J. Austral. Math. Soc. **23** (Series A) (1977), 147–165.

SOME CANCELLATION THEOREMS WITH APPLICATIONS TO NILPOTENT GROUPS

Dedicated to Wilhelm Magnus on his 70th birthday

R. HIRSHON

(Received 18 February 1975; revised 28 April 1976)

Abstract

If C is a group which satisfies the maximal condition for normal subgroups, then C may be cancelled from a group A in direct products if and only if the infinite cyclic group can be cancelled from A. Finitely generated torsion free nilpotent groups of class 2 satisfy a Remak Krull Schmidt condition.

1. Introduction

We consider the direct product decomposition $G = A \times C = B \times D$ where C and D are isomorphic groups. We say that C may be cancelled in direct products if $A \times C = B \times D$ implies $A \approx B$. If we fix A and C in the equations $A \times C = B \times D$, $C \approx D$, and if this equation implies $A \approx B$, we say C may be cancelled from A.

The author has shown in Hirshon (1970) that if C satisfies the maximal condition for normal subgroups and an arbitrary homomorphic image of C satisfies the minimal condition for normal subgroups isomorphic to itself, then C may be cancelled. However even though this theorem includes finite groups and other familiar groups, it excludes many familiar groups, the infinite cyclic group being the first that comes to mind. Indeed, an infinite cyclic group can not be cancelled (Hirshon (1969), Walker (1956)). However if $C \approx D$ and C is infinite cyclic, there exists a positive integer n depending on A and B such that the direct product of n copies of A is isomorphic to the direct product of n copies of B (Hirshon (1975)). Recently several authors have begun to investigate cancellation and non-cancellation phenomena an example of which can be found in Warfield (1975).

In Section (2), we show that in studying the cancellation properties of a group B which satisfies the maximal condition for normal subgroups it suffices to study an infinite cyclic group J. In particular we show that if $A \times B \approx A_1 \times D$, $B \approx D$ and B satisfies the maximal condition for normal subgroups then $A \times J \approx A_1 \times J$. In other words B can be cancelled from A if and only if J can be cancelled from A.

In Section 3 we study the equation $A \times J \approx A_1 \times J$ for special A. We show for example that J may be cancelled from torsion free nilpotent finitely generated groups of class 2 but not from torsion free nilpotent groups of class 3. In Section 4, we study how the cancellation properties of J from a certain kind of group can effect its decomposition into indecomposable factors. We obtain as a corollary of our observations here, the Remak Krull Schmidt type theorem for finitely generated torsion free nilpotent groups of class 2. In Section 5, we consider free products and in Section 6, we make some miscellaneous remarks.

We will use upper case letters $A, B, C \cdots$ to represent groups. Lower case letters $a, b, c \cdots$ will represent either group elements or integers. The expression a^c will usually designate the group element a to the cth power. $Z(A)$ will be the center of A and J will be the infinite cyclic group. The group generated by $e_1, e_2 \cdots e_r$ will be designated by $\langle e_1, e_2 \cdots e_r \rangle$.

2. How the cancellation properties of J affect more general groups

LEMMA 1. *If H and K are groups such that $J \times J \times H \approx J \times K$, then $J \times H \approx K$.*

PROOF. Write $\langle w \rangle \times \langle y \rangle \times H = \langle g \rangle \times K = G$. Then we may assume $g \in \langle y \rangle \times H$. For if $g \notin \langle y \rangle \times H$ and $g \notin \langle w \rangle \times H$ we may write $g = w^a y^b h$, $h \in H$, $a \neq 0$, $b \neq 0$. Let d be the greatest common divisor of the integers a, b and let $a = da_1$, $b = db_1$. Let $y_* = w^{a_1} y^{b_1}$ and write $\langle w \rangle \times \langle y \rangle = \langle w_* \rangle \times \langle y_* \rangle$ for suitable w_*. Hence $g = y_*^d h \in \langle y_* \rangle \times H$. Hence we assume that $g \in \langle y \rangle \times H$. Hence $\langle y \rangle \times H = \langle g \rangle \times K_1$, where $K_1 = K \cap (\langle y \rangle \times H)$. Hence $G = \langle w \rangle \times \langle g \rangle \times K_1 = \langle g \rangle \times K$. Hence $J \times H \approx \langle w \rangle \times K_1 \approx K$.

LEMMA 2. *If $A \times F \approx A_1 \times F$, F free abelian of finite rank, then $A \times J \approx A_1 \times J$.*

PROOF. This follows from Lemma 1.

LEMMA 3. *If $D \times B = D_1 \times B_1 = G$ are two decompositions of a group G with $B \approx B_1$, then if $F = B \cap D_1$, $K = B_1 \cap D$, then*

$$D \times (B/F) \times (B_1/K) \approx D_1 \times (B/F) \times (B_1/K)$$

where the above are external direct products.

PROOF. See p. 402 of Hirshon (1970).

LEMMA 4. *Let $D \times B = D_1 \times B_1$, $B \approx B_1$ and suppose B satisfies the maximal condition for normal subgroups. Then if $B \cap D_1 = 1$ or $B_1 \cap D = 1$, then B and B_1 have identical commutator subgroups.*

PROOF. If $B \cap D_1 = 1$ then B and D_1 commute pointwise so that $B' \subset B_1'$. But if θ is an isomorphism of B onto B_1, θ induces an isomorphism on B'. Furthermore,

$$B_1' = B' \times (B_1' \cap D') \tag{1}$$

so that (1) induces a projection homorphism γ of B_1' on B'. Hence $\alpha = \theta\gamma$ is a homorphism of B_1' onto B_1'. Furthermore, if $B_1' \cap D' \neq 1$, α is not an isomorphism. Also if L is a normal subgroup of B, $L \subset B'$ and L_1 is the inverse image of L under γ, then L_1 is a normal subgroup of B_1. Furthermore θ maps normal subgroups of B into normal subgroups of B_1. Note that the kernel of α^n is obtained by taking the inverse image of $B_1' \cap D'$ under α^{n-1}. It follows that the kernel of α^n is a normal subgroup of B_1 so that if α is not an isomorphism we would obtain a contradiction of the fact that B_1 satisfies the maximal condition for normal subgroups. Hence $B_1' = B'$ as asserted.

