\phi(r)} = \lim_{r\to 0+} \frac{\alpha}{\mathrm{E}\big[\Theta \mathrm{e}^{-r\Theta}\big]} = \begin{cases} \dfrac{\alpha}{\mathrm{E}[\Theta]} & \text{if (4.20) holds,} \\ 0 & \text{if } \alpha > 0 \text{ and } \int^{+\infty} r \log r\pi(\mathrm{d}r) = +\infty. \end{cases} \tag{4.24}$$
Combining (4.23), (4.24) with the fact that $\lim_{t\to+\infty} \varphi(t) = 0$, we get
$$\lim_{t\to+\infty} \psi(\varphi(t))\,\mathrm{e}^{\alpha t} = \lim_{t\to+\infty} \psi(\varphi(t))\,\mathrm{e}^{\alpha\phi(\varphi(t))} = \begin{cases} \dfrac{\alpha}{\mathrm{E}[\Theta]} & \text{if (4.20) holds,} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that
$$\lim_{t\to+\infty} \mathrm{I}(\alpha, t, s) = \begin{cases} \dfrac{\alpha}{\mathrm{E}[\Theta]\psi(\varphi(s))}\,\mathrm{e}^{-\alpha s} & \text{if (4.20) holds,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.25}$$

On the other hand, by Theorem 4.2,
$$\lim_{t\to+\infty} \mathrm{II}(\lambda, t, s) = \mathrm{E}\big[\mathrm{e}^{-(\lambda-\varphi(s))V_s}\big]. \tag{4.26}$$

Combining (4.22), (4.25), and (4.26), we have
$$\lim_{t\to+\infty} \mathbb{P}_x^{\uparrow}\big[\mathrm{e}^{-\lambda Z_t^{\uparrow}}\big] = \begin{cases} \dfrac{\alpha}{\mathrm{E}[\Theta]\psi(\varphi(s))}\,\mathrm{e}^{-\alpha s}\,\mathrm{e}\big[\mathrm{e}^{-(\lambda-\varphi(s))V_s}\big] & \text{if (4.20) holds,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence (4.21) follows by letting $s \to +\infty$, and we prove the first assertion. If (4.20) fails, we have $\lim_{t\to+\infty} \mathbb{P}_x^{\uparrow}\big[\mathrm{e}^{-\lambda Z_t^{\uparrow}}\big] = 0$ for all $\lambda > 0$. Thus, for any $M > 0$,
$$\mathbb{P}_x^{\uparrow}\big(Z_t^{\uparrow} \le M\big) = \mathbb{P}_x^{\uparrow}\big(\mathrm{e}^{-Z_t^{\uparrow}} \ge \mathrm{e}^{-M}\big) \le \mathrm{e}^{M}\mathbb{P}_x^{\uparrow}\big[\mathrm{e}^{-Z_t^{\uparrow}}\big] \to 0$$
as $t \to +\infty$. Consequently $\lim_{t\to+\infty} \mathbb{P}_x^{\uparrow}(Z_t^{\uparrow} > M) = 1$ for all $M > 0$, and so $Z_t^{\uparrow}$ converges to infinity in probability. Hence we prove the second assertion. □

One can see from Proposition 4.4 and Theorem 4.2 that the two double limits coincide:
$$\lim_{s\to+\infty}\lim_{t\to+\infty} \mathbb{P}_x(Z_t \in A \mid \zeta = t + s) = \lim_{t\to+\infty}\lim_{s\to+\infty} \mathbb{P}_x(Z_t \in A \mid \zeta = t + s)$$
for any Borel set $A \subset (0, +\infty)$ with $\mathrm{P}(V_\infty \in \partial A) = 0$, and any $x > 0$. Moreover, the limit is non-degenerate if and only if (4.20) holds.

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press

Appendix

Lemma A.1. (1) *Suppose ν_n, ν are finite measures on $(0, +\infty)$ with $\widehat{\nu}(0) > 0$. Then ν_n converges weakly to ν if, for all $\lambda \geq 0$,*

$$\widehat{\nu}(\lambda) < +\infty \quad \textit{and} \quad \widehat{\nu_n}(\lambda) \to \widehat{\nu}(\lambda) \quad \textit{as } n \to +\infty. \tag{A.1}$$

(2) *Suppose ν_n, ν are measures on $(0, +\infty)$ with $0 < \widehat{\nu}(\beta) < +\infty$ for some $\beta > 0$. Then ν_n converges vaguely to ν if (A.1) holds for all $\lambda \geq \beta$.*

Proof. (1) Without loss of generality we assume $\widehat{\nu_n}(0) > 0$ for every $n \geq 1$. Let $\rho_n(\cdot) := \nu_n(\cdot)/\widehat{\nu_n}(0)$ and $\rho(\cdot) := \nu(\cdot)/\widehat{\nu}(0)$. Then ρ_n and ρ are probability measures on $(0, +\infty)$ with $\widehat{\rho_n}(\lambda) = \widehat{\nu_n}(\lambda)/\widehat{\nu_n}(0)$ and $\widehat{\rho}(\lambda) = \widehat{\nu}(\lambda)/\widehat{\nu}(0)$ for all $\lambda \geq 0$. (A.1) implies that ρ_n converges weakly to ρ. The weak convergence of ν_n follows from the weak convergence of ρ_n immediately.

(2) Since ν_n and ν can be viewed as measures on $[0, +\infty)$ by setting $\nu_n(\{0\}) = \nu(\{0\}) = 0$, this assertion is a direct result of [6, Theorem 8.5.a]. □

Acknowledgement

The authors sincerely thank the referee for the valuable comments and suggestions that have led to the present improved version of the original manuscript.

