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ON THE TIME CONSTANT AND PATH LENGTH OF FIRST-PASSAGE PERCOLATION

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Abstract

Let U be the distribution function of the passage time of an individual bond of the square lattice, and let p_T be the critical probability above which the expected size of the open component of the origin (in the usual bond percolation) is infinite. It is shown that if (*)$U(0-)=0$, $U(0)<p_T$, then there exist constants $0<a$, $C_1<\infty$ such that $P\{\exists$ a self-avoiding path of at least n steps starting at the origin and with passage time $\leqq an\}\leqq 2\exp(-C_1 n)$.

From this it follows that under (*) the time constant $\mu(U)$ of first-passage percolation is strictly positive and that for each $c>0$ $\limsup (1/n)N_n(c)<\infty$, where $N_n(c)$ is the maximal number of steps in the paths starting at the origin with passage time at most cn.

FIRST-PASSAGE PERCOLATION; BERNOULLI PERCOLATION; TIME CONSTANT; ROUTE LENGTH

1. Statement of results

First-passage percolation was first studied by Hammersley and Welsh [2]. More recent results can be found in the monograph [5] of Smythe and Wierman. We generally follow the notation of [5]. The lattice L of integral points in the plane is viewed as a graph with vertices the points $v=(v', v'')$ of $\mathbb{Z}\times\mathbb{Z}$, and edges or bonds the line segments between adjacent points. Here v and w are called adjacent if $v'=w'$, $|v''-w''|=1$ or if $|v'-w'|=1$, $v''=w''$. All edges are undirected; e and f will denote generic edges. A path on L from v to w is a sequence $(v_0, e_1, v_1, \cdots, e_n, v_n)$ with each v_i a vertex in L and such that $v_0=v$, $v_n=w$ and v_{i+1} is adjacent to v_i, and with e_i the edge connecting v_{i-1} and v_i. The path is called self-avoiding if $v_i\neq v_j$ for $i\neq j$.

To each edge e is assigned a random variable $X(e)$, called the passage time of e. It is assumed that all $X(e)$, $e\in L$, are independent identically distributed with distribution function U, satisfying

$$U(0-)=0. \tag{1.1}$$

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Thus all $X(e)$ are non-negative w.p. 1. The passage time of the path $r = (v_0, e_1, \cdots, v_n)$ is defined as

$$t(r) = \sum_{i=1}^{n} X(e_i), \tag{1.2}$$

and its length, $l(r)$, as n. As in [5] we set

(1.3) $a_{0,n} = \inf\{t(r)$: r is a self-avoiding path from $(0, 0)$ to $(n, 0)\}$,

(1.4) $b_{0,n} = \inf\{t(r)$: r is a self-avoiding path from $(0, 0)$ to (n, m) for some $m\}$

(1.5) $t_{0,n} = \inf\{t(r)$: $r = (v_0, e_1, \cdots, e_k, v_k)$ is a self-avoiding path from $(0, 0)$ to $(n, 0)$ with $0 < v_i' < n$ for $1 \leqq i \leqq k-1\}$,

(1.6) $s_{0,n} = \inf\{t(r)$: $r = (v_0, e_1, \cdots, e_k, v_k)$ is a self-avoiding path from $(0, 0)$ to (n, m) for some m with $0 < v_i' < n$ for $1 \leqq i \leqq k-1\}$.

It is proved in [5], Chapter 5, that if (1.1) holds and

$$\int_{[0,\infty)} x \, dU(x) < \infty, \tag{1.7}$$

then there exists a constant $\mu(U)$ such that

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{0,n}}{n} = \lim_{n\to\infty} \frac{b_{0,n}}{n} = \lim_{n\to\infty} \frac{t_{0,n}}{n} = \lim_{n\to\infty} \frac{s_{0,n}}{n} \\ = \mu(U) \text{ w.p. 1 and in } L^1. \end{aligned} \tag{1.8}$$

Recently Wierman [6] and Cox and Durrett [1] have shown that even without any moment conditions on U there exists a constant $\mu(U)$ such that $(1/n)a_{0,n} \to \mu(U)$ in probability. $\mu(U)$ is called the time constant of U. Note that under (1.1) all edges have non-negative passage times so that from any path r we can 'remove loops' to obtain a self-avoiding path $\tilde{r}$ with the same initial and end point as r but $t(\tilde{r}) \leqq t(r)$. Thus under (1.1) the restriction to self-avoiding r in (1.3)–(1.6) is superfluous. However, in Section 4 we shall also need (1.8) for some distributions U which do not satisfy (1.1) and for such U the proper definitions are as given in (1.3)–(1.6) (cf. [5], p. 64).

To formulate our results we also need the critical probability p_T which is defined in terms of Bernoulli percolation as follows. For our purposes, a Bernoulli percolation is a family of independent identically distributed random variables $\{X(e): e \in L\}$ as above with U a Bernoulli distribution:

$$U(\{0\}) = p, \qquad U(\{1\}) = q = 1 - p \tag{1.9}$$

($U(A)$ is the mass assigned by U to the set A). It is customary to call e open (closed) if $X(e) = 0$ (respectively $X(e) = 1$). A path $r = (v_0, e_1, \cdots, e_k, v_k)$ is open if each e_i in r is open. Set

$V(p) = E\{$number of edges in L which belong to an open path starting at the origin$\}$

$$= \sum_{e \in L} P\{\exists \text{ open path starting at the origin and containing } e\}.$$

Note that in terms of passage times governed by the distribution (1.9) we can also write

$$V(p) = \sum_{e \in L} P\{\exists \text{ path } r \text{ starting at the origin and containing } e \text{ with } t(r) = 0\}. \tag{1.10}$$

Now

$$p_T = \inf\{p : V(p) = \infty\}. \tag{1.11}$$

Our principal result is the following.