THEOREM 1. *Let B satisfy the maximal condition for normal subgroups. Let $A \times B \approx A_1 \times B$. Then $A \times J \approx A_1 \times J$.*

PROOF. Suppose the assertion is false. Consider the pairs of groups $(B/K, A_K)$ such that

$$A_K \times B/K \approx A_k^* \times B/K$$

but $A_K \times J$ is not isomorphic to $A_k^* \times J$. This set is not empty so that we may choose a pair $(B/L, A_L)$ with L maximal. We may then write

$$E \times F = E_1 \times F_1$$

where $E \approx A_L$, $F \approx F_1 \approx B/L$ and $E \times J$ and $E_1 \times J$ are not isomorphic. Then if $F \cap E_1 = 1$ or $E \cap F_1 = 1$, we could conclude by Lemma 4 that $F = F_1'$ and hence

$$E \times (F/F') \approx E_1 \times (F_1/F_1') \tag{2}$$

But after cancelling the torsion part of F/F' from (2) we end up with

$$E \times L \approx E_1 \times L \tag{3}$$

where $L = 1$ or L is free abelian of finite rank. If we use Lemma 2, we obtain a contradiction of the fact that $E \times J$ is not isomorphic to $E_1 \times J$. Hence by Lemma 3 we may write

$$(E \times G_1) \times G_2 \approx (E_1 \times G_1) \times G_2 \tag{4}$$

where G_1 and G_2 are proper homomorphic images of F. If $E \times G_1$ and $E_1 \times G_1$ are not isomorphic, by the maximality of L,

$$E \times G_1 \times J \approx E_1 \times G_1 \times J \tag{5}$$

or

$$(E \times J) \times G_1 \approx (E_1 \times J) \times G_1 \tag{6}$$

Again by the maximality of L (6) implies

$$E \times J \times J \approx E_1 \times J \times J \tag{7}$$

which by Lemma 2 implies $E \times J \approx E_1 \times J$. Hence $E \times G_1 \approx E_1 \times G_1$. But then again by the maximality of L,

$$E \times J = E_1 \times J \tag{8}$$

which again contradicts our hypothesis and the theorem is proved.

COROLLARY 1. *Let $A \times G \approx A_1 \times G$ where G obeys the hypothesis of Theorem 2. Then there exists an integer n such that the direct product of n copies of A is isomorphic to the direct product of n copies of B.*

PROOF. Use the result of Hirshon (1975) mentioned in the introduction.

COROLLARY 2. *Let $A \times G \approx A_1 \times G$ where G is finitely generated. Then*

$$(A/A'') \times J \approx (A_1/A_1'') \times J.$$

PROOF. Write $A \times G = A_1 \times G_1 = L$ with $G \approx G_1$. Then

$$L/L'' = (A \times G)/(A'' \times G'') = (A_1 \times G_1)/(A_1'' \times G_1'').$$

Hence

$$L/L \approx (A/A'') \times (G/G'') \approx (A_1/A_1'') \times (G_1/G_1'').$$

By a theorem of P. Hall, finitely generated metabelian groups satisfy the maximal condition for normal subgroups (Hall (1954), Theorem 3). Hence by Theorem 1 we may replace G/G'' and G_1/G_1'' by J in the above decomposition of L/L''.

We note that the hypothesis of Corollary 2 does not imply in general, that $A/A'' \approx A_1/A_1''$ (see the example following Corollary 2 of Theorem 3).

In concluding this section we mention that using the methods presented here one can show

THEOREM 2. *If B satisfies the maximal condition for normal subgroups*

https://doi.org/10.1017/S1446788700018152 Published online by Cambridge University Press

and if the commutator subgroups of B is of finite index in B, then B may be cancelled in direct products.

3. Some cases in which J may and may not be cancelled

The result of the previous section naturally leads us to look for situations in which J may be cancelled. However, questions about J which are easily posed often seem to have answers which are elusive if answerable at all as any student of elementary number theory can attest to.

Some situations in which J may be cancelled from A are

(a) A has a periodic center (Walker (1956), p. 901)

(b) A has a divisible center

(c) A is a torsion free nilpotent group with a cyclic center (Hirshon (1975), Baumslag (1975))

(d) A is torsion free nilpotent of class 2 and finitely generated

(e) $A = H \times K$ where J may be cancelled from H and from K.

We will discuss (b) and (d) in this section. We prove (e) in Section 4. We begin with some general comments concerning the decomposition $\langle w\rangle \times A = \langle y\rangle \times B$. Let us note that if either A or B is a subgroup of the other then $A = B$. For example if $A \subset B$ then $J \approx (\langle w\rangle \times A)/A = (\langle y\rangle \times B)/A \approx J \times B/A$ which implies that $A = B$. Hence we may assume that AB contains A and B properly so that $AB = A \times C_1 = B \times D_1$ where C_1 and D_1 are infinite cyclic. Hence we may assume in the sequel that $G = AB = A \times C = B \times D$. Let $E = A \cap B$ and let w and y be generators of C and D respectively. Let aE and bE be generators of A/E and B/E respectively. We have

LEMMA 5. *We may assume without loss of generality that*

$$w = y^t b^z e \qquad y = w^s a^{-z} e^{-s}, \qquad z = st - 1 \tag{10}$$
$$a = yb^s \qquad b = w^{-1} a^t e$$

for some element e of E.

