This is a ``preproof'' accepted article for *Canadian Journal of Mathematics*
This version may be subject to change during the production process.
DOI: 10.4153/S0008414X25101703

LIMIT CYCLES NEAR A NILPOTENT CENTRE AND A HOMOCLINIC LOOP TO A NILPOTENT SINGULARITY OF HAMILTONIAN SYSTEMS

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ABSTRACT. For a planar analytic Hamiltonian system, which has a period annulus limited by a nilpotent centre and a homoclinic loop to a nilpotent singularity, we study its analytic perturbation to obtain the number of limit cycles bifurcated from the periodic orbits inside the period annulus. By characterizing the coefficients and their properties of the high order terms in the expansion of the first order Melnikov function near the loop, we provide a new way to find more limit cycles. Moreover, we apply these general results to concrete systems, for instance, an $(m+1)$th order generalized Liénard system, and an mth order near-Hamiltonian system with a hyperelliptic Hamiltonian of degree 6.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The second part of the Hilbert's 16th problem, one of the 23 problems posed by Hilbert in 1900, asks the maximum number of limit cycles that planar polynomial vector fields of a given degree can have and the topological structures of these maximum number of limit cycles. This problem is still open, even for quadratic differential systems, see e.g. [29, 34, 42]. In the 1970s Arnold [1, 2] suggested a weakened version of this problem, so-called *the weakened Hilbert's 16th problem*. This weakened version can be stated as follows: *Let ω be a real 1-form with polynomial coefficients of degree m, and let H be a real polynomial of degree $n+1$ in the plane, whose level curves $\{L_h : H = h\}$ in closed connected components, called ovals of H, form continuous families of periodic orbits of the Hamiltonian system associated to H. The question is what is the maximum number of isolated zeros of the Abelian integrals*

$$I(h) = \int_{L_h} \omega.$$

The Abelian integrals can be viewed in essence as the first order Melnikov (or Poincaré–Pontryagin-Melnikov) function relevant to a near-Hamiltonian system, obtained by polynomial perturbations of a polynomial Hamiltonian system. Poincaré–Pontryagin–Melnikov theorem shows that the number of simple isolated zeros of the Abelian integrals provides a lower bound on the maximum number of limit cycles of the near-Hamiltonian system. In this direction, there have appeared lots of published works, see e.g. [5, 7, 8, 15, 16, 18, 21, 28, 33, 36, 37, 50, 66] and the references therein. But in general this problem is far to be solved.

As we know, the Abelian integrals strongly depend on the families of ovals $\{L_h\}$. And the families can be limited in different regions. Here we study a continuous family of ovals, formed by level curves of a Hamiltonian, which are limited in a unique region bounded by a nilpotent centre and a homoclinic loop to a nilpotent singularity.

Consider an analytic differential system of the form:

$$\begin{cases} \dot{x} = \ \ H_y(x,y) + \varepsilon P_0(x,y,\varepsilon,\delta), \\ \dot{y} = -H_x(x,y) + \varepsilon Q_0(x,y,\varepsilon,\delta), \end{cases} \tag{1}$$

2010 *Mathematics Subject Classification.* 34C07; 37G15; 34C05 .

Key words and phrases. Melnikov function; limit cycle; nilpotent centre; homoclinic loop; nilpotent singularity.

with $H \in C^\omega(\mathbb{R}^2)$ and $P_0, Q_0 \in C^\omega(\mathbb{R}^2 \times I \times D)$, where $\varepsilon \in I = [0, \rho]$ with $\rho > 0$ small and $\delta \in D \subset \mathbb{R}^d$ a compact subset in the real d-dimensional space. The origin is *a nilpotent singularity* of the unperturbed Hamiltonian sytsem $(1)|_{\varepsilon=0}$, if the Hamiltonian H satisfies $H_x(0,0) = H_y(0,0) = 0$ and

$$\frac{\partial(H_y, -H_x)}{\partial(x,y)}(0,0) \neq 0, \qquad \det \frac{\partial(H_y, -H_x)}{\partial(x,y)}(0,0) = 0.$$

As it is well known, if a singularity of system (1) is nilpotent, there exists an invertible real linear change of variables under which system (1) is transformed to a new one, whose linear part has the matrix

$$\begin{pmatrix} 0 & 2h_{00} \\ 0 & 0 \end{pmatrix},$$

with h_{00} nonzero. Without loss of generality, we set the nilpotent singularity at the origin. And system (1) has its linear part in the above normal form. Then the unperturbed Hamiltonian is of the form $H(x,y) = h_{00}y^2 + \text{h.o.t.}$, where h.o.t. represents the higher order terms. By [11, 67], applying the implicit function theorem to $H_y(x,y) = 0$ in a neighborhood of the origin yields a unique analytic solution $y = \phi(x)$, defined in a neighborhood of $x = 0$. Therefore, the Hamiltonian H, without loss of generality, can be written in

$$H(x,y) = H_0(x) + y^2 \widetilde{H}(x,y), \tag{2}$$

where

$$H_0(x) = \sum_{j \geq k} h_j x^j, \qquad \widetilde{H}(x,y) = \sum_{j \geq 0} H_j^* y^j, \qquad H_j^* = \sum_{i \geq 0} h_{ij} x^i, \tag{3}$$

with h_k ($k \in \mathbb{N}$ and $k \geq 3$) and h_{00} nonzero constants provided that the origin is a nilpotent singularity. For more information on nilpotent singularities, see e.g. [11, 21, 40, 67]. Here we list the ones that we need. We remark that in the next Proposition 1 and Lemmas 1-3, we use h_{00} as a nonzero constant, but it was given by Han [21] in its original statement taking $h_{00} = 1/2$. We must say that this is not essentially different. In the lemmas the clockwise orientation of the homoclinic loop or periodic orbits near the nilpotent centre implies $h_{00} > 0$.

Proposition 1. [21] *Let the origin be a nilpotent singularity of the unperturbed system* $(1)|_{\varepsilon=0}$*, and the corresponding Hamiltonian has the form* (2) *with* (3)*. Then the origin is a nilpotent centre of order m for the unperturbed system* $(1)|_{\varepsilon=0}$ *if $k = 2m+2$ and $h_k h_{00} > 0$. The origin is a nilpotent saddle of order m if $k = 2m+2$ and $h_k h_{00} < 0$. The origin is a cusp of order m if $k = 2m+1$ and $h_k h_{00} \neq 0$.*

Associated to the forms (2) and (3) via Proposition 1, one has $k \geq 3$ (i.e. $m \geq 1$). When $k = 2$ (i.e. $m = 0$), the origin is a hyperbolic saddle if $h_k h_{00} < 0$, or else is an elementary centre. As mentioned before, we suppose that the unperturbed system $(1)|_{\varepsilon=0}$ has the structure on the orbits:

(a) A continuous family of periodic orbits $\{L_h^1\} \subset \{(x,y) : H(x,y) = h\}$, which forms a period annulus, denoted by U_1.
(b) A homoclinic loop L_s to a nilpotent singularity S, which forms the outer boundary of the annulus U_1.
(c) A nilpotent centre C, which is the inner boundary of U_1.

These assumptions together with the expression of H in (2) imply that the Hamiltonian system $(1)|_{\varepsilon=0}$ has the phase portrait as shown in Fig. 1.

In this paper, we study the limit cycle bifurcation for the first two cases in Fig. 1. The third one will be handled in a separate paper, because of the length of this paper and some technical

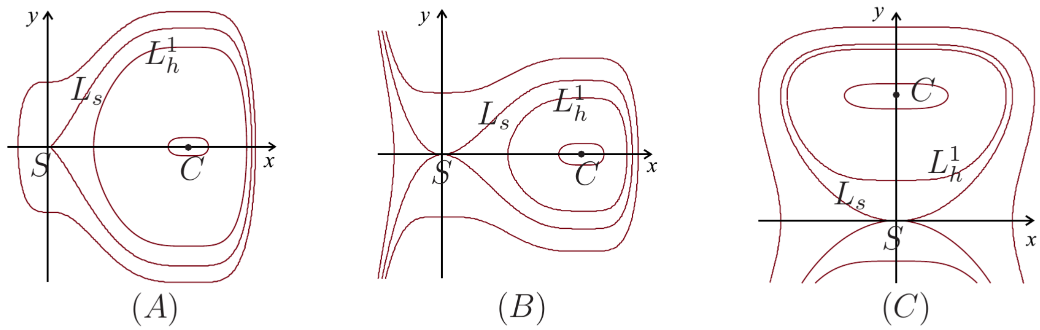

FIGURE 1. The phase portrait of the Hamiltonian system (1)$|_{\varepsilon=0}$ consisting of a nilpotent centre C and a homoclinic loop to a nilpotent singularity S enclosing a period annulus. (A): The nilpotent singularity S is a cusp. (B) and (C): The nilpotent singularity S is a nilpotent saddle.

difference. Without loss of generality, we suppose that the nilpotent saddle is $S = (0, 0)$, and the nilpotent centre is $C = (x_c, 0)$ with $x_c > 0$, and that the homoclinic loop L_s is oriented in clockwise in the positive sense. For $h_s := H(0, 0) = 0$ and $h_c := H(x_c, 0)$, one has $h_c < h_s$.

Associated to the family of periodic orbits $\{L_h^1,\ h \in (h_c, h_s)\}$ of the unperturbed system (1)$|_{\varepsilon=0}$, the first order Melnikov function is

$$M(h,\delta) = \oint_{L_h^1} (Q_0 \mathrm{d}x - P_0 \mathrm{d}y)\,|_{\varepsilon=0}, \qquad h \in (h_c, h_s). \tag{4}$$

It is well known that the number of limit cycles of system (1) could be found partially through the first order Melnikov function if it is not zero identically. The coefficients of the expansion of the Melnikov function in h play an essential role in determining the number of limit cycles.

The next three results exhibit the expressions of the expansions of the first order Melnikov function near a homoclinic loop to a nilpotent singularity or near a nilpotent centre. The first one is on the expansion near a homoclinic loop to a nilpotent saddle.

Lemma 1. [26, 65] *For the analytic near-Hamiltonian system* (1)*, whose Hamiltonian H has an oriented clockwise homoclinic loop and is of the form* (2) *with* (3)*, $k \geq 4$ even and $h_k < 0$, the first order Melnikov function near and inside the loop L_s has the asymptotic expression*

(5)
$$M(h,\delta) = -\frac{1}{2k} h \ln|h| I^*_{1,\frac{k}{2}-1}(h,\delta) + |h|^{\frac{1}{2}} \sum_{\substack{r=1 \\ r\neq \frac{k}{2}}}^{k-1} A_{r-1} I^*_{1,r-1}(h,\delta)|h|^{\frac{r}{k}} + \psi(h,\delta), \quad 0 < -h \ll 1,$$

where $\psi(h,\delta), I^*_{1r}(h,\delta), r = 0, 1, \cdots, k-2$, are analytic functions in (h,δ) for $|h|$ small and $\delta \in D$, and A_r's are constants for $0 \leq r \leq k-2$ and $r \neq \frac{k}{2} - 1$.

We remark that for concrete systems we needs to compute the coefficients in the asymptotic expansions in h of the analytic functions $\psi(h,\delta)$ and $I^*_{1r}(h,\delta)$'s. Of course, these coefficients are in general analytic functions in δ. For more information on $I^*_{1r}(h,\delta)$'s and A_r's, see page 7 of [26] or page 2731 of [52].

The second one is on the expansion of the Melnikov function near a homoclinic loop to a cusp.

Lemma 2. [27] *For the analytic near-Hamiltonian system* (1)*, whose Hamiltonian H has an oriented clockwise homoclinic loop and is of the form* (2) *with* (3)*, $k \geq 3$ odd and $h_k < 0$, the* *first order Melnikov function near and inside the loop L_s has the expansion*

$$M(h,\delta)=|h|^{\frac{1}{2}}\sum_{r=1}^{k-1}\varphi_r(h,\delta)|h|^{\frac{r}{k}}+\varphi(h,\delta),\qquad 0<-h\ll 1, \tag{6}$$

where $\varphi(h,\delta),\varphi_r(h,\delta),r=1,2,\cdots,k-1$, are analytic functions in (h,δ) for $|h|$ small and $\delta\in D$.

As remarked for Lemma 1, when applied to concrete systems we need to further compute the coefficients of asymptotic expansions in h for $\varphi(h,\delta)$ and $\varphi_r(h,\delta)$'s. In Lemmas 1 and 2, we have the condition $0<-h\ll 1$. It is due to the fact that the assumption 'L_s is oriented in clockwise' implies $h<0$.

The third one is on the expansion of the Melnikov function near a nilpotent center.

Lemma 3. [23] *For the analytic differential system (1) with the Hamiltonian H being of the form (2) with (3), $k\geq 4$ even and $h_k>0$, the first order Melnikov function near a nilpotent centre at the origin, with oriented clockwise periodic orbits nearby, has the expression*

$$M(h,\delta)=h^{\frac{2+k}{2k}}N(h^{\frac{2}{k}},\delta),\qquad 0<h\ll 1,$$

where $N(u,\delta)\in C^{\omega}$ is of the form

$$N(h^{\frac{2}{k}},\delta)=\sum_{r\geq 0}C_r^0(\delta)h^{\frac{2r}{k}}, \tag{7}$$

with

$$C_r^0(\delta)=\sum_{i+\frac{k}{2}j=r}r_{ij}(\delta)\beta_{ij},\quad \beta_{ij}=\int_0^1 u^{2i}(1-u^k)^{\frac{2j+1}{2}}\mathrm{d}u \tag{8}$$

and r_{ij}'s depending on δ are given in (26).

It is evident that the concrete coefficients in the expansion of the first order Melnikov function play a central role in studying limit cycles bifurcating from periodic orbits near a homoclinic loop or a centre. For the first order Melnikov function (5) in Lemma 1, Han et al. [26] in 2012 came up with an algorithm to compute the coefficients of the first two terms. in the expression (5) by virtue of Maple programme, and presented the expressions of the coefficients of h^0 and h^1 in the asymptotic expansion of $\psi(h,\delta)$ in (5) with $k=4$. Yang et al. [63] in 2019 indicated the form of the $(k+1)$th coefficient in the expansion of the Melnikov function (5) for even $k\geq 4$, and consequently the explicit expressions of the first $k+1$ coefficients can be theoretically obtained via the algorithm in [26] by using Maple programme. Wei and Zhang [52] in 2020 further displayed the definite expressions of the first $k+1$ coefficients in the expansion of the Melnikov function (5) for even $k\geq 4$. For the first order Melnikov function (6) in Lemma 2, the authors in [3, 27] exhibited the coefficients of the terms with degree less than 2 in the asymptotic expansion of the function (6) for $k=3,5$. Xiong [56] in 2015 obtained the properties of the coefficients of the first two terms in the expansion of $\varphi(h,\delta)$ in the Melnikov function (6) for odd $k\geq 7$, and pointed out that the coefficients of the other terms with degree less than 2, in the function (6), can be calculated using the deductive process in [27]. Wei and Zhang [52] in 2020 represented $\varphi_r(h,\delta)$ in (6) as $A_{r-1}I^*_{1,r-1}(h,\delta)$ to simplify the calculations (see Lemmas 4 and 8), and depicted the expressions of the first $k+1$ coefficients in the asymptotic expansion of the Melnikov function (6) for odd $k\geq 3$.