Theorem 1. *If* (1.1) *holds and*

$$U(0) < p_T, \tag{1.12}$$

then there exist constants $0 < a$, $C_1 < \infty$ *such that*

$$P\{\exists \text{ self-avoiding path } r \text{ starting at the origin with } l(r) \geqq n \text{ and } t(r) \leqq an\} \leqq 2e^{-C_1 n}. \tag{1.13}$$

Remark 1. Under (1.1), $U(0)$ is the atom of U at 0. Russo [3], Theorem 2 (see also Seymour and Welsh [4] and Smythe and Wierman [5], Section 3.4–3.6) states that for the Bernoulli percolation with U as in (1.9) and $p < p_T$ one has for each $\varepsilon > 0$

$$P\{\exists \text{ path } r = (v_0, e_1, \cdots, e_k, v_k) \text{ starting at } v_0(0,0) \text{ and ending at some } v_k \text{ with } |v_k'| \geqq 3^n \text{ or } |v_k''| \geqq 3^n\} \leqq \varepsilon^n \tag{1.14}$$

for n sufficiently large. (1.14) corresponds to the estimate $n^{(\log 3)^{-1} \log \varepsilon}$ for the left-hand side of (1.13) when $a = 0$. Clearly (1.13) is a considerable improvement of (1.14).

We also prove the following.

Theorem 2. *If* (1.1) *and* (1.12) *hold, then for* $\theta = a$, b, s *or* t

$$\liminf \frac{\theta_{0,n}}{n} \geqq b > 0 \text{ w.p. 1, } \textit{for some } b = b(U) > 0. \tag{1.15}$$

In particular, $\mu(U) \geqq b$.

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Of course Theorem 2 is immediate from Theorem 1, with $b = a$. However, our method of proof forces us to prove Theorem 2 first.

Remark 2. Theorem 2 extends the known result (cf. [5], Proposition 7.1) that $\mu(U) > 0$ if $U(0) < \lambda^{-1}$, where λ is the so-called connectivity constant (Section [5], p. 24). Wierman and Reh [7] (also [5], Theorem 7.3) showed that $\mu(U) = 0$ if $U(0) > p_T$ or if $U(0) = \frac{1}{2}$. It seems to be generally believed that $p_T = \frac{1}{2}$. If this is correct, then we have for any distribution U with finite mean that $\mu(U) = 0$ if and only if $U(0) \geqq \frac{1}{2}$. (See note on p. 863.)

Another immediate consequence of Theorem 1 concerns

(1.16) $N_n(c) = \sup\{l(r)$: r a self-avoiding path starting at the origin with $t(r) \leqq cn\}$.

Theorem 3. *If* (1.1) *and* (1.12) *hold, then*

$$P\left\{N_n(c) \geqq \frac{c}{a} n\right\} \leqq 2e^{-C_1 a^{-1} cn}, \tag{1.17}$$

where a and C_1 are as in (1.13) *and* $c \geqq 0$ *arbitrary.*

Remark 3. Smythe and Wierman [5] define

(1.18) $N_n^\theta = \min\{l(r)$: r is a self-avoiding path starting at the origin with $t(r) = \theta_{0,n}\}$,

$\theta = a, b, s$ or t, and show that

$$\limsup_{n\to\infty} \frac{1}{n} N_n^\theta < \infty \tag{1.19}$$

for $U(0) < \lambda^{-1}$ or $U(0) > p_H$ (see [5], Chapter 8; p_H is defined on p. 29 of [5]). They raise the question ([5], Section 10.5) whether (1.19) remains valid for $\lambda^{-1} \leqq U(0) \leqq p_H$. Theorem 3 answers this question affirmatively for $U(0) < p_T$. Again, one believes that $p_T = p_H$. If this is true only the case $U(0) = p_H$ is left open. (See note on p. 863.)

For $U(0) < \lambda^{-1}$ Smythe and Wierman [5], Theorem 8.2, actually prove

$$\mu^+(0) \leqq \liminf_{n\to\infty} \frac{1}{n} N_n^\theta \leqq \limsup \frac{1}{n} N_n^\theta \leqq \mu^-(0) < \infty \text{ w.p. } 1, \tag{1.20}$$

where $\mu^+(0)$ ($\mu^-(0)$) is the right (respectively left) derivative at 0 of the function $y \to \mu(U \oplus y)$ and

$$U \oplus y(x) = U(x - y)$$

($U \oplus y$ is the distribution U shifted y units to the right). From Theorem 3 one obtains the following corollary.

Corollary. *If* (1.1), (1.7) *and* (1.12) *hold then* (1.20) *remains valid for* $\theta = a$, *b, s, or t. Moreover*

$$\mu^-(0) \leqq \frac{1}{a}\mu(U). \tag{1.21}$$

We have been unable to prove that $\lim (1/n)N_n^\theta$ exists w.p. 1.

2. Proof of Theorem 2

Theorem 2 will follow from our next proposition.

Proposition 1. *If* (1.1) *and* (1.12) *hold, then there exist constants* $0 < b$, C_2, $C_3 < \infty$ *such that*

(2.1) $P\{\exists$ self-avoiding path r from the origin to (n, m) for some m with $t(r) \leqq bn\} \leqq C_2 e^{-C_3 n}$.