PROOF. Note that G/E is a free abelian group of rank two which may be generated by either of the pairs wE, aE or aE, bE or yE, bE. Now recall that any two sets of free generators of a free abelian group are connected by a unimodular matrix. Using this fact with respect to the three sets of generators of G/E above, we see that we may write

$$w = y^t b^z e_1 \qquad y = w^s a^{-z} e_3 \tag{11}$$
$$a = yb^s e_2 \qquad b = w^{-1} a^t e_4$$

where $ts - z = 1$ or $ts - z = -1$ and where the e_i are suitable elements in E. We may assume that $ts - z = 1$ for if $ts - z = -1$, we could replace y by y^{-1}. Also we may assume that $e_2 = 1$ for if not, we can replace the generator a by the generator $ae_2^{-1} = \bar{a}$. Once we assume that $e_2 = 1$, it easily follows by forming the composite relations induced by (11) that $e_1 = e_4$ and $e_3 = e_1^{-s}$.

LEMMA 6. *If* $s = 0$, *then* $A \approx B$.

PROOF. If $s = 0$, then $a = y$ so $A = \langle a \rangle \times E$ and $B \approx (\langle w \rangle \times A)/\langle y \rangle \approx \langle w \rangle \times (A/\langle y \rangle) \approx \langle a \rangle \times E$.

We assume in the sequel that $s \neq 0$.

LEMMA 7. *If* $Z(A)$ *is contained in* E, $A \approx B$.

PROOF. Note that from (10) $a^{-z}e^{-s}$ is central in G and that there exists an isomorphism α of A into B with $a\alpha = b^s$ and which fixes E pointwise. Hence if E contains $Z(A)$ then $z = 0$ so that $st = 1$ so that α is an isomorphism of A onto B. We assume in the sequel that E does not contain $Z(A)$.

COROLLARY 1. *If* $Z(A) = 1$ *then* $A \approx B$ (Walker (1956)).

COROLLARY 2. *We may assume* $z \neq 0$.

LEMMA 8. *If* A *is abelian*, $A \approx B$ (Walker (1956)).

PROOF. The map α which maps a into b and which fixes E pointwise induces an isomorphism of A onto B if A is abelian.

LEMMA 9. *Let* $M = Z(A) \cap E = Z(B) \cap E$. *Then we may assume* $Z(A)/M \approx Z(B)/M \approx J$.

PROOF. Since $A/E \approx J$ either $Z(A)/M \approx J$ or $Z(A)$ is contained in E. In the latter case we could apply Lemma 7.

In the sequel we assume that A is torsion free, finitely generated nilpotent of class 2.

LEMMA 10. $Z(A) \cap Z(B) = Z(E)$.

PROOF. If $Z(E)$ contains m, from (10) m commutes with $b^z e$ and e. Hence $[m, b^z] = 1$. In torsion free nilpotent groups of class 2, this implies $[m, b] = 1$. Hence $Z(E)$ is contained in M.

In the sequel $a_1, a_2, \cdots a_r$ will designate free generators of M and $a_1, a_2, \cdots a_r, a_*$ will be free generators of $Z(A)$ and $a_1, a_2, \cdots a_r, b_*$ will be free generators of $Z(B)$. (There are no a_i if $Z(A)$ is cyclic.)

We can now show

THEOREM 3. *An infinite cyclic group may be cancelled from a finitely generated torsion free nilpotent group of class 2.*

PROOF. We assume $\langle w\rangle\times A=\langle y\rangle\times B=G=AB$. Note that $A/Z(A)\approx G/Z(G)\approx B/Z(B)$ and in the present situation this group is a free abelian group. If $g\in A$ and $g=g^*\operatorname{mod}\langle y\rangle$, $g^*\in B$, then the map $gZ(A)\to g^*Z(B)$ induces an isomorphism from $A/Z(A)$ onto $B/Z(B)$. Let $g_1,g_2\cdots g_q$ be elements in A such that $g_1Z(A), g_2Z(A),\cdots g_qZ(A)$ are free generators of $A/Z(A)$.

Hence generators of A are

$$a_1,a_2\cdots a_r,a_*,g_1,g_2\cdots g_q$$

and generators of B are

$$a_1,a_2\cdots a_r,b_*,g_1^*,g_2^*\cdots g_q^*$$

Now we claim that the map θ of the above generators

$$a_i\to a_i,\quad g_i\to g_i^*,\quad a_*\to b_*$$

induces an isomorphism of A onto B. To see this let

$$\bar w=\bar w(a_1,a_2,\cdots a_r,a_*,g_1,g_2\cdots g_q)=1$$

be a word in the displayed generators of A which defines the identity. We claim then that $\bar w(a_1,a_2\cdots a_r,b_*,g_1^*,g_2^*\cdots g_q^*)=1$. To see this note that $[g_i,g_j]\in Z(A)\cap B=M$. Furthermore for each generator g_i that appears in $\bar w$, the exponent sum on g_i in $\bar w$ must be 0. It follows that $\bar w$ is freely equal to a product of elements of the form $[g_i^c,g_j^d]$, f, $g^{-1}fg$, where f is a word in the elements a_1, a_2, a_r, a_*, and g is a word in the g_i. Hence a_* must appear in $\bar w$ with exponent sum 0 (that is a_* can appear only in a trivial way) or else we could deduce from the above considerations that some nontrivial power of a_* is in $A\cap B$. But now using the relation $g_i=g_i^*\bmod Z(G)$ we note that each of $[g_i^c,g_j^d]$, f, $g^{-1}fg$ is unchanged under the map θ so that θ is a homomorphism of A onto B. By the symmetry of the situation we see that θ^{-1} defines a homomorphism of B onto A so that indeed θ is an isomorphism.