All the results listed above are on the coefficients of the terms having lower degrees in the expansion of the first order Melnikov functions. Wei and Zhang [52], and Yang et al. [64] characterized all coefficients in the asymptotic expansion of the Melnikov function (5) or (6)

https://doi.org/10.4153/S0008414X25101703 Published online by Cambridge University Press

under some conditions, one of which is that the Hamiltonian system (1)$|_{\varepsilon=0}$ has an elementary centre inside the loop. When the Hamiltonian system (1)$|_{\varepsilon=0}$ has a homoclinic loop enclosing a period annulus with its inner boundary a nilpotent centre, the results in [52] and [64] do not hold. One of our main goals in this paper is to tackle this problem. That is, when system (1)$|_{\varepsilon=0}$ has a homoclinic loop to a nilpotent singularity, which surrounds a nilpotent centre, we develop the existed techniques to obtain the expressions of the coefficients of the terms with higher degrees in the expansion of the first order Melnikov function near the homoclinic loop via the nilpotent centre. To achieve this goal, we need to combine the coefficients of the terms in the expansion of the first order Melnikov function near the nilpotent centre.

Han et al. [23] in 2008 provided an algorithm to compute the coefficients in the asymptotic expansion (7) of the first order Melnikov function near the centre and described the expressions of the first four coefficients with $k=4$. However, for even $k \geq 6$ there are no explicit expressions for the coefficients appearing in the function (7).

Our first result is on the first $k/2$ coefficients in the expansion of the first order Melnikov function near a nilpotent centre.

Theorem 1. *For the asymptotic expansion of the first order Melnikov function given in Lemma 3, the first $k/2$ coefficients C_r^0, $r \in \{0,1,\dots,k/2-1\}$, have the general formula*

$$C_r^0 = \frac{1}{k}\mathrm{B}\left(\frac{2r+1}{k},\frac{3}{2}\right)\sum_{l=0}^{2r}\frac{d_{rl}}{l!}\frac{\partial^l}{\partial x^l}\left(\frac{\partial P_0}{\partial x}+\frac{\partial Q_0}{\partial y}\right)\bigg|_{(C,\delta)},$$

where $\mathrm{B}(\cdot,\cdot)$ stands for the $Beta$ function, and d_{rl}'s, $l=0,1,\cdots,2r$, are given in Lemma 4 below.

On the appearance of the Beta function, see Proposition 2 and the remark under the proof of this proposition below. We must say that the expressions on the coefficients in Theorem 1 are the key point in the proofs of our main results in this paper.

Next we further investigate the coefficients of the terms with higher degrees in the asymptotic expansions of the first order Melnikov functions near a homoclinic loop to a nilpotent singularity and near a nilpotent centre presented in Lemmas 1–3. For the analytic near-Hamiltonian system (1), to convert the coefficients of the higher degree terms in the first order Melnikov function into the coefficients of the lower degree terms for new near-Hamiltonian systems, replacing the perturbations $P_0(x,y,\varepsilon,\delta)$ and $Q_0(x,y,\varepsilon,\delta)$ by new ones, saying the analytic perturbations $P_i(x,y,\delta)$ and $Q_i(x,y,\delta)$, $i\in\mathbb{N}$, one gets a new near-Hamiltonian system for each $i\in\mathbb{N}$

$$\begin{cases}\dot{x} = H_y(x,y)+\varepsilon P_i(x,y,\delta),\\ \dot{y} = -H_x(x,y)+\varepsilon Q_i(x,y,\delta).\end{cases} \tag{9}$$

Under these notations, we expand the analytic function I_{1r}^*'s, ψ,φ and φ_r's in Lemmas 1 and 2 in Taylor series in h, we get the next asymptotic expressions of the first order Melnikov function near and inside the homoclinic loop L_s either as

$$\begin{aligned} M^i(h,\delta) =& B_0^i + |h|^{\frac{1}{2}}\sum_{j\geq 0}\sum_{r=k_1j+1}^{k_1j+\frac{k_1}{2}-1} B_r^i|h|^{\frac{r}{k_1}} + \sum_{j\geq 0} B^i_{k_1j+\frac{k_1}{2}}h^{j+1}\ln|h| \\ &+ |h|^{\frac{1}{2}}\sum_{j\geq 0}\sum_{r=k_1j+\frac{k_1}{2}}^{(j+1)k_1-1} B_{r+1}^i|h|^{\frac{r}{k_1}}, \qquad 0<-h\ll 1, \end{aligned} \tag{10}$$

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for L_s to a nilpotent saddle S of order $[\frac{k_1-1}{2}]$ with even k_1, or as

$$\begin{aligned} M^i(h,\delta) = & B_0^i + |h|^{\frac{1}{2}} \sum_{j\geq 0} \sum_{r=k_1j+1}^{k_1j+\frac{k_1-1}{2}} B_r^i |h|^{\frac{r}{k_1}} + \sum_{j\geq 0} B^i_{k_1j+\frac{k_1+1}{2}} h^{j+1} \\ & + |h|^{\frac{1}{2}} \sum_{j\geq 0} \sum_{r=k_1j+\frac{k_1+1}{2}}^{(j+1)k_1-1} B_{r+1}^i |h|^{\frac{r}{k_1}}, \qquad 0 < -h \ll 1, \end{aligned} \tag{11}$$

for L_s to a nilpotent cusp S of order $[\frac{k_1-1}{2}]$ with odd k_1. Recall that $[\cdot]$ denotes the integer part function. In (10) and (11) the coefficients B_j^i's are functions of the parameters δ, which need to be computed. The first k_1+1 coefficients in the last two expressions had been got in [52], and they will be presented in Section 3 for our application.

As mentioned before, the nilpotent centre C is located on the x-axis, and the near-Hamiltonian system (1) has its unperturbed one having the Hamiltonian of the form (2) with (3). By Lemma 3 it follows that the first order Melnikov function near the nilpotent centre C, of order $k_2/2-1$ with even k_2, can have the asymptotic expansion

$$M^i(h,\delta) = (h-h_c)^{\frac{2+k_2}{2k_2}} \sum_{r\geq 0} C_r^i (h-h_c)^{\frac{2r}{k_2}}, \qquad 0 \leq h-h_c \ll 1. \tag{12}$$

The coefficients C_r^i's for $r=0,1,\cdots,\frac{k_2}{2}-1$ can be gained from Lemma 3 through a translation. It can be seen from the expressions of the coefficients C_r^0's in Theorem 1 that when the subscript r increases by 1, the number of the terms in the summation in C_r^0 increases by 2. To obtain our main results, we come up with the next new symbols

$$C_{r1}^i := \frac{1}{k} \mathrm{B}\left(\frac{2r+1}{k}, \frac{3}{2}\right) \sum_{l=0}^{r} \frac{d_{r,2l}}{(2l)!} \frac{\partial^{2l}}{\partial x^{2l}} \left(\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y}\right)\bigg|_{(C,\delta)}.$$

Set

$$\begin{aligned} \Delta^i := \Big\{ \delta : B_r^i(\delta) = 0, \quad & r = 1,2,\cdots,\left[\frac{k_1}{2}\right], \left[\frac{k_1}{2}\right]+2,\cdots,k_1, \\ & C_0^i(\delta) = C_j^i(\delta) = C_{j1}^i(\delta) = 0, \quad j = 1,2,\cdots,\frac{k_2}{2}-1 \Big\}, \end{aligned}$$

$$\overline{\Delta}^i := \bigcap_{s=0}^{i} \Delta^s.$$

Our second result is to characterize the coefficients of the terms with higher degrees in the asymptotic expansion of the first order Melnikov function (4) near a homoclinic loop to a nilpotent singularity for the near-Hamiltonian system (1). It combines the asymptotic expansion of the first order Melnikov function at the nilpotent centre.

Theorem 2. *For the analytic near-Hamiltonian system* (1)*, whose unperturbed Hamiltonian is of the form* (2) *with* (3)*, and has a nilpotent centre of order* $\frac{k_2}{2}-1$ *and an oriented clockwise homoclinic loop to a nilpotent singularity of order* $\left[\frac{k_1-1}{2}\right]$*, assume that the following conditions hold.*

(i) *The first order Melnikov function is of the form* (10) *or* (11) *with* $i=0$ *near the loop, or of the form* (12) *with* $i=0$ *near the centre.*

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(ii) *There exist analytic functions* $P_i(x,y,\delta)$ *and* $Q_i(x,y,\delta)$ *for* $i=1,2,\cdots,l$, *such that for* $\delta\in\overline{\Delta}^{i-1}$, *the following equalities inductively hold*

$$\frac{\partial H(x,y)}{\partial x}P_i(x,y,\delta)+\frac{\partial H(x,y)}{\partial y}Q_i(x,y,\delta)=\left(\frac{\partial P_{i-1}}{\partial x}+\frac{\partial Q_{i-1}}{\partial y}\right)(x,y,\delta), \tag{13}$$

over the period annulus $U:=\bigcup_{h_c\le h\le h_s}L_h$, *where* $P_0(x,y,\delta)$ *and* $Q_0(x,y,\delta)$ *are just* $P_0(x,y,0,\delta)$ *and* $Q_0(x,y,0,\delta)$ *in system* (1), *respectively.*

Then the coefficients in the first order Melnikov function, given in (10)–(12), *of system* (1) *satisfy the following formulas for* $\delta\in\overline{\Delta}^{i-1}$, *in which* B^i_j*'s are given in* (33) *below.*

(a) *For* $r=1,2,\cdots,[\frac{k_1-1}{2}]$,

$$B^0_{k_1i+r}=\frac{(-1)^i(2k_1)^i}{\prod_{j=1}^{i}((2j+1)k_1+2r)}B^i_r.$$

(b) *For* $r=[\frac{k_1+1}{2}]$,

$$B^0_{k_1i+r}=\frac{1}{(i+1)!}B^i_r.$$

(c) *For* $r=\frac{k_1}{2}+1$ (*in the case that* k_1 *is odd, this statement is void*)

$$B^0_{k_1i+r}=\frac{(-1)^i}{(i+1)!}B^i_r+\frac{(-1)^i}{(i+1)!}\left(\sum_{n=2}^{i+1}\frac{1}{n}\right)B^i_{r-1}.$$

(d) *For* $r=[\frac{k_1}{2}]+2,\cdots,k_1$,

$$B^0_{k_1i+r}=\frac{(-1)^i(2k_1)^i}{\prod_{j=1}^{i}((2j+1)k_1+2r-2)}B^i_r.$$

(e) *For* $j=0,1,\cdots,\frac{k_2}{2}-1$,

$$C^0_{\frac{k_2}{2}i+j}=\frac{(2k_2)^i}{\prod_{s=1}^{i}((2s+1)k_2+4j+2)}C^i_j.$$

We remark that the properties in the above theorem have been partly acquired by Tian and Han [48] and Wei et al. [51] for $k_1=k_2=2$, in which the centre is an elementary one and the singularity that the homoclinic loop approaches is a hyperbolic saddle, see Page 2. And the properties have been presented by Wei and Zhang [52] for $k_2=2, k_1\ge 3$, in which the centre in Theorem 2 is an elementary one and the loop is homoclinic to a nilpotent singularity of order $[\frac{k_1-1}{2}]$.

The result in Theorem 2 characterizes parts of the coefficients in the asymptotic expansion of the first order Melnikov function (4) near a homoclinic loop under some condition by utilizing the properties of the nilpotent centre and also parts of the coefficients in the expansion near the nilpotent centre. In other words, we could say that the result of the theorem indicates the properties on the coefficients in the expansion near a nilpotent centre via a homocinic loop. Following this idea, we can think only about the properties of the coefficients in the Melnikov function near a nilpotent centre without regard to a homoclinic loop.

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The next result presents the properties near a nilpotent centre. Its proof can be obtained in a similar way as that in the proof of Theorem 2 given in section 3. The details are omitted.

Set

$$\overline{\Lambda}^i := \bigcap_{s=0}^{i} \Lambda^s \quad \text{with} \quad \Lambda^i := \left\{\delta :\ C_0^i(\delta) = C_j^i(\delta) = C_{j1}^i(\delta) = 0, j = 1, 2, \cdots, \frac{k_2}{2} - 1\right\}.$$

Theorem 3. *For the analytic near-Hamiltonian system (1) with its unperturbed one being of the form (2) with (3) and having a nilpotent centre of order* $\frac{k_2}{2} - 1$ *with oriented clockwise periodic orbits nearby, assume that the following conditions hold.*

(i) *The first order Melnikov function has the expansion of the form (7) near the centre.*

(ii) *There exist analytic functions* $P_i(x, y, \delta)$ *and* $Q_i(x, y, \delta)$ *for* $i = 1, 2, \cdots, l$, *such that for* $\delta \in \overline{\Lambda}^{i-1}$ *the equations (13) hold over the period annulus* $V := \bigcup_{h_c \leq h \leq h_c + \tau} L_h$ *with* $0 < \tau \ll 1$.

Then the coefficients $C_j^0, j \geq k_2/2$, *in the expansion (12) of the first order Melnikov function satisfy the following relations.*

$$C^0_{\frac{k_2}{2}i+r}\Big|_{\delta \in \overline{\Lambda}^{i-1}} = \frac{(2k_2)^i}{\prod\limits_{s=1}^{i}((2s+1)k_2 + 4r + 2)} C_r^i,$$

for $r = 0, 1, \cdots, \frac{k_2}{2} - 1$, *where* C_r^i*'s are the coefficients in the expansion (12).*

We remark that the conclusion in Theorem 3 is not different from that in (e) of Theorem 2. The differences are in the following aspects: Theorem 3 focuses on the coefficients in the asymptotic expansion of the first order Melnikov function near a nilpotent centre, so the restriction $\delta \in \overline{\Lambda}^{i-1}$ there is only for the coefficients of the terms with lower degrees in the expansion just near the nilpotent centre. Whereas Theorem 2 handles the coefficients in the expansion of the Melnikov function near the homoclinic loop or near the nilpotent centre, in which the restriction $\delta \in \overline{\Delta}^{i-1}$ is on both coefficients of the terms with lower degrees in the expansions near the loop and near the center. It leads to interdependent relationships among the coefficients in the expansions near the loop and near the center.

The last two theorems display the relative expressions of parts of the coefficients in the asymptotic expansion of the first order Melnikov function under the prescribed conditions, which is crucial in finding more limit cycles of the system. The next result explains the realizability of the conditions in Theorem 2, especially the conditions (13) for $i \geq 1$, for a certain kind of analytic near-Hamiltonian systems. We remark that the condition $\delta \in \overline{\Delta}^{i-1}$ under which the equation (13) holds is different from that in Tian and Han [48], where the realization is explained under the condition $H(x, y) = \int_0^x q(x)\mathrm{d}x + \int_0^y p(y)\mathrm{d}y$. Here we do not have this limitation for the Hamiltonian.