(2.1) implies (1.15) because $\theta_{0,n} < bn$ implies the existence of a path from $(0, 0)$ to (n, m) for some m with $t(r) < bn$. We need some additional notation:

$$H_n = \{v = (v', v'') \in Z \times Z : v' \geqq n\},$$

$A(v, n, x) = \{\exists$ self-avoiding path r from v to some point in H_n with $t(r) \leqq x\}$,

(Note that the left-hand side of (2.1) is just $P\{A((0, 0), n, bn)\}$.) For a fixed integer N

$$S(v) = S(v, N) = \{w = (w', w'') \in \mathbb{Z} \times \mathbb{Z} : |w' - v'| = N \text{ and } |w'' - v''| \leqq N \text{ or } |w' - v'| \leqq N \text{ and } |w'' - v''| = N\}.$$

$S(v)$ is the circumference of a square of edge length $2N$ centered at v. We denote its interior by $\mathring{S}(v, N)$. For fixed N and $\Delta > 0$ we set

(2.2) $f(N, \Delta) = P\{\exists$ self-avoiding path r from v to some point of $S(v, N)$ with $t(r) \leqq \Delta$ and such that all vertices of r except the final one belong to $\mathring{S}(v, N)\}$.

The principal estimate is given by the following.

Lemma 1. *If* $v' < n - N$, *then for all* $\Delta > 0$

$$P\{A(v, n, x)\} \leqq \sum_{w \in S(v,N)} P\{A(w, n, x - \Delta)\} + f(N, \Delta) \sum_{w \in S(v, N)} P\{A(w, n, x)\}. \tag{2.3}$$

Proof. Let

$$E = \bigcup_{w \in S(v)} A(w, n, x - \Delta).$$

Clearly the first term in the right-hand side of (2.3) is an upper bound for $P\{E\}$, so that we only have to estimate $P\{A \setminus E\}$. Assume that $A(v, n, x) \setminus E$ occurs and that $r = (v, e_1, v_1, \cdots, e_k, v_k)$ is a self-avoiding path from v to $v_k \in H_n$ with $t(r) \leqq x$. Since $v' < n - N$, $S(v, N) \cap H_n = \varnothing$ and the endpoint v_k of r must lie outside $S = S(v, N)$. Let a be the smallest index with $v_a \in S$ and b the largest index with $v_b \in S$. Also let $r_0 = (v, e_1, v_1, \cdots, e_a, v_a)$ (the initial piece of r) and $r_2 = (v_b, e_{b+1}, \cdots, e_k, v_k)$ (the final piece of r), and finally $r_1 = (v_a, e_{a+1}, \cdots, e_k, v_k)$. Then $v_a \in S$ and since E does not occur neither does $A(v_a, n, x - \Delta)$ and necessarily $t(r_1) > x - \Delta$. Together with $t(r) \leqq x$ this implies

$$t(r_0) = t(r) - t(r_1) \leqq \Delta. \tag{2.4}$$

Thus r_0 is a path from v to a point of S with $t(r_0) \leqq \Delta$ and all vertices but the last one in $\mathring{S}$. Furthermore, r_2 is a path from v_b to H_n with $t(r_2) \leqq t(r) \leqq x$, and by definition of b, r_2 has no vertex other than v_b in $S \cup \mathring{S}$. Thus we proved

(2.5) $P\{A(v, n, x) \setminus E\} \leqq \sum_{w \in S} P\{\exists$ a self-avoiding path r_0 from v to S with $t(r_0) \leqq \Delta$ and all vertices but the last one in $\mathring{S}$, and $\exists$ a path r_2 from w to H_n with $t(r_2) \leqq x$ and no vertex of r_2 besides w lies in $S \cup \mathring{S}\}$
$= f(N, \Delta) \sum_{w \in S} P\{\exists$ a path r_2 from w to H_n with $t(r_2) \leqq x$ and no vertex of r_2 besides w lies in $S \cup \mathring{S}\}$
$\leqq f(N, \Delta) \sum_{w \in S} P\{A(w, n, x)\}$.

The first equality in (2.5) follows from the fact that passage times of paths with all edges in $\mathring{S}$ and of paths with all edges in the exterior of S are independent. (2.3) is now immediate from (2.5).

The rest of the proof of Proposition 1 is now routine. Now set $S = S((0, 0), N)$ and let μ be the measure on $S \times \{0, \Delta\}$ given by

$$\mu(w, \Delta) = 1 \quad \text{if} \quad w \in S,$$
$$\mu(w, 0) = f(N, \Delta) \quad \text{if} \quad w \in S.$$

Then (2.3) can be rewritten as

$$P\{A(v, n, x)\} \leqq \sum_{(w,y) \in S \times \{0, \Delta\}} \mu(w, y) P\{A(v + w, n, x - y), \quad v' < n - N. \tag{2.6}$$

But clearly

$$P\{A(v, n, x)\} = 0 \quad \text{for} \quad x < 0$$

and

$$P\{A(v, n, x)\} \leqq 1 \quad \text{always.}$$

Thus, by iteration of (2.6) we obtain for any $k \geqq 1$, and $v=(0,0)$, $n>N$,

$$
\begin{aligned}
P\{A((0,0),n,x)\} &\leqq \sum_{w_1,y_1} \mu(w_1,y_1)P\{A(w_1,n,x-y_1)\} \\
&\leqq \sum_{\substack{w_1 \geqq n-N \\ y_1 \leqq x}} \mu(w_1,y_1) \\
&\quad + \sum_{\substack{w_1<n-N \\ y_1 \leqq x}} \mu(w_1,y_1)P\{A(w_1,n,x-y_1)\} \qquad (2.7) \\
&\leqq \cdots \leqq \sum_{j=1}^{k} \sum^{(j)} \prod_{i=1}^{j} \mu(w_i,y_i) \\
&\quad + \sum_{\substack{w_1'+\cdots+w_k<n-N \\ y_1+\cdots+y_k \leqq x}} \prod_{i=1}^{k} \mu(w_i,y_i).
\end{aligned}
$$

Here $\sum^{(j)}$ is the sum over $w_1,\cdots,w_j,y_1,\cdots,y_j$ with $w_1'+\cdots+w_{j-1}'<n-N$, $w_1'+\cdots+w_j' \geqq n-N$ and $y_1+\cdots+y_j \leqq x$. Of course all sums in (2.7) are also restricted to $(w_i,y_i) \in S \times \{0,\Delta\}$.