COROLLARY 1. *Given the finitely generated torsion free nilpotent group G of class 2 with $G=\langle w\rangle\times A=\langle y\rangle\times B$ then either $w=y^d \bmod A\cap B$, $d=1$ or $d=-1$, or A and B both have infinite cyclic direct factors.*

PROOF. One easily verifies that we may assume $G=AB$. In the previous proof, $st=1$ is equivalent to $w=y^d\bmod A\cap B$, $d=1$ or -1. If $st=0$, and say $s=0$, then as we showed in the proof of Lemma 6, A and B

have infinite cyclic direct factors. In other cases we showed that any relation $\bar{w}(a_*, a_1 \cdots a_r, g_1 \cdots g_q) = 1$ implies that a_* appears trivially. Hence

$$A = \langle a_* \rangle \times \langle a_1 \cdots a_r, g_1 \cdots g_q \rangle$$

and

$$B = \langle b_* \rangle \times \langle a_1 \cdots a_r, g_1^* \cdots g_q^* \rangle$$

COROLLARY 2. *Let G be a torsion free finitely generated nilpotent group of class 2 and let $G = \langle w \rangle \times H = L \times M$, $w \neq 1$. Then either L has an infinite cyclic direct factor or M does.*

PROOF. We may assume L or M is not contained in H. Say L is not contained in H. Hence,

$$(H \times \langle w \rangle)/(H \cap L) = (L \times M)/(H \cap L) \tag{12}$$

or

$$(H/H \cap L) \times \langle w \rangle \approx (L/H \cap L) \times M$$

and $L/(H \cap L) \approx J$. Note in (12),

$$(H/H \cap L) \cap [M \times (H \cap L)]/(H \cap L) = [(H \cap L) \times (H \cap M)]/(H \cap L).$$

Hence if $L = \langle l \rangle (H \cap L)$, by the previous results, either M has an infinite cyclic direct factor, or $l = w^d h_1 h_2$, where $h_1 \in H \cap L$, $h_2 \in H \cap M$. But then, $L = \langle l h_1^{-1} \rangle H \cap L$ and $l h_1^{-1} = w^d h_2 \in Z(L)$ so $L = \langle l h_1^{-1} \rangle \times H \cap L$.

We note that the previous theorem can not be strengthened by considering torsion free nilpotent groups of class n in general. For there exists a finitely presented torsion free nilpotent group B of class 3 from which J may not be cancelled. We define B on the generators a_1, a_2, c_1, c_2, b, g_1, g_2, g_3 as follows:

$$[a_i, x] = 1, \quad x \in B, \ i = 1, 2, \quad [c_1, c_2] = 1, \quad [c_1^{-1}, g_1^{-1}] = a_1^p,$$

$$[c_1^{-1}, g_2^{-1}] = a_2^p, \quad [c_1, g_3] = 1, \quad [c_2, g_1] = a_1^p,$$

$$[c_2, g_2] = a_2^p, \quad [c_2, g_3] = 1, \quad [g_1, g_2] = c_1, \quad [g_1, g_3] = c_2,$$

$$[g_2, g_3] = 1, \quad [b, g_1^{-1}] = a_1, \quad [b, g_2^{-1}] = a_2, \quad [b, u] = 1 \quad \text{if}$$

$$u = c_1, c_2, g_3.$$

Note that $b_* = b^p c_1 \in Z(B)$. Let $G = \langle y \rangle \times B$. Let $a = yb^s$. Note $a_* = a^p c_1^s$ is central. Let $w = y^t b_*$. Then if $st - 1 = p$ and if $E = \langle a_1, a_2, c_1, c_2, g_1, g_2, g_3 \rangle$ then $G = \langle w \rangle \times (\langle a \rangle E) = \langle y \rangle \times (\langle b \rangle E)$. Furthermore if $|st| \neq 1$, $|p| \neq 1$, and s is not congruent to 1 or to -1 mod p then $A = \langle a \rangle E$, and $B = \langle b \rangle E$ are not isomorphic. One can see this as follows: If A were isomorphic to B we would be able to find an isomorphism θ of A onto B with $a_0\theta = b$ where a_0 has the form

$$a_0 = a^d c_1^i c_2^j a_1^k a_2^l$$

with $d = 1$ or $d = -1$. To see this note that $A' = B' = \langle a_1, a_2, c_1, c_2 \rangle$ and $A'\theta = B'$ so that θ maps the free abelian group A' on itself. Moreover if $B_2/Z(B)$ and $A_2/Z(A)$ are the center of $B/Z(B)$ and $A/Z(A)$ then

$$B_2 = \langle a_1, a_2, c_1, c_2, b \rangle$$

and

$$A_2 = \langle a_1, a_2, c_1, c_2, a \rangle$$

and

$$A_2\theta = B_2.$$

The above together with $A'\theta = A'$ easily implies that $d = 1$ or $d = -1$. If to be precise, $d = 1$, we see $a_0^{-1} g_1 a_0 = g_1 a_1^{s+pr}$, $a_0^{-1} g_2 a_0 = g_2 a_2^{s+rp}$ for some r. Hence if $\bar{A}$ is the normal subgroup of A generated by a_0, we see in $A/\bar{A}$, $a_1\bar{A}$, $a_2\bar{A}$ are nontrivial elements of finite order. But if $\bar{B}$ is the normal subgroup of B generated by b we must have since $\bar{A}\theta = \bar{B}$, $A/\bar{A} \approx B/\bar{B}$. This is impossible since $A/\bar{A} \approx B/\bar{B}$ and $B/\bar{B}$ is torsion free.

We will show in the sequel (Theorem 9) that if $A \times J \approx B \times J$ then A and B have the same finite homomorphic images no matter what the structure of A and B. Hence the last example gives an easy way of seeing that finitely generated torsion free nilpotent groups are not determined by their finite homomorphic images. Remeslennikov, Higman (see Baumslag, (1971), p. 8) Pickel (1971) and Baumslag (1974) are a few of several authors who have been concerned with this question.