Theorem 4. *Assume that the Hamiltonian* $H(x, y)$ *satisfies the condition* (i) *of Theorem 2, and* $\frac{\partial H}{\partial y}(x, y)/y$ *does not vanish in* U *defined in Theorem 2. For any analytic 1-form* $\omega = P\mathrm{d}y - Q\mathrm{d}x$, *if its exterior derivative* $\mathrm{d}\omega = f(x, y)\mathrm{d}x \wedge \mathrm{d}y$ *satisfies* $f(x, 0) = x^{k_1-1}(x - x_c)^{k_2-1}\mu(x)$ *with* $\mu(x)$ *analytic in* U, *then there exists an analytic 1-form* $\eta = A\mathrm{d}y - B\mathrm{d}x$ *such that*

$$\mathrm{d}\omega = \mathrm{d}H \wedge \eta.$$

This result is an extension of Theorem 6 in [52] from an elementary centre to a nilpotent centre. The proof is also similar to that in [52], and will be omitted. Analogously, the conditions in Theorem 3 are also realizable for some near-Hamiltonian systems.

Finally we apply our main results in Theorems 2–3 and the theories on limit cycle bifurcation to find more limit cycles via the number of simple isolated zeros of the Abelian integral (the first order Melnikov function), related to the two classes of near–Hamiltonian systems. Givental [17] found some nonoscillation-type properties of Abelian integrals in the hyperelliptic Hamiltonian $H(x,y) = y^2 + U_{n+1}(x)$ with a real polynomial $U_{n+1}(x)$ in x of degree $n+1$. Here our perturbed Hamiltonian systems are also defined by hyperelliptic Hamiltonians.

Hereafter we use the notion $\mathrm{mod}(a,b)$ to denote the remainder of a divided by b.

Consider a generalized polynomial Liénard differential system of the form

$$\dot{x} = y, \qquad \dot{y} = -g_n(x) + \varepsilon y f_m(x), \tag{14}$$

where $\varepsilon > 0$ is sufficiently small, g_n and f_m are real polynomials of degrees n and m, respectively. Denote by $Z(n,m)$ the maximum number of isolated zeros of the Abelian integral associated to the generalized Liénard system (14). On determination of $Z(n,m)$, there are rich results for $n = 1,2,3,4$, see e.g. [6, 9, 10, 12, 16, 22, 30, 38, 41, 46, 49, 50, 53, 54, 57] and the references therein.

For $n = 5$, Xu and Li [59] in 2012 got $Z(5,m) \geq m+1$ for $m = 2,4$ and $Z(5,m) \geq 10$ for $m = 6,8$, where the unperturbed Hamiltonian system includes two elementary centres and a double homoclinic loop surrounded by a heteroclinic to two hyperbolic saddles. Xu and Li [60] in 2013 gained $Z(5,m) \geq m + \mathrm{mod}(m/2,2)$ for $m = 4,6,10$, when the period annulus is bounded by a compound loop consisting of a heteroclinic loop and two homoclinic loops. Han and Romanovski [24] in 2013 achieved $Z(5,m) \geq 2[\frac{m-1}{3}] + [\frac{m-1}{2}]$ using a different approach for $m \geq 5$. Sun et al. [45, 47] presented $Z(5,4) \geq 3$ in the case that the period annulus is limited by an elementary centre and a heteroclinic loop to a hyperbolic saddle and a nilpotent saddle, respectively. Xiong [55] in 2014 showed $Z(5,m) \geq 2m+1$ for $m = 2,3,4$, and $Z(5,m) \geq m+5$ for $5 \leq m \leq 8$ in the case that the period annulus is limited by two elementary centres and a compound loop consisting of a cuspidal loop and a heteroclinic loop with a cusp and a hyperbolic saddle. In the same year, Xiong and Han [58] further got $Z(5,m) \geq 2[\frac{m-1}{3}]+[\frac{m}{2}]+2$ for $m \geq 5$. Li and Yang [35] in 2019 provided also the lower bounds of $Z(5,m)$ in the case that the unperturbed Hamiltonian system is the same as that of [55] for $10 \leq m \leq 20$. Yang et al. [64] in 2021 achieved that for $22 \leq m \leq 46$ even, $Z(5,m) \geq \frac{5}{4}m$ if $4|m$ and $Z(5,m) \geq \frac{5}{4}m - \frac{1}{2} + [\frac{\mathrm{mod}(m/2,4)}{2}]$ if $4 \nmid m$, under the conditions that the corresponding Liénard system (14) is centrally symmetric and has two centres and a double homoclinic loop with a nilpotent saddle, and that for $22 \leq m \leq 46$ odd, $Z(5,m) \geq Z(5,m-1)$. Yang and Han [62] in 2024 gave further in the case of an elementary centre and a cuspidal loop passing a cusp of order 2 that for $15 \leq m \leq 25$ a lower bound is $8[\frac{m}{6}] + \chi(m)$ with $\chi(m) = 2\mathrm{mod}(m,6)+1$ if $\mathrm{mod}(m,6) \leq 3$ and $\chi(m) = 8$ if $\mathrm{mod}(m,6) \geq 4$. For $n \geq 6$, there are also some works on the lower bound of the number of isolated zeros, see e.g. [4, 24, 25, 44, 56, 58] and the references therein.

Next we apply our results on the expressions of the first order Melnikov functions to study the Liénard system (14) with $g_5 = x^2(x-1)^3$ for obtaining a better low bound on $Z(5,m)$.

Theorem 5. *There exists a generalized Liénard system of the form* (14) *with $n = 5$ and $g_5 = x^2(x-1)^3$, which has at least $7 \cdot [\frac{m-1}{6}] + 2r - \left[\frac{r}{4}\right] \cdot (2\mathrm{mod}(r,3) - 1) - 1$ limit cycles with $r = \mathrm{mod}(m-1,6)$ and $4 \leq m \leq 300$.*

Recall that $\mathrm{mod}(a,b)$ denotes the remainder of a divided by b. This theorem provides a larger lower bound on $Z(5,m)$ than those in [24] with $Z(5,m) \geq 2[\frac{m-1}{3}] + [\frac{m-1}{2}]$.

For the Hamiltonian system having a nilpotent centre, Llibre and Zhang [39] in 2001 considered the degree m polynomial perturbation of the Hamiltonian centre with the Hamiltonian $H(x,y) = \frac{1}{2n}x^{2n} + \frac{1}{2l}y^{2l}$, and gained the number s if $n = l$; $\frac{s(s+3)}{2}$ if $s < k$ and $n \neq l$; and

https://doi.org/10.4153/S0008414X25101703 Published online by Cambridge University Press

$\frac{k(2s-k+3)}{2} - 1$ if $s \geq k$ and $n \neq l$, of isolated zeros of the Abelian integrals, where $s = [\frac{m-1}{2}]$ and $k = \max\{n/\gcd(n,l), l/\gcd(n,l)\}$ with the greatest common divisor $\gcd(n,l)$. Jiang et al. [32] in 2009 studied the same problem in a centrally symmetric case with $n = 2, l = 1, m = 5$, and got 10 isolated zeros by blowing up a nilpotent centre of order 1 to a double homoclinic loop. Su et al. [43] in 2012 showed that a generalized Liénard system of degree m can have 1, 3, 5 limit cycles near a nilpotent centre of order 1 for $m = 3, 4, 5$. In the same year, Yang and Han [61] figured out that a generalized Liénard system of degree m with the Hamiltonian $H(x,y) = \frac{1}{2}y^2 + \frac{1}{5}x^5 - \frac{3}{4}x^4 + x^3 - \frac{1}{2}x^2$ can have $m-1-[\frac{m}{5}]$ limit cycles for $2 \leq m \leq 13$, which include $m - 2 - [\frac{m}{5}]$ limit cycles near a nilpotent centre of order 1. Gasull et al. [14] in 2015 focused on an $\frac{n-1}{2}$ order nilpotent center with the perturbations, linear term plus homogeneous polynomials of odd degree n, whose Abelian integrals can have $\frac{n+1}{2}$ isolated zeros around the centre with their proof using the generalized Lyapunov polar coordinate change of variables. As far as we know, the limit cycle bifurcation near a nilpotent centre was rarely studied.

As a second application, we consider a near-Hamiltonian system (1) with the unperturbed Hamiltonian

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{4}x^4 - \frac{1}{5}x^5 - \frac{1}{6}x^6, \tag{15}$$

and the perturbed polynomials $P_0(x,y)$ and $Q_0(x,y)$ being of degree m. The unperturbed system has exactly three singularities: a nilpotent centre at the origin together with two hyperbolic saddles on its two sides.

Theorem 6. *There exists a near-Hamiltonian system* (1) *with the Hamiltonian* (15)*, which has at least* $2 \cdot [\frac{m+1}{2}] + 2 \cdot \left[\frac{2\cdot[\frac{m}{2}]-1}{3}\right] + \mathrm{mod}(2 \cdot [\frac{m}{2}] - 1, 3) - 3$ *small-amplitude limit cycles near the nilpotent centre for* $3 \leq m \leq 22$*.*

We remark that there is no a published paper which studied this near-Hamiltonian system. The related work is the book [13], in which a generic 3-parameter unfolding was investigated for planar vector fields with a nilpotent singularity.

The paper is organized as follows. In Section 2, we study the first order Melnikov function near a nilpotent centre and a homoclinic loop to a nilpotent singularity, and obtain the exact expressions and the properties of the coefficients in the corresponding asymptotic expansions. The proofs of Theorems 1–3 are also presented in this section. Section 3 studies the relation between the limit cycle bifurcation and the independence of the coefficients obtained in Theorems 1–3. The proofs of Theorems 5 and 6 are given in Section 4.

2. The first order Melnikov functions near a centre and a homoclinic loop

Han et al. [23, 31, 61] presented the asymptotic expansions of the first order Melnikov functions near a nilpotent centre, and established an algorithm to compute the coefficients of the Melnikov functions by Maple programme. In this section, we further study the coefficients and optimize their algorithm to simplify the calculations.

Yang and Han [61] in 2012 provided an algorithm to compute the coefficients in the expansion of the first order Melnikov function near a nilpotent centre of order $\frac{k}{2} - 1$ with the help of the Maple programme. There, they need to require a fixed integer k in the process of Maple programme, and it is hard to figure out the explicit expressions of the coefficients with higher degrees due to computer performance limitations. In the present paper, for any $k \geq 4$ even the expressions and properties of the first coefficients in the first order Melnikov function near a nilpotent centre are provided and they will play key roles in understanding the coefficients of the terms with higher degrees in the Melnikov function established near a nilpotent centre or

https://doi.org/10.4153/S0008414X25101703 Published online by Cambridge University Press

near a homoclinic loop to a nilpotent singularity. This will provide a nice tool to handle the limit cycles bifurcating from the periodic orbits near the nilpotent centre or near the homoclinic loop.

For simplicity to notations, we take $x_c = 0$ in the first subsection below, and suppose that the Hamiltonian of the unperturbed system (1)$|_{\varepsilon=0}$ is of the form (2) with (3) and $h_{00} > 0$. To study the asymptotic expression of the first order Melnikov functions, without loss of generality, we set the perturbation functions being of the form

$$P_0(x,y,0,\delta) = \sum_{i+j\geq 0} a_{ij}x^iy^j, \quad Q_0(x,y,0,\delta) = \sum_{i+j\geq 0} b_{ij}x^iy^j, \tag{16}$$

where a_{ij} and b_{ij} are analytic functions in the parameters $\delta \in D \subset \mathbb{R}^d$.

2.1. **The coefficients of the terms with lower degrees.** By the classical results [19, 42] on Melnikov functions for computing limit cycles bifurcating from period annulus, we get from Green's formula together with some manipulations that the first order Melnikov function (4) can be rewritten as

$$M(h,\delta) = \oint_{H(x,y)=h} \overline{q}(x,y,\delta)\mathrm{d}x, \tag{17}$$

where

$$\begin{aligned}\overline{q}(x,y,\delta) &= \int_0^y \left(\frac{\partial P_0}{\partial x}(x,v,0,\delta) + \frac{\partial Q_0}{\partial y}(x,v,0,\delta)\right)\mathrm{d}v \\ &= Q_0(x,y,0,\delta) - Q_0(x,0,0,\delta) + \int_0^y \frac{\partial P_0}{\partial x}(x,v,0,\delta)\mathrm{d}v \\ &= \sum_{j\geq 1} q_j(x)y^j\end{aligned}$$

with

$$q_j(x) = \sum_{i\geq 0} \overline{b}_{ij}x^i, \tag{18}$$

whose coefficients satisfy

$$\overline{b}_{ij} = b_{ij} + \frac{i+1}{j}a_{i+1,j-1}. \tag{19}$$

Following [23], we know that the Hamiltonian (2) contains the component $\widetilde{H}(x,y) = h_{00} + O(|x,y|)$. Then $H(x,y) = h$ is equivalent to $\omega = |y|(\widetilde{H}(x,y))^{\frac{1}{2}}$, where $\omega = \sqrt{h - H_0(x)}$. Applying the implicit function theorem to it, one gets two locally analytic solutions of $H(x,y) = h$, saying $y_1(x,\omega)$ and $y_2(x,\omega)$, which satisfy $y_1(x,\omega) = (h_{00})^{-\frac{1}{2}}\omega(1 + O(|x,\omega|)) = y_2(x,-\omega)$ near $(x,\omega) = (0,0)$, and $y_1(x,\omega)$ can be expanded in ω as follows

$$y_1(x,\omega) = \sum_{i\geq 1} a_i(x)\omega^i, \tag{20}$$

where

$$a_1(x) = \frac{1}{\sqrt{H_1^*}},$$

$$a_{i+1}(x) = -\frac{1}{2\sqrt{H_1^*}}\left(H_1^* \sum_{\substack{t_1+t_2=i+2,\\ t_1,t_2\geq 2}} a_{t_1}a_{t_2} + \sum_{j=1}^{i} H_{j+1}^* b_{i+2}^{j+2}\right), \quad \text{for} \quad i \geq 1,$$

with $b_i^j = \sum_{t_1+t_2+\cdots+t_j=i} a_{t_1}a_{t_2}\cdots a_{t_j}$, and $\sum_{\substack{t_1+t_2=i+2,\\ t_1,t_2\geq 2}} a_{t_1}a_{t_2} = 0$ if $i=1$.

Under these notations, the integral $M(h,\delta)$ in (17) can be written in

$$\begin{aligned} M(h,\delta) &= \int_{b(h)}^{a(h)} \left(\overline{q}(x,y_1(x,\omega),\delta) - \overline{q}(x,y_2(x,\omega),\delta)\right) \mathrm{d}x \\ &= \int_{b(h)}^{a(h)} \sum_{j\geq 1} q_j(x)(y_1^j - y_2^j)\mathrm{d}x, \end{aligned}$$

where $a(h)$ and $b(h)$ are two solutions of the equation $H_0(x) = h$ in a neighborhood of $x=0$. The symmetry of y_1 and y_2 with respect to the y-axis further simplifies the Melnikov function to

$$M(h,\delta) = \int_{b(h)}^{a(h)} \sum_{j\geq 0} \overline{q}_j(x)(h - H_0(x))^{\frac{2j+1}{2}}\mathrm{d}x, \tag{21}$$

where

$$\overline{q}_j(x) = 2\sum_{i=0}^{2j} q_{i+1} b_{2j+1}^{i+1}. \tag{22}$$

We note that the detalis about the derivations of (20) and (21), together with the expressions of a_1, a_2, a_3 in (20) and $\overline{q}_0, \overline{q}_1$ in (22), can be found on page 191 or on page 193 of book [21].