Next we show that N and Δ can be chosen such that

$$\sum_{w \in S((0,0),N)} \mu(w,0) \leqq \frac{3}{4}. \qquad (2.8)$$

To prove this, note that if there exists a path $r=((0,0),e_1,\cdots,e_k,v_k)$ from the origin to $S=S((0,0),N)$ which has all vertices but the last one in $\mathring{S}=\mathring{S}((0,0),N)$ and $t(r)=0$, then necessarily e_k is an edge between a point in $\mathring{S}$ and a point in S, and e_k is contained in the path r with $t(r)=0$. Thus

$$f(N,0) \leqq \sum_{\substack{e \text{ connects a point} \\ \text{of } \mathring{S} \text{ with a point of } S}} P\{\exists \text{ path } r \text{ starting at the origin and containing } e \text{ with } t(r)=0\}.$$

Consequently, by (1.10) and (1.11), if $U(0)<p_T$, then

$$
\begin{aligned}
\infty > V(U(0)) &= \sum_{e \in L} P\{\exists \text{ path } r \text{ starting at the origin and containing } e \text{ with } t(r)=0\} \\
&\geqq \sum_{N=0}^{\infty} f(N,0).
\end{aligned}
$$

In particular, we can fix $N>0$ such that

$$8Nf(N,0) = \sum_{w \in S((0,0),N)} f(N,0) \leqq \tfrac{1}{2}.$$

Since

$$f(N, \Delta) \downarrow f(N, 0) \quad \text{as} \quad \Delta \downarrow 0$$

we can next choose $\Delta > 0$ such that

$$\sum_{w \in S} f(N, \Delta) \leqq \tfrac{3}{4},$$

which is just (2.8).

Now

$$\varphi(\lambda) \equiv \sum_{(w,y) \in S \times \{0,\Delta\}} \mu(w, y) e^{-\lambda y} \leqq \tfrac{3}{4} + \sum_{w \in S} \mu(w, \Delta) e^{-\lambda \Delta} \leqq \tfrac{3}{4} + 4N e^{-\lambda \Delta},$$

and we can fix $\lambda > 0$ such that

$$\varphi(\lambda) \leqq \tfrac{7}{8}.$$

With these choices of N, Δ and λ

$$\sum_{\substack{w_i \in S \\ y_1 + \cdots + y_j \leqq x}} \prod_{i=1}^{j} \mu(w_i, y_i) \leqq e^{\lambda x} \sum_{\substack{(w_i, y_i) \in S \times \{0,\Delta\} \\ 1 \leqq i \leqq j}} \prod_{i=1}^{j} \{\mu(w_i, y_i) e^{-\lambda y_i}\}$$

$$\leqq e^{\lambda x} \{\varphi(\lambda)\}^j \leqq e^{\lambda x} (\tfrac{7}{8})^j.$$

If we also take into account that $w_j \in S$ implies $w_i' \leqq N$ so that $w_1' + \cdots + w_j' \geqq n - N$ can occur only for $j \geqq N^{-1} n - 1$, we obtain from (2.7)

$$P\{A((0,0), n, x)\} \leqq \lim_{k \to \infty} e^{\lambda x} \left\{ \sum_{(n/N) - 1 \leqq j \leqq k} (\tfrac{7}{8})^j + (\tfrac{7}{8})^k \right\}$$

$$\leqq 10 e^{\lambda x} (\tfrac{7}{8})^{n/N}, \quad n > N.$$

If we take $x = bn$ with b so small that

$$e^{\lambda b} (\tfrac{7}{8})^{1/N} = e^{-C_3}$$

for some $C_3 > 0$, then we obtain (2.1).

Remark 4. We proved above that (1.12) implies

$$8Nf(N, 0) < 1 \quad \text{for some} \quad N > 0. \tag{2.9}$$

Conversely, the proof of Proposition 1 goes through with the hypothesis (1.12) replaced by (2.9), and then the conclusion (2.1) of Proposition 1 directly implies $V(U(0)) < \infty$ and hence $U(0) \leqq p_T$. If (2.9) holds then it continues to hold even when $U(0)$ is slightly increased because for each fixed N, $f(N, 0)$ is a continuous function of $U(0)$. Thus (2.9) actually implies $U(0) < p_T$ and (1.12) and (2.9) are equivalent. The above argument also shows that the interval of p's with $V(p)<\infty$ is open on the right, so that $V(p_T)=\infty$.

3. Proof of Theorem 1

To estimate (1.13) we divide L into the $N\times N$ squares

$$S_{i,j}=\{v\in L: iN\leqq v'\leqq(i+1)N,\ jN\leqq v''\leqq(j+1)N\}.$$

N will be chosen later. Now consider a vertex v on the left boundary segment of $S_{i,j}$, i.e., $v'=iN$, $jN\leqq v''\leqq(j+1)N$, and let $s=(v,e_1,v_1,\cdots,e_k,v_k)$ be a self-avoiding path starting at v with length

$$l(s)=k\geqq(2N+1)(3N+1).$$

Since there are only $(2N+1)(3N+1)$ distinct vertices in the set

$$\tilde{S}_{i,j}=\bigcup_{|l-j|\leqq 1}(S_{i-1,l}\cup S_{i,l})$$

(which is a union of six of the $S_{i,j}$), there must be some $b\leqq k$ with v_b on the circumference of $\tilde{S}_{i,j}$. Let b be the smallest index with this property, and for the sake of definiteness let v_b lie on the right edge of $\tilde{S}_{i,j}$, i.e. $v_b'=(i+1)N$, $(j-1)N\leqq v_b''\leqq(j+1)N$. Next let a be the largest index less than b with $v_a'=iN$. Then $\tilde{s}=(v_a,e_{a+1},\cdots,v_b)$ is a self-avoiding path (it is part of s) which lies entirely in