We complete this section with the observation that if $Z(A)$ is divisible, then J may be cancelled from A. For by using the notation of Lemma 5, by Corollary 2 of Lemma 7 we may suppose that $z \neq 0$. From (10), $a^{-z}e^{-s} \in Z(A)$, so that we may write

$$a^{-z}e^{-s} = \bar{a}^z, \qquad \bar{a} \in Z(A)$$

But this implies that

$$\bar{a} = a^{-1} \operatorname{mod} A \cap B$$

so that

$$A = \langle a \rangle A \cap B = \langle \bar{a} \rangle \times A \cap B$$

so that

$$A \approx J \times A \cap B$$

Similarly

$$B \approx J \times A \cap B$$

so that

$$A \approx B$$

4. Decompositions into indecomposable factors

We say a group G can be decomposed in an essentially unique way as a direct product of indecomposable groups if we may write $G = G_1 \times G_2 \times \cdots \times G_r$ where $G_i \neq 1$, $1 \leqq i \leqq r$, and each G_i is indecomposable as a direct product and up to an isomorphism of terms, this is the only way of decomposing G as a direct product of indecomposable groups. In some cases, the presence of a cancellation theorem implies an essentially unique factorization in terms of indecomposable groups. For an illustration of this statement see Hirshon (1971) and Walker (1956). It seems natural to wonder what can be said about the decompositions of a group into indecomposable factors if J can be cancelled from the group.

Due to our limited knowledge in this area we shall consider the small class of groups G which satisfy the conditions:

(a) G obeys the maximal conditions for normal subgroups and G is torsion free.

(b) If G/N is torsion free, then J may be cancelled from G/N.

From example, by Theorem 3, a finitely generated torsion free nilpotent group of class two is in this class.

The following theorem will imply that finitely generated torsion free nilpotent groups of class 2 may be decomposed in an essentially unique way.

THEOREM 4. *Let B obey* (a) *and* (b) *and suppose that B may not be decomposed in an essentially unique way. Let $G = B/N$ be torsion free and suppose G does not decompose in an essentially unique way and let N be maximal with respect to this property. Then G has exactly two distinct decompositions (up to isomorphism) into indecomposable factors. In one of these decompositions G is expressed as the direct product of an indecomposable group, which is not J, with a nontrivial free abelian group. In the other decomposition G is the direct product of indecomposable groups none of which is J.*

PROOF. We may write two distinct (up to isomorphism) decompositions of G as

$$G = L_1 \times L_2 \times \cdots \times L_k \times F_1 \times F_2 \times \cdots \times F_r \quad \text{and} \tag{13}$$

$$G = H_1 \times H_2 \times \cdots \times H_t \tag{14}$$

where $L_i \approx J$ or $L_i = 1$, $1 \leqq i \leqq k$ and none of the H's or F's are 1 or infinite cyclic, but are indecomposable. For if (13) and (14) are really distinct up to isomorphism all infinite cyclic factors if present must occur in one decomposition (13) or (14) but not in both or else by our cancellation hypothesis we could cancel one infinite cyclic factor in each, obtaining a contradiction of the maximality of N. Also we may assume

$$L_j \cap H_i = 1, \quad 1 \leqq j \leqq k, \quad 1 \leqq i \leqq t$$

for if say $L_1 \cap H_1 \neq 1$, then since G/H_1 is torsion free we can conclude that $L_1 \subset H_1$. Since H_1 is indecomposable, this implies that $H_1 = L_1$, contrary to the fact that no H is isomorphic to J.

Now we note,

$$\text{No } F_i \text{ is isomorphic to } H_j. \tag{15}$$

For if say $F_1 \approx H_1$, by Theorem 1 and our cancellation hypothesis, we may cancel F_1 and H_1 from (13) and (14). This yields two decompositions of G/H_1 which are not essentially unique contradicting the maximality of N.

Now we assert that either some F_j cannot be mapped homomorphicly onto any of the H's or some H_j cannot be mapped homomorphicly onto any of the F's. For otherwise we can write a sequence of groups beginning with

$$F_1 = F_{i_1}, H_{j_1}, F_{i_2}, H_{j_2}, F_{i_3}, H_{j_3} \cdots$$

such that each group in the sequence is a homomorphic image of the preceding one.

But then we may choose $i_k = i_f$ for some $k > f$ so that by considering the composition of homomorphisms, we may obtain an endomorphism α of F_{i_k} onto $F_{i_f} = F_{i_k}$. But any group which obeys the maximal condition for normal groups is hopfian. Hence α is an automorphism so that the homomorphism of F_{i_k} onto H_{j_k} must be an isomorphism contrary to (15).

Now we claim that it is actually some H that cannot be mapped homomorphicly onto any F. For say the opposite situation occurs and say to be definite that F_1 does not have any H as a homomorphic image. Since F_1 is not abelian, F_1 intersects some H nontrivially. Say $F_1 \cap H_1 \neq 1$. But then we may consider $G/(F_1 \cap H_1)$ and the two decompositions of $G/(F_1 \cap H_1)$ into indecomposable factors induced from (13) and (14) by breaking $F_1/F_1 \cap H_1$

and $H_1/F_1 \cap H_1$ into indecomposable factors. By using the maximality of N, we see that these decompositions are unique up to isomorphism. But since no H is infinite cyclic H_2 would have to be isomorphic to some F, $F \neq F_1$ contrary to (15). Hence we may assume H_1 has no F as a homomorphic image. Again since H_1 is not abelian, H_1 intersects some F, say $H_1 \cap F_1 \neq 1$. Again by considering the decompositions of $G/F_1 \cap H_1$ into indecomposable factors induced by (13) and (14) and the fact that these two must be essentially unique by the maximality of N, we see if F_2 were present it would have to be isomorphic to some H contrary to (15). Hence $r = 1$. This implies that some L is not 1 or G would be indecomposable and hence could not have two decompositions. Hence we may write

$$G = L_1 \times L_2 \times \cdots \times L_k \times F \quad \text{and} \tag{16}$$

$$G = H_1 \times H_2 \times \cdots \times H_t \tag{17}$$

where each H_i is noncyclic, each L_i is infinite cyclic and all groups are indecomposable as a direct product. Moreover there can not be a decomposition of G into indecomposable factors which is not the same (up to isomorphism) as (16) or (17). For let

$$G = M_1 \times \cdots \times M_n \tag{18}$$

be a decomposition of G into indecomposable factors, $M_i \neq 1$. If some M_i is infinite cyclic, say M_1 is cyclic we could cancel M_1 and L_1 and then use the maximality of N to deduce that the decomposition (18) is essentially the same as that of (16). If no M_i is cyclic then the decomposition (18) must be essentially the same as that of (17) for our previous argument shows that given two distinct (up to isomorphism) decompositions of G, one of them must have an infinite cyclic factor. This leads to