Introducing a new variable

$$u = \operatorname{sgn}(x)\cdot |H_0(x)|^{\frac{1}{k}} := \Phi(x), \tag{23}$$

the Melnikov function can be expressed as

$$\begin{aligned} M(h,\delta) &= \sum_{j\geq 0} \int_{-h^{\frac{1}{k}}}^{h^{\frac{1}{k}}} \widetilde{q}_j(u)(h-u^k)^{\frac{2j+1}{2}}\mathrm{d}u \\ &= \sum_{j\geq 0} \int_0^{h^{\frac{1}{k}}} (\widetilde{q}_j(u) + \widetilde{q}_j(-u))(h-u^k)^{\frac{2j+1}{2}}\mathrm{d}u, \end{aligned} \tag{24}$$

where

$$\widetilde{q}_j(u) = \frac{\overline{q}_j(x)}{\Phi'(x)}\Big|_{x=\Phi^{-1}(u)}, \tag{25}$$

can be expanded as a power series in u in a neighborhood of $u=0$, and then

$$\widetilde{q}_j(u) + \widetilde{q}_j(-u) = \sum_{i\geq 0} r_{ij}u^{2i}, \tag{26}$$

with the constants r_{ij}'s, $i,j\geq 0$, depending on the coefficients of the perturbations P_0 and Q_0 of the analytic near-Hamiltonian system (1). As in [61], substituting (26) into (24) yields the asymptotic expression in h as in Lemma 3, of the first order Melnikov function (4) near the nilpotent centre.

The next result characterizes the coefficients appearing in Lemma 3.

Proposition 2. *The constants β_{ij}'s given in Lemma 3 have the next expressions*

$$\beta_{ij} = \frac{1}{k}\mathrm{B}\left(\frac{2i+1}{k}, j+\frac{3}{2}\right).$$

Proof. The expression β_{ij} in (8) can be deformed as

$$\beta_{ij} = \int_0^1 (u^k)^{\frac{2i}{k}}(1-u^k)^{\frac{2j+1}{2}}\mathrm{d}u.$$

Then one has

$$\beta_{ij} = \frac{1}{k}\int_0^1 t^{\frac{2i+1}{k}-1}(1-t)^{j+\frac{1}{2}}\mathrm{d}t.$$

The proof completes via the definition of the Beta function. □

We remark that $\beta_{00}, \beta_{10}, \beta_{20}, \beta_{30}, \beta_{01}, \beta_{11}$ can be obtained from page 198 of [21] when $k=4$. Although the expressions are different from those of Proposition 2, they coincide by some calculations. From Lemma 3, one needs to calculate the coefficients C_r^0 of the first order Melnikov function, which is the key element in the expressions of r_{ij}'s.

Lemma 4. *For the formula (8) given in Lemma 3, there exist constants d_{il}'s, $l=0,1,\cdots,2i$, depending on the coefficients of the unperturbed Hamiltonian one of system (1), such that for all nonnegative integers i's,*

$$r_{i0} = \sum_{l=0}^{2i} d_{il}\left((l+1)a_{l+1,0}+b_{l1}\right) = \sum_{l=0}^{2i}\frac{d_{il}}{l!}\frac{\partial^l}{\partial x^l}\left(\frac{\partial P_0}{\partial x}+\frac{\partial Q_0}{\partial y}\right)\bigg|_{(C,\delta)}.$$

In particular, $d_{i,2i} = 4|h_k|^{-\frac{2i+1}{k}}(h_{00})^{-\frac{1}{2}}$, $d_{i,2i-1} = -\frac{2}{k}((4i+2)h_{k+1}+kh_kh_{10})|h_k|^{-\frac{2i+3}{k}}(h_{00})^{-\frac{1}{2}}$.

Proof. In order to get the expression of r_{i0}, we have to know the expansions of the level curves defining the ovals. According to the expression of $H_0(x)$ in (3) and the definition of $\Phi(x)$ in (23), one has

$$u = \Phi(x) = |h_k|^{\frac{1}{k}}\sum_{i\geq 1}\mu_i x^i, \tag{27}$$

whose coefficients μ_i's, $i\geq 2$ depend on h_{j+k}, $j=1,\cdots,i-1$, and have the expressions

$$\mu_i = \begin{cases} 1, & i=1,\\ \sum\limits_{m=1}^{i-1}\frac{(-1)^m}{m!(kh_k)^m}\prod\limits_{j=0}^{m-1}(jk-1)\sum\limits_{j_1+\cdots+j_m=i-1} h_{j_1+k}\times\cdots\times h_{j_m+k}, & i\geq 2.\end{cases} \tag{28}$$

Then it follows from (27) that

$$\frac{1}{\Phi'(x)} = |h_k|^{-\frac{1}{k}}\left(1+\sum_{i\geq 1}(i+1)\mu_{i+1}x^i\right)^{-1} = |h_k|^{-\frac{1}{k}}\sum_{i\geq 0}n_i x^i, \tag{29}$$

where $n_i, i \geq 1$, are functions in $\mu_2, \cdots, \mu_{i+1}$, and have the expressions

$$n_i = \begin{cases} 1, & i = 0, \\ \sum\limits_{m=1}^{i} (-1)^m \sum\limits_{j_1+\cdots+j_m=i} (j_1+1) \times \cdots \times (j_m+1) \times \mu_{j_1+1} \times \cdots \times \mu_{j_m+1}, & i \geq 1, \end{cases}$$

with the positive integers j_l's.

Let $x = \Phi^{-1}(u)$ in (25) have the power series in a neighborhood of $u = 0$

$$x = \Phi^{-1}(u) =: \sum_{i\geq 1} \overline{\mu}_i u^i. \tag{30}$$

By the expression $u = \Phi(x)$ in (27), equating the next equality

$$u = \Phi(\Phi^{-1}(u)) = |h_k|^{\frac{1}{k}} \sum_{i\geq 1} u^i \sum_{m=1}^{i} \mu_m \sum_{j_1+\cdots+j_m=i} \overline{\mu}_{j_1} \times \cdots \times \overline{\mu}_{j_m}$$

yields

$$\overline{\mu}_i = \begin{cases} |h_k|^{-\frac{1}{k}} \mu_1^{-1} = |h_k|^{-\frac{1}{k}}, & i = 1, \\ -\sum\limits_{m=2}^{i} \mu_m \sum\limits_{j_1+\cdots+j_m=i} \overline{\mu}_{j_1} \times \cdots \times \overline{\mu}_{j_m}, & i \geq 2, \end{cases} \tag{31}$$

with μ_m's given in (28). Now $a_1(x)$, as the first coefficient of $y_1(x,\omega)$ in (20), via the expression of H_0^* in (3), has the asymptotic expansion

$$\begin{aligned} a_1(x) &= (h_{00})^{-\frac{1}{2}} \left(1 + \sum_{i\geq 1} \frac{-\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-i+1)}{i!(h_{00})^i} \left(\sum_{j\geq 1} h_{j0} x^j \right)^i \right) \\ &= (h_{00})^{-\frac{1}{2}} \sum_{i\geq 0} x^i m_i, \end{aligned} \tag{32}$$

where m_i's have the expressions

$$m_i = \begin{cases} 1, & i = 0, \\ \sum\limits_{l=1}^{i} \frac{-\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-l+1)}{l!} \sum\limits_{j_1+\cdots+j_l=i} h_{j_1,0} \times \cdots \times h_{j_l,0}, & i \geq 1. \end{cases}$$

Substituting (18), (22), (29) and (32) into (25), we get

$$\begin{aligned} \widetilde{q}_0(\Phi(x)) &= 2|h_k|^{-\frac{1}{k}} (h_{00})^{-\frac{1}{2}} \left(\sum_{i\geq 0} \overline{b}_{i1} x^i \right) \left(\sum_{i\geq 0} n_i x^i \right) \left(\sum_{i\geq 0} m_i x^i \right) \\ &= 2|h_k|^{-\frac{1}{k}} (h_{00})^{-\frac{1}{2}} \sum_{i\geq 0} x^i t_i, \end{aligned}$$

where $t_i = \sum\limits_{i_1+i_2+i_3=i} \overline{b}_{i_3 1} n_{i_1} m_{i_2}$, $i_1, i_2, i_3 \geq 0$ with $\overline{b}_{j1}$, n_j and m_j appearing respectively in (19), (29) and (21). In particular, $t_0 = \overline{b}_{01}$. Substituting (30) into the above expression gives

$$\widetilde{q}_0(u) = 2|h_k|^{-\frac{1}{k}} (h_{00})^{-\frac{1}{2}} \left(t_0 + \sum_{i\geq 1} u^i \sum_{m\geq 1}^{i} t_m \sum_{j_1+\cdots+j_m=i} \overline{\mu}_{j_1} \times \cdots \times \overline{\mu}_{j_m} \right).$$

Combining this expression with (26) yields

$$\begin{aligned}
r_{00} &= 4|h_k|^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}}t_0 = d_{00}\bar{b}_{01},\\
r_{i0} &= 4|h_k|^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}}\sum_{m\geq 1}^{2i}\sum_{l=0}^{m}\bar{b}_{l1}\Big(\sum_{i_1+i_2=m-l}n_{i_1}m_{i_2}\Big)\sum_{j_1+\cdots+j_m=2i}\overline{\mu}_{j_1}\times\cdots\times\overline{\mu}_{j_m}\\
&= \sum_{l\geq 0}^{2i} d_{il}\bar{b}_{l1}, \qquad i\geq 1,
\end{aligned}$$

where

$$\begin{aligned}
d_{00} &= 4|h_k|^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}},\\
d_{i0} &= 4|h_k|^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}}\sum_{m=1}^{2i}\left(\sum_{i_1+i_2=m}n_{i_1}m_{i_2}\right)\sum_{j_1+\cdots+j_m=2i}\overline{\mu}_{j_1}\times\cdots\times\overline{\mu}_{j_m},\\
d_{il} &= 4|h_k|^{-\frac{1}{k}}(h_{00})^{-\frac{1}{2}}\sum_{m=l}^{2i}\left(\sum_{i_1+i_2=m-l}n_{i_1}m_{i_2}\right)\sum_{j_1+\cdots+j_m=2i}\overline{\mu}_{j_1}\times\cdots\times\overline{\mu}_{j_m},\ l=1,\cdots,2i,
\end{aligned}$$

for $i\geq 1$, where $\overline{\mu}_j$'s are given in (31) through (28). Note that d_{il}'s, $l=0,1,\cdots,2i$, depend only on the coefficients of the Hamiltonian $H(x,y)$ given in (2). The precise expressions of $d_{i,2i-1}$ and $d_{i,2i}$ are presented in Lemma 4.

This proves the lemma using the notations in (16) and (19). □

Proof of Thorem 1. The proof follows from the formula (8) and Lemma 4. □

2.2. **The coefficients of the terms with higher degree.** For the analytic near-Hamiltonian system (1), the authors in [52] showed that the first k_1 coefficients in the asymptotic expansion of the first order Melnikov function (10) or (11) satisfy the following expressions.

$$\begin{aligned}
&B_0^i = \oint_{L_s} Q_i\mathrm{d}x - P_i\mathrm{d}y|_{\varepsilon=0},\\
&B_r^i = A_{r-1}\sum_{l=0}^{r-1}\frac{\tilde{d}_{r-1,l}}{l!}\frac{\partial^l}{\partial x^l}\left(\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}\right)(S,\delta), \qquad r=1,\cdots,\left[\frac{k_1-1}{2}\right],\\
&B_{\frac{k_1}{2}}^i = -\frac{1}{2k_1}\sum_{l=0}^{\frac{k_1}{2}-1}\frac{\tilde{d}_{\frac{k_1}{2}-1,l}}{l!}\frac{\partial^l}{\partial x^l}\left(\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}\right)(S,\delta),\\
&B_r^i = A_{r-2}\sum_{l=0}^{r-2}\frac{\tilde{d}_{r-2,l}}{l!}\frac{\partial^l}{\partial x^l}\left(\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}\right)(S,\delta), \qquad r=\left[\frac{k_1}{2}\right]+2,\cdots,k_1,
\end{aligned}\tag{33}$$

and

$$\begin{aligned}
B_{[\frac{k_1}{2}]+1}^i =&(-1)^{k_1-1}\oint_{L_s}\left(\left(\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}\right)\Big|_{\varepsilon=0} - \sum_{l=0}^{[\frac{k_1}{2}]-1}\frac{1}{l!}\frac{\partial^l}{\partial x^l}\left(\frac{\partial P_i}{\partial x}+\frac{\partial Q_i}{\partial y}\right)(S,\delta)x^l\right)\mathrm{d}t\\
&+\sum_{l=1}^{[\frac{k_1}{2}]}\tilde{T}_l B_l^i,
\end{aligned}$$

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where the concrete expressions of the coefficients A_i's and $\tilde{d}_{ij}$ can be obtained from Lemmas 8–9 in [52]. Set

$$\Omega^i := \left\{\delta : B_r^i(\delta) = 0, r = 1, 2, \cdots, \left[\frac{k_1}{2}\right], \left[\frac{k_1}{2}\right] + 2, \cdots, k_1,\ C_j^i(\delta) = 0, j = 0, 1, \cdots, \frac{k_2}{2} - 1\right\}$$

and $\overline{\Omega}^i := \bigcap_{s=0}^i \Omega^s$, where C_j^i's are given in (12). We mention the difference of Ω^i with Δ^i in Theorem 2.

Lemma 5. *Let the basic conditions on the system and the homoclinic loop and the assumption* (i) *in Theorem 2 hold. Assume that there exist analytic functions* $P_i(x, y, \delta)$ *and* $Q_i(x, y, \delta)$ *for* $i = 1, 2, \cdots, l$, *such that for* $\delta \in \overline{\Omega}^{i-1}$ *the equalities (13) hold over* U. *Then for* $\delta \in \overline{\Omega}^{i-1}$ *the conclusions* (a)–(e) *of Theorem 2 hold.*

Proof. Here we just prove this lemma in the case that the unperturbed system $(1)|_{\varepsilon=0}$ has a nilpotent centre and a homoclinic loop to a nilpotent cusp. The proof as the homoclinic loop to a nilpotent saddle can be handled in a similar way as that in the proof for the nilpotent cusp.