$$R_{i,j}=\{v\in L: iN\leqq v'\leqq(i+1)N,\ (j-1)N\leqq v''\leqq(j+1)N\}.$$

Moreover, only its initial and final vertex lie on $\partial R_{i,j}$, the circumference of $R_{i,j}$, and these points lie on the opposite long sides of $\partial R_{i,j}$ ($R_{i,j}$ is an $N\times 3N$ rectangle). We call such a path $\tilde{s}$ a 'crossing of $R_{i,j}$'. Had v_b been on the left edge of $\tilde{S}_{i,j}$, i.e. $v_b'=(i-1)N$, $(j-1)N\leqq v_b''\leqq(j+1)N$, then $\tilde{s}$ would have been a crossing of $R_{i-1,j}$. Similarly, if v_b is in the top (bottom) edge of $\tilde{S}_{i,j}$ then we find that part of s is a crossing of the rectangle

$$T_{i,j+1}=\{v\in L:(i-1)N\leqq v'\leqq(i+1)N,\ (j+1)N\leqq v''\leqq(j+2)N\}$$

(respectively $T_{i,j-1}$).

Now set $M=(2N+1)(3N+1)$ and let $r=((0,0),e_1,v_1,\cdots,e_n,v_n)$ be a self-avoiding path of length $l(r)=n$ with $n\geqq M$, starting at the origin. Then some piece $\tilde{r}=(v_a,e_{a+1},\cdots,e_b,v_b)$ with $a<b\leqq M$ is a crossing of one of the rectangles $R_{0,0}$, $R_{-1,0}$, $T_{0,1}$, $T_{0,-1}$. Applying the above reasoning to the remaining part $(v_b,e_{b+1},\cdots,e_n,v_n)$ of r we see that some piece $\tilde{\tilde{r}}=(v_c,e_{c+1},\cdots,v_d)$ with $b\leqq c<d\leqq b+M$ is the next crossing of some rectangle R_{ij} or T_{ij}. Moreover given which rectangle was crossed by $\tilde{r}$, there are at most 19

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possibilities for the rectangle which is crossed by $\tilde{\tilde{r}}$. In this way we find that there is a string of $[nM^{-1}]$ rectangles $D_1, D_2, \cdots, D_K$, $K = [nM^{-1}]$, each D_l equal to some R_{ij} or T_{ij}, such that r successively crosses each one of the D_l. Also given D_l there are no more than 19 possibilities for D_{l+1} so that there are at most 19^K such strings. It follows that the left-hand side of (1.13) is bounded by

$$\sum_{D_1, \cdots, D_K} P\{\exists \text{ self-avoiding path } r = ((0,0), e_1, v_1, \cdots, e_n, v_n) \text{ which crosses } D_1, \cdots, D_K \text{ and with } t(r) \leqq an\}, \tag{3.1}$$

and the sum in (3.1) runs over at most 19^K possible strings. We now fix $D_1, \cdots, D_K$ and estimate the corresponding terms in (3.1). There are only $2(3N+1)$ possible initial points for a crossing of any given D_l. Moreover, since r is self-avoiding, all its crossings of a given D_l must be disjoint (i.e. pass through disjoint sets of vertices). Thus the same D_l can be crossed at most $(6N+2)$ times by r. In other words, any given rectangle D can occur at most $(6N+2)$ times among the D_l. Let $\gamma > 0$ (its precise value will be determined below) and let

ν = number of distinct D_l which occur at most γN times among the $D_1, \cdots, D_K$,
ρ = number of distinct D_l which occur more than γN times among the $D_1, \cdots, D_K$.

Then, by the above

$$\nu\gamma N + \rho(6N+2) \geqq K. \tag{3.2}$$

In addition each D_l intersects exactly 45 other rectangles R_{ij} or T_{ij}. Thus we can find at least $\nu_1 = \nu/46$ $(\rho_1 = \rho/46)$ disjoint D_l which occur at most γN (respectively more than γN) times among the $D_1, \cdots, D_K$. We now also fix subsets $\{D(i_1), \cdots, D(i_{\nu_1})\}$ and $\{D(j_1), \cdots, D(j_{\rho_1})\}$ of disjoint rectangles from the $D_1, \cdots, D_K$ with

$$\gamma\nu_1 + 8\rho_1 \geqq \frac{1}{46N} K, \tag{3.3}$$

and only consider those self-avoiding paths r which cross each $D(i_p)$ at most γN times and each $D(j_q)$ more than γN times. There are at most 3^K choices for these subsets, $\{i_1, \cdots, i_{\nu_1}\}$ and $\{j_1, \cdots, j_{\rho_1}\}$, since we already fixed $D_1, \cdots, D_K$ before.

So far the analysis has been purely deterministic. We now bring in the random passage times. For any rectangle R_{ij} or T_{ij}, D say, let $Y = Y(D)$ be the minimal passage time of any crossing of D. Then by Proposition 1

$$E\{e^{-Y}\} \leqq e^{-bN} + \sum_v P\{\exists \text{ crossing } s \text{ of } D \text{ which starts at } v \text{ and with } t(s) \leqq bN\} \leqq e^{-bN} + (6N+2)C_2 e^{-C_3 N}. \tag{3.4}$$

Indeed, as observed above there are at most $(6N+2)$ possible starting points v for crossings of D, and if for instance $D = R_{i,j}$ and v is on its left edge then a crossing of D starting at v is a self-avoiding path from v to some w with $w' = v' + N$. (3.4) will suffice for all the rectangles $D(i_p)$. For the rectangles $D(j_q)$ we need a stronger estimate. Let D be a rectangle as above and denote by $Z = Z(D)$ the minimum over all sets of γN disjoint crossings of D of the sum of the passage times of the γN crossings.