THEOREM 5. *A finitely generated torsion free nilpotent group of class* 2 *may be decomposed in an essentially unique way.*

PROOF. If G does not decompose uniquely choose N maximal such that G/N is torsion free and G does not decompose uniquely. But then there are two decompositions of G/N exactly one of which has a J term. This contradicts Corollary 2 of Theorem 3.

Our example following Corollary 2 of Theorem 3 shows that Theorem 5 can not be extended to nilpotent groups of class larger than two. In fact Baumslag (1975) has shown the surprising result:

Let m and n be given integers with $m > 1$, $n > 1$. There exists a finitely generated torsion free nilpotent group which can be expressed as a direct product of m directly indecomposable groups and also as a direct product of n directly indecomposable groups in such a way that no factor in the first decomposition is isomorphic to a factor in the second.

All of Baumslag's groups in this example are of class 3.

5. Cancelling free products

In considering the cancellation properties of a group one is tempted to feel (and probably quite properly so) that the center of the group is of paramount importance. However even if $Z(G) = 1$, G is not necessarily cancellable in direct products. For example if A is a group with $Z(A) = 1$, and $A \times A$ and A are not isomorphic and G is the direct product of a countable number of copies of A, then

$$G \times A \approx G \times A \times A$$

so that G is not cancellable. The next theorem is one of several available examples which illustrate the point of view that in certain cases free products are easier to handle than direct products whose behavior can often be quite unpredictable.

THEOREM 6. *A nontrivial free product is cancellable in direct products.*

PROOF. Let $A \times F = A_1 \times F_1$ where $F \approx F_1$ and F is a nontrivial free product. Hence

$$F \approx K_1 L_1$$

where

$$K_1 = (A_1 A)/A, \quad L_1 = (F_1 A)/A.$$

Hence we may write

$$F = KL$$

where $K \approx K_1$, $L \approx L_1$ and K and L, commute elementwise. Now let us suppose $K \neq 1$, $L \neq 1$. Then exactly as in the proof that a free product can not decompose as a direct product (Kurosh (1956), p. 28), we can show L and K have a trivial intersection with any factor appearing in the decomposition of F as a free product and that consequently L and K are free groups. But if $F = K_2 * L_2$ where $K_2 \neq 1$, $L_2 \neq 1$, and if k is in K, l is in L and since k and l commute, either k and l are powers of the same element or are in the same conjugate of K_2 or L_2 (Magnus, Karrass, Solitar (1966), p. 247 exercise 11). The latter is impossible since K and L intersect conjugates of K_2 and L_2 trivially for the same reason that they intersect K_2 and L_2 trivially. Hence k

https://doi.org/10.1017/S1446788700018152 Published online by Cambridge University Press

and l are powers of the same element. This implies K and L each have rank 1. But then F is abelian which is impossible. Hence either $K = 1$ or $L = 1$. If $K = 1$, then $K_1 = 1$ and A_1 is contained in A. But then

$$F_1 \approx (A_1 \times F_1)/A_1 \approx (A/A_1) \times F$$

and since a free product may not decompose as a direct product we have $A = A_1$. If $L = 1$, then $L_1 = 1$ and F_1 is contained in A. Hence

$$A = F_1 \times A \cap A_1$$

$$A \times F = F_1 \times A \cap A_1 \times F = F_1 \times A_1$$

so that dividing by F_1 above we see

$$A_1 \approx F \times A \cap A_1$$

so that $A \approx A_1$ which completes the proof.

6. Some miscellaneous remarks

We terminate this paper with some miscellaneous observations concerning the decomposition $A \times C = B \times D$ with $C \approx D \approx J$.

THEOREM 7. *If $C \approx D \approx J$ but A and B are not isomorphic, there exists a sequence $p_1, p_2, p_3 \cdots$ of distinct primes and a sequence $q_1, q_2, q_3 \cdots$ of distinct primes and a sequence $A_1, A_2, A_3 \cdots$ of normal subgroups in A and a sequence $B_1, B_2, B_3 \cdots$ of normal subgroups in B with*

$$\begin{aligned} &A_i \approx B, [A : A_i] = p_i \\ (19) \qquad &B_i \approx A, [B : B_i] = q_i. \end{aligned}$$

In addition we can find normal subgroups L_i of A, with $L_i \approx A$ and $[A : L_i] = p_i q_i$.