We first prove the result for $l = 1$. Differentiating $M^0(h, \delta)$ in the expansions (11) and (12) with respect to h, together with the condition $\delta \in \overline{\Omega}^0$, one has for $0 < -h \ll 1$,

$$\begin{aligned}\frac{\partial M^0(h,\delta)}{\partial h} =& B^0_{\frac{k_1+1}{2}} - |h|^{\frac{1}{2}} \sum_{j\geq 0} \sum_{r=k_1 j+1}^{k_1 j + \frac{k_1-1}{2}} \left(\frac{r}{k_1} + \frac{3}{2}\right) B^0_{k_1+r} |h|^{\frac{r}{k_1}} \\ &+ \sum_{j\geq 0} (j+2) B^0_{k_1(j+1)+\frac{k_1+1}{2}} (h - h_s)^{j+1} \\ &- |h|^{\frac{1}{2}} \sum_{j\geq 0} \sum_{r=k_1 j + \frac{k_1+1}{2}}^{(j+1)k_1 - 1} \left(\frac{r}{k_1} + \frac{3}{2}\right) B^0_{k_1+r+1} |h|^{\frac{r}{k_1}},\end{aligned} \tag{34}$$

and for $0 \leq h - h_c \ll 1$,

$$\frac{\partial M^0(h,\delta)}{\partial h} = (h - h_c)^{\frac{2+k_2}{2k_2}} \sum_{r\geq 0} \frac{2 + 3k_2 + 4r}{2k_2} C^0_{r+\frac{k_2}{2}} (h - h_c)^{\frac{2r}{k_2}}. \tag{35}$$

Set $C_r^1 := \frac{2+3k_2+4r}{2k_2} C^0_{r+\frac{k_2}{2}}, r \geq 0$. Recall that the derivative of the first order Melnikov function $M(h, \delta)$ defined by (4) in h has the following expression, see [20].

$$\frac{\partial M(h,\delta)}{\partial h} = \oint_{L_h^1} \left(\frac{\partial P_0}{\partial x} + \frac{\partial Q_0}{\partial y}\right)\bigg|_{\varepsilon=0} \mathrm{d}t.$$

Substituting the equation (13) with $i = 1$ into the above derivative, a simple computation shows

$$\frac{\partial M^0(h,\delta)}{\partial h} = \oint_{L_h^1} Q_1 \mathrm{d}x - P_1 \mathrm{d}y =: M^1(h, \delta), \tag{36}$$

which has the same integral form as the first order Melnikov function (4). Then it can be viewed as the first order Melnikov function $M^1(h, \delta)$ of a new near-Hamiltonian system (9) with $i = 1$, and it has the expansion (11) near the homoclinic loop and the expansion (12) near the nilpotent centre. It follows from the equation (36) that the expression (34) and the expansion (11) with $i = 1$ are the same near the loop L_s, and that the expression (35) is consistent with the expansion (12) with $i = 1$ near the centre.

Comparing the coefficients in the expression (34) and in the expansion (11) with $i = 1$, as well as the coefficients in the expression (35) and in (12) with $i = 1$, one has for $\delta \in \overline{\Omega}^0$

(a) when $r = 1, 2, \cdots, [\frac{k_1-1}{2}]$,

$$B^1_{k_1 j+r} = -\left(\frac{r}{k_1} + \frac{3}{2} + j\right) B^0_{k_1(j+1)+r},$$

(b) when $r = [\frac{k_1+1}{2}]$,

$$B^1_{k_1 j+r} = (j+2)\, B^0_{k_1(j+1)+r},$$

(c) when $r = [\frac{k_1}{2}] + 2, \cdots, k_1$,

$$B^1_{k_1 j+r} = -\left(\frac{r-1}{k_1} + \frac{3}{2} + j\right) B^0_{k_1(j+1)+r},$$

(d)

$$C^1_j = -\left(\frac{2r+1}{k_2} + \frac{3}{2}\right) C^0_{\frac{k_2}{2}+j},$$

for all nonnegative integers j. Obviously, the conclusion (e) in Theorem 2 holds, which can be obtained from (d) above. Taking $j = 0$ for (a)–(c) above, one gets the proof of (a)–(b) and (d) in Theorem 2 for $l = 1$.

For $l > 1$, according to the foregoing proof and the conditions of Lemma 5, for $\delta \in \overline{\Omega}^{i-1}$ the conclusions (a)–(d) hold with the superscripts i and $i-1$ instead of the superscripts 1 and 0, respectively.

By induction, one can further prove that for $\delta \in \overline{\Omega}^{i-1}$

(a) when $r = 1, 2, \cdots, [\frac{k_1-1}{2}]$,

$$B^0_{k_1(i+j)+r} = \frac{(-1)^i (2k_1)^i}{\prod\limits_{s=j+1}^{i+j} (k_1 + 2(k_1 s + r))} B^i_{k_1 j+r},$$

(b) when $r = [\frac{k_1+1}{2}]$,

$$B^0_{k_1(i+j)+r} = \frac{1}{\prod\limits_{s=j+2}^{i+j+1} s} B^i_{k_1 j+r},$$

(c) when $r = [\frac{k_1}{2}] + 2, \cdots, k_1$,

$$B^0_{k_1(i+j)+r} = \frac{(-1)^i (2k_1)^i}{\prod\limits_{s=j+1}^{i+j} (k_1 + 2(k_1 s + r - 1))} B^i_{k_1 j+r},$$

(d)

$$C^0_{\frac{k_2}{2}i+j} = \frac{(2k_2)^i}{\prod\limits_{s=1}^{i} ((2s+1)k_2 + 4j + 2)} C^i_j,$$

with nonnegative integers j.

Statement (d) above implies the conclusion (e) in Theorem 2. Taking $j = 0$ in the above conclusions (a)–(c), it follows the conclusions (a)–(b) and (d) in Theorem 2. The proof is completed. □

Proof of Theorem 2. It follows from Lemma 5, together with the fact $\overline{\Delta}^{i-1} \subseteq \overline{\Omega}^{i-1}$ for $i = 1, 2, \cdots, l$. □

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Next we turn to think about the coefficients of the terms with higher degrees in the expansion of the first order Melnikov function near the homoclinic loop to the nilpotent cusp, called a cuspidal loop. Han et al. [27] first depicted the expansions of the first order Melnikov function near this kind of loop, and the first coefficients in the expansions for $k = 3$. In [3, 52, 56] the authors further studied the coefficients with lower degrees of the expansions for $k \geq 5$. The results are summarized as follows.

Lemma 6. *For the analytic near-Hamiltonian system* (9)*, assume that the unperturbed Hamiltonian has an oriented clockwise homoclinic loop to a nilpotent cusp at the origin, and satisfies* (2) *with $h_k < 0$, then the first order Melnikov function near and inside the cuspidal loop is given in* (11)*, and it near and outside the loop (see Fig. 1(A)) can be written in the form*

$$\begin{aligned} M^i(h,\delta) =& b_0^i + \sum_{j\geq 0}\sum_{r=kj+1}^{kj+\frac{k-1}{2}} b_r^i h^{\frac{r}{k}+\frac{1}{2}} + \sum_{j\geq 0} b^i_{kj+\frac{k+1}{2}} h^{j+1} \\ &+ \sum_{j\geq 0}\sum_{r=kj+\frac{k+1}{2}}^{(j+1)k-1} b^i_{r+1} h^{\frac{r}{k}+\frac{1}{2}}. \qquad 0 < h \ll 1, \end{aligned} \tag{37}$$

where

$$\begin{aligned} & b_0^i = B_0^i, && b_r^i = \frac{A^*_{r-1}}{A_{r-1}} B_r^i, \quad r = 1, \cdots, \left[\frac{k}{2}\right], \\ & b^i_{[\frac{k}{2}]+1} = B^i_{[\frac{k}{2}]+1} + \sum_{l=1}^{[\frac{k}{2}]} T_l^* B_l^i, && b_r^i = \frac{A^*_{r-2}}{A_{r-2}} B_r^i, \quad r = \left[\frac{k}{2}\right] + 2, \cdots, k, \end{aligned} \tag{38}$$

with the constants T_l^*, $A_r^* > 0$ for $0 \leq r < \frac{k}{2} - 1$ and $A_r^* < 0$, for $\frac{k}{2} - 1 < r < k - 1$.

Yang and Han [62] in 2024 provided further the relation between the coefficients of the terms with the same order in the asymptotic expansions of the first order Melnikov functions inside or outside the cuspidal loop. Moreover, the constants T_l^*'s are zeros in Lemma 6.

This lemma implies the information

$$\left\{\delta : B_r^i(\delta) = 0, 1 \leq r \leq k, r \neq \left[\frac{k}{2}\right] + 1\right\} = \left\{\delta : b_r^i(\delta) = 0, 1 \leq r \leq k, r \neq \left[\frac{k}{2}\right] + 1\right\}.$$

In the light of the idea in Theorem 2 and its proof, together with this last lemma, we can characterize the coefficients of the expansions (11) and (37) of the first order Melnikov functions near the cuspidal loop, as shown in the next results. Their proofs will be omitted.

Theorem 7. *For the analytic near-Hamiltonian system* (1)*, the following assumptions hold.*

- (i) *The unperturbed Hamiltonian one of system* (1) *has a nilpotent centre of order $\left[\frac{k_2}{2}\right] - 1$ and an oriented clockwise homoclinic loop to a nilpotent cusp of order $\left[\frac{k_1-1}{2}\right]$.*
- (ii) *The first order Melnikov function is of the form* (11) *with $i = 0$ near the loop, and is of the form* (12) *with $i = 0$ near the centre.*
- (iii) *There exist analytic functions $P_i(x, y, \delta)$ and $Q_i(x, y, \delta)$ for $i = 1, 2, \cdots, l$, such that for $\delta \in \overline{\Delta}^{i-1}$ the equations* (13) *hold over $U_2 := \bigcup_{h_c \leq h \leq h_s + \tau} L_h$ with a positive number $\tau \ll 1$.*

Then the statements (a), (b), (d) and (e) in Theorem 2 hold, and the coefficients of the expansions have the next relations.

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(a) *For* $r = 1, 2, \cdots, \frac{k_1-1}{2}$,

$$b^0_{k_1 i+r}\Big|_{\overline{\Delta}^{i-1}} = \frac{(2k_1)^i}{\prod_{j=1}^{i}((2j+1)k_1+2r)} b^i_r.$$

(b) *For* $r = \frac{k_1+1}{2}$,

$$b^0_{k_1 i+r}\Big|_{\overline{\Delta}^{i-1}} = \frac{1}{(i+1)!} b^i_r.$$

(c) *For* $r = \frac{k_1+3}{2}, \cdots, k_1$,

$$b^0_{k_1 i+r}\Big|_{\overline{\Delta}^{i-1}} = \frac{(2k_1)^i}{\prod_{j=1}^{i}((2j+1)k_1+2r-2)} b^i_r.$$

3. LIMIT CYCLE BIFURCATIONS NEAR A HOMOCLINIC LOOP AND A CENTRE

This section considers the limit cycle bifurcations of the families of periodic orbits near a centre and a homoclinic loop with a nilpotent singularity. If the singularity is a nilpotent saddle, the Hamiltonian system $(1)|_{\varepsilon=0}$ could have a family of periodic orbits either inside or outside the homoclinic loop. If the singularity is a nilpotent cusp, there could have two families of periodic orbits locating at both sides of the cuspidal loop, see Fig.1(A). In this last case we study limit cycle bifurcations in both sides of the cuspidal loop and near the centre for the near-Hamiltonian system (1), utilizing the properties of the coefficients of the first order Melnikov functions estabilished in Theorems 2 and 3 and Lemma 7. On the limit cycle bifurcation in a single side of a homoclinic loop and near a nilpotent centre will be directly demonstrated without a proof, since they can be obtained in a similar way as that in the case of the cuspidal loop.

Set $B_0 := B^0_0$. For $0 \le j \le \frac{k_2}{2} - 1$, $1 \le r \le k_1$ and $r \ne [\frac{k_1}{2}] + 1$, set

(39) $$C_{\frac{k_2}{2}i+j} := \begin{cases} C^0_j, & i = 0, \\ C^0_{\frac{k_2}{2}i+j}|_{\overline{\Delta}^{i-1}}, & i > 0, \end{cases} \qquad B_{k_1 i+r} := \begin{cases} B^0_r, & i = 0, \\ B^0_{k_1 i+r}|_{\overline{\Delta}^{i-1}}, & i > 0. \end{cases}$$

For $r = [\frac{k_1}{2}] + 1$, set

(40)
$$B_{k_1 i+r} := \frac{(-1)^{k_1-1}}{(i+1)!} \oint_{L_s} \left(\left(\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y} \right) - \sum_{l=0}^{[\frac{k_1}{2}]-1} \frac{1}{l!} \frac{\partial^l}{\partial x^l} \left(\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y} \right)(S, \delta) x^l \right)\Bigg|_{\varepsilon=0} \mathrm{d}t.$$

Denote

$$\mathbf{F}^i := \left(C^i_{11}, C^i_{21}, \cdots, C^i_{\frac{k_2}{2}-1,1} \right), \qquad \mathbf{F}_i := (\mathbf{F}^0, \mathbf{F}^1, \cdots, \mathbf{F}^i), \qquad \mathbf{F}_{-1} := \varnothing,$$

$$\mathbf{A}^i := \left(B^i_1, B^i_2, \cdots, B^i_{[\frac{k_1}{2}]}, B^i_{[\frac{k_1}{2}]+2}, \cdots, B^i_{k_1}, C^i_0, C^i_1, \cdots, C^i_{\frac{k_2}{2}-1} \right),$$

$$\mathbf{A}_i := \left(B_{k_1 i+1}, B_{k_1 i+2}, \cdots, B_{k_1 i+[\frac{k_1}{2}]}, B_{k_1 i+[\frac{k_1}{2}]+2}, \cdots, B_{k_1 i+k_1}, C_{\frac{k_2}{2}i}, \cdots, C_{\frac{k_2}{2}(i+1)-1} \right),$$

$$O_i := O\left(\left| B_{k_1 i+1}, B_{k_1 i+2}, \cdots, B_{k_1 i+[\frac{k_1}{2}]} \right| \right),$$

and

$$O^j := \begin{cases} O(|\mathbf{A}_0, \mathbf{A}_1, \cdots, \mathbf{A}_{j-1}, \mathbf{F}_{j-1}|), & j > 0, \\ 0, & j = 0. \end{cases}$$

Theorem 8. *Let the assumptions (i)–(ii) in Theorem 7 hold. If there exist nonnegative integers $n(\leq m)$, $r_1 \in [1, k_1], r_2 \in [0, \frac{k_2}{2} - 1]$ and $\delta_0 \in \mathbb{R}^d$, such that for B_j and C_j defined in (39)–(40),*

$$B_{k_1 n+r_1}(\delta_0) C_{\frac{k_2}{2}n+r_2}(\delta_0) \neq 0, \qquad \mathbf{F}_{n-1}(\delta_0) = \mathbf{0},$$

$$B_i(\delta_0) = C_j(\delta_0) = 0, \quad i = 0, 1, \cdots, k_1 n + r_1 - 1,\ j = 0, 1, \cdots, \frac{k_2}{2}n + r_2 - 1,$$

$$\text{Rank} \frac{\partial\left(B_0, B_1, \cdots, B_{k_1 n+r_1-1}, C_0, \cdots, C_{\frac{k_2}{2}n+r_2-1}, \mathbf{F}_{n-1}\right)}{\partial(\delta_1, \cdots, \delta_s)}(\delta_0) = (k_1 + k_2 - 1)n + r_1 + r_2.$$

Then system (1) can have $(2k_1 + \frac{k_2}{2} - 1)n + 2r_1 + r_2 - 1$ limit cycles near either the centre or the cuspidal loop for some (ε, δ) near $(0, \delta_0)$, being comprised of $(2k_1 - 1)n + 2r_1 - 1$ limit cycles near the cuspidal loop and $\frac{k_2}{2}n + r_2$ limit cycles near the centre.