Formally

$$Z(D) = \min_{\substack{r_1, \cdots, r_{\gamma N} \\ \text{disjoint crossings of } D}} \sum_{j=1}^{\gamma N} t(r_j). \tag{3.5}$$

We shall prove below that for each fixed γ and Γ we have for all large N

$$E\{e^{-Z(D)}\} \leqq e^{-\Gamma N}. \tag{3.6}$$

Before proving (3.6) we show that Theorem 1 follows from (3.4) and (3.6). Indeed for each choice of $D(i_1), \cdots, D(i_{\nu_1})$ and $D(j_1), \cdots, D(j_{\rho_1})$ we have

$$P\{\exists \text{ self-avoiding path } r \text{ which crosses each } D(i_p) \text{ and crosses each } D(j_q) \text{ at least } \gamma N \text{ times and has } t(r) \leqq an\} \leqq e^{an} E\left\{\exp - \sum_{p=1}^{\nu_1} Y(D(i_p)) - \sum_{q=1}^{\rho_1} Z(D(j_q))\right\} \tag{3.7}$$

because

$$t(r) \geqq \sum_{p=1}^{\nu_1} Y(D(i_p)) + \sum_{q=1}^{\rho_1} Z(D(j_q))$$

for any r which crosses each $D(i_p)$ at least once and each $D(j_q)$ at least γN times. Since all $D(i_p)$ and all $D(j_q)$ are disjoint, all the

$$Y(D(i_p)), \quad 1 \leqq p \leqq \nu_1, \quad \text{and} \quad Z(D(j_q)), \quad 1 \leqq q \leqq \rho_1,$$

are independent. Thus, by (3.4) and (3.6) the right-hand side of (3.7) is at most

$$e^{an}[e^{-bN} + (6N+2)C_2 e^{-C_3 N}]^{\nu_1} e^{-\rho_1 \Gamma N}. \tag{3.8}$$

Now set $C_4 = \frac{1}{2} \min(b, C_3)$ and choose γ, Γ, N and a such that

$$60 \exp - \frac{C_4}{92\gamma} \leqq \frac{1}{2}, \quad \Gamma = \frac{8C_4}{\gamma}, \tag{3.9}$$

N so large that (3.6) holds as well as

$$e^{-bN}+(6N+2)C_2e^{-C_3N}\leqq e^{-C_4N},$$

and lastly

$$a=\frac{1}{4\gamma}\{(2N+1)(3N+1)\}^{-1}\frac{C_4}{46}\leqq\frac{1}{92\gamma}C_4\frac{K}{n}\quad(\text{for } n\geqq M).$$

With these choices we see for any ν_1 and ρ_1 satisfying (3.3) that (3.8) (and hence (3.7)) is bounded by

$$\exp\{an-N(C_4\nu_1+\Gamma\rho_1)\}=\exp\left\{an-\frac{C_4N}{\gamma}(\gamma\nu_1+8\rho_1)\right\}\leqq\exp-\frac{C_4}{92\gamma}K$$
$$\leqq\left(\frac{1}{120}\right)^K,\quad n\geqq M.$$

As observed before there are at most 19^K choices for $D_1,\cdots,D_K$, and given $D_1,\cdots,D_K$ there are at most 3^K choices for the subsets $D(i_1),\cdots,D(i_{\nu_1})$ and $D(j_1),\cdots,D(j_{\rho_1})$ so that (3.1), and hence the left-hand side of (1.13), is for $n\geqq M$ bounded by

$$19^K3^K(120)^{-K}\leqq 2^{-K}\leqq 2.2^{-n/M}.$$

Obviously this bound also holds for $n\leqq M$. Thus (1.13) will follow with

$$C_1=\frac{1}{M}\log 2=\{(2N+1)(3N+1)\}^{-1}\log 2$$

once we prove the following.

Lemma 2. *If* (1.1) *and* (1.12) *hold, then for each fixed* γ *and* Γ (3.6) *holds for all large* N.

Proof. Without loss of generality we take

$$D=\{v:0\leqq v'\leqq N, 0\leqq v''\leqq 3N\}.$$

Then

$$E\{e^{-Z(D)}\}\leqq e^{-2\Gamma N}+P\{Z(D)\leqq 2\Gamma N\}\tag{3.10}$$
$$\leqq e^{-2\Gamma N}+P\left\{\exists\ \gamma N \text{ disjoint crossings } r_1,\cdots,r_{\gamma N} \text{ of } D \text{ with } \sum_1^{\gamma N}t(r_i)\leqq 2\Gamma N\right\}.$$

To estimate the probability in the last member of (3.10) we proceed in a manner very similar to the beginning of the proof of Theorem 1. For some integer Q, which will be determined later, we divide D into the $Q\times Q$ squares

$$D_{i,j}=\{v:iQ\leqq v'\leqq(i+1)Q, jQ\leqq v''\leqq(j+1)Q\},$$
$$0\leqq i\leqq Q^{-1}N-1, 0\leqq j\leqq 3Q^{-1}N-1.$$