PROOF. Assume $G = A \times C = B \times D = AB$ and note that if $C = \langle w \rangle$ and $D = \langle y \rangle$ we may write the equations (10) and assume that $s \neq 0$, $t \neq 0$ and $z \neq 0$. Let α and β be the projection homomorphisms defined as follows:

$$w\alpha = 1, \ \alpha \text{ fixes } A \text{ pointwise}$$

$$y\beta = 1, \ \beta \text{ fixes } B \text{ pointwise.}$$

Note that $A\beta = \langle b^s \rangle E \approx A$ and $B\alpha = \langle a^t \rangle E \approx B$ and $A\beta\alpha = \langle a^{st} \rangle E \approx A$ and $B\alpha\beta = \langle b^{st} \rangle E \approx B$. Now the idea of the proof is to switch the generators w and y in a suitable way and hence the maps α and β. Hence note

LEMMA 11. *Let* $w_r = w(a^{-z}e^{-s})^r$ *and let* $y_r = y(b^z e)^r$.
Then the relations (10) *in terms of* w_r, a, y, b *are*

$$w_r = y^{t-rz}b^{z-srz}e_r, \qquad y = w_r^s a^{-z+rzs}e_r^{-s}$$

$$a = yb^s, \qquad b = w_r^{-1}a^{t-rz}e_r$$

where $e_r = e^{1-sr}$. The relations (10) in terms of w, ae^r, y_r, b become

$$w = y_r^t b^{-zrt+z}e^{1-rt}, \qquad ae^r = y_r b^{s-zr}.$$

Now note that t and z are relatively prime and that s and z are relatively prime so that the arithmetic progression $t - rz$, $r = 0, 1 \cdots$ contains infinitely many primes. Similarly the arithmetic progression $s - rz$, $r = 0, 1 \cdots$ contains infinitely many primes. Note that in passing from w, y, a, b to w_r, y, a, b, the "t" exponent changes and the "s" exponent remains fixed. In passing from w, y, a, b, to w, y_r, ae^r, b the "t" exponent remains fixed while the "s" exponent varies. Hence by changing generators by using composite changes of the type indicated in Lemma 11, we can arrive at new generators replacing the original w and y where the new t and s exponents call them $\bar{t}$ and $\bar{s}$ are of the form

$$\bar{t} = t - r_1 z, \ \bar{t} \text{ a prime}$$

$$\bar{s} = s - r_2 z, \ \bar{s} \text{ a prime}.$$

Thanks to Dirichlet, we know there are infinitely many ways we can do this. Now apply the associated projections to get the desired groups.

THEOREM 8. *If* A *is a group whose center is finitely generated modulo its torsion subgroup we can embed* A *as a subgroup of finite index in a group* H *such that* J *may be cancelled from* H.

PROOF. Note that if J is cancellable from a group H then J is cancellable from $H \times F$ where F is any finitely generated abelian group. Now say that J is not cancellable from A. Then we may write $A \times C = B \times D$, A not isomorphic to B and the equations (10) where $z \neq 0$. Then

$$(A \times C/\langle wy^{-1}\rangle) \approx (B/\langle b^z e\rangle) \times D \tag{20}$$

Now the left hand side of (20) is a finite (cyclic) extension of A obtained by adjoining a central element w_1 to A. Hence if H_1 is the extension of A in (20) $Z(H_1) \approx Z(A)$. Hence the torsion free rank of $B_1 = (B/\langle b^z e\rangle$ is one less than that of H_1. If J is cancellable from B_1, then we may take $H = H_1$. Otherwise we may repeat the above process (to B_1) and find a group

$$H_2 = \langle w_2\rangle B_1 \approx J \times B_2$$

https://doi.org/10.1017/S1446788700018152 Published online by Cambridge University Press

where w_2 is central, H_2/B_1 finite, $Z(H_2) \approx Z(B_1)$ and the torsion free rank of B_2 is one less than that of B_1.

Now if J is cancellable from B_2, it is cancellable from H_2 and from $H_2 \times J = H_3$ and we can easily verify that H_3 is a finite extension of A and we may take $H_3 = H$. Otherwise, we may continue the process and write

$$H_3 = \langle w_3 \rangle B_2 \approx J \times B_3.$$

Ultimately one of the B's obtained in this manner must allow cancellation of J for at each step the torsion free rank of each B is reduced. If the torsion free rank of A is n, the process terminates in at most n steps. For then we arrive at B_n which has a periodic center and we can invoke the result (a) mentioned in the beginning of Section 3 to see that J is cancellable from B_n.

COROLLARY. *If the torsion free rank of $Z(A)$ is n, H may be obtained by adjoining n or fewer central elements of H to A.*

THEOREM 9. *$A \times J \approx B \times J$ implies that A and B have the same finite homomorphic images.*

PROOF. Write $A \times \langle w \rangle = B \times \langle y \rangle = G = AB$. Let A/K be finite. Then dividing both decompositions of G by K we obtain

$$\bar{A} \times \langle \bar{w} \rangle = \bar{B} \langle \bar{y} \rangle = \bar{A}\bar{B} = \bar{G} \tag{21}$$

where $\bar{A} \approx A/K$, $\langle \bar{w} \rangle \approx J$, $\bar{B} \approx (BK)/K \approx B/B \cap K$ and $\langle \bar{y} \rangle \approx (\langle y \rangle K)/K$. Now write $y = w^s \bmod A$. Then if A and B are not isomorphic, as we have seen in the proof of Theorem 7 (by changing generators if necessary) we have an infinite number of primes which are possible values of s. Hence we may choose y and w, so that $y = w^s \bmod A$, s a prime and s and $[A:K:]$ are relatively prime. But then by examining the way $\bar{G}$ is obtained from G we see

$$\bar{y} = \bar{w}^s \bar{a} \qquad \bar{a} \in \bar{A}.$$

Hence

$$\bar{\bar{A}} = (\bar{A} \times \langle \bar{w} \rangle)/\langle \bar{y} \rangle \approx \bar{B}/\bar{B} \cap \langle \bar{y} \rangle \tag{22}$$

Note that $\bar{\bar{A}}$ has a direct factor isomorphic to $\bar{A}$. For $\bar{\bar{A}} = A_1 \langle w_1 \rangle$ where $A_1 = (\bar{A} \times \langle \bar{y} \rangle)/\langle \bar{y} \rangle \approx \bar{A}$ and where $\bar{w}_1 = \bar{w} \langle \bar{y} \rangle$. But $w_1^s \in A_1$ and since each element in A_1 has an sth root, we may write $w_1^s = a_1^s$ with $a_1 \in Z(A_1)$. This implies $\bar{\bar{A}} = \langle w_1 a_1^{-1} \rangle \times A_1$. Since $A_1 \approx \bar{A}$ it easily follows that $\bar{A}$ is a homomorphic image of B.