Proof. Without loss of generality, we assume that $(-1)^{k_1 n+r_1}\ B_{k_1 n+r_1}(\delta_0) > 0$ for $r_1 < \frac{k_1+3}{2}$ and $(-1)^{\frac{k_2}{2}n+r_2} C_{\frac{k_2}{2}n+r_2}(\delta_0) > 0$. For unifying the notations, we denote by $B^i_{jl} := B^i_j$ for $l = 1$ and $B^i_{jl} := b^i_j$ for $l = 2$. It follows from the expansion (11) and Lemma 6 that the asymptotic expansions of the first order Melnikov function near the cuspidal loop can be uniformly written in

$$\begin{aligned} M^0_l(h, \delta) =& B^0_{0l} + |h|^{\frac{1}{2}} \sum_{j\geq 0} \sum_{r=k_1 j+1}^{k_1 j+\frac{k_1-1}{2}} B^0_{rl} |h|^{\frac{r}{k_1}} + \sum_{j\geq 0} B^0_{k_1 j+\frac{k_1+1}{2},l} h^{j+1} \\ &+ |h|^{\frac{1}{2}} \sum_{j\geq 0} \sum_{r=k_1 j+\frac{k_1+1}{2}}^{(j+1)k_1-1} B^0_{r+1,l} |h|^{\frac{r}{k_1}} \end{aligned} \tag{41}$$

for $0 < (-1)^l h \ll 1$ with $l \in \{1, 2\}$, and that the asymptotic expansion near the centre is given in (12) with $i = 0$.

To study the number of isolated zeros of the first order Melnikov function, treating B_i, C_j and $\mathbf{F}_{n-1}$ with $i = 0, 1, \cdots, k_1 n + r_1 - 1, j = 0, 1, \cdots, \frac{k_2}{2}n + r_2 - 1$ as free parameters, and write B^0_{il}'s and C^0_i's in terms of B_i's and C_i's. Then $B^0_{01} = B^0_{02} = B_0$, $C^0_0 = C_0$, and

$$\begin{aligned} B^0_{k_1 j+r,l} &= B^0_{k_1 j+r,l}\big|_{\overline{\Delta}^{j-1}} + O\left(|\mathbf{A}^0, \mathbf{A}^1, \cdots, \mathbf{A}^{j-1}, \mathbf{F}^0, \mathbf{F}^1, \cdots, \mathbf{F}^{j-1}|\right) \\ &= B^0_{k_1 j+r,l}\big|_{\overline{\Delta}^{j-1}} + O^j, \\ C^0_{\frac{k_2}{2}j+t} &= C^0_{\frac{k_2}{2}j+t}\Big|_{\overline{\Delta}^{j-1}} + O^j. \end{aligned}$$

Together with the properties in Theorem 7 and the relations (38), one further has

$$B^0_{k_1 j+r,l} = \begin{cases} B_{k_1 j+r} + O^j, & l = 1, \\ (-1)^j \frac{A^*_{r-1}}{A_{r-1}} B_{k_1 j+r} + O^j, & l = 2 \end{cases}$$

for $1 \leq r \leq [\frac{k_1}{2}]$,

$$B^0_{k_1 j+r,l} = B_{k_1 j+r} + O_j + O^j$$

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for $r = [\frac{k_1}{2}] + 1$,

$$B^0_{k_1 j + r, l} = \begin{cases} B_{k_1 j + r} + O^j, & l = 1, \\ (-1)^j \frac{A^*_{r-2}}{A_{r-2}} B_{k_1 j + r} + O^j, & l = 2 \end{cases}$$

for $[\frac{k_1}{2}] + 2 \le r \le k_1$, and

$$C^0_{\frac{k_2}{2} j + t} = C_{\frac{k_2}{2} j + t} + O^j$$

for $0 \le t \le \frac{k_2}{2} - 1$.

Therefore, the expansions (41) of the first order Melnikov function can be respectively written in

$$\begin{aligned} M^0_1(h, \delta) = B_0 &+ \sum_{j=0}^{n-1} \sum_{r=k_1 j+1}^{k_1 j + \frac{k_1-1}{2}} \left(B_r + O^j\right) |h|^{\frac{r}{k_1} + \frac{1}{2}} + \sum_{j=0}^{n-1} \left((-1)^{j+1} B_{k_1 j + \frac{k_1+1}{2}} + O_j + O^j\right) \\ &\cdot |h|^{j+1} + \sum_{j=0}^{n-1} \sum_{r=k_1 j + \frac{k_1+1}{2}}^{(j+1)k_1 - 1} \left(B_{r+1} + O_j\right) |h|^{\frac{r}{k_1} + \frac{1}{2}} + \sum_{r=k_1 n+1}^{k_1 n + r_1 - 1} \left(B_r + O^n\right) \\ &\cdot |h|^{\frac{r}{k_1} + \frac{1}{2}} + B^0_{k_1 n + r_1} |h|^{n + \frac{r_1}{k_1} + \frac{1}{2}} + o\left(|h|^{n + \frac{r_1}{k_1} + \frac{1}{2}}\right) \end{aligned}$$

near and inside the loop L_s,

$$\begin{aligned} M^0_2(h, \delta) = B_0 &+ \sum_{j=0}^{n-1} \sum_{r=k_1 j+1}^{k_1 j + \frac{k_1-1}{2}} \left((-1)^{j+1} \left|\frac{A^*_{r-1}}{A_{r-1}}\right| B_r + O^j\right) h^{\frac{r}{k_1} + \frac{1}{2}} + \sum_{j=0}^{n-1} \Big(B_{k_1 j + \frac{k_1+1}{2}} \\ &+ O_j + O^j\Big) h^{j+1} + \sum_{j=0}^{n-1} \sum_{r=k_1 j + \frac{k_1+1}{2}}^{(j+1)k_1 - 1} \left((-1)^{j+1} \left|\frac{A^*_{r-1}}{A_{r-1}}\right| B_{r+1} + O^j\right) \cdot h^{\frac{r}{k_1} + \frac{1}{2}} \\ &+ \sum_{r=k_1 n+1}^{k_1 n + r_1 - 1} \left((-1)^{n+1} \left|\frac{A^*_{r-1}}{A_{r-1}}\right| B_r + O^n\right) h^{\frac{r}{k_1} + \frac{1}{2}} \\ &+ b^0_{k_1 n + r_1} h^{n + \frac{r_1}{k_1} + \frac{1}{2}} + o\left(h^{n + \frac{r_1}{k_1} + \frac{1}{2}}\right) \end{aligned}$$

near and outside the loop, and

$$\begin{aligned} M^0(h, \delta) = &(h - h_c)^{\frac{k_2+2}{2k_2}} \left(\sum_{j=0}^{n-1} \sum_{r=\frac{k_2}{2} j}^{\frac{k_2}{2}(j+1) - 1} \left(C_r + O^j\right) (h - h_c)^{\frac{2r}{k_2}} + \sum_{r=\frac{k_2}{2} n}^{\frac{k_2}{2} n + r_2 - 1} \left(C_r + O^n\right) (h - h_c)^{\frac{2r}{k_2}} \right. \\ &\left. + C^0_{\frac{k_2}{2} n + r_2} (h - h_c)^{1 + \frac{2n + r_2}{k_2}} + o\left(|h - h_c|^{1 + \frac{2n + r_2}{k_2}}\right) \right) \end{aligned}$$

near the nilpotent centre for $0 < h - h_c \ll 1$, where $B^0_{k_1 n + r_1} = B_{k_1 n + r_1}(\delta_0) + \overline{O}$, $b^0_{k_1 n + r_1} = (-1)^n \frac{A^*_{k_1 n + r_1 - 2}}{A_{k_1 n + r_1 - 2}} B_{k_1 n + r_1}(\delta_0) + \overline{O}$, and $C^0_{\frac{k_2}{2} n + r_2} = C_{\frac{k_2}{2} n + r_2}(\delta_0) + \overline{O}$ with $\overline{O} = O(|B_0, B_1, \cdots,$

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$B_{k_1n+r_1-1}, C_0, C_1, \cdots, C_{\frac{k_2}{2}n+r_2-1}, \mathbf{F}_{n-1}|)$. Take the free parameters B_i's and C_i's such that

$$\begin{aligned}0 <&(-1)^{k_1j}B_{k_1j} \ll (-1)^{k_1j+1}B_{k_1j+1} \ll \cdots \ll (-1)^{k_1j+\frac{k_1-1}{2}}B_{k_1j+\frac{k_1-1}{2}} \ll \\ &(-1)^{k_1j+\frac{k_1-1}{2}+j}B_{k_1j+\frac{k_1+1}{2}} \ll (-1)^{k_1j+\frac{k_1+3}{2}}B_{k_1j+\frac{k_1+3}{2}} \ll \cdots \ll (-1)^{k_1j+k_1}B_{k_1j+k_1} \\ &\ll (-1)^{k_1n+1}B_{k_1n+1} \ll \cdots \ll (-1)^{k_1n+r_1-1}B_{k_1n+r_1-1} \ll (-1)^{k_1n+r_1}B_{k_1n+r_1}(\delta_0), \\ 0 <&(-1)^{r-1}C_{r-1} \ll (-1)^rC_r \ll (-1)^{\frac{k_2}{2}n+r_2}C_{\frac{k_2}{2}n+r_2}(\delta_0)\end{aligned}$$

satisfying $\max\left\{|B_{k_1(j+1)}|, |C_{\frac{k_2}{2}j+\frac{k_2}{2}-1}|\right\} \ll \min\left\{|B_{k_1(j+1)+1}|, |C_{\frac{k_2}{2}(j+1)}|\right\}$ with $0 \le j \le n-1$ and $1 \le r \le \frac{k_2}{2}n + r_2 - 1$. Then the first order Melnikov function $M(h, \delta)$ can have $k_1n + r_1$ simple zeros in a right neighborhood of $h = 0$ and $(k_1 - 1)n + r_1 - 1$ simple zeros in a left neighborhood of $h = 0$, and $\frac{k_2}{2}n + r_2$ simple zeros near $h = h_c$. Hence the near-Hamiltonian system (1) can have $(2k_1 + \frac{k_2}{2} - 1)n + 2r_1 + r_2 - 1$ limit cycles near either the cuspidal loop or the nilpotent centre, obtained by applying the implicit function theorem to the Poincaré map.

The proofs of the other cases are similar as those given above, and so are omitted. We finish the proof of the theorem. □

We remark that in practice, it is quite troublesome to find δ_0 that meets the conditions in Theorem 8. If the perturbations P_0 and Q_0 in system (1) are linear in the parameter δ, the Jacobian matrix in the theorem is independent of the parameter δ. Then we can simplify the conditions in Theorem 8 even further.

Corollary 1. *Let the assumptions (i) and (ii) in Theorem 7 hold. Suppose that there exist nonnegative integers $n(\le m)$, $r_1 \in [1, k_1]$ and $r_2 \in [0, \frac{k_2}{2} - 1]$, such that the Jacobian matrix*

$$\frac{\partial\left(B_0, B_1, \cdots, B_{k_1n+r_1-1}, C_0, \cdots, C_{\frac{k_2}{2}n+r_2-1}, \mathbf{F}_{n-1}, b\right)}{\partial(\delta_1, \cdots, \delta_s)}$$

with $b \in \left\{B_{k_1n+r_1}, C_{\frac{k_2}{2}n+r_2}\right\}$ is of full rank. Then system (1) can have $(2k_1 + \frac{k_2}{2} - 1)n + 2r_1 + r_2 - 1$ limit cycles near either the nilpotent centre or the cuspidal loop for suitable choices of the values of (ε, δ).

Based on the ideas of the proof to Theorem 8, one can establish similar results on limit cycle bifurcations near and inside a homoclinic loop to a nilpotent singularity and near a nilpotent centre. We present them here without a proof, because of the similarity and cumbersome.

Theorem 9. *Let the assumptions (i) and (ii) in Theorem 2 hold. Suppose that there exist nonnegative integers $n(\le m)$, $r_1 \in [1, k_1], r_2 \in [0, \frac{k_2}{2} - 1]$ and $\delta_0 \in \mathbb{R}^s$, such that the B_j's and C_j's defined in (39)–(40) satisfy the following conditions:*

$$B_{k_1n+r_1}(\delta_0)C_{\frac{k_2}{2}n+r_2}(\delta_0) \ne 0, \qquad \mathbf{F}_{n-1}(\delta_0) = \mathbf{0},$$

$$B_i(\delta_0) = C_j(\delta_0) = 0, \quad i = 0, 1, \cdots, k_1n + r_1 - 1,\ j = 0, 1, \cdots, \frac{k_2}{2}n + r_2 - 1,$$

$$\text{Rank}\frac{\partial\left(B_0, B_1, \cdots, B_{k_1n+r_1-1}, C_0, \cdots, C_{\frac{k_2}{2}n+r_2-1}, \mathbf{F}_{n-1}\right)}{\partial(\delta_1, \cdots, \delta_s)}(\delta_0) = (k_1 + k_2 - 1)n + r_1 + r_2.$$

Then system (1) can have $(k_1 + \frac{k_2}{2})n + r_1 + r_2$ limit cycles near either the centre or the homoclinic loop for suitable choices of the values of (ε, δ) near $(0, \delta_0)$.

Corollary 2. *Let the assumptions* (i) *and* (ii) *in Theorem 2 hold. If there exist nonnegative integers* $n(\leq m)$*,* $r_1 \in [1, k_1]$ *and* $r_2 \in [0, \frac{k_2}{2} - 1]$*, such that the Jacobian matrix*

$$\frac{\partial\left(B_0, B_1, \cdots, B_{k_1 n + r_1 - 1}, C_0, \cdots, C_{\frac{k_2}{2}n + r_2 - 1}, \mathbf{F}_{n-1}, b\right)}{\partial(\delta_1, \cdots, \delta_s)}$$

with $b \in \left\{B_{k_1 n + r_1}, C_{\frac{k_2}{2}n + r_2}\right\}$ *is of full rank. Then system (1) can have* $(k_1 + \frac{k_2}{2})n + r_1 + r_2$ *limit cycles near either the centre or the homoclinic loop for suitable choices of the values of* (ε, δ) *near* $(0, \delta_0)$*.*

For $0 \leq j \leq \frac{k_2}{2} - 1$, set

$$C_{\frac{k_2}{2}i+j} := \begin{cases} C_j^0, & i = 0, \\ C^0_{\frac{k_2}{2}i+j}|_{\overline{\Lambda}^{i-1}}, & i > 0. \end{cases} \tag{42}$$

Theorem 10. *Let the assumptions* (i) *and* (ii) *in Theorem 3 hold. Suppose that there exist nonnegative integers* $n(\leq m)$*,* $r(\leq \frac{k_2}{2} - 1)$ *and* $\delta_0 \in \mathbb{R}^s$*, such that the* C_j*'s defined in (40) satisfy*

$$\begin{aligned} &C_{\frac{k_2}{2}n+r}(\delta_0) \neq 0, \qquad \mathbf{F}_{n-1}(\delta_0) = \mathbf{0}, \\ &C_l(\delta_0) = 0, \quad l = 0, 1, \cdots, \frac{k_2}{2}n + r - 1, \\ &\mathrm{Rank}\frac{\partial\left(C_0, \cdots, C_{\frac{k_2}{2}n+r-1}, \mathbf{F}_{n-1}\right)}{\partial(\delta_1, \cdots, \delta_s)}(\delta_0) = (k_2 - 1)n + r. \end{aligned}$$

Then system (1) can have $\frac{k_2}{2}n + r$ *limit cycles near the centre for suitable choices of the values of* (ε, δ) *near* $(0, \delta_0)$*.*

Corollary 3. *Let the assumptions* (i) *and* (ii) *in Theorem 3 hold. If there exist nonnegative integers* $n(\leq m)$ *and* $r(\leq \frac{k_2}{2} - 1)$*, such that the Jacobian matrix*

$$\frac{\partial\left(C_0, \cdots, C_{\frac{k_2}{2}n+r-1}, \mathbf{F}_{n-1}, C_{\frac{k_2}{2}n+r}\right)}{\partial(\delta_1, \cdots, \delta_s)}$$

is of full rank, then system (1) can have $\frac{k_2}{2}n + r$ *limit cycles near the centre for some* (ε, δ)*.*

Denoted by $C^i_{r2} := C_{\frac{k_2}{2}i+r} - C^i_{r1}$, $\widetilde{\mathbf{F}}^i := \left(C^i_{1,s_{i1}}, C^i_{2,s_{i2}}, \cdots, C^i_{\frac{k_2}{2}-1, s_{i,\frac{k_2}{2}-1}}\right)$, $\widetilde{\mathbf{F}}_i := \left(\widetilde{\mathbf{F}}^0, \widetilde{\mathbf{F}}^1, \cdots, \widetilde{\mathbf{F}}^i\right)$ for $i \geq 0$ and $\widetilde{\mathbf{F}}_{-1} = \varnothing$.