For convenience let us take $Q^{-1}N$ an integer, so that D equals the union of the $D_{i,j}$. Now let $r_1, \cdots, r_{\gamma N}$ be disjoint crossings of D. Since D contains only $3N(N+1)$ vertical edges, each of which can belong to at most one r_i, there are at least $\frac{1}{2}\gamma N$ of the r_i which contain $\leqq 12\gamma^{-1}N$ vertical edges. Number the r_i so that $r_1, \cdots, r_{(1/2)\gamma N}$ have at most $12\gamma^{-1}N$ vertical edges. Now consider any r_p, $p \leqq \frac{1}{2}\gamma N$ and let $r_p = (w_0, e_1, w_1, \cdots, e_l, w_l)$. For the sake of definiteness let $w_0(w_l)$ be on the left (respectively right) edge of D. Then r_p must cross each of the columns

$$C_i = \{v : iQ \leqq v' \leqq (i+1)Q, 0 \leqq v'' \leqq 3N\}$$

$$= \bigcup_{0 \leqq j < 3NQ^{-1}} D_{i,j}, \; 0 \leqq i < NQ^{-1}.$$

More precisely, for each such column i we can first find the minimal $b \leqq l$ with $w'_b = (i+1)Q$ and then the maximal $a < b$ with $w'_a = iQ$. Then $(w_a, e_{a+1}, \cdots, e_b, w_b)$ is a self-avoiding path from the left edge of C_i to the right edge of C_i, and $w_j \in C_i$ for all $a \leqq j \leqq b$. We call it a proper crossing of C_i if there exists a $Q \times (24\gamma^{-1}+1)Q$ rectangle

$$B_{i,j} = \{w : iQ \leqq w' \leqq (i+1)Q, jQ \leqq w'' \leqq (j+1+24\gamma^{-1})Q\}$$

in C_i which contains $(w_a, e_{a+1}, \cdots, e_b, w_b)$. Note that if $(w_a, \cdots, w_b)$ is not a proper crossing of C_i, then it must have more than $24\gamma^{-1}Q$ vertical edges, for if this is not true, then the highest and lowest values of $w''_a, \cdots, w''_b$ differ by at most $24\gamma^{-1}Q$ and $(w_a, \cdots, w_b)$ lies in some $B_{i,j}$. Now r_p must cross each of the NQ^{-1} columns $C_0, \cdots, C_{NQ^{-1}-1}$, and for $p \leqq \frac{1}{2}\gamma N$ r_p has at most $12\gamma^{-1}N$ vertical edges. Thus at least $\frac{1}{2}NQ^{-1}$ of the column crossings must be proper crossings. Altogether $r_1, \cdots, r_{(\frac{1}{2})\gamma N}$ have at least $\frac{1}{4}\gamma N^2 Q^{-1}$ proper crossings all of which are disjoint. One $B_{i,j}$ can contain at most $(24\gamma^{-1}+2)Q$ disjoint proper crossings since each proper crossing must contain one of the $(24\gamma^{-1}+1)Q+1$ points on the left edge of $B_{i,j}$. Consequently at least

$$\tfrac{1}{4}\gamma(24\gamma^{-1}+2)^{-1}Q^{-2}N^2$$

of the $B_{i,j}$ contain a proper crossing by one of the $r_1, \cdots, r_{\frac{1}{2}\gamma N}$. Also each $B_{i,j}$ intersects fewer than $3(48\gamma^{-1}+3)$ other $B_{k,l}$, so that there must at least be

$$\nu_0 \equiv \tfrac{1}{24}\gamma(24\gamma^{-1}+2)^{-2}Q^{-2}N^2$$

disjoint $B_{i,j}$ which contain at least one proper crossing. There are fewer than $NQ^{-1}3NQ^{-1} = 3Q^{-2}N^2$ rectangles $B_{i,j}$ in D, and thus at most

$$\binom{3Q^{-2}N^2}{\nu_0}$$

ways to select ν_0 disjoint $B_{i,j}$'s.

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As in the proof of Theorem 1 the above ends the deterministic part of the analysis. For given i,j, let $U=U(i,j)$ be the minimal passage time of any self-avoiding path in $B_{i,j}$ which starts on the left edge and ends on the right edge. Exactly as in (3.4)

$$E\{e^{-U}\}\leqq e^{-bQ}+(24\gamma^{-1}Q+2)C_2e^{-C_3Q} \leqq e^{-C_4Q}, \tag{3.11}$$

again with $C_4=\frac{1}{2}\min(b, C_3)$ and $Q\geqq Q_0$, where Q_0 depends on b, γ, C_2 and C_3 only. Thus, by the independence of the $U(i_p, j_p)$ for disjoint B_{i_p,j_p}.

$$P\Big\{\exists\ \gamma N \text{ disjoint crossing } r_1,\cdots, r_{\gamma N} \text{ of } D \text{ with } \sum_{i=1}^{\gamma N} t(r_i)\leqq 2\Gamma N\Big\}$$

$$\begin{aligned} &\leqq \sum\nolimits^{\nu_0} P\{\exists \text{ proper crossings of } B_{i_1,j_1},\cdots, B_{i_{\nu_0},j_{\nu_0}} \\ &\qquad\qquad \text{whose total passage time is } \leqq 2\Gamma N\} \\ &\leqq \sum\nolimits^{\nu_0} P\Big\{\sum\nolimits_1^{\nu_0} U(i_p, j_p)\leqq 2\Gamma N\Big\} \\ &\leqq \binom{3Q^{-2}N^2}{\nu_0} e^{2\Gamma N}e^{-C_4Q\nu_0} \end{aligned} \tag{3.12}$$

(compare (3.7)). Here $\sum^{\nu_0}$ is the sum over all sets of ν_0 pairs (i,j), $0\leqq i<NQ^{-1}$, $0\leqq j<3NQ^{-1}$ for which the B_{i_p,j_p}, $1\leqq p\leqq \nu_0$ are disjoint. The last member of (3.12) is at most

$$2^{3Q^{-2}N^2}e^{2\Gamma N}e^{-C_4Q\nu_0}\leqq e^{-2\Gamma N} \tag{3.13}$$

as soon as

$$C_4Q\nu_0\geqq 4\Gamma N+3Q^{-2}N^2,$$

or

$$\tfrac{1}{24}C_4\gamma(24\gamma^{-1}+2)^{-2}\geqq 4\Gamma QN^{-1}+3Q^{-1}. \tag{3.14}$$

Clearly (3.14) holds for

$$Q\geqq Q_1\equiv 4.24C_4^{-1}\gamma^{-1}(24\gamma^{-1}+2)^2$$

and $N\geqq 4\Gamma Q^2$. Now fix $Q\geqq \max(Q_0, Q_1)$. Then (3.6) follows for $N\geqq 4\Gamma Q^2$ from (3.10), (3.12) and (3.13).