COROLLARY. *If $A \times B \approx A_1 \times B_1$ and $B \approx B_1$ and B satisfies the maximal condition for normal subgroups then A and A_1 have the same finite homomorphic images.*

PROOF. This follows from Theorem 1.

THEOREM 10. *If J can be cancelled from H and from K then J may be cancelled from $A = H \times K$.*

PROOF. Again we write

$$\langle w \rangle \times A = \langle y \rangle \times B = G. \tag{23}$$

As in the proof of Lemma 11 we may assume without loss of generality,

$$y = w^s \bmod A, \ s \text{ a prime.}$$

Moreover we may assume neither H nor K is contained in B. For if say B contains H, from (23) we could write

$$B = H \times L, \qquad L = B \cap (K \times \langle w \rangle). \tag{24}$$

But then

$$\langle w \rangle \times H \times K = \langle y \rangle \times H \times L = G.$$

Hence

$$\langle w \rangle \times K \approx \langle y \rangle \times L. \tag{25}$$

Hence since J is cancellable from K, (25) implies $K \approx L$ so that from (24),

$$B = H \times L \approx H \times K \times A.$$

Hence we may assume

$$HB/B \approx H/H \cap B \approx J$$

and

$$KB/B \approx K/K \cap B \approx J.$$

Let

$$H = \langle h \rangle H \cap B, \qquad K = \langle k \rangle K \cap B.$$

Note from (23), we may observe that if $M = H \cap B \times K \cap B$, G/M is a free abelian group of rank 3 with free generators wM, hM, kM. Also from (23) we see that yM generates a free factor of G/M so that

$$y = w^s h^i k^j \bmod M$$

where the integers s, i, j have greatest common divisor 1. Since s is a prime, either s and i are relatively prime or s and j are relatively prime. Say $(s, i) = 1$. Hence we may choose integers u, v, so that the determinant

https://doi.org/10.1017/S1446788700018152 Published online by Cambridge University Press

$$\begin{vmatrix} s & i & j \\ 0 & 0 & 1 \\ u & v & 0 \end{vmatrix} = 1.$$

However this implies that free generators of G/M are

$$yM, \quad kM, \quad w^u h^v M.$$

Hence G is generated by

$$\langle y \rangle, K, \langle w^u h^v \rangle H \cap B = N$$

and it easily follows that

$$G = \langle y \rangle \times K \times N = \langle y \rangle \times B = \langle w \rangle \times H \times K. \tag{26}$$

Hence by cancelling $\langle y \rangle$ we obtain

$$B \approx K \times N. \tag{27}$$

But also by cancelling K in (26) we obtain $\langle y \rangle \times N \approx \langle w \rangle \times H$ so that since J may be cancelled from H we see $N \approx H$ so that (27) shows $A \approx B$.

Acknowledgements

The author would like to thank Professor Gilbert Baumslag for his question, "Can J be cancelled from a finitely generated torsion free nilpotent group?". The author gave at least two "proofs" that the answer is yes. Professor Fred Pickel kindly pointed out the errors in these proofs. Thanks are due to the referee for several helpful suggestions.

Added in Proof (*7 April* 1977). Since the completion of this manuscript the author has obtained some generalizations which will appear in the Journal of Algebra with the title 'Cancellation and Hopficity in direct products'.

References

Gilbert Baumslag (1971), *Lecture Notes on Nilpotent Groups* (Regional Conference Series in Mathematics, **2**, Amer. Math. Soc. Providence, Rhode Island, (1971).

Gilbert Baumslag (1974), 'Some metacyclic groups with the same finite images', *Compositio Math.* **29**, 249–252.

Gilbert Baumslag (1975), 'Direct decompositions of finitely generated torsion-free nilpotent groups', *Math. Z.* **145**, 1–10.

P. Hall (1954), 'Finite conditions for soluble groups', *Proc. London Math. Soc.* (3) **4**, 419–436.

R. Hirshon (1969), 'On cancellation in groups', *Amer. Math. Monthly* **76**, 1037–1039.

R. Hirshon (1970), 'Cancellation of groups with maximal condition', *Proc. Amer. Math. Soc.* **24**, 401–403.

R. Hirshon (1971), 'Some new groups admitting essentially unique directly indecomposable decompositions', *Math. Ann.* **194**, 123–125.

R. Hirshon (1975), 'The cancellation of an infinite cyclic group in direct products', *Arch. Math. (Basel)* **26**, 134–138.

A. G. Kurosh (1956), *The Theory of Groups* (translated by K. A. Hirsch from the second Russian edition. Chelsea, New York, 1956).

Wilhelm Magnus, Abraham Karrass, Donald Solitar (1966), *Combinatorial Group Theory: Presentations of groups in terms of generators and relations* (Pure and Applied Mathematics, **13**. Interscience [John Wiley & Sons], New York, London, Sydney, 1966).

P. F. Pickel (1971), 'Finitely generated nilpotent groups with isomorphic finite quotients', *Trans. Amer. Math. Soc.* **160**, 327–341.

В. Н. Ремесленников [V. N. Remeslennikov], 'Сопряженность подгрупп в нильпотентных группах' [Conjugacy of subgroups in nilpotent groups], *Algebra i Logika Sem.* **6**, no. **2**, **61–76**.

E. A. Walker (1956), 'Cancellation in direct sums of groups', *Proc. Amer. Math. Soc.* **7**, 898–902.

R. B. Warfield, Jr. (1975), 'Genus and cancellation for groups with finite commutator subgroup', *J. Pure Appl. Algebra* **6**, 125–132.

Mathematics Dept.,
Polytechnic Institute of New York,
333 Jay Street,
Brooklyn 11201,
U.S.A.