Funding information

The research of Rongli Liu is supported by NSFC (grant 12271374). The research of Yan-Xia Ren is supported by NSFC (grants 12071011 and 12231002) and the Fundamental Research Funds for Central Universities, Peking University LMEQF. The research of Ting Yang is supported by NSFC (grants 12271374 and 12371143).

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

Data

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

[1] Abraham, R. and Delmas, J. F. (2009). Williams' decomposition of the Lévy continuum random tree and simultaneous extinction probability for populations with neutral mutations. *Stoch. Process. Appl.* **119**, 1124–1143.

[2] Alsmeyer, G. and Rösler, U. (2006). The Martin entrance boundary of the Galton–Watson process. *Ann. Inst. H. Poincaré Prob. Statist.* **42**, 591–606.

[3] Asmussen, S. and Hering, H. (1983). *Branching Processes*. Birkhäuser, Boston.

[4] Athreya, K. B. and Ney, P. E. (1972). *Branching Processes* (Grundlehren der mathematischen Wissenschaften **196**). Springer, New York.

[5] Bertoin, J. (1998). *Lévy Processes*. Cambridge University Press, Cambridge.

[6] Bhattacharya, R. and Waymire, E. C. (2007). *A Basic Course in Probability Theory*, 1st edn. Springer, New York.

[7] BINGHAM, N. H. (1976). Continuous branching processes and spectral positivity. *Stoch. Process. Appl.* **4**, 217–242.
[8] BINGHAM, N. H., GOLDIE C. M. AND TEUGELS, J. L. (1987). *Regular Variation*. Cambridge University Press, Cambridge.
[9] CHAZAL, M., LOEFFEN, R. AND PATIE, P. (2015). Smoothness of continuous state branching with immigration semigroups. *J. Math. Anal. Appl.* **459**, 619–660.
[10] DELMAS, J. F. AND HÉNARD, O. (2013). A Williams decomposition for spatially dependent super-processes. *Electron. J. Prob.* **18**, 1–43.
[11] ESTY, W. W. (1976). Diffusion limits of critical branching processes conditioned on extinction in the near future. *J. Appl. Prob.* **13**, 247–254.
[12] GREY, D. R. (1974). Asymptotic behaviour of continuous time continuous state space branching processes. *J. Appl. Prob.* **11**, 669–677.
[13] HARRIS, T. E. (1963). *The Theory of Branching Processes*. Springer, Berlin.
[14] HOPPE, F. M. (1977). Representations of invariant measures on multitype Galton–Watson processes. *Ann. Prob.* **5**, 291–297.
[15] IMOMOV, A. A. (2014). Limit properties of transition functions of continuous-time Markov branching processes. *Int. J. Stoch. Anal.* **2014**, 409345.
[16] KALLENBERG, O. (2017). *Random Measures, Theory and Applications*. Springer, Cham.
[17] KELLER-RESSEL, M. AND MIJATOVIĆ, A. (2012). On the limit distributions of continuous-state branching processes with immigration. *Stoch. Process. Appl.* **122**, 2329–2345.
[18] KUZNETSOV, A., KYPRIANOV, A. E. AND RIVERO, V. (2012). The theory of scale functions for spectrally negative Lévy processes. In *Lévy Matters II* (Lecture Notes Math. **2061**), pp. 97–186. Springer, Berlin.
[19] KYPRIANOU, A. E. (2014). *Fluctuations of Lévy Processes with Applications* (Universitext), 2nd edn. Springer, Heidelberg.
[20] LAMBERT, A. (2007). Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct. *Electron. J. Prob.* **12**, 420–446.
[21] LI, Z. (2000) Asymptotic behaviour of continuous time and state branching processes. *J. Austral. Math. Soc. (Ser. A)* **68**, 68–84.
[22] LI, Z. (2011). *Measure-Valued Branching Markov Processes* (Probability and its Applications). Springer, Heidelberg.
[23] MAILLARD, P. (2018). The λ-invariant measures of subcritical Bienaymé–Galton–Watson processes. *Bernoulli* **24**, 297–315.
[24] NAKAGAWA, T. (1984). On the reverse process of a critical multitype Galton–Watson process without variances. *J. Multivariate Anal.* **14**, 94–100.
[25] OGURA, Y. (1969). Spectral representation for branching processes on the real half line. *Publ. Res. Inst. Math. Sci.* **5**, 423–441.
[26] OGURA, Y. (1970). Spectral representation for branching processes with immigration on the real half line. *Publ. Res. Inst. Math. Sci.* **6**, 307–321.
[27] OGURA, Y. (1974). Spectral representation for continuous state branching processes. *Publ. Res. Inst. Math. Sci.* **10**, 51–75.
[28] OGURA, Y. AND SHIOTANI, K. (1976). On invariant measures of critical multitype Galton–Watson processes. *Osaka J. Math.* **13**, 83–98.
[29] PAKES, A. G. (1999). Revisiting conditional limit theorems for the mortal simple branching process. *Bernoulli* **5**, 969–998.
[30] PAKES, A. G. (2008). Conditional limit theorems for continuous time and state branching process. In *Records and Branching Processes*, ed. M. Ahsanullah and G. P. Yanev, pp. 63–103. Nova Science Publishers.
[31] PAPANGELOU, F. (1968). A lemma on the Galton–Watson process and some of its consequences. *Proc. Amer. Math. Soc*. **19**, 1469–1479.
[32] REN, Y.-X., SONG, R. AND ZHANG, R. (2018). Williams decomposition for superprocesses. *Electron. J. Prob.* **23**, 1–33.
[33] REN, Y.-X., YANG, T. AND ZHAO, G.-H. (2014). Conditional limit theorems for critical continuous-state branching processes. *Sci. China Math.* **57**, 2577–2588.

https://doi.org/10.1017/jpr.2024.75 Published online by Cambridge University Press