4. Proof of Theorem 3 and corollary

Theorem 3 is immediate from (1.13) since $N_n(c) \geqq (c/a)n$ implies that there exists a self-avoiding path r of length at least $(c/a)n$ with passage time $t(r) \leqq cn = a\,.\,(c/a)n$.

As for the corollary, let $\theta^y_{0,n}\,.\;\theta = a$, b, t or s, denote the respective infima in (1.3)–(1.6) when $X(e)$ is replaced by $X(e)+y$. Then (1.20) follows in exactly the same way as in Theorem 8.2 of [5] once it is established that

$$\sup_{n \geqq 1} E\left\{\frac{1}{n} N^\theta_n\right\} < \infty \tag{4.1}$$

and

$$\lim \frac{1}{n}\,\theta^y_{0,n} = \mu(U \oplus y) > -\infty, \; -y_0 \leqq y < 0, \tag{4.2}$$

for some $y_0 > 0$ (see (1.18) for N^θ_n).

To prove (4.1) observe that on the set $\{\theta_{0,n} < axn\}$ any self-avoiding path r with $t(r) = \theta_{0,n}$ has length $\leqq N_n(ax)$ (see (1.16)). Thus, by (1.17)

$$\begin{aligned} P\{N^\theta_n \geqq xn\} &\leqq P\{\theta_{0,n} \geqq axn\} + P\{N_n(ax) \geqq xn\} \\ &\leqq P\{\theta_{0,n} \geqq axn\} + 2e^{-C_1 xn}. \end{aligned} \tag{4.3}$$

Consequently

$$E\left\{\frac{1}{n} N^\theta_n\right\} \leqq \frac{1}{a} E\left\{\frac{1}{n}\,\theta_{0,n}\right\} + 2C_1^{-1} \leqq \frac{1}{a}\int_0^\infty x\,dU(x) + 2C_1^{-1}.$$

(The last inequality follows from the fact that $\theta_{0,n} \leqq t(r_n)$, where r_n is the path along the x-axis from $(0, 0)$ to $(n, 0)$; compare [5], Theorem 7.4.) This proves (4.1).

As for (4.2), this follows as in [5], Chapter 5.4 with the following replacement of Lemma 5.14. Let

$$A^y = \sup\{(t(r) + yl(r))^- : r \text{ a self-avoiding path starting at } (0, 0)\}.$$

Then if (1.1) and (1.12) hold, and $-a \leqq y < 0$,

$$P\{A^y \geqq x\} \leqq 2\{1 - e^{-C_1}\}^{-1} e^{-C_1 x|y|^{-1}}, \tag{4.4}$$

and A^y has all moments. To obtain (4.4) observe that $t(r) \geqq 0$ so that

$$\{A^y \geqq x\} \subset \left\{\exists \text{ self-avoiding path } r \text{ starting at } (0, 0) \text{ with } l(r) \geqq \frac{x}{|y|} \right.$$
$$\left. \text{and } t(r) \leqq -x + |y|\, l(r)\right\}$$
$$\subset \bigcup_{n \geqq c|y|^{-1}} \{\exists \text{ self-avoiding path } r \text{ starting at } (0, 0) \text{ with } l(r) = n \text{ and } t(r) \leqq an\}.$$

Thus (4.4) follows from (1.13).

We omit further details of (1.20). Finally we prove (1.21). Observe that if

$$\frac{-\varepsilon}{2} \leqq y < 0 \quad \text{and} \quad \varepsilon < \min\{a, a^{-2}\} \tag{4.5}$$

then

$$t(r) > \max\{\mu(U)(1+\varepsilon^2 y)n, al(r)\} \quad \text{and} \quad t(r) + yl(r) \leqq \mu(U)\left(1 + \frac{y}{a-\varepsilon}\right)n$$

cannot occur simultaneously. Therefore, if (4.5) holds

$$\begin{aligned} &P\left\{\theta^y_{0,n} \leqq \mu(U)\left(1+\frac{y}{a-\varepsilon}\right)n\right\} \\ &\leqq P\left\{\exists \text{ self-avoiding path } r \text{ from } (0,0) \text{ to } (n,m) \text{ with } l(r) \geqq n \text{ and} \right. \\ &\qquad \left. t(r) + yl(r) \leqq \mu(U)\left(1+\frac{y}{a-\varepsilon}\right)\right\} \\ &\leqq P\{\exists \text{ self-avoiding path } r \text{ from } (0,0) \text{ to} (n,m) \text{ for some } m, \text{ with} \\ &\qquad t(r) \leqq \mu(U)(1+\varepsilon^2 y)n\} \\ &\quad + \sum_{l=n}^{\infty} P\{\exists \text{ self-avoiding path } r \text{ with } l(r) = l \text{ and } t(r) \leqq al\} \\ &\leqq P\left\{\frac{1}{n} b_{0,n} \leqq \mu(U)(1-\varepsilon^2|y|)\right\} + 2\{1-e^{-C_1}\}^{-1} e^{-C_1 n} \text{ (by (1.13)).} \end{aligned} \tag{4.6}$$

By virtue of (1.8) the last member of (4.6) tends to 0, so that

$$\mu(U \oplus y) = \lim_{n\to\infty} \frac{1}{n} \theta^y_{0,n} \geqq \mu(U)\left(1+\frac{y}{a-\varepsilon}\right) \tag{4.7}$$

whenever (4.5) holds. (1.21) is immediate from (4.7).

Note added in proof. It has now been proved that indeed $P_T = P_H = \frac{1}{2}$. See Kesten, H. (1980) *Commun. Math. Phys.* **24,** 41–59.

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