We remark if there exists a maximal linearly independent group $(C_{\frac{k_2}{2}i+r}, C^i_{r,s_{ir}})$ with $s_{ir} \in \{0, 1, 2\}$, where $C^i_{r0} = \varnothing$, then Theorems 8-10 and Corollaries 1-3 still hold with the symbol $\widetilde{\mathbf{F}}$ instead of the symbol $\mathbf{F}$.

4. Applications

This section is a proof to Theorems 5 and 6, which are the applications of our results obtained in the previous sections to two concrete systems.

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4.1. **Proof of Theorem 5.** Consider the generalized Liénard system of the form:

$$\dot{x} = y, \qquad \dot{y} = -x^2(x-1)^3 + \varepsilon y f_m(x). \tag{43}$$

with a polynomial $f_m(x) = \sum_{i=0}^{m} b_i x^i$. Obviously, the Hamiltonian of the unperturbed system (43)$|_{\varepsilon=0}$ is $H(x,y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 + \frac{3}{4}x^4 - \frac{3}{5}x^5 + \frac{1}{6}x^6$, and its associated system has a nilpotent centre of order 1 at the point $(1,0)$ and a cuspidal loop L_s, which is contained in the level set $H(x,y) = 0$ and is homoclinic to the cusp of order 1 at the origin. By Theorem 1, the coefficients of the first order Melnikov function (11) with $i = 0$ and (33) satisfy

$$\begin{aligned}
&B_0 = \sum_{i=0}^{m} b_i I_i, && C_0 = 2\mathrm{B}\left(\frac{1}{4},\frac{3}{2}\right)\sum_{i=0}^{m} b_i,\\
&B_1 = 2^{\frac{3}{2}}3^{\frac{1}{3}}A_0 b_0, && C_1 = 2\mathrm{B}\left(\frac{3}{4},\frac{3}{2}\right)\sum_{i=0}^{m}\left(i^2+\frac{1}{5}i+\frac{41}{25}\right)b_i,\\
&B_2 = \sum_{i=1}^{m} b_i J_i, && C_{11}^0 = 2\mathrm{B}\left(\frac{3}{4},\frac{3}{2}\right)\sum_{i=0}^{m}\left(i(i-1)+\frac{41}{25}\right)b_i,\\
&B_3 = 2^{\frac{1}{2}}3^{\frac{2}{3}}A_1(3b_0+2b_1), &&
\end{aligned} \tag{44}$$

where the constants A_0, A_1 are given in Lemma 2, and

$$I_i = \oint_{L_s} y x^i \mathrm{d}x = 2\sqrt{2}\int_0^{x_0} x^{i+1}\sqrt{\frac{1}{3}x - \frac{3}{4}x^2 + \frac{3}{5}x^3 - \frac{1}{6}x^4}\,\mathrm{d}x, \qquad i = 0,1,\cdots,m,$$

$$J_i = \oint_{L_s} x^i \mathrm{d}t = \sqrt{2}\int_0^{x_0} \frac{x^{i-1}}{\sqrt{\frac{1}{3}x - \frac{3}{4}x^2 + \frac{3}{5}x^3 - \frac{1}{6}x^4}}\mathrm{d}x, \qquad i = 1,2,\cdots,m,$$

with $x_0 = \frac{(28+10\sqrt{10})^{\frac{1}{3}}}{10} - \frac{3}{5(28+10\sqrt{10})^{\frac{1}{3}}} + \frac{6}{5}$. Write the expressions (44) in the compact form

$$(B_0, B_1, B_2, B_3, C_0, C_1, C_{11}^0)^T = R_0(b_0, b_1, \cdots, b_m)^T, \tag{45}$$

with R_0 a $7\times(m+1)$ matrix of the form

$$R_0 = \begin{pmatrix}
I_0 & I_1 & I_2 & \cdots & I_m\\
2^{\frac{3}{2}}3^{\frac{1}{3}}A_0 & 0 & 0 & \cdots & 0\\
0 & J_1 & J_2 & \cdots & J_m\\
2^{\frac{1}{2}}3^{\frac{5}{3}}A_1 & 2^{\frac{3}{2}}3^{\frac{2}{3}}A_1 & 0 & \cdots & 0\\
2\mathrm{B}(\frac{1}{4},\frac{3}{2}) & 2\mathrm{B}(\frac{1}{4},\frac{3}{2}) & 2\mathrm{B}(\frac{1}{4},\frac{3}{2}) & \cdots & 2\mathrm{B}(\frac{1}{4},\frac{3}{2})\\
2\mathrm{B}(\frac{3}{4},\frac{3}{2})\cdot\frac{41}{25} & 2\mathrm{B}(\frac{3}{4},\frac{3}{2})\cdot\frac{71}{25} & 2\mathrm{B}(\frac{3}{4},\frac{3}{2})\cdot\frac{151}{25} & \cdots & 2\mathrm{B}(\frac{3}{4},\frac{3}{2})\cdot\left(m^2+\frac{1}{5}m+\frac{41}{25}\right)\\
2\mathrm{B}(\frac{3}{4},\frac{3}{2})\cdot\frac{41}{25} & 2\mathrm{B}(\frac{3}{4},\frac{3}{2})\cdot\frac{41}{25} & 2\mathrm{B}(\frac{3}{4},\frac{3}{2})\cdot\frac{91}{25} & \cdots & 2\mathrm{B}(\frac{3}{4},\frac{3}{2})\cdot\left(m(m-1)+\frac{41}{25}\right)
\end{pmatrix}.$$

The condition $\overline{\Delta}^0 = \Delta^0$ in Theorem 2 yields $x^2|f_m(x)$ and $(x-1)^3|f_m(x)$. It follows from

$$P_0 = 0 \quad \text{and} \quad Q_0 = y f_m(x)$$

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that equation (13) with $i = 1$ holds by ordering

$$P_1 = \frac{f_m(x)|_{\overline{\Delta}^0}}{x^2(x-1)^3},$$

which implies the expression

$$P_1 = \sum_{j=0}^{m-5} b_j^1 x^j,$$

where

$$b_j^1 = \sum_{l=5}^{m-j} \frac{(l-3)(l-4)}{2} b_{l+j} \quad \text{for} \quad j = 0, 1, 2, \cdots, m-5.$$

Set

$$P_{i-1} := \sum_{j=0}^{n_{i-1}} b_j^{i-1} x^j \quad \text{and} \quad Q_{i-1} := 0,$$

with $n_{i-1} = m + 1 - 6(i-1)$ for $i \geq 2$. Manipulating as the above paragraph, it follows from the condition $\overline{\Delta}^i$ and the equality (13) that

$$P_i = \sum_{j=0}^{n_i} b_j^i x^j \quad \text{and} \quad Q_i = 0,$$

where

$$b_j^i = \sum_{l=6}^{n_i - j + 6} \frac{(l-4)(l-5)}{2} (l+j) b_{l+j}^{i-1}$$

for $j = 0, 1, 2, \cdots, n_i$ and $n_i = n_{i-1} - 6 = m + 1 - 6i$.

Moreover, one has the next relations on the coefficients

$$(46) \qquad (b_1^i, b_2^i, \cdots, b_{m-6i+1}^i)^T = \prod_{j=1}^{i} A_j^i (b_{6i}, b_{6i+1}, \cdots, b_m)^T, \quad i = 1, 2, \cdots, \left[\frac{m-1}{6}\right]$$

with A_j^i being the upper triangular matrices of order $m - 6i + 1$

$$A_j^i = \begin{pmatrix} 1 \cdot (6j+1) & 3 \cdot (6j+2) & \cdots & (m-6(i-j)+1) \cdot \frac{(m-6i+2)(m-6i+1)}{2} \\ 0 & 1 \cdot (6j+1) & \cdots & (m-6(i-j)+1) \cdot \frac{(m-6i+1)(m-6i)}{2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (m-6(i-j)+1) \end{pmatrix}$$

for $1 \leq j \leq i-1$, and

$$A_i^i = \begin{pmatrix} 1 & 3 & 6 & \cdots & \frac{1}{2}(m-6i+2)(m-6i+1) \\ 0 & 1 & 3 & \cdots & \frac{1}{2}(m-6i+1)(m-6i) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

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On the other hand, we get from (12) and (33) that for $1 \le i \le \left[\frac{m-1}{6}\right]$,

$$B_1^i = 2^{\frac{3}{2}} 3^{\frac{1}{3}} A_0 b_1^i, \qquad C_0^i = 2\mathrm{B}\left(\frac{1}{4}, \frac{3}{2}\right) \sum_{j=1}^{m-6i+1} j b_j^i,$$

$$B_2^i|_{B_1^i=0} = \sum_{j=2}^{m-6i+1} j J_{j-1} b_j^i, \qquad C_1^i = 2\mathrm{B}\left(\frac{3}{4}, \frac{3}{2}\right) \sum_{j=1}^{m-6i+1} j \left(j^2 - \frac{9}{5}j + \frac{61}{25}\right) b_j^i,$$

$$B_3^i = 2^{\frac{1}{2}} 3^{\frac{2}{3}} A_1 (3b_1^i + 4b_2^i), \qquad C_{11}^i = 2\mathrm{B}\left(\frac{3}{4}, \frac{3}{2}\right) \sum_{j=1}^{m-6i+1} j \left(j^2 - 3j + \frac{91}{25}\right) b_j^i.$$

Namely, for $1 \le i \le \left[\frac{m-1}{6}\right]$,

$$(B_1^i, B_2^i|_{B_1^i=0}, B_3^i, C_0^i, C_1^i, C_{11}^i)^T = V^i (b_1^i, b_2^i, \cdots, b_{m-6i+1}^i)^T, \tag{47}$$

with the $6 \times (m - 6i + 1)$ matrices

$$V^i = \begin{pmatrix} 2^{\frac{3}{2}} 3^{\frac{1}{3}} A_0 & 0 & 0 & \cdots & 0 \\ 0 & 2J_1 & 3J_2 & \cdots & n_i J_{m-6i} \\ 2^{\frac{1}{2}} 3^{\frac{5}{3}} A_1 & 2^{\frac{5}{2}} 3^{\frac{2}{3}} A_1 & 0 & \cdots & 0 \\ 2\mathrm{B}\left(\frac{1}{4}, \frac{3}{2}\right) & 4\mathrm{B}\left(\frac{1}{4}, \frac{3}{2}\right) & 6\mathrm{B}\left(\frac{1}{4}, \frac{3}{2}\right) & \cdots & 2n_i\mathrm{B}\left(\frac{1}{4}, \frac{3}{2}\right) \\ \frac{82}{25}\mathrm{B}\left(\frac{3}{4}, \frac{3}{2}\right) & \frac{284}{25}\mathrm{B}\left(\frac{3}{4}, \frac{3}{2}\right) & \frac{906}{25}\mathrm{B}\left(\frac{3}{4}, \frac{3}{2}\right) & \cdots & 2n_i(n_i^2 - \frac{9}{5}n_i + \frac{61}{25})\mathrm{B}\left(\frac{3}{4}, \frac{3}{2}\right) \\ \frac{82}{25}\mathrm{B}\left(\frac{3}{4}, \frac{3}{2}\right) & \frac{164}{25}\mathrm{B}\left(\frac{3}{4}, \frac{3}{2}\right) & \frac{546}{25}\mathrm{B}\left(\frac{3}{4}, \frac{3}{2}\right) & \cdots & 2n_i(n_i^2 - 3n_i + \frac{91}{25})\mathrm{B}\left(\frac{3}{4}, \frac{3}{2}\right) \end{pmatrix}.$$

According to Theorem 2, together with the definitions of B_{3i+r} and C_i in (39)–(40), one gets for $1 \le i \le \left[\frac{m-1}{6}\right]$,

$$(B_{3i+1}, B_{3i+2}, B_{3i+3}, C_{2i}, C_{2i+1}, C_{11}^i)^T = U^i (B_1^i, B_2^i|_{B_1^i=0}, B_3^i, C_0^i, C_1^i, C_{11}^i)^T, \tag{48}$$

with the diagonal matrices of order 6

$$U^i = \mathrm{diag}\left(\frac{(-6)^i}{\prod_{j=1}^{i}(6j+5)}, \frac{1}{(i+1)!}, \frac{(-6)^i}{\prod_{j=1}^{i}(6j+7)}, \frac{4^i}{\prod_{j=1}^{i}(4j+3)}, \frac{4^i}{\prod_{j=1}^{i}(4j+5)}, 1\right).$$

Substituting (46) and (47) into (48) gives for $1 \le i \le \left[\frac{m-1}{6}\right]$,

$$(B_{3i+1}, B_{3i+2}, B_{3i+3}, C_{2i}, C_{2i+1}, C_{11}^i)^T = U^i V^i \prod_{j=1}^{i} A_j^i (b_{6i}, b_{6i+1}, \cdots, b_m)^T. \tag{49}$$

Set $m - 1 = 6n + r$ with $r \le 6$ and n nonnegative integers. According to (45) and (49) one gets by means of Python programme that all Jacobian matrices $\frac{\partial G}{\partial(b_0, b_1, \cdots, b_m)}$ are of full rank with $m \le 300$, where

- for $1 \le r \le 3$, $G = (B_0, B_1, \cdots, B_{3n+r-1}, C_{11}^0, C_{11}^1, \cdots, C_{11}^{n-1}, C_0, C_1, \cdots, C_{2n-1}, b)$ with $b \in \{B_{3n+r}, C_{2n}\}$,
- for $4 \le r \le 6$, $G = (B_0, B_1, \cdots, B_{3n+2}, C_{11}^0, C_{11}^1, \cdots, C_{11}^{n-1}, C_0, C_1, \cdots, C_{2n}, b)$ with $b \in \{B_{3n+3}, C_{2n+1}\}$.

In the light of Theorem 8 or Corollary 1, the Liénard differential system (43) can have

- $7n+2r-1$ limit cycles for $1 \le r \le 3$, obtained by taking $r_1 = r$ and $r_2 = 0$ in Theorem 8;
- $7n+6$ limit cycles for $4 \le r \le 6$, obtained by taking $r_1 = 3$ and $r_2 = 1$ in Theorem 8.

The proof is completed. □

4.2. **Proof of Theorem 6.** Consider the near-Hamiltonian system of the form:

$$\dot{x} = y + \varepsilon P_0(x,y), \qquad \dot{y} = -x^3(1-x-x^2) + \varepsilon Q_0(x,y). \tag{50}$$

with the perturbed polynomials $P_0(x,y)$ and $Q_0(x,y)$ of degree m. The corresponding Hamiltonian function is $H(x,y) = \frac{1}{2}y^2 + \frac{1}{4}x^4 - \frac{1}{5}x^5 - \frac{1}{6}x^6$, whose associated system has a nilpotent centre of order 1 at the origin.

Set $G^0(x,y) := \frac{\partial P_0}{\partial x} + \frac{\partial Q_0}{\partial y} = \sum_{i+j=0}^{m-1} c_{ij}x^i y^j$. It follows from Theorem 1 and (39) that

$$C_0 = 2\mathrm{B}\left(\frac{1}{4},\frac{3}{2}\right)c_{00}, \qquad C_1 = 2\mathrm{B}\left(\frac{3}{4},\frac{3}{2}\right)\left(\frac{46}{25}c_{00} - \frac{3}{5}c_{10} + 2c_{20}\right),$$

$$C_{11}^0 = 2\mathrm{B}\left(\frac{3}{4},\frac{3}{2}\right)\left(\frac{46}{25}c_{00} + 2c_{20}\right),$$

which give

$$(C_0, C_1, C_{11}^0)^T = R_0(c_{00}, c_{10}, c_{20})^T, \tag{51}$$

with R_0 a 3×3 matrix of the form

$$R_0 = \begin{pmatrix} 2\mathrm{B}\left(\frac{1}{4},\frac{3}{2}\right) & 0 & 0 \\ \frac{92}{25}\mathrm{B}\left(\frac{3}{4},\frac{3}{2}\right) & -\frac{6}{5}\mathrm{B}\left(\frac{3}{4},\frac{3}{2}\right) & 4\mathrm{B}\left(\frac{3}{4},\frac{3}{2}\right) \\ \frac{92}{25}\mathrm{B}\left(\frac{3}{4},\frac{3}{2}\right) & 0 & 4\mathrm{B}\left(\frac{3}{4},\frac{3}{2}\right) \end{pmatrix}. \tag{52}$$

The condition $\overline{\Lambda}^0 = \Lambda^0$ of Theorem 3 implies $\frac{\partial^l}{\partial x^l}G^0(0,0) = l!c_{l0} = 0$ for $l = 0,1,2$. It follows from the expression of G^0 that the equality (13) with $i = 1$ holds by ordering

$$P_1 = \frac{G^0(x,y)|_{\overline{\Lambda}^0}}{x^3(1-x-x^2)} \quad \text{and} \quad Q_1 = \frac{1}{y}(G^0(x,y) - G^0(x,0)),$$

which implies the expressions

$$P_1 = \sum_{i=0}^{\infty} a_{i0}^1 x^i \quad \text{and} \quad Q_1 = \sum_{i+j=0}^{m-2} b_{ij}^1 x^i y^j,$$

with $a_{i0}^1 = \sum_{s=0}^{i} c_{s+3,0}d_{i-s}$ for $s \le m-4$, $a_{i0}^1 = \sum_{s=0}^{m-4} c_{s+3,0}d_{i-s}$ for $s \ge m-3$, and $b_{ij}^1 = c_{i,j+1}$ for $0 \le i+j \le m-2$, where we have used the expansion

$$\frac{1}{1-x-x^2} = \sum_{i=0}^{\infty} x^i d_i, \quad d_i = \sum_{j=[\frac{i+1}{2}]}^{i} \frac{j!}{(i-j)!(2j-i)!}. \tag{53}$$

Set

$$P_{l-1} := \sum_{i=0}^{\infty} a_{i0}^{l-1}x^i \quad \text{and} \quad Q_{l-1} := \sum_{i+j=0}^{n_{l-1}} b_{ij}^{l-1}x^i y^j$$

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with $n_{l-1} = m - 2l + 2$ for $l \geq 2$. Simple calculations show that

$$G^{l-1}(x,y) := \frac{\partial P_{l-1}}{\partial x} + \frac{\partial Q_{l-1}}{\partial y} = \sum_{i+j=0}^{n_{l-1}-1} c_{ij}^{l-1} x^i y^j + \sum_{i=n_{l-1}}^{\infty} c_{i0}^{l-1} x^i,$$

where

$$c_{ij}^{l-1} = \begin{cases} (i+1)a_{i+1,0}^{l-1} + \bar{b}_{i1}^{l-1}, & j = 0, \\ (j+1)b_{i,j+1}^{l-1}, & j \geq 1,\ 1 \leq i+j \leq n_{l-1} - 1, \end{cases}$$

with $\bar{b}_{i1}^{l-1} = b_{i1}^{l-1}$ for $0 \leq i \leq n_{l-1} - 1$ and $\bar{b}_{i1}^{l-1} = 0$ for $i \geq n_{l-1}$.

Recursive manipulation as above applies to the condition $\overline{\Lambda}^l$ and the equality (13), one gets

$$P_l = \sum_{i=0}^{\infty} a_{i0}^l x^i \quad \text{and} \quad Q_l = \sum_{i+j=0}^{n_l} b_{ij}^l x^i y^j,$$

where $a_{i0}^l = \sum_{s=0}^{i} c_{s+3,0}^{l-1} d_{i-s}$ with d_i given in (53) for all i's and $b_{ij}^l = (j+2)b_{i,j+2}^{l-1}$ for $0 \leq i+j \leq n_l$. Further calculations via induction yield

$$(c_{00}^1, c_{10}^1, c_{20}^1)^T = \begin{cases} (c_{02}, c_{12}, c_{22})^T, & m = 3, \\ a_1(c_{30}, c_{40}, \cdots, c_{m-1,0})^T + (c_{02}, c_{12}, c_{22})^T, & 4 \leq m \leq 6, \\ A_1(c_{30}, c_{40}, c_{50}, c_{60})^T + (c_{02}, c_{12}, c_{22})^T, & m \geq 7, \end{cases} \tag{54}$$

and for $l = 2, 3, \cdots, \left[\frac{2[\frac{m}{2}]-2}{3}\right] + \left[\frac{m+1}{2}\right] - 1$,

$$\begin{aligned} (c_{00}^l, c_{10}^l, c_{20}^l)^T =& \sum_{i=1}^{l-1} (2i-1)!! \prod_{j=0}^{l-i} A_j (c_{3,2i}, c_{4,2i}, \cdots, c_{4(l-i)+2,2i})^T \\ &+ (2i-1)!!(c_{0,2l}, c_{1,2l}, c_{2,2l})^T \\ &+ \begin{cases} \prod_{j=0}^{l} A_j (c_{30}, c_{40}, \cdots, c_{4l+2,0})^T, & 4l \leq m-3, \\ \prod_{j=0}^{l-1} A_j a_l (c_{30}, c_{40}, \cdots, c_{m-1,0})^T, & 4l \geq m-2, \end{cases} \end{aligned} \tag{55}$$

where A_0 is the identity matrix of order 3, A_1 is the 3×4 matrix

$$A_1 = \begin{pmatrix} d_1 & d_0 & 0 & 0 \\ 2d_2 & 2d_1 & 2d_0 & 0 \\ 3d_3 & 3d_2 & 3d_1 & 3d_0 \end{pmatrix},$$

A_j's are the $(4j-4) \times 4j$ matrices

$$A_j = \begin{pmatrix} 4d_4 & 4d_3 & 4d_2 & \cdots & 0 \\ 5d_5 & 5d_4 & 5d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (4j-1)d_{4j-1} & (4j-1)d_{4j-2} & (4j-1)d_{4j-3} & \cdots & (4j-1)d_0 \end{pmatrix}$$

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for $2 \le j \le l$, and a_l's are the $(4l-4)\times(m-3)$ matrices

$$a_l = \begin{pmatrix} 4d_4 & 4d_3 & 4d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (m-4)d_{m-4} & (m-4)d_{m-5} & (m-4)d_{m-6} & \cdots & (m-4)d_0 \\ (m-3)d_{m-3} & (m-3)d_{m-4} & (m-3)d_{m-5} & \cdots & (m-3)d_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (4l-1)d_{4l-1} & (4l-1)d_{4l-2} & (4l-1)d_{4l-3} & \cdots & (4l-1)d_{4l-m+3} \end{pmatrix}$$

for $l \ge 2$, and $a_1 = (a^1_{ij})$ is the $3\times(m-3)$ matrix with $a^1_{ij} = id_{i+1-j}$ for $i+1 \ge j$, and $a^1_{ij} = 0$ for $i+1 < j$. We remark that for compactness to notations, we have introduced $c_{ij} = 0$ for $i+j \ge m$ in (54)–(55).

By Theorem 1 and (12), one has for $l = 1, 2, \cdots, \left[\frac{2[\frac{m}{2}]-2}{3}\right] + \left[\frac{m+1}{2}\right] - 1$,

$$(C^l_0, C^l_1, C^l_{11})^T = R_0(c^l_{00}, c^l_{10}, c^l_{20})^T \tag{56}$$

with the matrix R_0 given in (52). According to Theorem 3 and the definitions of C_{2i+j}'s in (42), one further has for $1 \le l \le \left[\frac{2[\frac{m}{2}]-2}{3}\right] + \left[\frac{m+1}{2}\right] - 1$,

$$(C_{2l}, C_{2l+1}, C^l_{11})^T = U_l(C^l_0, C^l_1, C^l_{11})^T \tag{57}$$

with the diagonal matrices of order 3

$$U_l = \text{diag}\left(\frac{4^i}{\prod_{j=1}^{i}(4j+3)}, \frac{4^i}{\prod_{j=1}^{i}(4j+5)}, 1\right).$$

Substituting (54) and (55) into (56) and (57) gives

$$(C_2, C_3, C^1_{11})^T = U_1R_0(c_{02}, 0, 0)^T \tag{58}$$

for $m = 3$, and

$$\begin{aligned}(C_{2l}, C_{2l+1}, C^l_{11})^T =& \sum_{i=1}^{l-1}(2i-1)!!U_lR_0\prod_{j=0}^{l-i}A_j(c_{3,2i}, c_{4,2i}, \cdots, c_{4(l-i)+2,2i})^T \\ &+ (2i-1)!!U_lR_0(c_{0,2l}, c_{1,2l}, c_{2,2l})^T \\ &+ \begin{cases} U_lR_0\prod_{j=0}^{l}A_j(c_{30}, c_{40}, \cdots, c_{4l+2,0})^T, & 4l \le m-3, \\ U_lR_0\prod_{j=0}^{l-1}A_ja_l(c_{30}, c_{40}, \cdots, c_{m-1,0})^T, & 4l \ge m-2 \end{cases}\end{aligned} \tag{59}$$

for $m \ge 4$ and $1 \le l \le \left[\frac{2[\frac{m}{2}]-2}{3}\right] + \left[\frac{m+1}{2}\right] - 1$, where the the summation $\sum_{i=1}^{l-1}$ is null if $l = 1$, and $c_{ij} = 0$ if $i+j \ge m$.

One can check easily from the formulas (51) and (58)–(59) that C_l's and C^l_{11}'s depend merely on the parameters $c_{i,2j}$ with $i+2j \le m-1$. Set $E_m := \{c_{i,2j} : i+2j \le m-1\}$. Observe that E_m contains $[\frac{m+1}{2}][\frac{m+2}{2}]$ elements. Set $3[\frac{m+1}{2}] + 2[\frac{m}{2}] - 4 = 3n + r$ with $r \le 2$ and n nonnegative integers.

For $m = 1, 2$, it follows from (51) that the Jacobian matrices $\frac{\partial(C_0,\cdots,C_m)}{\partial E_m}$ are of full rank.

https://doi.org/10.4153/S0008414X25101703 Published online by Cambridge University Press

For $m \geq 3$, it follows from (51) and (57), together with Python programme that all Jacobian matrices $\frac{\partial G}{\partial E_m}$ are of full rank with $m \leq 22$, where $G = (C_0, C_1, \cdots, C_{2n+r-1}, C_{11}^0, C_{11}^1, \cdots, C_{11}^{n-1})$. These together with Theorem 10 or Corollary 3 prove that the near-Hamiltonian system (50) can have $2n + r - 1$ limit cycles.

This proves the theorem. □

We remark that Theorems 5 and 6 both have limitations on the degree m of the perturbed polynomials. These have been used in calculations of the ranks of the Jacobian matrices, supported by two Python programs available in our GitHub repository: https://github.com/grass2sturdy/Rank. Theoretically, these calculations work by Python programmes for any m. But in practice it depends on memory of computers.

5. Conclusions

A lot of papers studied the coefficients in the expansion of the first order Melnikov function near a homoclinic loop with a nilpotent singularity in a near-Hamiltonian system. Based on the existing results, Yang et al. [63] and Wei and Zhang [52] presented the coefficients of the terms with degree less than 2 in the case of a nilpotent singularity of arbitrary order. Moreover, Yang and Han [62] pushed further the relation between the coefficients of the terms with the same order in the expansions inside or outside a cuspidal loop. The characteristics of the coefficients, of the terms with degree greater than or equal to 2, were developed further and exhibited in [52, 64] provided that the centre is elementary. If the centre of the unperturbed Hamiltonian system is nilpotent, as well as the homoclinic loop approaches also to a nilpotent singularity, the problem remains open. This paper tackles this problem.

The work in this paper presents the accurate expressions of the first $\frac{k}{2}$ coefficients in the asymptotic expansion of the first order Melnikov function near a nilpotent centre of order k, and characterizes its higher order coefficients, which are given in Theorems 1 and 3, respectively. Based on these results, our primary motivation is to depict the features of the higher order coefficients in the first order Melnikov function near a homoclinic loop to a nilpotent singularity of arbitrary order, in virtual of the nilpotent centre of arbitrary order inside the loop that the unperturbed Hamiltonian system has. The related results are given in Theorem 2. On account of these established results, we further investigate the limit cycle bifurcations near either a nilpotent centre or a homoclinic loop to a nilpotent singularity.

At last, we apply our results to two concrete systems, an $(m+1)$th order Liénard differential system and an mth order near-Hamiltonian differential system with a hyperelliptic Hamiltonian, for achieving more number of limit cycles, which are bifurcated from periodic orbits either near a nilpotent centre or near a homoclinic loop to a nilpotent singularity.

Acknowledgments

This work is partially supported by National Key R&D Program of China grant number 2022YFA1005900.

The first author is partially supported by NSF of Zhejiang Province Grant Number LY24A010009, by NNSF of China Grant Numbers 12471156 and 12371167, and by the Shanghai Frontier Research Center of Modern Analysis. The second author is partially supported by NNSF of China Grant Number 11931016. The fourth author is partially supported by NNSF of China Grant Number 12471169.

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https://doi.org/10.4153/S0008414X25101703 Published online by Cambridge University Press